Separability of symmetric states and moment problem

Marcin Marciniak
(joint work with Adam Rutkowski and Michał Banacki)

Institute of Theoretical Physics and Astrophysics
University of Gdańsk

Mathematical Aspects in Current Quantum Information Theory 2019
Seoul, Korea
May 20 – 24, 2019

Contents

1. Introduction
   - (bosonic) symmetric states
   - separability, PPT property
   - Dicke states, states diagonal in Dicke states
   - Known results
2. D-symmetry
   - D-symmetric states
   - restricted Dicke states and states diagonal in restricted Dicke states
   - separable D-symmetric states
   - entanglement witnesses for D-symmetric systems
3. Moment problem
4. Results
   - PPT property vs moment problem
   - Separability vs moment problem
5. Generalized CCR relation of Bożejko-Speicher type

[A. Rutkowski, M. Banacki, M. M., Phys. Rev A 99 (2019)]

Symmetric states for N qudits

- Let $H = \mathbb{C}^d$ and let us fix a basis $|0\rangle, |1\rangle, \ldots, |d-1\rangle$.
- Symmetrizer $P_{\mathrm{S}} \in B\left(H^{\otimes N}\right)$

$$P_{\mathrm{S}}|\mathbf{i}\rangle = \frac{1}{N!} \sum_{\sigma \in S_N} |\sigma(\mathbf{i})\rangle$$

$$|\mathbf{i}\rangle := |i_1, i_2, \ldots, i_N\rangle, \quad i_1, i_2, \ldots, i_N \in \{0, 1, \ldots, d-1\}.$$

$$|\sigma(\mathbf{i})\rangle := |i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \ldots, i_{\sigma^{-1}(N)}\rangle$$

- Permutationally symmetric state ρ

$$\sigma\rho = \rho\sigma, \qquad \sigma \in S_N$$

- (Bosonic) symmetric state ρ

$$P_{\mathrm{S}}\rho = \rho P_{\mathrm{S}}$$

Problem of separability of symmetric states

In the past decade, the problem of separability of permutationally symmetric states has been intensively analyzed
[O. Guhne and G. Toth, Phys. Rep. 474, 1 (2009)]
[G. Toth and O. Guhne, Phys. Rev. Lett. 102, 170503 (2009)]
[G. Toth and O. Guhne, Appl. Phys. B 98, 617 (2010)]
[E. Wolfe and S. F. Yelin, Phys. Rev. Lett. 112, 140402 (2014)]
[N. Yu, Phys. Rev. A 94, 060101(R) (2016)]

Dicke states for qubits

- Fix $d = 2$.
- Basis of Dicke (unnormalized) states:

$$|D_{N;k}\rangle := \binom{N}{k} P_{\mathrm{S}} |\underbrace{0,\ldots,0}_{N-k},\underbrace{1,\ldots,1}_{k}\rangle, \quad k = 0, 1, \ldots, N$$

[Wolfe at al. (2014), Yu (2016)]

- It has been observed by several authors that there is a strong connection between separability and the PPT property for mixtures of Dicke states.

[Toth at al. (2009), Wolfe at al. (2014)]

Separability and PPT property

- $\rho \in B(H_1) \otimes \ldots \otimes B(H_N)$ a (nonnormalized) state i.e. positive semidefinite, $\mathrm{Tr}\rho = 1$ (but not necessarily)
- (Full) sperability

$$\rho = \sum_\alpha \lambda_\alpha \rho_\alpha^1 \otimes \ldots \otimes \rho_\alpha^N$$

$$\rho_\alpha^i \in B(H_i), \qquad \lambda_\alpha \geq 0$$

- $(m_1, \ldots, m_N)$-PPT property

$$(T_1^{m_1} \otimes \ldots \otimes T_N^{m_N})\rho \quad \text{is a state,}$$

  - T_j the transposition on $B(H_j)$,
  - $(m_1, \ldots, m_N) \in \{0, 1\}^N$
  - $T_j^0 = \mathrm{id}_j$ and $T_j^1 = T_j$, i.e. all 1's in the system $(m_1, \ldots, m_N)$ mark subsytems which are transposed.

Separability and PPT property

- Clearly, if a state ρ is separable then it has a $(m_1, \ldots, m_n)$-PPT property for every binary system $(m_1, \ldots, m_n)$.
- In general, the converse implication is not true unless $N = 2$ and the pair (H_1, H_2) is one of the following: $(\mathbb{C}^2, \mathbb{C}^2), (\mathbb{C}^2, \mathbb{C}^3), (\mathbb{C}^3, \mathbb{C}^2)$.
- In spite of this general statement, there are classes of states such that the PPT property implies separability within them

States diagonal in Dicke basis (for qubits)

- Dicke diagonal states

$$\rho_{(p_k)} = \sum_{k=0}^{N} p_k |D_{N;k}\rangle\langle D_{N;k}|, \qquad p_k \geq 0$$

- For fixed $m := m_1 + \ldots + m_N$ all $(m_1, \ldots, m_N)$-PPT conditions are equivalent for symmetric states. Thus, it is enough to consider only PPT conditions with first m subsystems transposed, where $m \leq \lfloor N/2 \rfloor$, denoted by m-PPT.

Theorem (Yu (2016))

Let $(p_k)_{0 \leq k \leq N}$ be a sequence of non-negative numbers. Then the following conditiions are equivalent:

1. *The state $\rho_{(p_k)}$ is separable*
2. *The state $\rho_{(p_k)}$ has $\lfloor N/2 \rfloor$-PPT property.*

Qudit case - stright generalization

- Dicke states for qudits, $d \geq 2$ arbitrary

$$|D_{N,d;k_0,k_1,\ldots,k_{d-1}}\rangle = \binom{N}{k_0,\ldots,k_{d-1}} P_S \left(|0\rangle^{\otimes k_0} \otimes \ldots \otimes |d-1\rangle^{\otimes k_{d-1}} \right)$$

$$k_i \geq 0, \qquad k_0 + k_1 + \ldots + k_{d-1} = N$$

[T.-C. Wei at al., Quantum Inf. Comput. 4, 252 (2004)]
[N. Ananth and M. Senthilvelan, Int. J. Theor. Phys. 55, 1854 (2016)]
[J. Tura at al., Quantum 2, 45 (2018)]

- Dicke diagonal states for qudits

$$\rho = \sum p_{k_0,\ldots,k_{d-1}} |D_{N,d;k_0,\ldots,k_{d-1}}\rangle\langle D_{N,d;k_0,\ldots,k_{d-1}}|$$

In general, PPT does not imply separability.
[Tura at al. (2018)]

D-symmetry of states

- D-binomial coefficients

$$\mathbf{i} = (i_1, \ldots, i_N), \quad 0 \le i_1, \ldots, i_n \le d-1, \quad |\mathbf{i}| = i_1 + \ldots + i_N$$

$$\binom{N}{k}_d := \#\{\mathbf{i} : |\mathbf{i}| = k\}, \quad 0 \le k \le N(d-1), \quad \binom{N}{k}_2 = \binom{N}{k}$$

Generalized property of binomial coefficients

$$\binom{N}{k}_d = \sum_{j=0}^{\min\{k,d-1\}} \binom{N-1}{k-j}_d$$

D-symmetry of states

- D-symmetrizer

$$P_{\mathrm{D}}|\mathbf{i}\rangle = \binom{N}{|\mathbf{i}|}_d^{-1} \sum_{\mathbf{j}\colon |\mathbf{j}|=|\mathbf{i}|} |\mathbf{j}\rangle$$

- P_{D} is a projection.
- $P_{\mathrm{D}} P_{\mathrm{S}} = P_{\mathrm{S}} P_{\mathrm{S}} = P_{\mathrm{D}}$,
- $P_{\mathrm{D}}\left((\mathbb{C}^d)^{\otimes N}\right) \subset P_{\mathrm{S}}\left((\mathbb{C}^d)^{\otimes N}\right)$, i.e. D-symmetric vectors are permutationally symmetric.
- D-symmetric states

$$\rho P_{\mathrm{D}} = P_{\mathrm{D}} \rho$$

- Restricted Dicke states

$$|R_{N,d;k}\rangle = |R_k\rangle := \sum_{i_1+i_2+\ldots+i_N=k} |i_1, i_2, \ldots, i_N\rangle$$

Restricted Dicke states

$$|R_k\rangle := \sum_{i_1+i_2+\ldots+i_N=k} |i_1, i_2, \ldots, i_N\rangle$$

Assume that a system is composed of N bosons with d levels of excitation each. We make an assumption that subsequent levels differ by a fixed value. Then $|R_{N,d;k}\rangle$ can be interpreted as such a state of the system that the total number of excitations in all bosons is equal to k. It can be used to model systems of bosons concentrated in a small area which behave as single particle and only total energy can be recognized. Such models were used to explain the notion of superradiance in quantum optics.

[R. H. Dicke, Phys. Rev. 93, 99 (1954)]

[M. Gross and S. Haroche, Phys. Rep. 93, 301 (1982)]

Restricted Dicke diagonal states

$$\rho_{(p_k)} = \sum_{k=0}^{N(d-1)} p_k |R_{N,d;k}\rangle\langle R_{N,d;k}| \qquad p_0, p_1, \ldots, p_{N(d-1)} \geq 0.$$

Problem

What is the relationship between PPT property and separability for resrticted Dicke diagonal states?

Separable S-symmetric states

$$\rho = \sum_{\alpha} \lambda_\alpha \rho_\alpha^1 \otimes \ldots \otimes \rho_\alpha^N, \qquad \rho_\alpha^i = |\xi_\alpha^i\rangle\langle\xi_\alpha^i|$$

Proposition

Assume that ρ is symmetric, i.e. $\rho = P_{\mathrm{S}}\rho P_{\mathrm{S}}$. If all coefficients λ_α are strictly positive then $\rho_\alpha^i = \rho_\alpha^j$ for every $\alpha = 1, \ldots, n$ and $i, j = 1, \ldots, N$.

Can assume $|\xi_\alpha^i\rangle = |\xi_\alpha^j\rangle$ for $i, j = 1, \ldots, N$.

Separable D-symmetric states

$$\rho = \sum_\alpha \lambda_\alpha \rho_\alpha \otimes \ldots \otimes \rho_\alpha, \qquad \rho_\alpha = |\xi_\alpha\rangle\langle\xi_\alpha|$$

Proposition

Assume that ρ is D-symmetric, i.e. $\rho = P_{\mathrm{D}}\rho P_{\mathrm{D}}$. Then for each $\alpha = 1, \ldots, n$, either

$$|\xi_\alpha\rangle = |d-1\rangle$$

or there is a number $z \in \mathbb{C}$ such that

$$|\xi_\alpha\rangle = C_z \sum_{i=0}^{d-1} z^i |i\rangle,$$

where C_z is a normalization.

Entanglement witnesses for D-symmetric systems

Definition

A Hermitian operator $W \in B((\mathbb{C}^d)^{\otimes N})$ *is an entaglement witness for the D-symmetric system if*

1. $W = P_{\mathrm{D}} W P_{\mathrm{D}}$
2. $\mathrm{Tr}(W\sigma) \geq 0$ *for all pure separable D-symmetric states*

Proposition

A D-symmetric state ρ *is separable if and only if* $\mathrm{Tr}(W\rho) \geq 0$ *for every entanglement witness* W *for the D-symmetric system.*

A simple consequence of the hyperplane separation theorem.

[Yu (2016)]

Entanglement witnesses for D-symmetric systems

$$|\widetilde{R_k}\rangle = \binom{N}{k}_d^{-1} \sum_{|\mathbf{i}|=k} |\mathbf{i}\rangle = \binom{N}{k}_d^{-1} |R_k\rangle, \qquad \langle \widetilde{R_k} | R_l \rangle = \delta_{kl}.$$

Proposition

Let $n_1 = \left\lfloor \frac{N(d-1)}{2} \right\rfloor$ *and* $n_2 = \left\lfloor \frac{N(d-1)-1}{2} \right\rfloor$. *Let two systems* $(s_k)_{0\le k\le n_1}$ *and* $(t_k)_{0\le k\le n_2}$ *of complex numbers be given. Define*

$$V_{(s)} = \sum_{k,l=0}^{n_1} s_k \overline{s_l} |\widetilde{R_{k+l}}\rangle\langle\widetilde{R_{k+l}}|$$

$$U_{(t)} = \sum_{k,l=0}^{n_2} t_k \overline{t_l} |\widetilde{R_{k+l+1}}\rangle\langle\widetilde{R_{k+l+1}}|.$$

Then $V_{(s)}$ *and* $U_{(t)}$ *are entanglement witnesses for D-symmetric systems.*

Moment problem – definition

Definition

Let $(p_k)_{k=0}^{n}$ be a finite sequence of real numbers. We say that the sequence (p_k) is a solution of the generalized moment problem on the interval $[0, \infty)$ if there exists a positive measure σ with support contained in $[0, \infty)$ such that

$$p_k = \begin{cases} \int_0^\infty t^k d\sigma(t), & k = 0, 1, \ldots, n-1, \\ \int_0^\infty t^n d\sigma(t) + M, & k = n, \end{cases}$$

where $M \geq 0$. Alternatively, we say that it is a solution of the strict moment problem on the interval $[0, \infty)$ if it is a solution of the generalized moment problem with $M = 0$.

[M. G. Krein and A. A. Nudelman, The Markov Moment Problem and Extremal Problems (AMS, Providence, RI, 1977), Russian ed. in 1973]

Hankel matrices

$$n_0 = \left\lfloor \frac{n}{2} \right\rfloor, \qquad n_1 = \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$(p_{k+l})_{k,l=0}^{n_0} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{n_0} \\ p_1 & p_2 & p_3 & \cdots & p_{n_0+1} \\ p_2 & p_3 & p_4 & \cdots & p_{n_0+2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n_0} & p_{n_0+1} & p_{n_0+2} & \cdots & p_{2n_0} \end{pmatrix},$$

$$(p_{k+l+1})_{k,l=0}^{n_1} = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{n_1+1} \\ p_2 & p_3 & p_4 & \cdots & p_{n_1+2} \\ p_3 & p_4 & p_5 & \cdots & p_{n_1+3} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n_1+1} & p_{n_1+2} & p_{n_1+3} & \cdots & p_{2n_1+1} \end{pmatrix}$$

Moment problem – characterization

Theorem

A sequence $(p_k)_{k=0}^{n}$ *is a solution of the generalized moment problem if and only if both Hankel matrices* $(p_{k+l})_{k,l=0}^{n_0}$ *and* $(p_{k+l+1})_{k,l=0}^{n_1}$ *are positive semidefinite. If both matrices are strictly positive definite then the sequence is a solution of the strict moment problem.*

[M. G. Krein and A. A. Nudelman, The Markov Moment Problem and Extremal Problems (AMS, Providence, RI, 1977)]

Example: $n = 9$, $n_0 = 4$, $n_1 = 4$

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 & p_4 \\ p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2 & p_3 & p_4 & p_5 & p_6 \\ p_3 & p_4 & p_5 & p_6 & p_7 \\ p_4 & p_5 & p_6 & p_7 & p_8 \end{pmatrix} \quad \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2 & p_3 & p_4 & p_5 & p_6 \\ p_3 & p_4 & p_5 & p_6 & p_7 \\ p_4 & p_5 & p_6 & p_7 & p_8 \\ p_5 & p_6 & p_7 & p_8 & p_9 \end{pmatrix}$$

Restricted Dicke diagonal states with PPT

Theorem

Let $m \leq N/2$. *The state* $\rho_{(p_k)}$ *is* m*-PPT if and only if*

(a) *matrices* $(p_{i+j})_{i,j=0}^{m(d-1)}$ *and* $(p_{i+j+1})_{i,j=0}^{m(d-1)-1}$ *are positive definite, when* $N = 2m$,

(b) *matrices* $(p_{i+j+l})_{i,j=0}^{m(d-1)}$, $l = 0, \ldots, (N-2m)(d-1)$, *are positive definite, when* $2m < N$.

Restricted Dicke diagonal states with PPT

Theorem

Let $m \leq N/2$. The state $\rho_{(p_k)}$ is m-PPT if and only if

(a) *matrices $(p_{i+j})_{i,j=0}^{m(d-1)}$ and $(p_{i+j+1})_{i,j=0}^{m(d-1)-1}$ are positive definite, when $N = 2m$,*

(b) *matrices $(p_{i+j+l})_{i,j=0}^{m(d-1)}$, $l = 0, \ldots, (N-2m)(d-1)$, are positive definite, when $2m < N$.*

Example: $d = 3$, $N = 3$, $m = 1$

$$\rho_{(p_k)} = \sum_{k=0}^{6} p_k |R_k\rangle\langle R_k|$$

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{pmatrix} \quad \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \\ p_3 & p_4 & p_5 \end{pmatrix} \quad \begin{pmatrix} p_2 & p_3 & p_4 \\ p_3 & p_4 & p_5 \\ p_4 & p_5 & p_6 \end{pmatrix}$$

Restricted Dicke diagonal states with PPT

Theorem

Let $m \leq N/2$. The state $\rho_{(p_k)}$ is m-PPT if and only if

(a) *matrices $(p_{i+j})_{i,j=0}^{m(d-1)}$ and $(p_{i+j+1})_{i,j=0}^{m(d-1)-1}$ are positive definite, when $N = 2m$,*

(b) *matrices $(p_{i+j+l})_{i,j=0}^{m(d-1)}$, $l = 0, \ldots, (N-2m)(d-1)$, are positive definite, when $2m < N$.*

Example: $d = 3$, $N = 4$, $m = 2$, $\rho_{(p_k)} = \sum_{k=0}^{9} p_k |R_k\rangle\langle R_k|$

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 & p_4 \\ p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2 & p_3 & p_4 & p_5 & p_6 \\ p_3 & p_4 & p_5 & p_6 & p_7 \\ p_4 & p_5 & p_6 & p_7 & p_8 \end{pmatrix} \quad \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2 & p_3 & p_4 & p_5 & p_6 \\ p_3 & p_4 & p_5 & p_6 & p_7 \\ p_4 & p_5 & p_6 & p_7 & p_8 \\ p_5 & p_6 & p_7 & p_8 & p_9 \end{pmatrix}$$

PPT vs moment problem

Corollary

Assume that N is even and let $(p_k)_{0 \leq k \leq N(d-1)}$ be a sequence of nonnegative numbers. The following are equivalent:

1. $\rho_{(p_k)}$ *is* $N/2$*-PPT,*
2. *The sequence* (p_k) *is a solution of generalized moment problem*

Moreover, if $d = 2$ and N is odd, then the following are equivalent

1. $\rho_{(p_k)}$ *is* $(N-1)/2$*-PPT,*
2. *The sequence* (p_k) *is a solution of generalized moment problem*

PPT vs moment problem

$d = 2,\ N = 5,\ \rho_{(p_k)} = \sum_{k=0}^{5} p_k |R_k\rangle\langle R_k|$

- 2-PPT:

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{pmatrix} \quad \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \\ p_3 & p_4 & p_5 \end{pmatrix}$$

- Moment problem: $n = 5,\ n_0 = 2,\ n_1 = 2.$ The above matrices are precisely the two Hankel matrices from the theorem.

Separability vs moment problem

Proposition

If $(p_k)_{k=0,1,\dots,N(d-1)}$ *is a geometric sequence then* $\rho_{(p_k)}$ *separable.*

Proof. Let $p_k = t^k$ for some $t > 0$.

$$\omega = \exp\left(\frac{2\pi i}{N(d-1)+1}\right)$$

$$|\hat{\alpha}\rangle = \sum_{j=0}^{d-1} t^{j/2}\omega^{\alpha j}|j\rangle, \quad \alpha = 0, 1, \dots, N(d-1).$$

Then

$$\rho_{(t^k)} = \frac{1}{N(d-1)+1} \sum_{\alpha=0}^{N(d-1)} |\hat{\alpha}\rangle\langle\hat{\alpha}|^{\otimes N}$$

Separability vs moment problem

Theorem

Let $d \geq 2$ and N be arbitrary. The state $\rho_{(p_k)}$ is fully separable if and only the sequence $(p_k)_{k=0}^{N(d-1)}$ is a solution of the generalized moment problem.

Proof of necessity. Since $\rho_{(p_k)}$ is separable, $\mathrm{Tr}(\rho_{(p_k)} W) \geq 0$ for every entanglement witness for D-symmetric systems. In particular, for any sequence $(s_k)_{0 \leq k \leq \lfloor N(d-1)/2 \rfloor}$

$$\sum_{k,l=0}^{\lfloor N(d-1)/2 \rfloor} s_k \overline{s_l} p_{k+l} = \mathrm{Tr}(\rho_{(p_k)} V_{(s)}) \geq 0$$

what means that $(p_{k+l})_{0 \leq k,l \leq n_0}$ is positive semidefinte. Similarly, for the second Hankel matrix. Hence (p_k) is a solution of moment problem.

Separability vs moment problem

Theorem

Let $d \geq 2$ and N be arbitrary. The state $\rho_{(p_k)}$ is fully separable if and only the sequence $(p_k)_{k=0}^{N(d-1)}$ is a solution of the generalized moment problem.

Proof of sufficiency. Since (p_k) is a solution of the generalized moment problem, there are a positive measure σ supported on $[0, \infty)$ and $M \geq 0$ such that

$$p_k = \int_0^\infty t^k d\sigma(t) + \delta_{k,N(d-1)} M.$$

Then

$$\rho_{(p_k)} = \int_0^\infty \rho_{(t^k)} d\sigma(t) + M|R_{N(d-1)}\rangle\langle R_{N(d-1)}|.$$

$|R_{N(d-1)}\rangle\langle R_{N(d-1)}| = |d-1\rangle\langle d-1|^{\otimes N}$, so it is separable. According to Proposition from the previous slide, each $\rho_{(t^k)}$ is also a separable state. Consequently, $\rho_{(p_k)}$ is separable too.

Main result

Theorem (Rutkowski, Banacki, M.)

Assume that $d \geq 2$ *is arbitrary and* N *is even. Let* $(p_k)_{0\leq k\leq N(d-1)}$ *be a sequence of nonnegative numbers. The following conditions are equivalent:*

(a) $\rho_{(p_k)}$ *is fully separable.*

(b) $\rho_{(p_k)}$ *is* $N/2$*-PPT*

(c) *The sequence* (p_k) *is a solution of the generalized moment problem.*

Moreover, if $d = 2$ *and* N *is odd the following conditions are equivalent:*

(a) $\rho_{(p_k)}$ *is fully separable.*

(b) $\rho_{(p_k)}$ *is* $(N-1)/2$*-PPT*

(c) *The sequence* (p_k) *is a solution of the generalized moment problem.*

Let us note that for $d = 2$, i.e. for qubits, the above equivalence was proved in [Yu (2016)].

The case $d \geq 3$ and N odd

On the contrary to the case $d = 2$, if N is odd then $\frac{N-1}{2}$-PPT property does not imply separability of $\rho_{(p_k)}$ for $d \geq 3$.
Let $N = 3$ and $d = 3$ and let

$$(p_k)_{0 \leq k \leq 6} = (1, 1/4, 1/8, 1/9, 1/8, 1/4, 1)\,.$$

ρ is a 1-PPT state. Indeed, one can easily check that matrices

$$\begin{pmatrix} 1 & 1/4 & 1/8 \\ 1/4 & 1/8 & 1/9 \\ 1/8 & 1/9 & 1/8 \end{pmatrix} \quad \begin{pmatrix} 1/4 & 1/8 & 1/9 \\ 1/8 & 1/9 & 1/8 \\ 1/9 & 1/8 & 1/4 \end{pmatrix} \quad \begin{pmatrix} 1/8 & 1/9 & 1/8 \\ 1/9 & 1/8 & 1/4 \\ 1/8 & 1/4 & 1 \end{pmatrix}$$

are positive semidefinite. On the other hand the determinant of the Hankel matrix ($n_0 = 3$)

$$(p_{k+l})_{0 \leq k,l \leq 3} = \begin{pmatrix} 1 & 1/4 & 1/8 & 1/9 \\ 1/4 & 1/8 & 1/9 & 1/8 \\ 1/8 & 1/9 & 1/8 & 1/4 \\ 1/9 & 1/8 & 1/4 & 1 \end{pmatrix}$$

is negative, hence it is not positive semidefinite.

q-CCR relations

- Interpolating family of relations

$$a : \mathcal{H} \to B(H) \text{ antilinear}, \qquad q \in [-1, 1]$$

$$a(f)a(g)^\dagger - qa(g)^\dagger a(f) = \langle f|g\rangle \mathbb{I}$$

[M. Bożejo, R. Speicher, Comm. Math. Phys. 137 (1991)]

- q-statistics: $q = 1$ bosonic, $q = -1$ fermionic, $q = 0$ Boltzman statistics
- Noncommutative probability: $q = 1 \;\to$ classical independence, $q = 0 \;\to$ freeness,
  q-probability for general q (CLT, Poisson limit theorem, q-Brownian processes)
  [M. Bożejo, B. Kummerer, R. Speicher, Comm. Math. Phys. 185 (1997)]

q_{ij}-CCR relations

- Generalizations of q-relations: $q_{ij} \in [-1,1]$, $i,j = 1,\ldots,d$

$$|f_1\rangle, |f_2\rangle, \ldots, |f_d\rangle \qquad \text{orthonormal basis of } \mathcal{H}$$

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij} \mathbb{I}$$

[M. Bożejo, B. Kummerer, R. Speicher, Comm. Math. Phys. 185 (1997)]

q-CCR – second quantization

- q-relations are obtained by the second quantization procedure. [O. Bratteli, D. Robinson, Operator Algebras and Quantum Statistical Mechanics, vol. 2, Springer, 2003]
- q-deformed Fock space $\mathcal{F}_q(\mathcal{H}) = \bigoplus_{N=0}^{\infty} \mathcal{H}^{\otimes N}$

$$\langle f_1 \otimes \ldots \otimes f_N | g_1 \otimes \ldots \otimes g_N \rangle_q = \sum_{\sigma \in S_n} q^{l(\sigma)} \prod_{i=1}^{N} \langle f_i | q_{\sigma(i)} \rangle$$

- q-symmetrizer $P_q = \bigoplus_{N=0}^{\infty} P_q^{(N)}, \qquad P_q^{(N)} : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes N}$

$$P_q^{(N)} | f_1 \otimes \ldots \otimes f_N \rangle = \sum_{\sigma \in S_n} q^{l(\sigma)} | f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(N)} \rangle$$

$$\langle \xi | \eta \rangle_q = \langle \xi | P_q \eta \rangle_0$$

D-CCR relations

$$C_N > 0, \quad P_D^{\infty} = \sum_{N=0}^{\infty} C_N P_D^{(N)}, \quad \mathcal{F}_{\mathrm{D}}(\mathbb{C}^d) = P_D^{\infty}\left(\mathcal{F}(\mathbb{C}^d)\right)$$

Second quantization: $a_i^{\dagger} = P_{\mathrm{D}}^{\infty} b_i^{\dagger} P_{\mathrm{D}}^{\infty}$ where $b_i|i_1, \ldots, i_N\rangle = |i, i_1, \ldots, i_N\rangle$.

Theorem (RBM)

There are

- *a sequence* $(C_N)_{N\geq 0}$ *of positive numbers*
- *numbers* $q_{ij}(d) \in [-1, 1]$, $0 \leq i, j \leq d-1$
- *an invertible operator* $J : \mathcal{F}_{\mathrm{D}}(\mathbb{C}^d) \to \mathcal{F}_{\mathrm{D}}(\mathbb{C}^d)$

such that

$$a_i a_j^{\dagger} - q_{i,j}(d) a_j^{\dagger} a_i = \delta_{ij} J$$

Possible ways to proceed: D-statistics (?), D-probaility: CLT, D-Gaussian (?)

Conclusions

- We introduced the notion of D-symmetry for multipartite states which is stronger then bosonic symmetry.
- We considered D-symmetric analogs of Dicke states: restricted Dicke states.
- We proved that for even number N of systems $N/2$-PPT property is equivalent to separability. It was done using classical results on moment problem.
- We constructed concrete model satisfying q_{ij}-CCR relations of Bożejko-Speicher type.

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THANK YOU!