# Divergence radii and classical-quantum channel coding


Milán Mosonyi

Mathematical Institute, Budapest University of Technology and Economics

MTA-BME Lendület Quantum Information Theory Research Group


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*MOMENTUM OF INNOVATION*

MAGYAR TUDOMÁNYOS AKADÉMIA 1825

Lendület program

# Prelude

- Shannon 1948: *A Mathematical Theory of Communication*

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Information measures quantify the ultimate achievable performance of protocols in information theoretic tasks.

# Classical-quantum channels

- Classical-quantum (cq) channel: $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$

  $\mathcal{X}$: input alphabet, $\mathcal{S}(\mathcal{H}) :=$ {density operators on $\mathcal{H}$}

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- product channel: $W_k : \mathcal{X}_k \to \mathcal{B}(\mathcal{H}_k)_+, \qquad k = 1, \ldots, n$

  $$(W_1 \otimes \ldots \otimes W_n)(\underline{x}) := W_1(x_1) \otimes \ldots \otimes W_n(x_n), \qquad \underline{x} \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$$

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  average success probability:

$$P_s(\mathcal{E}_n, \mathcal{D}_n) := \frac{1}{M_n} \sum_{m=1}^{M_n} \operatorname{Tr} W^{\otimes n}(\mathcal{E}_n(m))\mathcal{D}_n(m)$$

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- HSW theorem:

$$\lim_{n \to +\infty} \max\{P_s(\mathcal{E}_n, \mathcal{D}_n) : M_n \geq 2^{nR}\} = \begin{cases} 1, & R < \chi(W), \\ 0, & R > \chi(W). \end{cases}$$

$$\chi(W) := \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi(W, P)$$ Holevo capacity

# Holevo quantity

$H(\varrho) := -\operatorname{Tr} \varrho \log \varrho$ von Neumann entropy

$$\chi(W, P) \quad := \quad H(W(P)) - \sum_x P(x) H(W(x))$$

# Holevo quantity

$$\chi(W,P) := H(W(P)) - \sum_x P(x) H(W(x)) = D\left(\widehat{W}(P) \| P \otimes W(P)\right)$$

$D(\varrho\|\sigma) := \mathrm{Tr}\,\varrho(\log\varrho - \log\sigma)$ relative entropy

$\widehat{W}(P) := \sum_x P(x)|x\rangle\langle x| \otimes W(x)$ joint input-output state

# Holevo quantity

$$\chi(W,P) := H(W(P)) - \sum_x P(x)H(W(x))$$

$$= D\left(\widehat{W}(P) \| P \otimes W(P)\right)$$ mutual information

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$$I_\Delta(W,P) := \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \Delta\left(\widehat{W}(P)\|P\otimes\sigma\right)$$ $\Delta$-mutual information

$$\chi_\Delta(W,P) := \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \sum_x P(x)\,\Delta(W(x)\|\sigma)$$ $P$-weighted $\Delta$-radius

$\Delta:\ \mathcal{B}(\mathcal{H})_+\times\mathcal{B}(\mathcal{H})_+\to\mathbb{R}\cup\{\pm\infty\}$ divergence

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# Mutual information and $P$-weighted radius

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Which of these quantities are relevant (if any)?

# Rényi $(\alpha, z)$-divergence

$\alpha \in (0, +\infty) \setminus \{1\}$, $z \in (0, +\infty)$

- $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{++}$

$$Q_{\alpha,z}(\varrho \| \sigma) := \operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \varrho^{\frac{\alpha}{2z}}\right)^{z}$$

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$$Q^{*}_{\alpha}(\varrho\|\sigma) := Q_{\alpha,\alpha}(\varrho\|\sigma) = \operatorname{Tr}\left(\varrho^{\frac{1}{2}}\sigma^{\frac{1-\alpha}{\alpha}}\varrho^{\frac{1}{2}}\right)^{\alpha}$$ sandwiched

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- $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{+}$

$$Q_{\alpha,z}(\varrho\|\sigma) := \lim_{\varepsilon\searrow 0} Q_{\alpha,z}(\varrho+\varepsilon I\|\sigma+\varepsilon I)$$

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- All of them coincide when $\varrho\sigma = \sigma\varrho$.

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log-Euclidean

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$$s(\alpha) := \operatorname{sgn}(\alpha - 1) = \begin{cases} -1, & \alpha < 1, \\ 1, & \alpha > 1 \end{cases}, \qquad \overline{Q}_{\alpha,z} := s(\alpha)Q_{\alpha,z}.$$

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- Rényi $(\alpha, z)$-divergence: [Audenaert, Datta 2013]

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# Mutual information and radius

$$I_\Delta(W,P) \;=\; \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \Delta\left(\hat{W}(P)\|P\otimes\sigma\right)$$

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Examples: $\overline{Q}_{\alpha,z}$, relative entropy $D$

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but not $D_{\alpha,z}$ with $\alpha\neq 1$

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&= \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \frac{1}{\alpha-1}\log\sum_x P(x)Q_{\alpha,z}(W(x)\|\sigma)
\end{aligned}
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# Rényi mutual information

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\left\{\begin{array}{ll} \leq & \alpha<1 \\ \geq & \alpha>1 \end{array}\right\} &\quad \chi_{D_{\alpha,z}(W,P)} =: \chi_{\alpha,z}(W,P)
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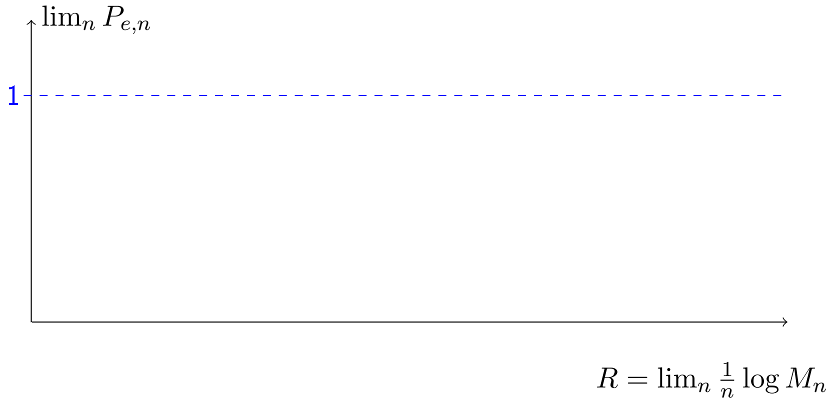
&= \frac{1}{\alpha-1}\log s(\alpha)\chi_{\overline{Q}_{\alpha,z}(W,P)}
\end{aligned}
$$

# Strong converse exponent



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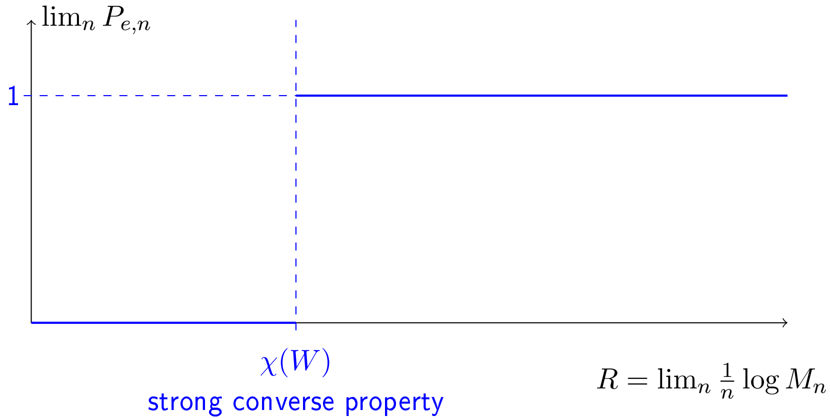

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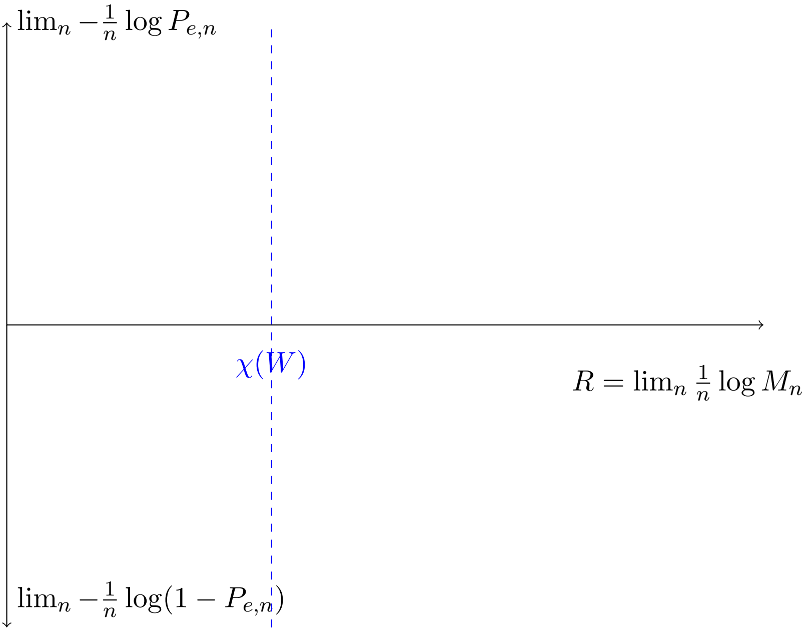

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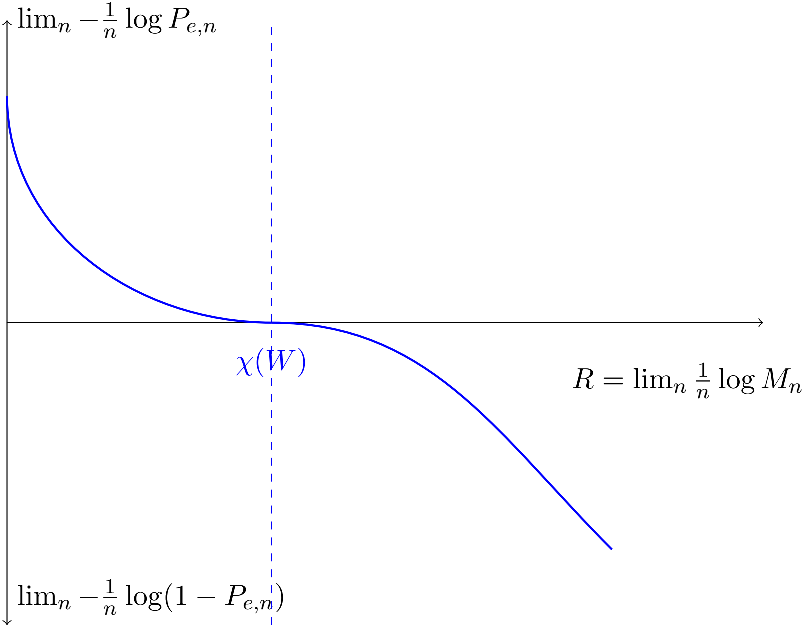

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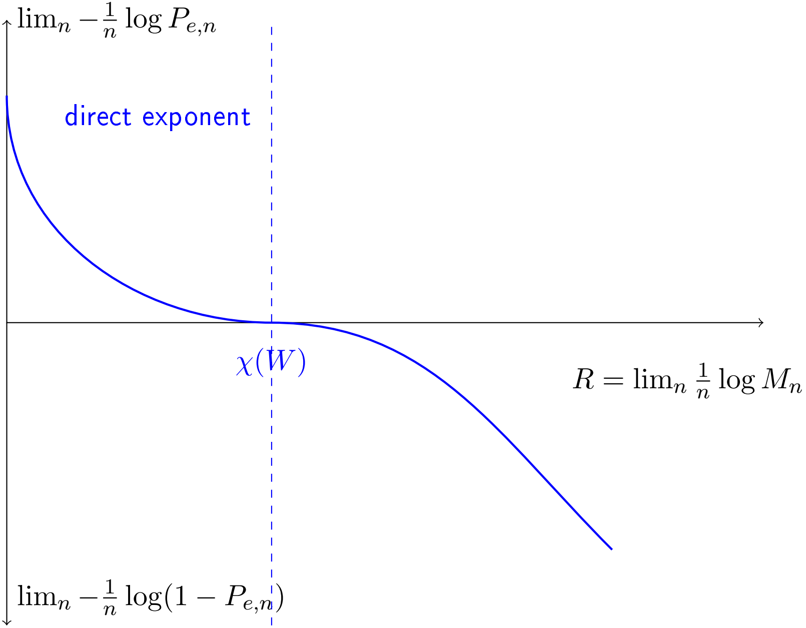

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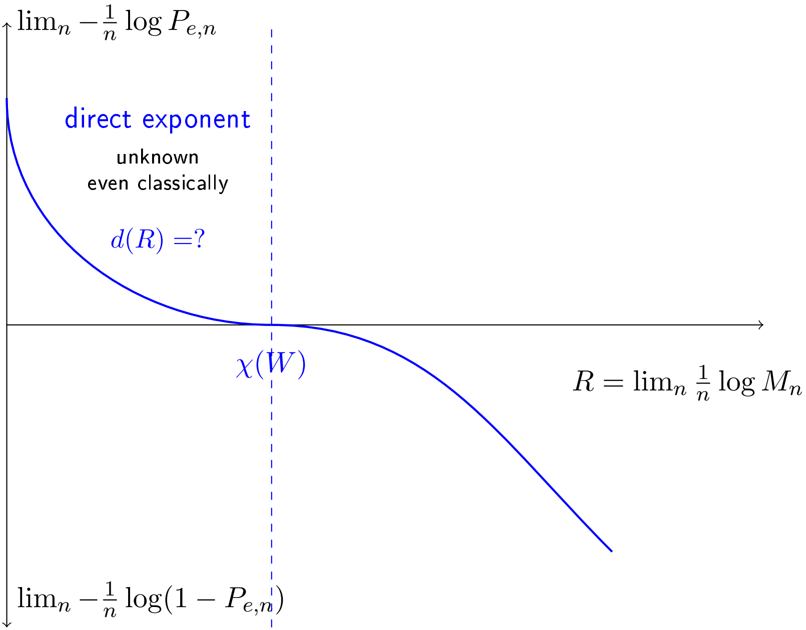

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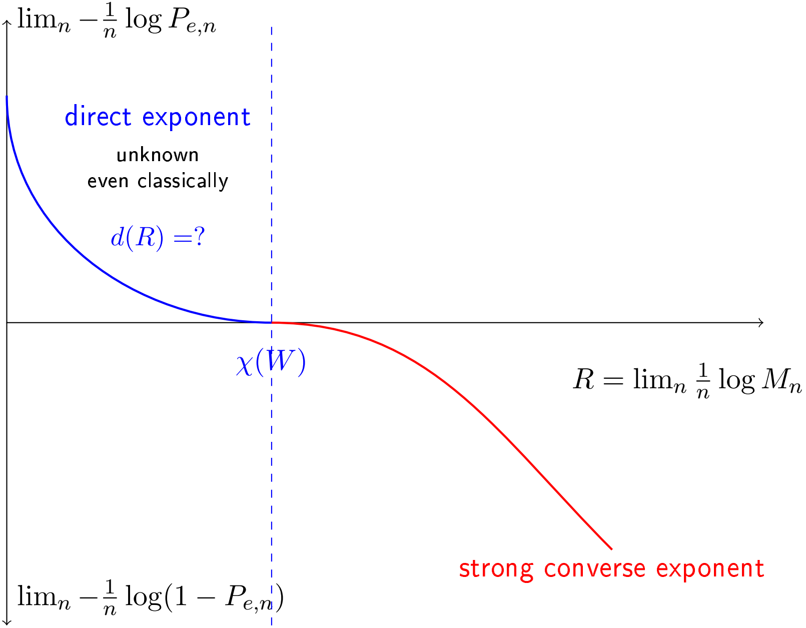

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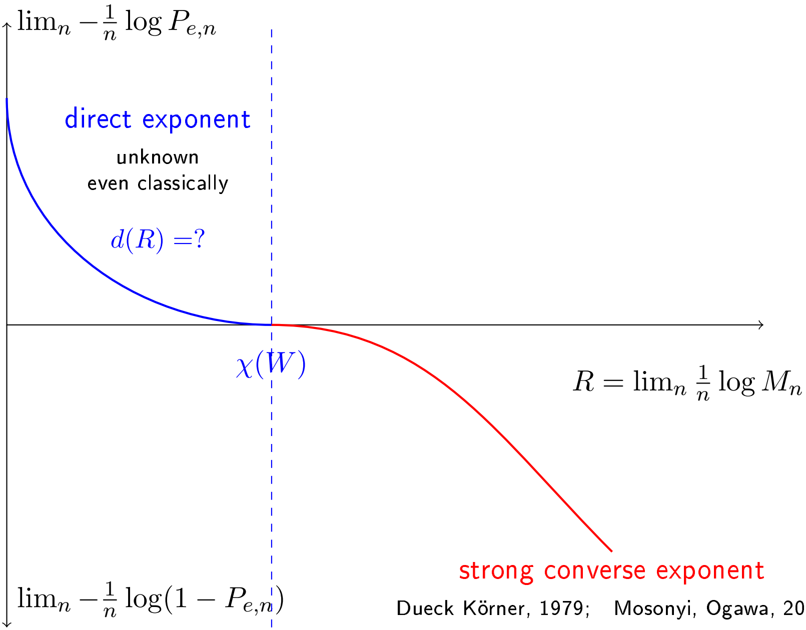


Dueck Körner, 1979; Mosonyi, Ogawa, 2014

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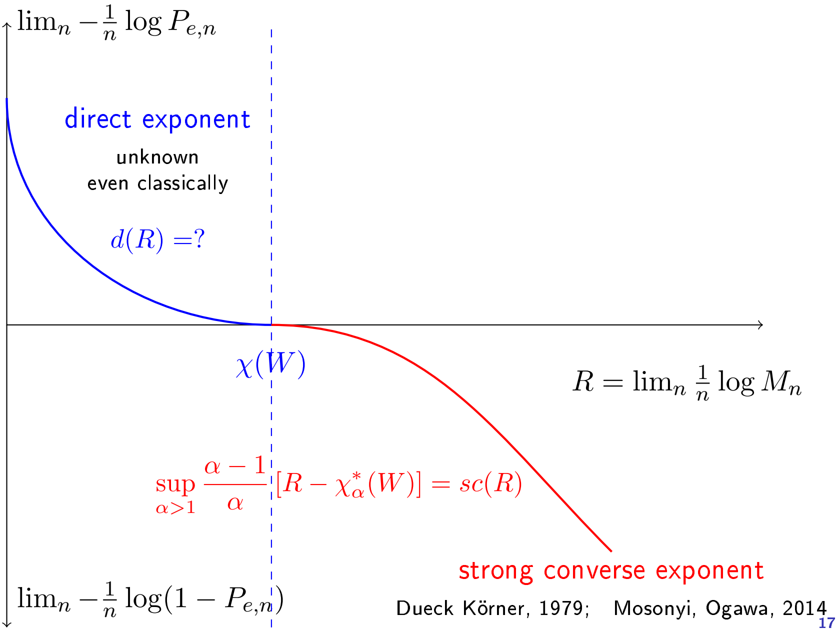

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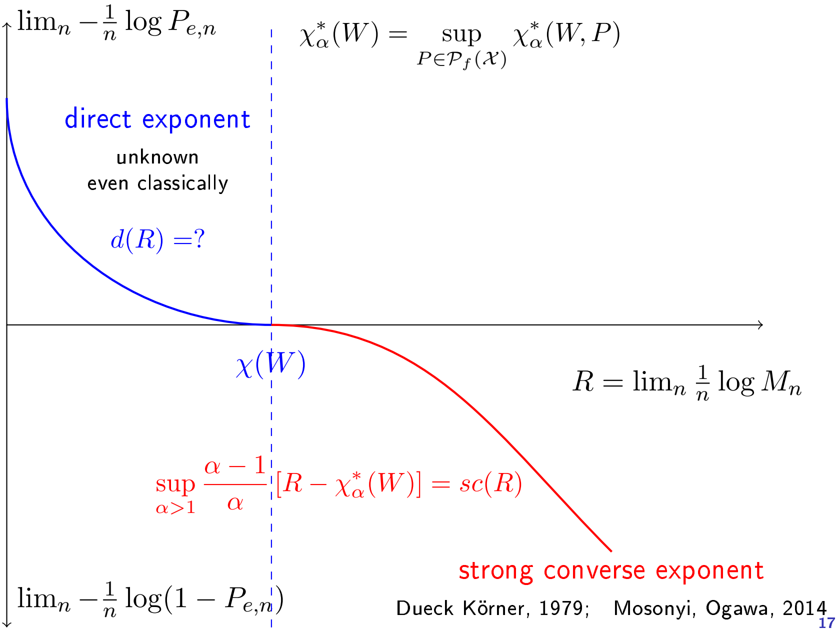

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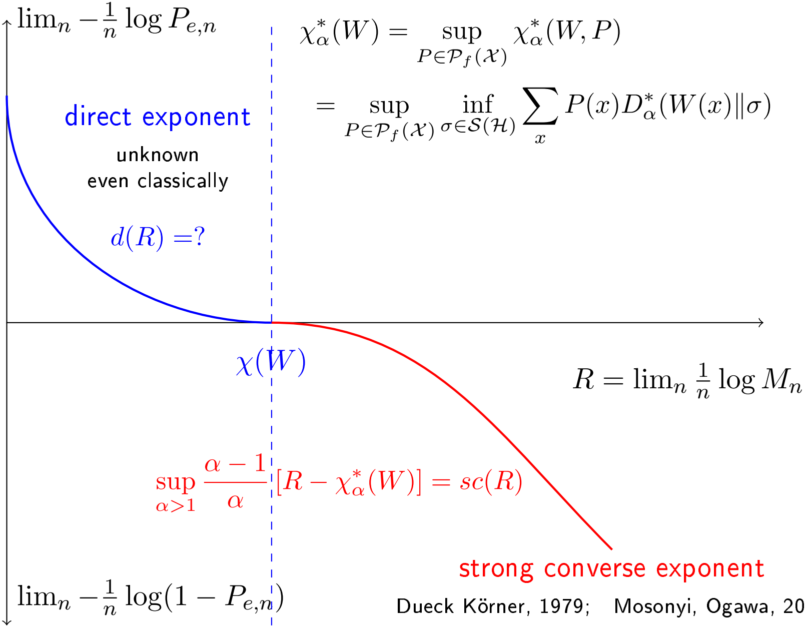


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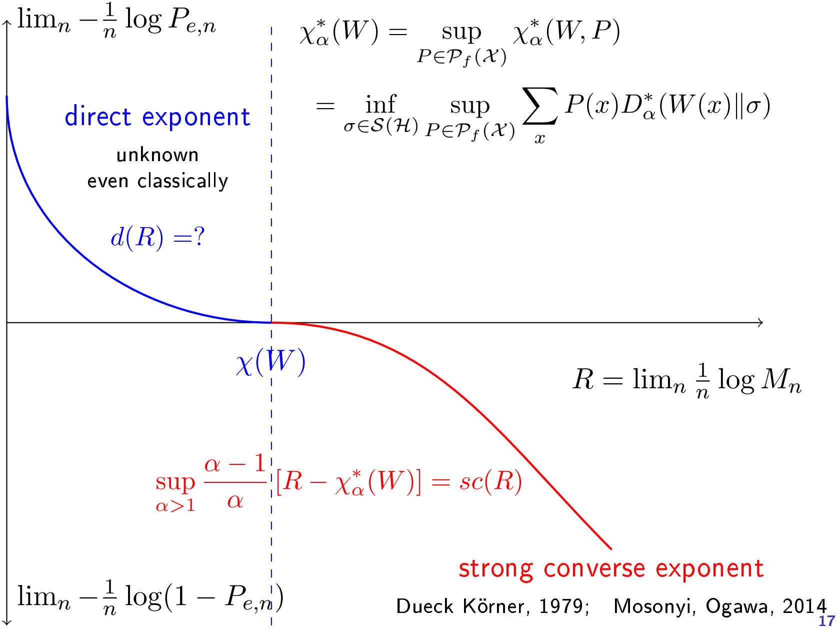


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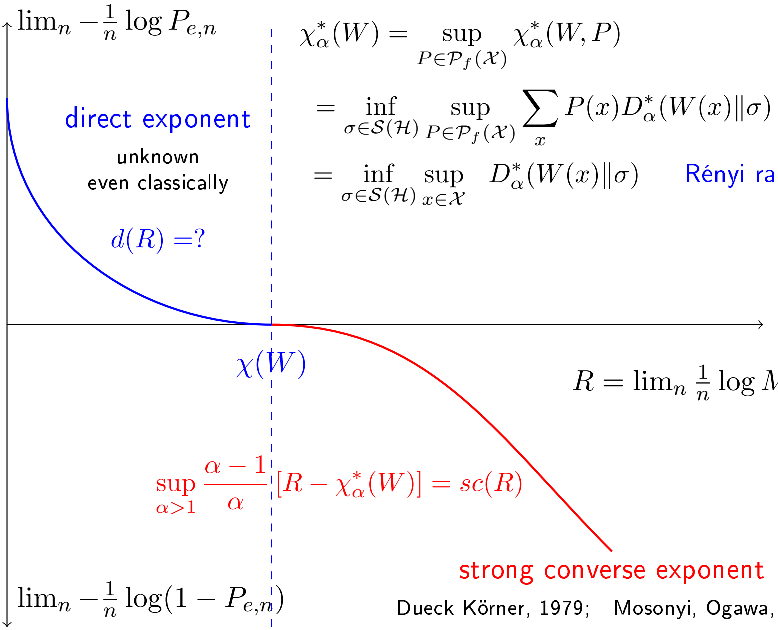


$$\chi_\alpha^*(W) = \sup_{P\in\mathcal{P}_f(\mathcal{X})}\chi_\alpha^*(W,P)$$

$$= \inf_{\sigma\in\mathcal{S}(\mathcal{H})}\sup_{P\in\mathcal{P}_f(\mathcal{X})}\sum_x P(x)D_\alpha^*(W(x)\|\sigma)$$

$$= \inf_{\sigma\in\mathcal{S}(\mathcal{H})}\sup_{x\in\mathcal{X}} D_\alpha^*(W(x)\|\sigma)$$ Rényi radius

Dueck Körner, 1979; Mosonyi, Ogawa, 2014

# Strong converse exponent

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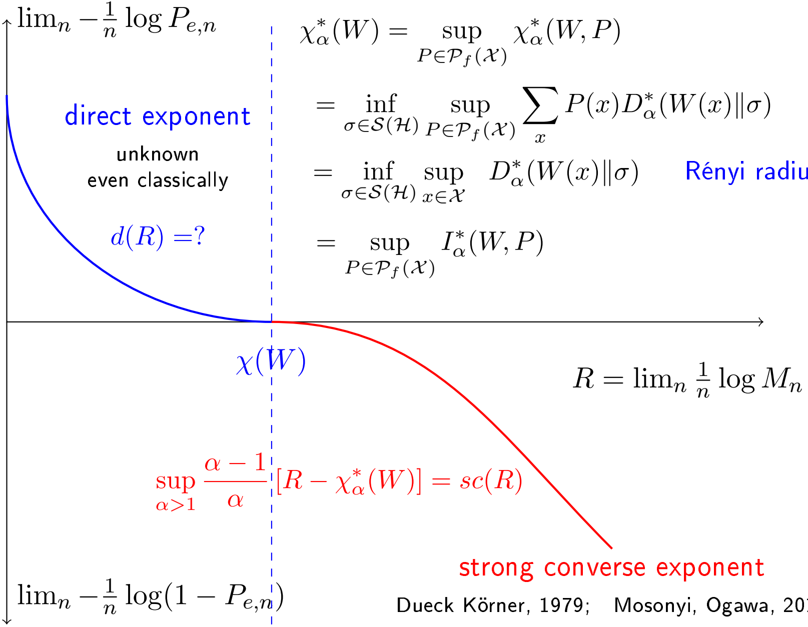
$$= \inf_{\sigma\in\mathcal{S}(\mathcal{H})} \sup_{x\in\mathcal{X}} D_\alpha^*(W(x)\|\sigma) \quad \text{Rényi radius}$$

$$= \sup_{P\in\mathcal{P}_f(\mathcal{X})} I_\alpha^*(W,P)$$




Dueck Körner, 1979; Mosonyi, Ogawa, 2014

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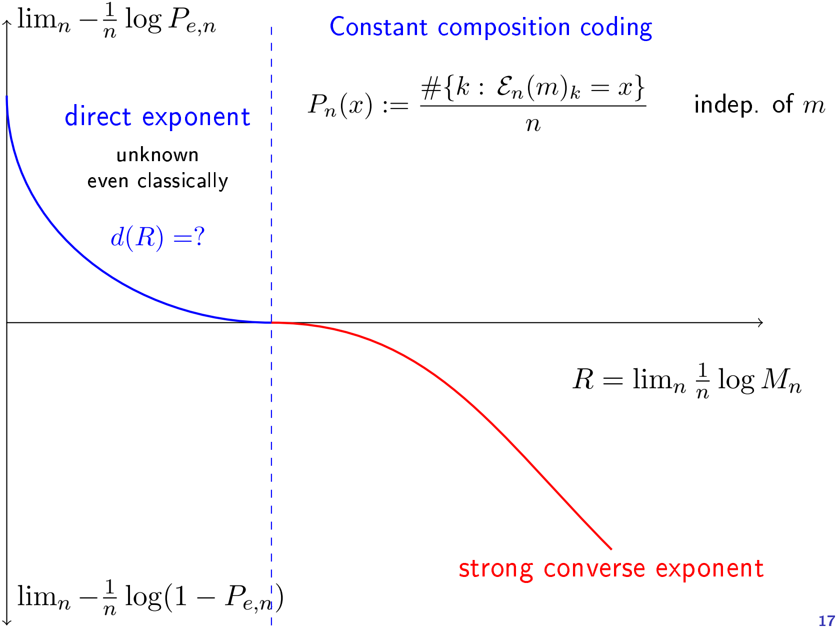

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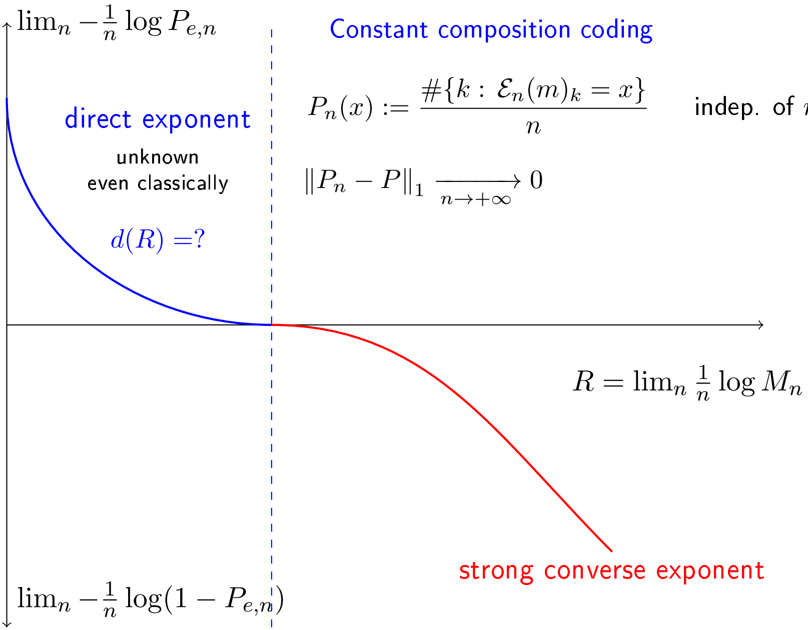


Constant composition coding

$$P_n(x) := \frac{\#\{k : \mathcal{E}_n(m)_k = x\}}{n}$$ indep. of $m$

$$\|P_n - P\|_1 \xrightarrow[n\to+\infty]{} 0$$

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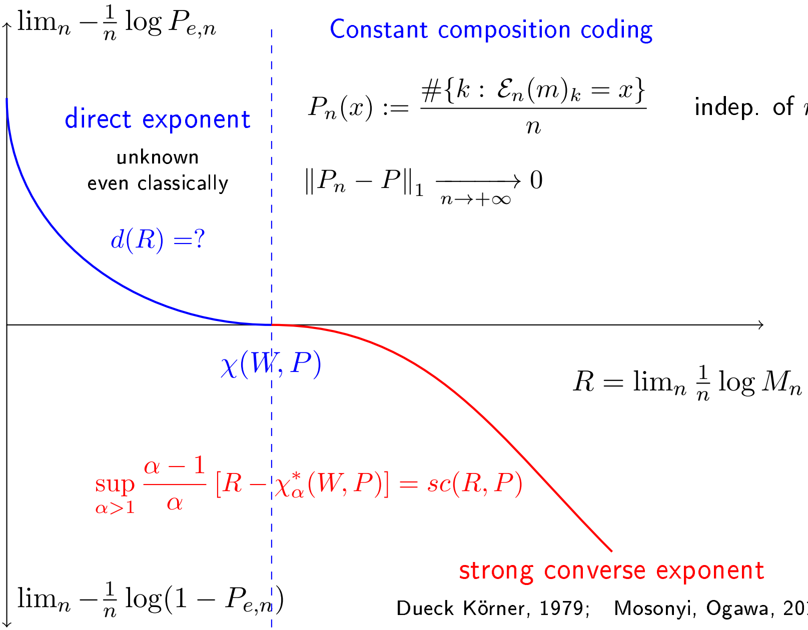
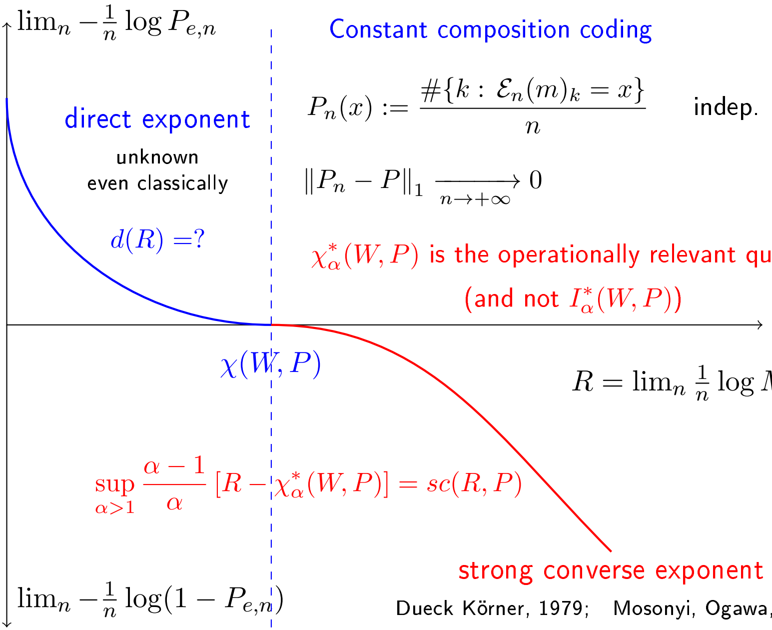



Dueck Körner, 1979; Mosonyi, Ogawa, 2018

# Strong converse exponent

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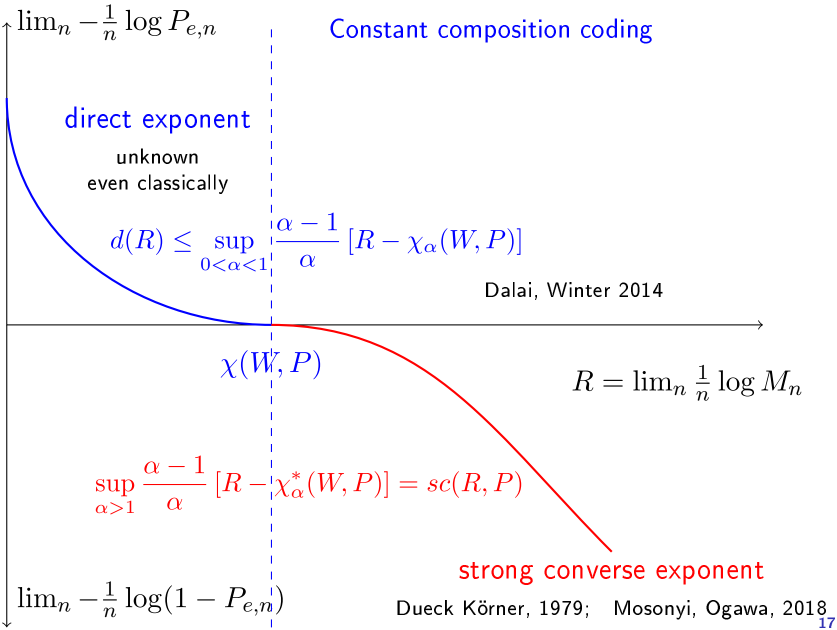
$\chi_\alpha^*(W,P)$ is the operationally relevant quantity (and not $I_\alpha^*(W,P)$)

Dueck Körner, 1979; Mosonyi, Ogawa, 2018

# Strong converse exponent



# Strong converse exponent

$$\underline{sc}(R,P) := \inf\left\{\liminf_{n\to+\infty} -\frac{1}{n}\log P_s(W^{\otimes n},\mathcal{C}_n) :\ \liminf_{n\to+\infty}\frac{1}{n}\log M_n \geq R\right\},$$

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Lemma: [Nagaoka 2000, Cheng et. al 2018, Mosonyi, Ogawa 2018]
For any $R > 0$,

$$\sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left[R-\chi_\alpha^*(W,P)\right] \leq \underline{sc}(R,P).$$

Proof: Easy from the monotonicity of $D_\alpha^*$.

# Strong converse exponent

Theorem: [Dueck, Körner 1979; Mosonyi, Ogawa 2014]

$$\overline{sc}(R,P) \le \inf_{V:\mathcal{X}\to\mathcal{S}(\mathcal{H})} \left\{ D(\widehat{V}(P)\|\widehat{W}(P)) + |R-\chi(V,P)|^+ \right\}$$

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Proof idea:
$$\operatorname{Tr}\left(V^{\otimes n}(\mathcal{E}_n(k)) - e^{na}\, W^{\otimes n}(\mathcal{E}_n(k))\right)_+ \ge \operatorname{Tr}\left(V^{\otimes n}(\mathcal{E}_n(k)) - e^{na}\, W^{\otimes n}(\mathcal{E}_n(k))\right) D_n(k),$$

# Strong converse exponent

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$$\ge e^{-na}\left\{ P_s(V^{\otimes n},\mathcal{C}_n) - \frac{1}{M_n}\sum_{k=1}^{M_n} \mathrm{Tr}\left(V^{\otimes n}(\mathcal{E}_n(k)) - e^{na}\,W^{\otimes n}(\mathcal{E}_n(k))\right)_{+} \right\}.$$

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Theorem: [Dueck, Körner 1979; Mosonyi, Ogawa 2014]

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[Hayashi 2009; Cheng et al. 2018]

Assume: $\chi(V,P) > R$

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Theorem: [Dueck, Körner 1979; Mosonyi, Ogawa 2014]

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[Hayashi 2009; Cheng et al. 2018] [Mosonyi, Ogawa 2018]

Assume: $\chi(V,P) > R$ $\qquad D(\hat{V}(P)\|\hat{W}(P)) < a$

Theorem: [Dueck, Körner 1979; Mosonyi, Ogawa 2014]

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$$= \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left[R-\chi_{\alpha}^{\flat}(W,P)\right]$$

# Pinching

- $\forall W, P, R \;\; \exists$ codes $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ with rate $R$ s.t.

$$\liminf_k \frac{1}{k} \log P_s(W^{\otimes k}, \mathcal{C}_k) \geq -\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left[R - \chi_\alpha^\flat(W,P)\right]$$

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- Let $\sigma_m$ be a universal symmetric state on $\mathcal{H}^{\otimes m}$

$$\forall \;\; \omega \in \mathcal{S}_{symm}(\mathcal{H}^{\otimes m}): \qquad \omega \leq v_{m,d}\sigma_m, \qquad v_{m,d} \leq (m+1)^{\frac{(d+2)(d-1)}{2}}$$

[Hayashi 2002]

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- pinched channel:

$$W_m : \underline{x} \mapsto \mathcal{E}_{\sigma_m}(W(x_1) \otimes \ldots \otimes W(x_m)), \qquad \underline{x} \in \mathcal{X}^m$$

# Pinching

- $\forall W, P, R \;\; \exists$ codes $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ with rate $Rm$ s.t.

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- Construct codes $\{\tilde{\mathcal{C}}_n\}_{n\in\mathbb{N}}$ with rate $R$ s.t.

$$\liminf_n \frac{1}{n} \log P_s(W^{\otimes n}, \tilde{\mathcal{C}}_n) = \frac{1}{m} \liminf_m \frac{1}{k} \log P_s(W_m^{\otimes k}, \mathcal{C}_k)$$

# Pinching

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&\geq -\sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left[R - \frac{1}{m}\chi_\alpha^\flat(W_m, P^{\otimes m})\right] \\
&\geq -\sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left[R - \frac{1}{m}\chi_\alpha^*(W^{\otimes}, P^{\otimes m})\right] - f(m)
\end{aligned}$$

# Pinching

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$$\liminf_{m} \frac{1}{k} \log P_s(W_m^{\otimes k}, \mathcal{C}_k) \geq -\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left[ Rm - \chi_\alpha^\flat(W_m, P^{\otimes m}) \right]$$

- Construct codes $\{\tilde{\mathcal{C}}_n\}_{n\in\mathbb{N}}$ with rate $R$ s.t.

$$\begin{aligned}
&\liminf_{n} \frac{1}{n} \log P_s(W^{\otimes n}, \tilde{\mathcal{C}}_n) \\
&= \quad \frac{1}{m} \liminf_{m} \frac{1}{k} \log P_s(W_m^{\otimes k}, \mathcal{C}_k) \\
&\geq \quad -\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left[ R - \frac{1}{m} \chi_\alpha^\flat(W_m, P^{\otimes m}) \right] \\
&\geq \quad -\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left[ R - \frac{1}{m} \chi_\alpha^*(W^\otimes, P^{\otimes m}) \right] - f(m) \\
&\overset{\text{additivity?}}{=} \quad -\sup_{\alpha>1} \frac{\alpha-1}{\alpha} \left[ R - \chi_\alpha^*(W, P) \right] - \underbrace{f(m)}_{\xrightarrow[m\to+\infty]{} 0}
\end{aligned}$$

# Additivity

- $Q_{\alpha,z}$ multiplicative $\Longrightarrow$ $D_{\alpha,z}$ additive $\Longrightarrow$

$$\chi_{\alpha,z}(W_1 \otimes W_2, P_1 \otimes P_2) \leq \chi_{\alpha,z}(W_1, P_1) + \chi_{\alpha,z}(W_2, P_2)$$
$$I_{\alpha,z}(W_1 \otimes W_2, P_1 \otimes P_2) \leq I_{\alpha,z}(W_1, P_1) + I_{\alpha,z}(W_2, P_2)$$

  by restricting the minimization to $\sigma_{12} = \sigma_1 \otimes \sigma_2$.

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**Theorem** (Mosonyi, Ogawa 2018)

If $D_{\alpha,z}$ monotone under CPTP and convex in the 2nd variable, $\sigma^0 = W(P)^0$ then

(1) $\sigma$ is a minimizer for $\chi_{\alpha,z}$ iff

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Proof: Just take the derivative to be $0$.

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Extends results by Beigi 2013 for $I_{\alpha,z}$ with $z=\alpha>1$;

Nakiboglu 2018 for classical.

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