Subspaces of maximal dimension with bounded Schmidt rank

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Introduction (Schmidt rank)

Let $\mathcal{H}$ denote the bipartite Hilbert space $\mathbf{C}^m \otimes \mathbf{C}^n$. By Schmidt decomposition theorem, any pure state $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{j=1}^{k} \alpha_j \, |u_j\rangle \otimes |v_j\rangle \tag{1}$$

for some $k \leq \min\{m, n\}$, where $\{|u_j\rangle : 1 \leq j \leq k\}$ and $\{|v_j\rangle : 1 \leq j \leq k\}$ are orthonormal sets in $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively, and α_j's are nonnegative real numbers satisfying $\sum_j \alpha_j^2 = 1$.

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Definition

In the Schmidt decomposition (1) of a pure bipartite state $|\psi\rangle$ the minimum number of terms required in the summation is known as the Schmidt rank of $|\psi\rangle$, and it is denoted by $SR(|\psi\rangle)$.

In [T. Cubitt-A. Montanaro-A. Winter, 2008]
it was proved that for a bipartite system $\mathbf{C}^m \otimes \mathbf{C}^n$, the dimension of any subspace of Schmidt rank greater than or equal to k is bounded above by $(m-k+1)(n-k+1)$.

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We construct subspaces $\mathcal{T}$ of dimension $(m-k+1)(n-k+1)$ of bipartite finite dimensional Hilbert space $\mathbf{C}^m \otimes \mathbf{C}^n$ such that any vector in $\mathcal{T}$ has Schmidt rank greater than or equal to k where $k = 2, 3$ and 4.

Unlike [T. Cubitt-A. Montanaro-A.Winter, 2008], the subspaces $\mathcal{T}$ of $\mathbf{C}^m \otimes \mathbf{C}^n$ that we construct also have bases consisting of elements of Schmidt rank k.

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Note that For the case when a subspace of $\mathbf{C}^m \otimes \mathbf{C}^n$ is of Schmidt rank greater than or equal to 2 (that is, the subspace does not contain any product vector), the maximum dimension of that subspace is $(m-1)(n-1)$, and this was first proved in [K. R. Parthasarathy, 2004], and [N. R. Wallach, 2002] (cf. [B. V. R. Bhat, 2006]).

Introduction (Schmidt number)

In the bipartite Hilbert space $\mathbf{C}^m \otimes \mathbf{C}^n$, for any $1 \le r \le \min\{m, n\}$, there is at least some state $|\psi\rangle$ with $SR(|\psi\rangle) = r$. Any state ρ on a finite dimensional Hilbert space $\mathcal{H}$ can be written as

$$\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j| , \tag{2}$$

where $|\psi_j\rangle$'s are pure states in $\mathcal{H}$ and $\{p_j\}$ forms a probability distribution. The following notion was introduced in [B. M. Terhal and P. Horodecki, 2000].

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Definition

The Schmidt number of a state ρ on a bipartite finite dimensional Hilbert space $\mathcal{H}$ is defined to be the least natural number k such that ρ has a decomposition of the form given in (2) with $SR(|\psi_j\rangle) \leq k$ for all j. The Schmidt number of ρ is denoted by $SN(\rho)$.

Schmidt number of a state on a bipartite Hilbert space is a measure of entanglement. Entanglement is the key property of quantum systems which is responsible for the higher efficiency of quantum computation and tasks like teleportation, super-dense coding, etc (cf. [R., P., M., K., Horodecki, 2009]). The following proposition establishes an important relation between Schmidt number of a state and the lower bound of Schmidt rank of any vector in the supporting subspace of the state. It should be well known.

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Proposition

Let $\mathcal{S}$ be a subspace of $\mathcal{H} = \mathbf{C}^m \otimes \mathbf{C}^n$ which does not contain any vector of Schmidt rank lesser or equal to k. Then any state ρ supported on $\mathcal{S}$ has Schmidt number at least $k+1$.

Idea

Let $\phi\colon \mathbf{C}^m \otimes \mathbf{C}^n \to M_{m\times n}(\mathbf{C})$ be defined by for each $\mathbf{C}^m \otimes \mathbf{C}^n \ni |\eta\rangle = \sum_{i,j} c_{ij}|e_i\rangle \otimes |f_j\rangle$, $\phi(|\eta\rangle) = [c_{ij}]$. Then $|\eta\rangle$ has Schmidt rank at least r if and only if the corresponding matrix $[c_{ij}]$ is of rank at least r.

Usiing this correspondence, we find a basis of $\{|\eta_i\rangle\}_{i=1}^d$ such that all non-zero linear combination C of $\{\phi(|\eta_i\rangle)\}_{i=1}^d$ has at least rank 4.

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[Example]

$$|e_0\rangle \otimes |f_3\rangle - a|e_1\rangle \otimes |f_2\rangle + a|e_2\rangle \otimes |f_1\rangle - |e_3\rangle \otimes |f_0\rangle$$

$$\xrightarrow{\phi}$$

$$\left(\begin{array}{ccccc} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & -a & \cdot & \cdot \\ \cdot & a & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right.$$

The order-n minors of $(n+k) \times n$ matrixes

For any $n \in \mathbf{N}$ and positive numbers a and b in $\mathbf{R}$, set $(n+1) \times n$ matrix

$$E_n(a,b) = \begin{pmatrix} -a & b & 0 & 0 & 0 & \cdots & 0 \\ a & -a & b & 0 & 0 & \cdots & 0 \\ -b & a & -a & b & 0 & \cdots & 0 \\ 0 & -b & a & -a & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & 0 & -b & a & -a \\ 0 & \cdots & \cdots & \cdots & 0 & -b & a \end{pmatrix}. \quad (3)$$

For $1 \le k \le n+1$ let $E_n^k(a,b)$ be a matrix which is obtained by deleting the k-th row of $E_n(a,b)$.

We would like to determine when $E_n^k(a, b)$ is invertible for any $1 \leq k \leq n+1$.

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For any $n \in \mathbf{N}$ and positive numbers a and b in $\mathbf{R}$ set the $n \times n$ matrix

$$D_n(a,b) = \begin{pmatrix} -a & b & 0 & 0 & 0 & \cdots & 0 \\ a & -a & b & 0 & 0 & \cdots & 0 \\ -b & a & -a & b & 0 & \cdots & 0 \\ 0 & -b & a & -a & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & 0 & -b & a & -a \end{pmatrix}.$$

We also define the $n \times n$ matrix $F_n(a, b)$ as follows:

$$F_n(a,b) = \begin{pmatrix} a & -a & b & 0 & 0 & 0 & \cdots & 0 \\ -b & a & -a & b & 0 & 0 & \cdots & 0 \\ 0 & -b & a & -a & b & 0 & \cdots & 0 \\ 0 & 0 & -b & a & -a & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & \cdots & 0 & -b & a & -a \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -b & a \end{pmatrix} = D_n(-a, -b)^t.$$

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Proposition

For $1 \leq k \leq n+1$ we have

$$\begin{aligned} |E_n^k(a,b)| &= |D_{k-1}(a,b)||F_{n-(k-1)}(a,b)| + |D_{k-2}(a,b)||F_{n-k}(a,b)|b^2 \\ &= (-1)^{n-k+1}(|D_{k-1}(a,b)||D_{n-k+1}(a,b)| \\ &- |D_{k-2}(a,b)||D_{n-k}(a,b)|b^2). \end{aligned}$$

If $b = 0$ and $1 \leq k \leq n+1$,

$$|E_n^k(a,b)| = (-1)^{n-k+1}|D_{k-1}(a,0)||D_{n-k+1}(a,0)| = (-1)^{k-1}a^n.$$

If $b \neq 0$, since $|E_n^k(a,b)| = b^k \left| E_n^k\left(\frac{a}{b}, 1\right)\right|$, we may assume that $b = 1$.

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Theorem

For $n \in \mathbf{N} \cup \{-1, 0\}$ and a positive number $a \in \mathbf{R}$, let $d_{-1} = 0$, $d_0 = 1$ and $d_n = |D_n(a,1)|$. Then for $1 \le k \le n+1$ we have

$$|E_n^k(a,1)| = (-1)^{n-k+1}(d_{k-1}d_{n-k+1} - d_{k-2}d_{n-k})$$

$$d_k = \frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{1}{1+ax+ax^2+x^3}\right)\Bigg|_{x=0}.$$

Moreover, if $x^3 + ax^2 + ax + 1 = 0$ has 3 different solutions α, β, γ, then we have for $1 \le k$

$$d_k = \frac{1}{\alpha^{k+1}(\alpha-\beta)(\gamma-\alpha)} + \frac{1}{\beta^{k+1}(\alpha-\beta)(\beta-\gamma)} + \frac{1}{\gamma^{k+1}(\gamma-\alpha)(\beta-\gamma)}.$$

Examples

Example

Set $a = 3$ and $b = 1$. Then we have

$$d_n = |D_n(3,1)| = \frac{1}{n!}\frac{d^n}{dx^n}(1+x)^{-3}|_{x=0} = \frac{(-1)^n}{2}(n+1)(n+2)$$

and

$$|E_n^k(3,1)| = \frac{(-1)^{k-1}}{2}k(n+2)(n-k+2).$$

Note that if $|E_n^k(3,1)| = 0$, then $k = n+2$. Therefore, for any $1 \leq k \leq n+1$ we have $|E_n^k(3,1)| \neq 0$, that is, all order-n minors of $E_n(3,1)$ are non-zero.

Example

Set $n = 10$ and $a = 2$. Then $|E^5_{10}(2,1)| = 0$.
Indeed, the equation $x^3 + 2x^2 + 2x + 1 = (x+1)(x^2+x+1) = 0$ has solutions $-1, \omega, \omega^2$, where $\omega = \dfrac{-1+\sqrt{3}\iota}{2}$. Then we have

$$\begin{aligned} d_k &= \frac{1}{(-1)^{k+1}(-1-\omega)(\omega^2+1)} + \frac{1}{\omega^{k+1}(-1-\omega)(\omega-\omega^2)} \\ &+ \frac{1}{\omega^{2(k+1)}(\omega^2+1)(\omega-\omega^2)} \\ &= (-1)^k - \frac{\omega^{k+2}(1-\omega^k)}{1-\omega}. \end{aligned}$$

Since $d_4 = d_5 = 0$, we have $|E^5_{10}(2,1)| = d_4 d_6 - d_3 d_5 = 0$.

Theorem

For $a \in \mathbf{R}$ with $a > 5$ and $1 \leq k \leq n+1$ the matrix $E_n^k(a,1)$ is invertible, that is, all order-n minors of $E_n(a,1)$ are non-zero.

Theorem

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Idea is as follows: Suppose that $E_n^k \mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \in \mathbf{C}^n$. If $x_1 = 0$ or $x_n = 0$, it is easy to show that $\mathbf{x} = \mathbf{0}$. Hence we may assume that $\mathbf{x} \neq \mathbf{0}$.
Let $a(i)$, $b(i)$, $c(i)$, $d(i) \in \mathbf{C}$ with $|a(i)| = |d(i)| = 1$ and $|b(i)| = |c(i)| = \alpha > 5$ $(i = 1, 2, \ldots, k-1)$. Since $x_1 \neq 0$ and

$$\begin{aligned}
&c(1)x_1 + d(1)x_2 = 0\\
&b(2)x_1 + c(2)x_2 + d(2)x_3 = 0\\
&a(3)x_1 + b(3)x_2 + c(3)x_3 + d(3)x_4 = 0\\
&a(4)x_2 + b(4)x_3 + c(4)x_4 + d(4)x_5 = 0\\
&\cdots\cdots\\
&a(k-1)x_{k-3} + b(k-1)x_{k-2} + c(k-1)x_{k-1} + d(k-1)x_k = 0,
\end{aligned}$$

then $|x_{l+1}| \geq (\alpha - 2)|x_l|$ $(l = 1, 2, \ldots, k-1)$, and $|x_k| > |x_{k-1}|$.
Conversely, since $x_n \neq 0$, we have $|x_{k-1}| \geq (\alpha - 2)|x_k| > |x_k|$.

Subspaces of maximal dimension with bounded Schmidt rank

For $n \in \mathbf{N}$ and a number $a \in \mathbf{R}$, we consider the $(n+3) \times n$ matrix $B_n(a,1)$ as follows:

$$B_n(a,1) = \begin{pmatrix} b_1 \\ E_n(a,1) \\ b_{n+3} \end{pmatrix}$$

where $b_1 = (1, 0, \ldots, 0)$, $b_{n+3} = (0, \ldots, 0, -1)$.

Proposition

For $1 \leq i < j < k \leq n+3$ and $a = 3$ or $a > 5$, let $B_n(a,1)^{i,j,k}$ be a matrix which is obtained by deleting the i, j, k-th rows of $B_n(a,1)$. Then $|B_n(a,1)^{i,j,k}| \neq 0$.

The following obsrvation is used to get our main results.

Corollary

For $a = 3$ or $a > 5$, the columns of $B_n(a, 1)$ are linearly independent such that any linear combination of these columns has at least 4 non-zero entries.

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Theorem

Let m and n be natural numbers such that $4 \le \min\{m, n\}$. Let $N = n + m - 2$, and $\{|e_i\rangle\}_{i=0}^{m-1}$ (resp. $\{|f_j\rangle\}_{j=0}^{n-1}$) be the canonical basis for $\mathbf{C}^m$ (resp. $\mathbf{C}^n$). For $3 \le d \le N - 3$ define

$$\mathcal{S}^{(d)} = \mathrm{span}\{|e_{i-2}\rangle \otimes |f_{j+1}\rangle - a|e_{i-1}\rangle \otimes |f_j\rangle + a|e_i\rangle \otimes |f_{j-1}\rangle - |e_{i+1}\rangle \otimes |f_{j-2}\rangle : \\ 2 \le i \le m-2, 2 \le j \le n-2, i + j = d + 1\},$$

$$\mathcal{S}^{(0)} = \mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(N-2)} = \mathcal{S}^{(N-1)} = \mathcal{S}^{(N)} = \{0\}$$

and $\mathcal{S} = \bigoplus_{d=0}^{N} \mathcal{S}^{(d)}$. If $a = 3$ or $a > 5$, then $\mathcal{S}$ does not contain any vector of Schmidt rank ≤ 3 and $\dim \mathcal{S} = (m-3)(n-3)$.

Scketch of the proof:
Let ϕ: $\mathbf{C}^m \otimes \mathbf{C}^n \to M_{m\times n}(\mathbf{C})$ be defined by for each $\mathbf{C}^m \otimes \mathbf{C}^n \ni |\eta\rangle = \sum_{i,j} c_{ij}|e_i\rangle \otimes |f_j\rangle$, $\phi(|\eta\rangle) = [c_{ij}]$. Then $|\eta\rangle$ has Schmidt rank at least r if and only if the corresponding matrix $[c_{ij}]$ is of rank at least r.

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Then we have the conclusion from the observation in $B_n(a, 1)$ and the following calculation:

$$|e_0\rangle \otimes |f_3\rangle - a|e_1\rangle \otimes |f_2\rangle + a|e_2\rangle \otimes |f_1\rangle - |e_3\rangle \otimes |f_0\rangle$$

$$\xrightarrow{\phi}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & -a & \cdot & \cdot \\ \cdot & a & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Remark

From the above theorem it follows that for $a = 3$ or $a > 5$, all the elements of the basis

$$B = \bigcup_{d=3}^{N-3} \{|e_{i-2}\rangle \otimes |f_{j+1}\rangle - a|e_{i-1}\rangle \otimes |f_j\rangle + a|e_i\rangle \otimes |f_{j-1}\rangle - |e_{i+1}\rangle \otimes |f_{j-2}\rangle : \\ 2 \leq i \leq m-2, 2 \leq j \leq n-2, i+j = d+1\},$$

of $\mathcal{S}$ have Schmidt rank 4.

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From the above theorem it follows that for $a = 3$ or $a > 5$, all the elements of the basis

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Remark: Let m and n be natural numbers such that $4 \leq \min\{m, n\}$. Let $N = n + m - 2$, and $\{|e_i\rangle\}_{i=0}^{m-1}$ (resp. $\{|f_j\rangle\}_{j=0}^{n-1}$) be the canonical basis for $\mathbf{C}^m$ (resp. $\mathbf{C}^n$) and set $g(i,j) = |e_i\rangle \otimes |f_j\rangle + a|e_{i-1}\rangle \otimes |f_{j+1}\rangle$. Consider

$$\mathcal{S} = \mathrm{span}\{g(i,j)\colon 1 \leq i \leq m-1, 0 \leq j \leq n-2\},$$

$$\mathcal{T} = \mathrm{span}\{g(i,j) + \frac{1}{a}g(i-1,j+1)\colon 2 \leq i \leq m-1, 0 \leq j \leq n-3\},$$

$$\mathcal{U} = \mathrm{span}\{(g(i,j) + \frac{1}{a}g(i-1,j+1)) + (g(i-1,j+1) + \frac{1}{a}g(i-2,j+2))\colon 3 \leq i \leq m-1, 0 \leq j \leq n-4\}.$$

When $a > 0$ and $a + \frac{1}{a} > 4$, we have

1. $\mathcal{U} \subset \mathcal{T} \subset \mathcal{S}$,
2. Any element in $\mathcal{S}$ has Schmidt rank ≥ 2, any generator in $\mathcal{S}$ has Schmidt rank 2, and $\dim \mathcal{S} = (m-1)(n-1)$,
3. Any element in $\mathcal{T}$ has Schmidt rank ≥ 3, any generator in $\mathcal{T}$ has Schmidt rank 3, and $\dim \mathcal{T} = (m-2)(n-2)$,
4. Any element in $\mathcal{U}$ has Schmidt rank ≥ 4, any generator in $\mathcal{U}$ has Schmidt rank 4, and $\dim \mathcal{U} = (m-3)(n-3)$.

Concluding remark

Question

Are there k-positive maps $\phi : M_m(\mathbf{C}) \to M_n(\mathbf{C})$ with $1 < k < m$ which are not decomposable ?

Concluding remark

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Are there k-positive maps $\phi : M_m(\mathbf{C}) \to M_n(\mathbf{C})$ with $1 < k < m$ which are not decomposable ?

Theorem (Terhal 2001)

Let S be a product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in $\mathbf{C}^m \otimes \mathbf{C}^n$ and suppose that the complementary subspace $H_S^\perp$ of a proper subspace H_S generated by S in $\mathbf{C}^m \otimes \mathbf{C}^n$, contains no product states. Then $\rho = \frac{1}{nm-|S|}(id - \sum_i |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i|)$ is entangled.

Using this observation Terhal constructed a family of indecomposable maps. We hope that after modifying our subspaces we could construct 2-positive map : $M_m(\mathbf{C}) \to M_n(\mathbf{C})$ with $\max\{n, m\} < 10$ which is not decomposable. Note that when $\min\{n, m\} \geq 10$ there are such examples by [Huber, Lami, Lancien, and Müller-Hermes, 2018].

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Thank you for your attention !!