Evaluating the robustness of **k**-coherence and **k**-entanglement

Yiu-Tung Poon

Iowa State University
Peng Cheng Laboratory

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Joint work with:
Nathaniel Johnston ,Chi-Kwong Li, Sarah Plosker and Bartosz Regula

Outline

1. Notations

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2. Robustness of k-coherence

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2. Robustness of *k*-coherence
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6. Conclusion

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7. Write $\mathcal{I} = \mathcal{I}_1$, the set of diagonal density matrices. Note that $\mathcal{I}_1 \subsetneq \mathcal{I}_2 \subsetneq \cdots \mathcal{I}_k \subsetneq \mathcal{I}_{k+1} \subsetneq \cdots \subsetneq \mathcal{I}_n = \mathcal{D}_n$.

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3) $R_1^g(\rho) = R(\rho)$ but $R_k^s(\rho)$ is not defined for $k = 1$ because $\mathcal{I}_1$ does not span M_n.

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$$\begin{aligned} R_k^s(|v\rangle\langle v|) = &\quad R_k^g(|v\rangle\langle v|) \\ = &\quad \frac{s_\ell^2}{k-\ell+1} - \sum_{i=\ell}^n v_i^2 \\ := &\quad S(k, \ell). \end{aligned}$$

Connection with k-support norm

The expression

$$S(k,\ell) = \frac{s_\ell^2}{k-\ell+1} - \sum_{i=\ell}^{n} v_i^2$$

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and $S(k,\ell) = \| |v\rangle \|_{(k)}^2 - 1$. We have $\|\mathbf{x}\|_{(1)} = \sum_i |x_i|$ and $\|\mathbf{x}\|_{(n)} = \sqrt{\mathbf{x}^\dagger \mathbf{x}}$. Hence, the k-support norm can be seen as a natural way to interpolate between the ℓ_1 and ℓ_2 norms.

Lower bound via semidefinite programming duality

Dual program formulation

Let $\mathcal{I}_k^\circ$ be the dual cone of $\mathcal{I}_k$ defined by

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Lower bound via semidefinite programming duality

Dual program formulation

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Hence, $R_k^g(\rho) \geq S(k, \ell)$.

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This allows the search of σ using *linear* programming, instead of semidefinite programming.

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Johnston and Kribs had conjectured that for any pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ and $k = 1, \ldots, \min\{m, n\}$, $R_k^{E,s}(|v\rangle\langle v|) = \||v\rangle\langle v|\|_{\gamma,k} - 1$.

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More generally, we have

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Theorem 2

Let $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ be a pure state with non-zero Schmidt coefficients $\lambda_1, \lambda_2, \dots, \lambda_r$ and define $|\lambda\rangle := (\lambda_1, \lambda_2, \dots, \lambda_r)^t$. Then

$$R_k^{E,s}(|v\rangle\langle v|) = R_k^s(|\lambda\rangle\langle\lambda|) = \||v\rangle\langle v|\|_{\gamma,k} - 1 = R_k^{E,g}(|v\rangle\langle v|).$$

Conclusion

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Thank you for your attention!

Reference

Some notes on the robustness of k-coherence and k-entanglement, Physical Review A, 98 (2018) 022328.