Multipartite entanglement and multipartite correlation

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MOMENTUM OF INNOVATION

Introduction

Bipartite correlation and entanglement

- classification/qualification/quantification: (S)LOCC
- uncorrelated/correlated
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Permutation invariant properties

- k-partitionability (k-separability, etc.)
- k-producibility (entanglement depth, etc.)
- duality

Quantum states

States of discrete finite quantum systems

- *state vector:* $|\psi\rangle \in \mathcal{H}$ (normalized) superposition
- *pure state:* $\pi = |\psi\rangle\langle\psi| \in \mathcal{P}$
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Measure of mixedness:

- von Neumann entropy: $S(\varrho) = -\operatorname{Tr}\varrho \ln \varrho$
- concave, nonnegative, vanishes iff ϱ pure
- Schur-concavity: *entropy* = *mixedness*
- increasing in bistochastic quantum channels
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Measure of distinguishability:

- (Umegaki's) quantum relative entropy: $D(\varrho||\sigma) = \operatorname{Tr}\varrho(\ln\varrho - \ln\sigma)$
- jointly convex, nonnegative, vanishes iff $\varrho = \omega$
- quantum Stein's lemma: *relative entropy* $=$ *distinguishability*
  (rate of decaying of the probability of error in hypothesis testing, Hiai & Petz)
- decreasing in quantum channels

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- in q.m. there are many (nontrivially) different observables in a system
- Γ remains meaningful even if there are no values, only events
- the state is *uncorrelated* iff $\mathrm{COV}(A,B) = 0$ for all A, B,
  iff $\langle AB\rangle = \langle A\rangle\langle B\rangle$ for all A, B, iff $\varrho = \varrho_1 \otimes \varrho_2$, iff $\Gamma = 0$

Bipartite correlation and entanglement

Pure states

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- state of subsystem (e.g., $\operatorname{Tr}_2 \pi \in \mathcal{D}_1$) not necessarily pure
- π is entangled if (and only if) $\operatorname{Tr}_2 \pi$ and $\operatorname{Tr}_1 \pi$ are mixed
  In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)

Bipartite correlation and entanglement

Mixed states: correlation

- *uncorrelated:* $\Gamma = 0$ (product), $\varrho = \varrho_1 \otimes \varrho_2 \in \mathcal{D}_{\text{unc}}$, else *correlated* $(\mathcal{D} \setminus \mathcal{D}_{\text{unc}})$
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  preparable by Local Operations and Classical Communication (LOCC),
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- classically correlated sources produce states of this kind (Werner) preparable by Local Operations and Classical Communication (LOCC), else *entangled* ($\mathcal{D} \setminus \mathcal{D}_{\text{sep}}$)
- the decomposition is not unique
- deciding separability is difficult

Bipartite correlation and entanglement – measures

- correlation "of the state itself": $\Gamma := \varrho - \varrho_1 \otimes \varrho_2$
  then $\mathrm{COV}(\varrho; A, B) = \langle AB \rangle - \langle A \rangle \langle B \rangle = \operatorname{Tr} \Gamma A \otimes B = \langle \Gamma | A \otimes B \rangle_{\mathrm{HS}}$
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- correlation might not be seen well from COV, but for all A, B,

$$\frac{1}{2}\mathrm{COV}(\varrho; \hat{A}, \hat{B})^2 \leq C(\varrho), \qquad \hat{A} = A/\|A\|_\infty, \hat{B} = B/\|B\|_\infty$$

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- faithful: $C(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\text{unc}}$, $E(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\text{sep}}$
- $E(\varrho)$ is hard to calculate

Multipartite correlation and entanglement – structure

Level 0.: subsystems — Boolean lattice structure: $P_0 = 2^L$

- whole system: $L = \{1, 2, \dots, n\}$
- subsystem: $X \subseteq L$, then $\mathcal{H}_X$, $\mathcal{P}_X$, $\mathcal{D}_X$

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)
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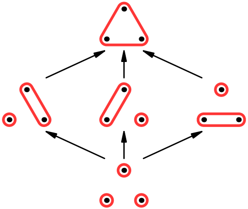

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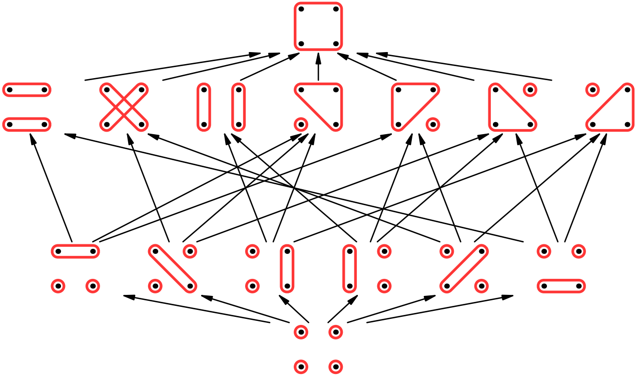

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- *ξ-uncorrelated states:* $\mathcal{D}_{\xi\text{-unc}} = \left\{ \bigotimes_{X \in \xi} \varrho_X \right\}$
  LO-closed
  $\upsilon \preceq \xi \Leftrightarrow \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}$
- *ξ-separable states:* $\mathcal{D}_{\xi\text{-sep}} = \operatorname{Conv} \mathcal{D}_{\xi\text{-unc}}$
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- *ξ-entanglement (of formation)*:

$$E_\xi(\pi) = C_\xi|_{\mathcal{P}}(\pi), \qquad E_\xi(\varrho) = \min\Big\{\sum\nolimits_i p_i E_\xi(\pi_i) \;\Big|\; \sum\nolimits_i p_i\pi_i = \varrho\Big\}$$

LOCC-monotone (proper entanglement measure)

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Multipartite correlation and entanglement – measures

Level I.: partitions lattice structure: $P_{\text{I}} = \Pi(L)$

- *ξ-correlation* (ξ-mutual information):

$$C_\xi(\varrho) = \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D(\varrho||\sigma) = \sum\nolimits_{X \in \xi} S(\varrho_X) - S(\varrho)$$

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- faithful: $C_\xi(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\xi\text{-unc}}$, $E_\xi(\varrho) = 0 \Leftrightarrow \varrho \in \mathcal{D}_{\xi\text{-sep}}$

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Multipartite correlation and entanglement – structure

Level II.: multiple partitions   lattice structure: $P_{\text{II}} = \mathcal{O}_{\downarrow}(P_{\text{I}}) \setminus \{\emptyset\}$

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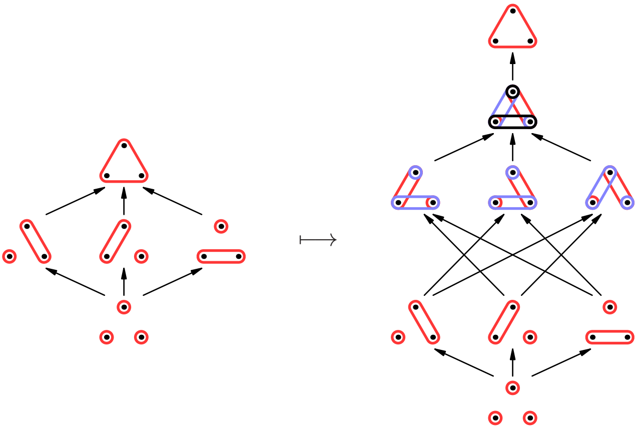

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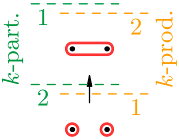

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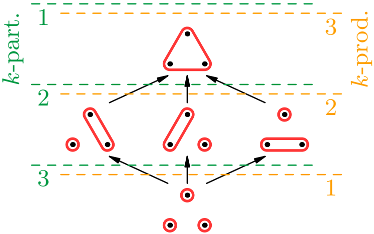
$n = 3$:



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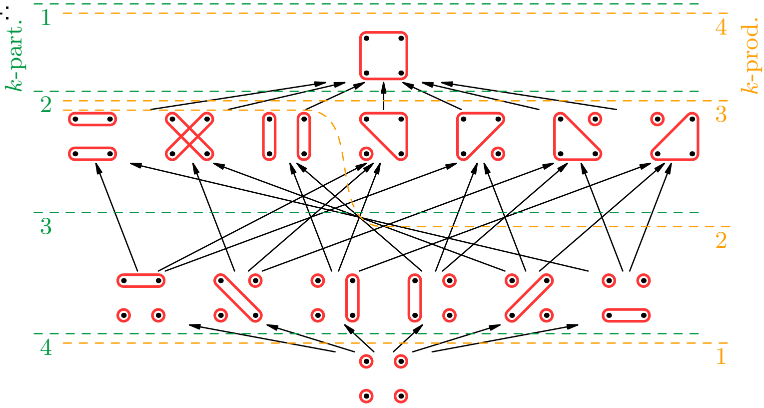

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- with these:
  k-partitionably and *k'-producibly uncorrelated*
  k-partitionably and *k'-producibly separable* states

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  k-partitionability and *k'-producibility entanglement*

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Example: Electron system of molecules

- elementary subsystems: localized atomic orbitals (Pipek-Mezey)
- "atomic split": $\alpha = \{A_1, A_2, \dots, A_{|\alpha|}\}$ (blue)
- "bond split": $\beta = \{B_1, B_2, \dots, B_{|\beta|}\}$ (red)

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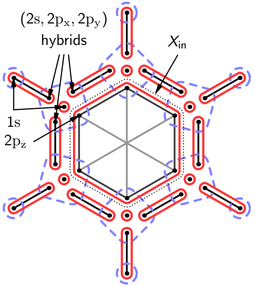
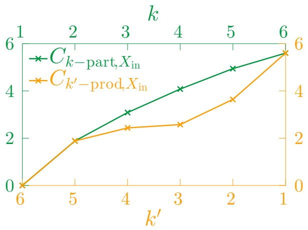
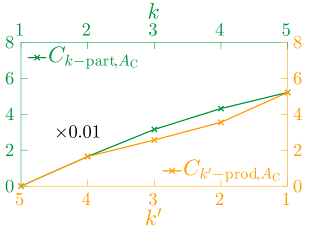
benzene (C_6H_6): $C_\alpha = 29.52$, $C_\beta = 2.33$







(in units $\ln 4$)

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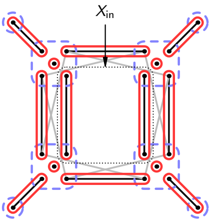
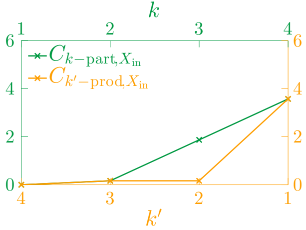
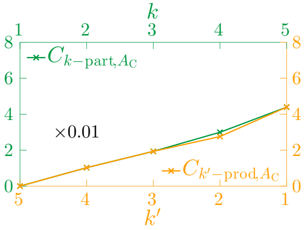
cyclobutadiene (C_4H_4): $C_\alpha = 19.48$, $C_\beta = 3.17$







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Entanglement classes

Level III: Entanglement classes   lattice structure: $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$

- ideal filter: $\underline{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{|\underline{\boldsymbol{\xi}}|}\} \subseteq P_{\mathrm{II}}$ (closed upwards w.r.t. $\preceq$)
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Entanglement classes

Level III: Entanglem

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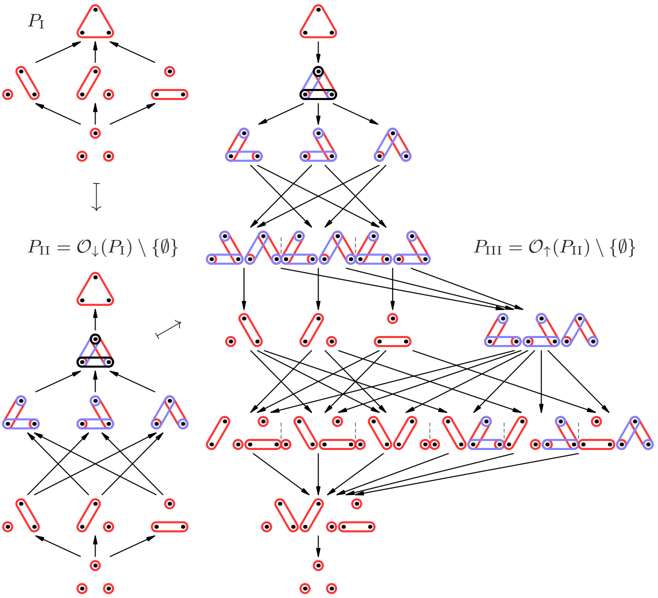

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$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}} := \bigcap_{\boldsymbol{\xi} \notin \underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} \cap \bigcap_{\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$$

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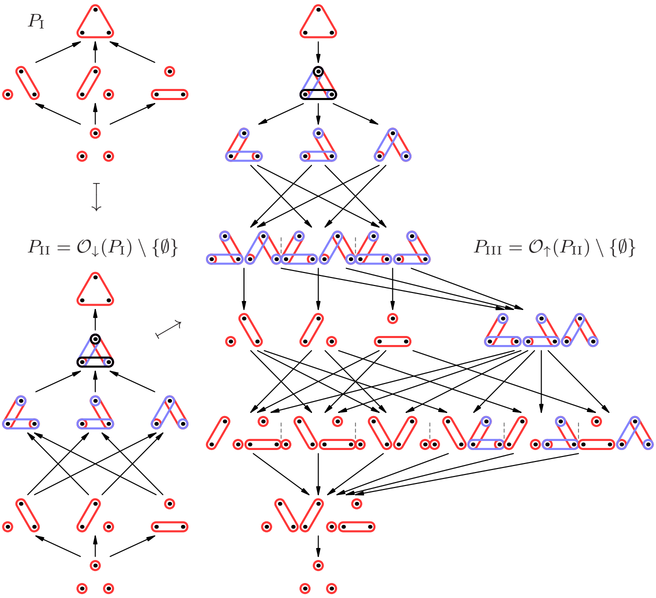

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- LOCC convertibility:
  if $\exists \varrho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}}$, $\exists \Lambda$ LOCC map s.t. $\Lambda(\varrho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$

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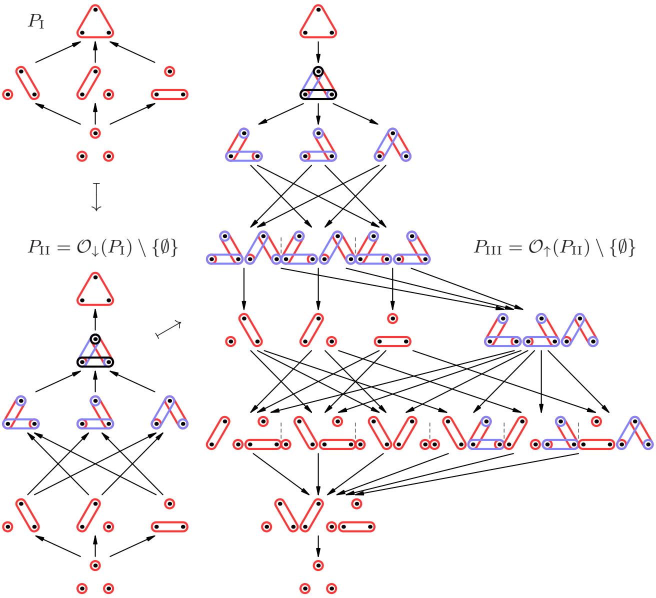

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proven constructively for $n = 3$ Han, Kye, PRA **99**, 032304 (2019)

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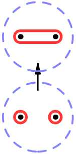
Permutation symmetric correlation and entanglement

Level I.: splitting type of the system of n elementary subsystems

$n = 2$:



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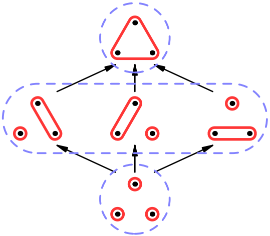

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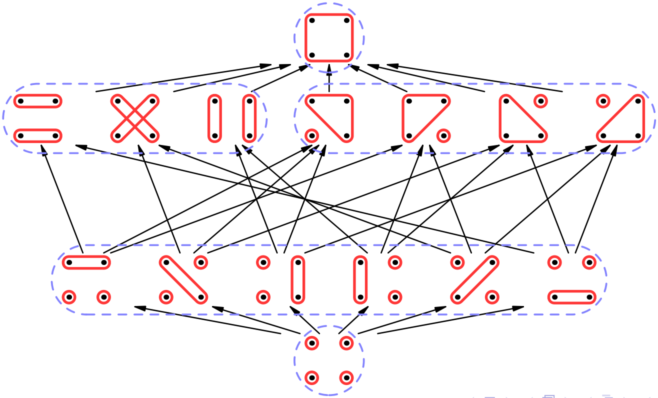

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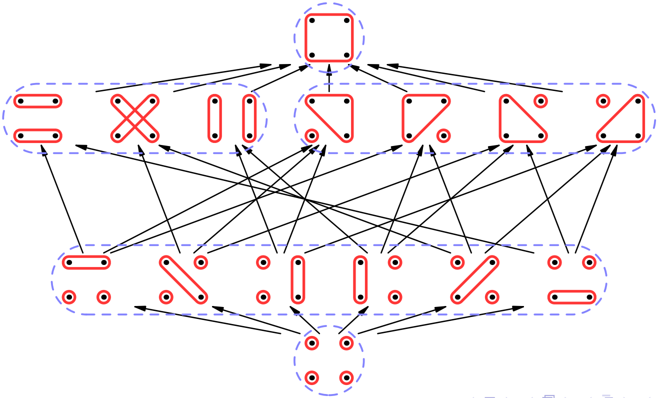

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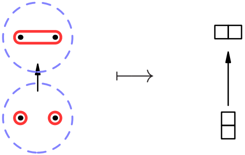

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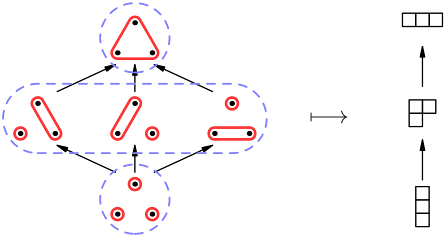

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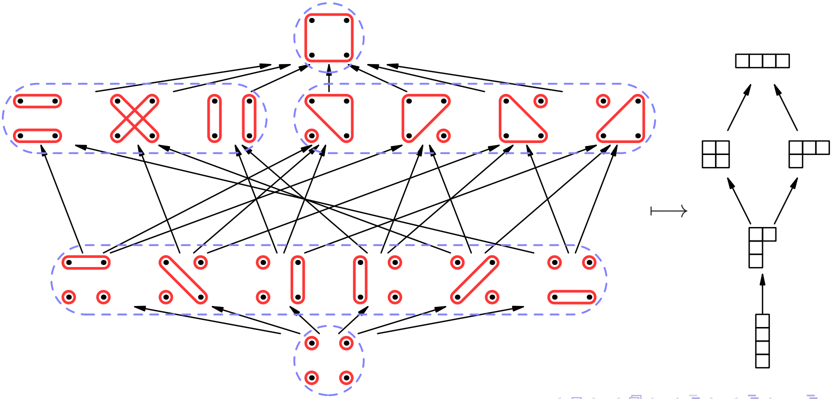

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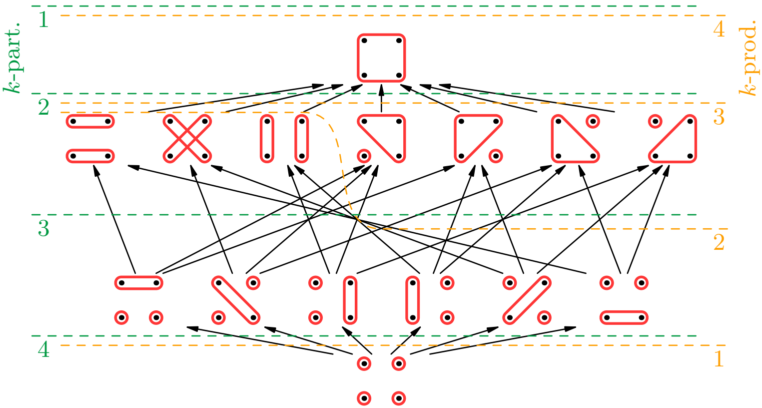
Structure of k-partitionability and k'-producibility

- P_{I} graded lattice, gradation = partitionability



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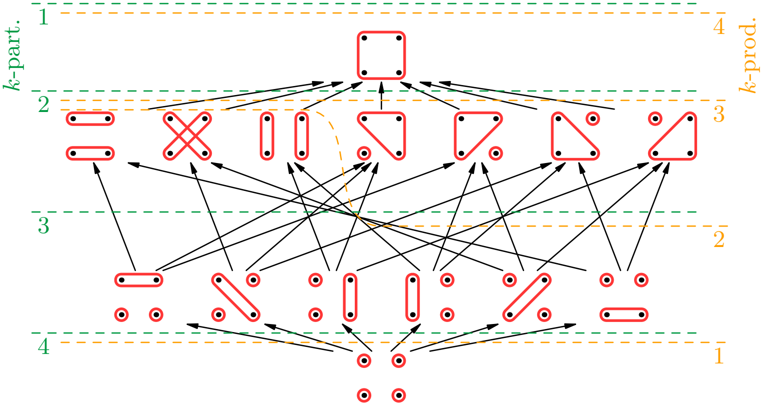

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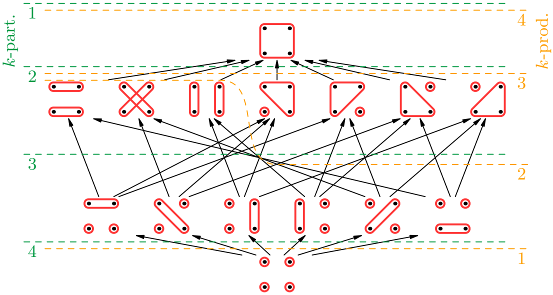

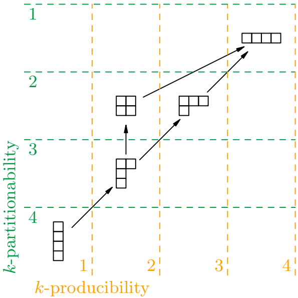

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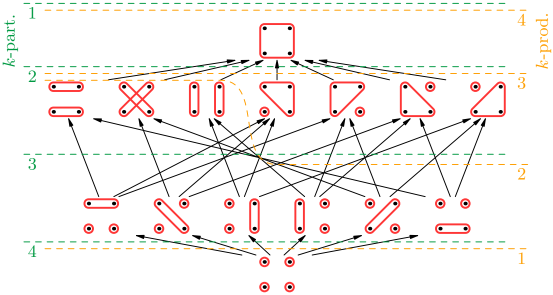

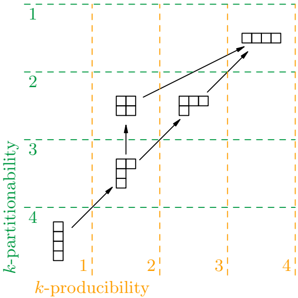

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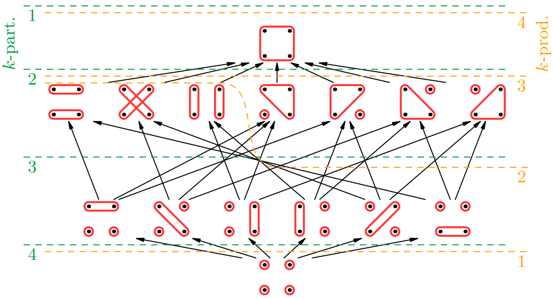

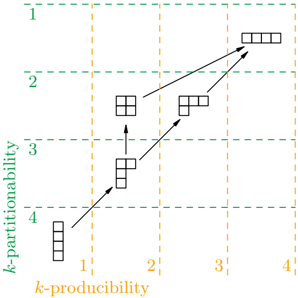

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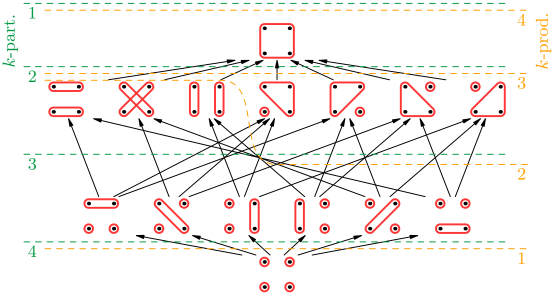

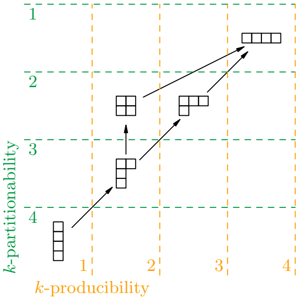

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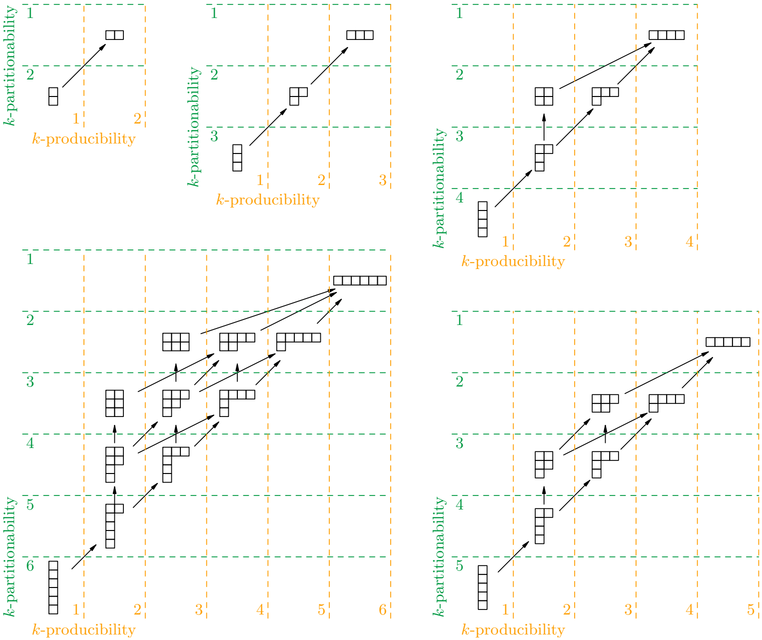
k-partitionability
k-producibility
1
2
3
4
5
6

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Permutation symmetric notions

- k-partitionability and k'-producibility are *prototypes* of them
- moreover, for $n < 7$, the pairs (k, k') of partitionability and producibility parameters are *sufficient* for the parametrization of them

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Notions of correlations:

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Partitionability/producibility: Young diagram min. heigth/max. width, dual

Thank you for your attention!

Szalay, JPhysA **51**, 485302 (2018)

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)

Szalay, PRA **92**, 042329 (2015)

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MINISTRY OF HUMAN CAPACITIES

NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE HUNGARY

PROJECT FINANCED FROM THE NRDI FUND

MOMENTUM OF INNOVATION

Correlation-based clustering – Overview

Bipartite correlation clustering (for treshold T_{b}): split $\gamma = C_1|C_2|\dots|C_{|\gamma|}$, the connectivity clustering of the graph $\big(L, \{(i,j)\}_{T_{\mathrm{b}} \leq C_{i|j}}\big)$,

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Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)

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We have a method to handle these.

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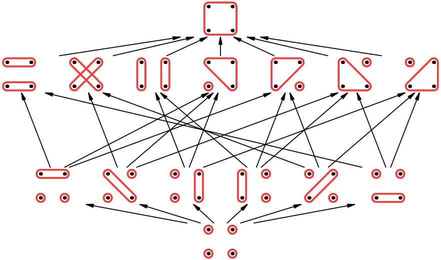

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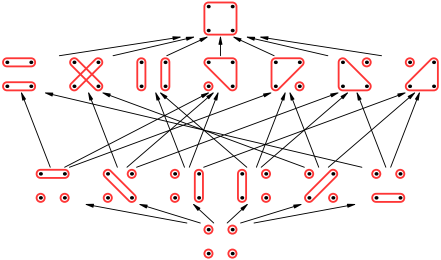

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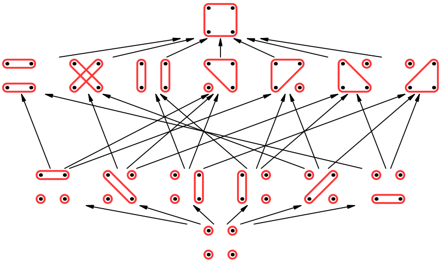
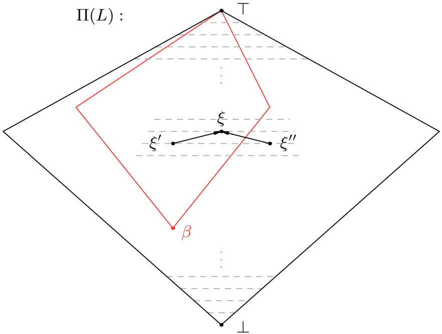
- reformulation: $\exists T_\mathrm{m} > 0$, such that
  $\forall \xi, \xi' \in \Pi(L)$ such that $\xi' \prec\!\cdot\, \xi$, and $\beta \preceq \xi$, then
  $\beta \preceq \xi' \;\Leftrightarrow\; C_{\xi'}(\varrho_L) - C_\xi(\varrho_L) \leq T_\mathrm{m}$





Correlation-based clustering – Properties

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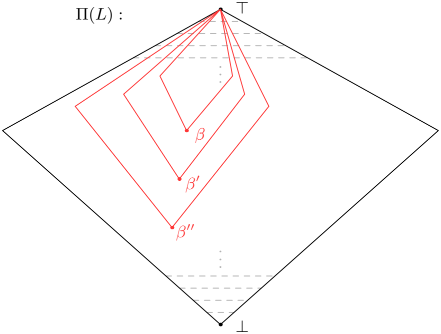
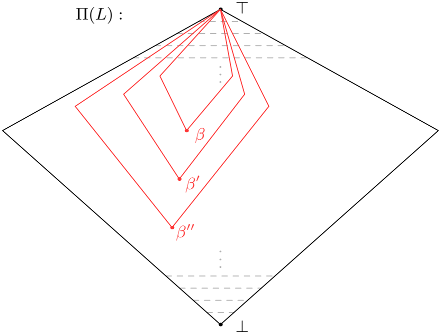
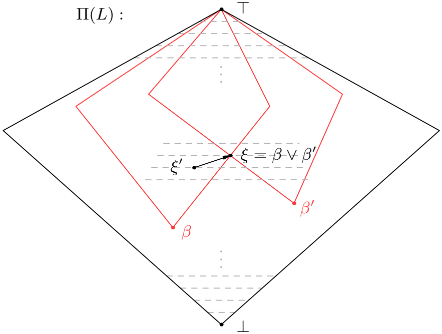


Correlation-based clustering – Properties

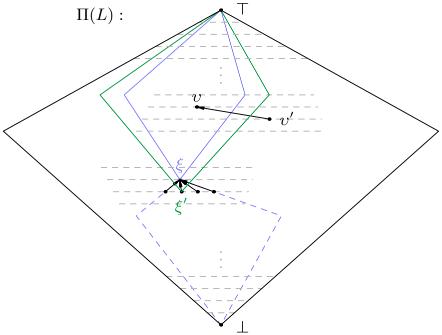
- there might not exist such clustering
- there may exist compatible clusterings (of different T_{m}s),
  but there exist no contradictory ones:

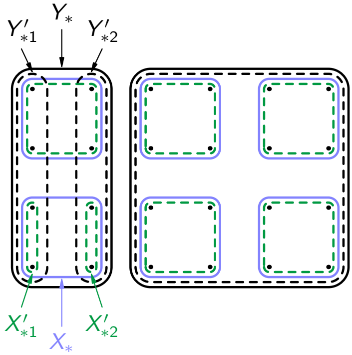




Correlation-based clustering – Finding β

- successive refinement from $\top$ to $\bot$ (taking the smallest step):
  $\forall \upsilon, \upsilon' \in \Pi(L)$ s.t. $\upsilon' \prec \upsilon$, and $\forall \xi \in \Pi(L)$ s.t. $\xi \preceq \upsilon$ but $\xi \not\preceq \upsilon'$,
  then $\min_{\xi' \prec\!\cdot\, \xi} C_{\xi'}(\varrho_L) - C_{\xi}(\varrho_L) \leq C_{\upsilon'}(\varrho_L) - C_{\upsilon}(\varrho_L)$





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- hint: does not dissect $G \in \gamma$ (bipart. corr. clustering), since
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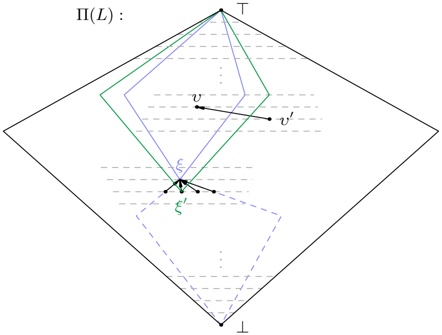

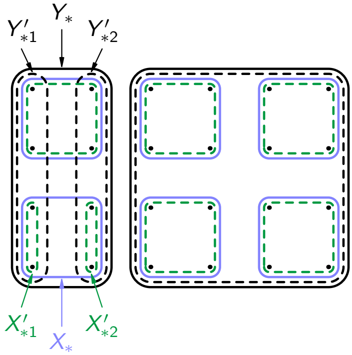

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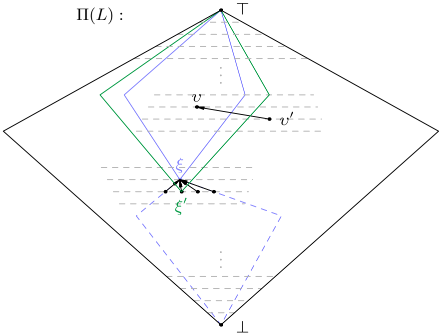
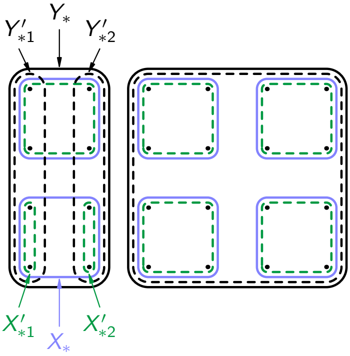
- hidden correlation: $\gamma \prec \beta$



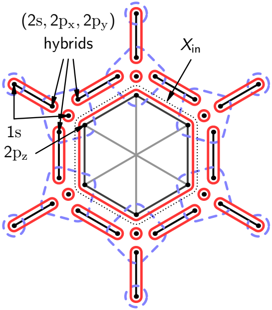


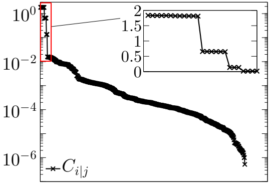
Example: Electron system of molecules

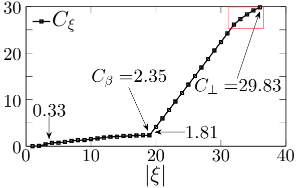
benzene (C_6H_6)

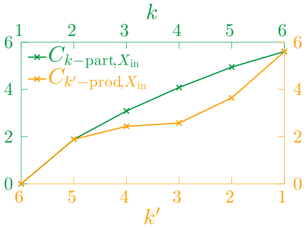
$C_\alpha = 29.52$

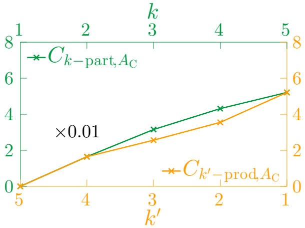
$C_\beta = 2.33$











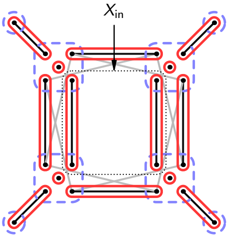
(in units $\ln 4$)

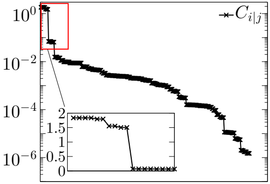
Example: Electron system of molecules

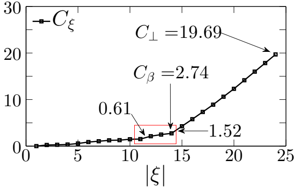
cyclobutadiene (C_4H_4)

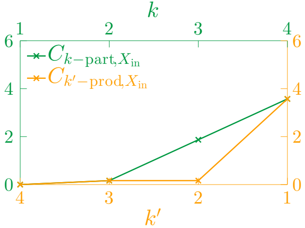
$C_\alpha = 19.48$

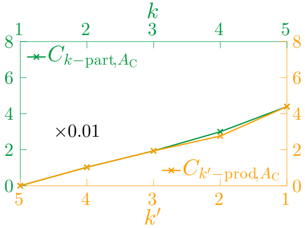
$C_\beta = 3.17$









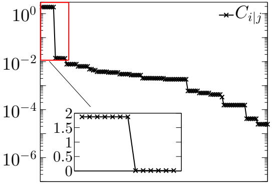


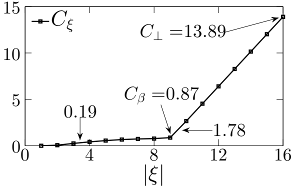
(in units $\ln 4$)

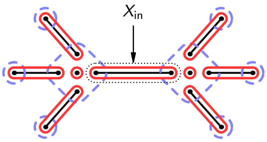
ethane (C_2H_6)

$C_\alpha = 13.84$

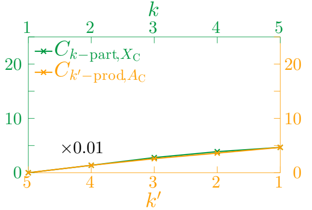
$C_\beta = 0.90$







$$C_{\text{2-part},X_{\text{in}}} = C_{\text{1-prod},X_{\text{in}}} = C_{\perp,X_{\text{in}}} = 1.796$$



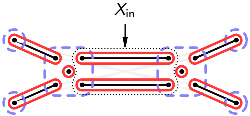
(in units ln 4)

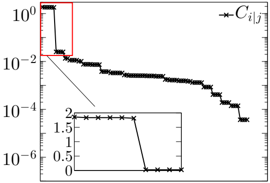
Example: Electron system of molecules

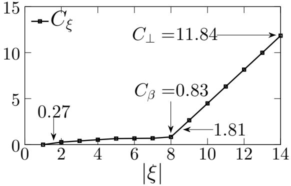
ethylene (C_2H_4)

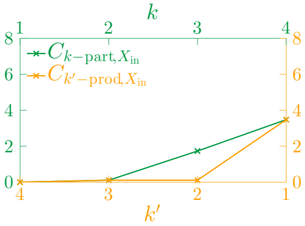
$C_\alpha = 11.76$

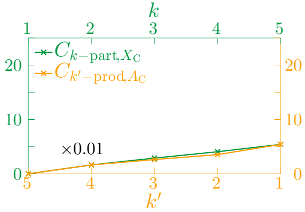
$C_\beta = 1.00$











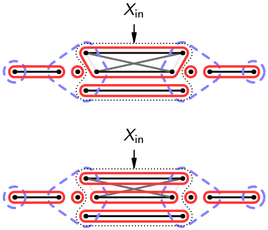
(in units ln 4)

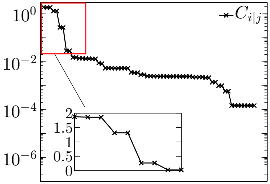
Example: Electron system of molecules

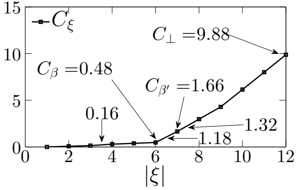
acetylene (C_2H_2)

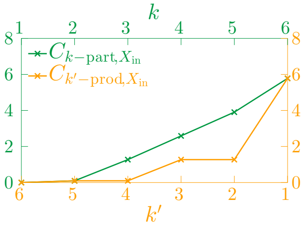
$C_\alpha = 9.74$

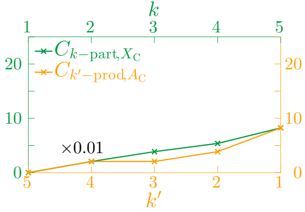
$C_\beta = 0.45, 1.30$











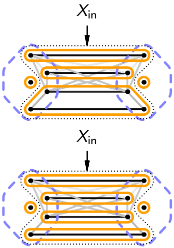
(in units ln 4)

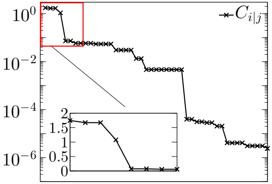
Example: Electron system of molecules

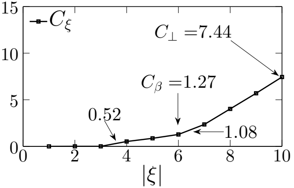
dicarbon (C_2H_0)

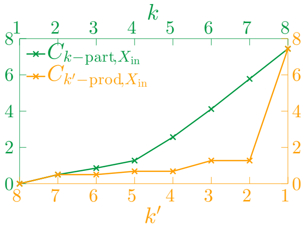
$C_\alpha = 7.06$

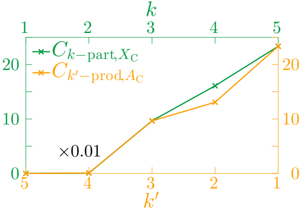
$C_\beta = 0.89, 1.51$











(in units $\ln 4$)