α-Logarithmic negativity

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Motivation: Entanglement and its manipulation

- Separable state: $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$.
- Entangled state: $\rho_{AB} \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$.
- The most natural set of free operations for entanglement manipulation consists of **local operations and classical communication (LOCC)**, which has a complex structure [CLM+14].
- Entangled states cannot be created by LOCC.
- Inspired the resource theory framework: free states + free operations.
- The seminal ideas coming from it are influencing diverse areas: quantum thermodynamics, quantum coherence and superposition, non-Gaussianity, magic states...

Quantifying entanglement

- Entanglement is a key physical resource in quantum information, quantum computation, and quantum cryptography.
- A quantitative theory is highly desirable to fully exploit the power of entanglement.
- Entanglement measure E
  - **Faithfulness**: $E(\rho) = 0$ if and only if ρ is separable.
  - **LOCC monotonicity**: $E(\Lambda(\rho)) \leq E(\rho)$ for any $\Lambda \in \mathrm{LOCC}$.
  - **Entanglement monotone**, **convexity**, **additivity**, etc.
- Zoo of ent. measures [PV07, Chr06].
- Information-processing task gives precise and operationally meaningful way to quantify a given physical resource.

Beyond LOCC

- Due to the mathematical difficulty of dealing with separability and LOCC, it can be helpful to move beyond it.
- The **positive partial transpose** (PPT) criterion was proposed early [Per96, HHH96]
- A state ρ_{AB} is PPT if

$$T_B(\rho_{AB}) \geq 0,$$

  where partial transpose map T_B for orthonormal basis $\{|i\rangle_B\}_i$ is defined as

$$T_B(Y_{AB}) \equiv \sum_{i,j} (I_A \otimes |i\rangle\langle j|_B) \, Y_{AB} \, (I_A \otimes |i\rangle\langle j|_B),$$

- If a state is separable, then it is PPT. Converse is not true in general.

Resource theory of NPT entanglement

- Rains proposed the **resource theory of non-positive partial transpose entanglement** [Rai99, Rai01]. That is, PPT states are free and non-PPT (NPT) states are resourceful.

- Free operations are the completely PPT preserving (C-PPT-P) channels. A bipartite channel $\mathcal{N}_{AB\to A'B'}$ is C-PPT-P if

$$T_{B'} \circ \mathcal{N}_{AB\to A'B'} \circ T_B \in \mathrm{CP}$$

- Equivalently, the Choi operator $J^{\mathcal{N}}_{AA'BB'}$ for $\mathcal{N}_{AB\to A'B'}$ is PPT:

$$T_{BB'}(J^{\mathcal{N}}_{AA'BB'}) \geq 0$$

- Key result: All LOCC bipartite channels are C-PPT-P. So then we can use this relationship to find bounds for tasks in resource theory of entanglement by using resource theory of NPT entanglement.

Logarithmic negativity

- One of the most popular measures for resource theory of NPT entanglement is the **logarithmic negativity** [VW02, Ple05]:

$$E_N(\rho_{AB}) \equiv \log_2 \left\| T_B(\rho_{AB}) \right\|_1 ,$$

- Entanglement monotone:

$$E_N(\rho_{AB}) \geq \sum_x p(x) E_N(\rho_{AB}^x)$$

where $\{p(x), \rho_{AB}^x\}_x$ is ensemble resulting from C-PPT-P instrument.

- Faithful on PPT states:

$$E_N(\rho_{AB}) = 0 \qquad \Longleftrightarrow \qquad \rho_{AB} \in \mathrm{PPT}$$

- It is neither convex nor monogamous.

κ-entanglement

- Recently, the **κ-entanglement measure** was defined as [WW18]

$$E_\kappa(\rho_{AB}) \equiv \log_2 \inf\{\operatorname{Tr}[S_{AB}] : \\ -S_{AB} \leq T_B(\rho_{AB}) \leq S_{AB},\ T_B(S_{AB}) \geq 0\}.$$

- Can be computed by semi-definite programming [WW18].
- Has a direct operational meaning in the resource theory of NPT entanglement, being equal to the exact PPT entanglement cost.
- It is an entanglement monotone, faithful on PPT states, but it is neither convex nor monogamous.
- Reduces to logarithmic negativity for all bipartite two-qubit states and bipartite bosonic Gaussian states.

Relation between log. negativity and κ-entanglement?

- Basic question is whether there is some relation between the two entanglement measures.
- **Result:** logarithmic negativity and κ-entanglement are extremes of a family of entanglement measures

Key quantities

- Consider
  1. $\alpha \geq 1$,
  2. a Hermitian operator $X \neq 0$, and
  3. a positive semi-definite operator $\sigma \neq 0$:

- Then define

$$\mu_\alpha(X\|\sigma) \equiv \begin{cases} \left\| \sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha & \text{if } \operatorname{supp}(X) \subseteq \operatorname{supp}(\sigma) \\ +\infty & \text{else} \end{cases},$$

$$\nu_\alpha(X\|\sigma) \equiv \log_2 \mu_\alpha(X\|\sigma),$$

- Key properties of $\mu_\alpha(X\|\sigma)$ and $\nu_\alpha(X\|\sigma)$ can be derived from [Bei13] and [Hia16]

α-Logarithmic negativities

Definition (α-logarithmic negativity)

Let ρ_{AB} be a bipartite state. Its **α-logarithmic negativity** defined as

$$E_N^\alpha(\rho_{AB}) \equiv \inf_{\sigma_{AB} \in \mathrm{PPT}(A:B)} \nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}),$$

where $\mathrm{PPT}(A:B)$ is the set of PPT states:

$$\mathrm{PPT}(A:B) \equiv \{\sigma_{AB} : \sigma_{AB}, T_B(\sigma_{AB}) \geq 0,\ \mathrm{Tr}[\sigma_{AB}] = 1\}.$$

We now discuss properties of the α-logarithmic negativities.

Ordering

Proposition

Let ρ_{AB} *be a bipartite quantum state, and let* $1 \leq \alpha \leq \beta$*. Then*

$$E_N(\rho_{AB}) \leq E_N^{\alpha}(\rho_{AB}) \leq E_N^{\beta}(\rho_{AB}).$$

Ordering

Follows from a simple generalization of [Bei13, Theorem 7]

Lemma

Let $X \neq 0$ be a Hermitian operator, and let σ be a positive definite operator. Then the following inequality holds for all $\beta > \alpha > 1$:

$$\frac{\alpha}{\alpha - 1}\left[\nu_\alpha(X\|\sigma) - \log_2 \|X\|_1\right] \leq \frac{\beta}{\beta - 1}\left[\nu_\beta(X\|\sigma) - \log_2 \|X\|_1\right]$$

which implies

$$\nu_\alpha(X\|\sigma) \leq \nu_\beta(X\|\sigma).$$

Limits

Proposition

Let ρ_{AB} be a bipartite quantum state. Then

$$\lim_{\alpha \to 1} E_N^{\alpha}(\rho_{AB}) = E_N(\rho_{AB}).$$

Limits

Definition (Max-logarithmic negativity)

For bipartite state ρ_{AB}, max-logarithmic negativity $E_N^{\max}(\rho_{AB})$ defined as

$$E_N^{\max}(\rho_{AB}) \equiv \inf_{\sigma_{AB} \in \mathrm{PPT}(A:B)} \nu_\infty(T_B(\rho_{AB}) \| \sigma_{AB}),$$

and

$$\nu_\infty(X\|\sigma) = D_{\max}(X\|\sigma) = \log_2 \left\| \sigma^{-1/2} X \sigma^{-1/2} \right\|_\infty .$$

Proposition

Let ρ_{AB} be a bipartite quantum state. Then

$$E_\kappa(\rho_{AB}) = E_N^{\max}(\rho_{AB}) = \lim_{\alpha \to \infty} E_N^\alpha(\rho_{AB}).$$

Collapse

Proposition

If ρ_{AB} satisfies the condition $T_B(|T_B(\rho_{AB})|) \geq 0$, then all α-logarithmic negativities are equal; i.e., the following equality holds for all $\alpha \geq 1$:

$$E_N^\alpha(\rho_{AB}) = E_N(\rho_{AB}).$$

All of the following satisfy $T_B(|T_B(\rho_{AB})|) \geq 0$:

1. pure states [ADMVW02],
2. two-qubit states [Ish04],
3. Werner states [APE03], and
4. bosonic Gaussian states [APE03]

Thus, conclude that collapse above holds for such states.

C-PPT-P quantum instruments

Definition

A **C-PPT-P quantum instrument** consists of the collection

$$\{\mathcal{N}^x_{AB\to A'B'}\}_x,$$

where

1. each $\mathcal{N}^x_{AB\to A'B'}$ is CP,
2. the map $T_{B'} \circ \mathcal{N}^x_{AB\to A'B'} \circ T_B$ is CP, and
3. the sum map $\sum_x \mathcal{N}^x_{AB\to A'B'}$ is TP.

Entanglement monotone

Theorem (Entanglement monotone)

Let $\{\mathcal{N}^x_{AB\to A'B'}\}_x$ be a C-PPT-P quantum instrument, and ρ_{AB} a bipartite state. Then α-logarithmic negativity is an **entanglement monotone***; i.e., the following inequality holds for all $\alpha \geq 1$:*

$$E_N^\alpha(\rho_{AB}) \geq \sum_{x:p(x)>0} p(x) E_N^\alpha(\rho^x_{A'B'}),$$

where

$$p(x) \equiv \mathrm{Tr}[\mathcal{N}^x_{AB\to A'B'}(\rho_{AB})],$$
$$\rho^x_{A'B'} \equiv \frac{1}{p(x)} \mathcal{N}^x_{AB\to A'B'}(\rho_{AB}).$$

Entanglement monotone proof

Follows from a simple generalization of [Bei13, Theorem 6]

Lemma

Let $X \neq 0$ be a Hermitian operator, and let σ be a positive definite operator. Let $\mathcal{P}$ be a positive and trace-non-increasing map. Then the following inequality holds for all $\alpha \geq 1$:

$$\nu_\alpha(X\|\sigma) \geq \nu_\alpha(\mathcal{P}(X)\|\mathcal{P}(\sigma)).$$

Entanglement monotone proof

Entanglement monotone property also follows from

Lemma

Let $$Y_{XB} \equiv \sum_x p(x)|x\rangle\langle x|_X \otimes Y_B^x, \qquad \sigma_{XB} \equiv \sum_x q(x)|x\rangle\langle x|_X \otimes \sigma_B^x,$$

where $\{Y_B^x\}_x$ *is a set of Hermitian operators,* $\{p(x)\}_x$ *is a probability distribution,* $\{\sigma_B^x\}$ *is a set of positive definite operators, and* $\{q(x)\}_x$ *is a set of strictly positive reals. Then for* $\alpha \geq 1$*,*

$$\nu_\alpha(Y_{XB}\|\sigma_{XB}) \geq \sum_x p(x)\nu_\alpha(Y_B^x\|\sigma_B^x) + \left(\frac{\alpha-1}{\alpha}\right) D(p\|q),$$

where $D(p\|q) := \sum_x p(x)\log_2(p(x)/q(x))$ *is the classical relative entropy.*

Computable by convex optimization

Proposition

Let ρ_{AB} be a bipartite state and $\alpha \geq 1$. Then the α-logarithmic negativity $E_N^\alpha(\rho_{AB})$ can be calculated by convex optimization.

The above proposition follows from

Lemma

Let $X \neq 0$ be a Hermitian operator, and let σ be a positive definite operator. Then for all $\alpha \geq 1$, the following function is convex:

$$\sigma \mapsto [\mu_\alpha(X\|\sigma)]^\alpha .$$

The above lemma follows directly from [Hia16, Theorem 5.2] (see also brief remarks stated before [Hia16, Theorem 5.3])

Faithfulness

Proposition (Faithfulness)

Let ρ_{AB} *be a bipartite quantum state, and let* $\alpha \geq 1$. *Then*

$$E_N^{\alpha}(\rho_{AB}) \geq 0$$

and

$$E_N^{\alpha}(\rho_{AB}) = 0 \qquad \Longleftrightarrow \qquad \rho_{AB} \in \mathrm{PPT}(A:B).$$

No convexity

By picking

$$\rho^1_{AB} \equiv \Phi^2_{AB}, \qquad \rho^2_{AB} \equiv \frac{1}{2}\left(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}\right),$$
$$\overline{\rho}_{AB} \equiv \frac{1}{2}\left(\rho^1_{AB} + \rho^2_{AB}\right),$$

we find for all $\alpha \in [1, \infty]$ that

$$E^\alpha_N(\rho^1_{AB}) = 1, \qquad E^\alpha_N(\rho^2_{AB}) = 0,$$
$$E^\alpha_N(\overline{\rho}_{AB}) = \log_2 \frac{3}{2},$$

which implies no convexity:

$$E^\alpha_N(\overline{\rho}_{AB}) > \frac{1}{2}\left[E^\alpha_N(\rho^1_{AB}) + E^\alpha_N(\rho^2_{AB})\right].$$

Definition of monogamy

An entanglement measure E is monogamous [CKW00, Ter04, KW04] if the following inequality holds for all tripartite states ρ_{ABC}:

$$E(\rho_{A:B}) + E(\rho_{A:C}) \leq E(\rho_{A:BC}),$$

where the bipartition is indicated by a colon.

No monogamy

- As a consequence of the counterexample given in [WW18, Proposition 7], it follows that the α-logarithmic negativity is not generally monogamous for any choice of $\alpha \in [1, \infty]$.

- Indeed, consider the following state of three qubits:

$$|\psi\rangle_{ABC} \equiv \frac{1}{2}\left(|000\rangle_{ABC} + |011\rangle_{ABC} + \sqrt{2}|110\rangle_{ABC}\right).$$

  Then

$$E_N^{\alpha}(\psi_{A:B}) + E_N^{\alpha}(\psi_{A:C}) > E_N^{\alpha}(\psi_{A:BC}),$$

Additivity?

Given tensor-product state $\omega_{A_1A_2B_1B_2} = \rho_{A_1B_1} \otimes \tau_{A_2B_2}$.

- Logarithmic negativity is **additive** [VW02]:

$$E_N(\omega_{A_1A_2:B_1B_2}) = E_N(\rho_{A_1:B_1}) + E_N(\tau_{A_2:B_2}).$$

- So is κ-entanglement [WW18]:

$$E_\kappa(\omega_{A_1A_2:B_1B_2}) = E_\kappa(\rho_{A_1:B_1}) + E_\kappa(\tau_{A_2:B_2}).$$

- For α-logarithmic negativity, subadditivity holds for $\alpha \in (1, \infty)$:

$$E_N^\alpha(\omega_{A_1A_2:B_1B_2}) \leq E_N^\alpha(\rho_{A_1:B_1}) + E_N^\alpha(\tau_{A_2:B_2}).$$

What about superadditivity?

Generalization to channels

- **Logarithmic negativity of a channel** defined as [HW01]

$$E_N(\mathcal{N}) \equiv \log_2 \left\| T_B \circ \mathcal{N}_{A\to B} \right\|_\diamond .$$

- **Diamond norm** of Herm.-preserving map $\mathcal{M}_{A\to B}$ defined as [Kit97]

$$\left\| \mathcal{M}_{A\to B} \right\|_\diamond \equiv \sup_{\psi_{RA}} \left\| \mathcal{M}_{A\to B}(\psi_{RA}) \right\|_1 ,$$

- Can write the logarithmic negativity of a quantum channel as an optimized version of the logarithmic negativity of quantum states:

$$E_N(\mathcal{N}) = \sup_{\psi_{RA}} E_N(\omega_{RB}),$$

where $\omega_{RB} \equiv \mathcal{N}_{A\to B}(\psi_{RA})$.

α-Logarithmic negativity of a channel

Definition (α-log. negativity of a channel)

The α-logarithmic negativity of a quantum channel is defined for $\alpha \geq 1$ as

$$E_N^\alpha(\mathcal{N}) = \sup_{\psi_{RA}} E_N^\alpha(\omega_{RB}),$$

with $\omega_{RB} \equiv \mathcal{N}_{A\to B}(\psi_{RA})$.

Recover **κ-entanglement of a channel** as a special case ($\alpha \to \infty$), which is equal to the exact PPT simulation cost of a quantum channel (parallel or sequential simulation) [WW18]

Conclusion

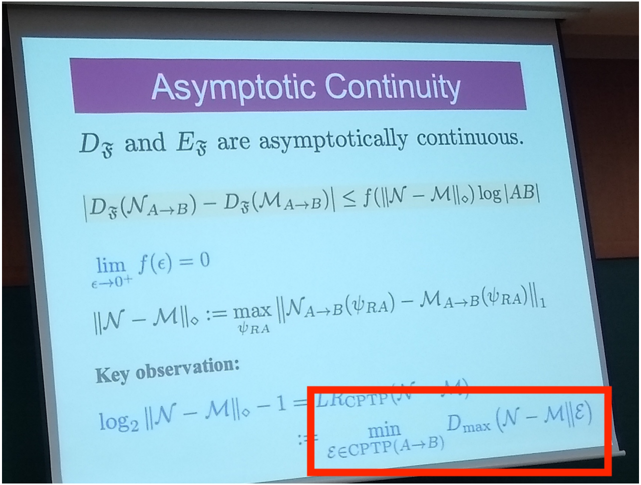
- α-logarithmic negativities are a family of entanglement monotones that include logarithmic negativity and κ-entanglement
- Entanglement monotone property follows from techniques of [Bei13]
- They are ordered, faithful, and computable by convex optimization
- They are neither convex nor monogamous

Going forward

- The concept put forward here can be generalized.
- Idea is to compare an unphysical object versus a physical one to obtain a useful information, distinguishability, or entanglement measure
- In our case, we compare the partial transpose of ρ_{AB} to the set of PPT states using $\nu_\alpha(X\|\sigma)$.

Going forward

Yesterday, we saw an interesting re-expression of diamond norm, using the same concept:



Thanks!

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