Temperley-Lieb quantum channels

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- (Claim 1) Representation theory is a new source to produce ‘interesting’ quantum channels
- (Claim 2) The channels have complicated structures!
- (Claim 3) But some important informational quantities are computable.

Definition

1. H = a finite dimensional Hilbert space.
2. $B(H)$ = the set of all linear maps from H into H.
3. Assicuated to an isometry $V : H_A \to H_B \otimes H_E$ are the following quatum channels

$$\Phi(\rho) : B(H_A) \to B(H_B), \rho \mapsto (id \otimes tr)(V\rho V^*) \ and$$
$$\widetilde{\Phi}(\rho) : B(H_A) \to B(H_E), \rho \mapsto (tr \otimes id)(V\rho V^*).$$

Here, $\widetilde{\Phi}$ is called the complementary channel of Φ.

Temperley-Lieb Channels 1/3

From the representation theory of $SU(2)$ and O_N^+, we have a family of isometries

$$V^{l,m}_{k,SU(2)},\ V^{l,m}_{k,O_N^+} : H_k \hookrightarrow H_l \otimes H_m,$$

where $l, m \in \{0\} \cup \mathbb{N}$ and k is one of $\begin{cases} l+m \\ l+m-2 \\ \quad\vdots \\ |l-m| \end{cases}$.

For $\mathbb{G} = SU(2)$ or O_N^+, we define channels

$$\Phi^{l,\overline{m}}_{k,\mathbb{G}} : B(H_k) \to B(H_l),\ \rho \mapsto (id \otimes tr)(V^{l,m}_{k,\mathbb{G}} \rho (V^{l,m}_{k,\mathbb{G}})^*) \text{ and}$$
$$\Phi^{\bar{l},m}_{k,\mathbb{G}} : B(H_k) \to B(H_m),\ \rho \mapsto (tr \otimes id)(V^{l,m}_{k,\mathbb{G}} \rho (V^{l,m}_{k,\mathbb{G}})^*).$$

Note that the channels above are complementary to each other.

Temperley-Lieb Channels 2/3

Let $G = SU(2)$. Then $H_l = \mathbb{C}^{l+1}$ and $\Phi^{l,\overline{m}}_{k,SU(2)} : B(H_k) \to B(H_l)$.

Example

For $(l, m) = (1, 1)$, possible k is either 0 or 2:

$$\Phi^{1,\overline{1}}_{0,SU(2)} : a \mapsto \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{a}{2} \end{bmatrix}$$

$$\Phi^{1,\overline{1}}_{2,SU(2)} : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} + \frac{a_{22}}{2} & \frac{a_{12}+a_{23}}{\sqrt{2}} \\ \frac{a_{21}+a_{32}}{\sqrt{2}} & \frac{a_{22}}{2} + a_{33} \end{bmatrix}$$

Temperley-Lieb Channels 3/3

Example

For $(l, m) = (3, 2)$, possible k is one of $1, 3$ or 5:

$$\Phi^{3,\overline{2}}_{1,SU(2)} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{a}{2} & \frac{b}{\sqrt{12}} & & \\ \frac{c}{\sqrt{12}} & \frac{a}{3} + \frac{d}{6} & \frac{b}{3} & \\ & \frac{c}{3} & \frac{a}{6} + \frac{d}{3} & \frac{b}{\sqrt{12}} \\ & & \frac{c}{\sqrt{12}} & \frac{d}{2} \end{bmatrix}$$

$$\Phi^{3,\overline{2}}_{3,SU(2)} : M_4(\mathbb{C}) \to M_4(\mathbb{C})$$

$$\Phi^{3,\overline{2}}_{5,SU(2)} : M_6(\mathbb{C}) \to M_4(\mathbb{C}).$$

G-covariant quantum channels 1/3

Example

Suppose that a q.channel $\Phi : M_2 \to M_2$ satisfies

$$\Phi(U\rho U^*) = U\Phi(\rho)U^* \ \forall U \in SU(2).$$

Then Φ is a convex combination of $\Phi_1^{1,\bar{0}}$ and $\Phi_1^{1,\bar{2}}$, where

$$\Phi_{1,SU(2)}^{1,\bar{0}} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\Phi_{1,SU(2)}^{1,\bar{2}} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{a+2d}{3} & -\frac{b}{3} \\ -\frac{c}{3} & \frac{2a+d}{3} \end{bmatrix}.$$

We are going to consider all dynamical transformations

$$\rho \mapsto \pi(x)\rho\pi(x)^{-1}$$

arising from irreducible unitary representations π!

G-covariant quantum channels 2/3

Definition

1. A unitary representation is a continuous ft $\pi : G \to B(H)$ s.t.
   1. $\pi(x)$ is unitary $\forall x$ and
   2. $\pi(xy) = \pi(x)\pi(y)$ $\forall x, y$.
2. A unitary rep. $\pi : G \to B(H)$ is called irreducible if there do not exist other two reps π_1, π_2 s.t.

$$\pi(x) \cong \begin{bmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{bmatrix}.$$

G-covariant quantum channels 3/3

Definition

Let $\pi_A : G \to B(H_A)$ and $\pi_B : G \to B(H_B)$ be irr. unitary reps. Then a quantum channel $\Phi : B(H_A) \to B(H_B)$ is called G-irreducibly covariant w.r.t. (π_A, π_B) if

$$\Phi\left(\pi_A(x)\rho\pi_A(x^{-1})\right) = \pi_B(x)\Phi(\rho)\pi_B(x^{-1})\ \forall x \in G.$$

Remark

- *The channels $\Phi^{l,\overline{m}}_{k,SU(2)}$ and $\Phi^{\overline{l},m}_{k,SU(2)}$ are $SU(2)$-irreducibly covariant.*
- *Within the framework of compact quantum groups $\mathbb{G}$, the notion of $\mathbb{G}$-covaraince can be defined naturally. Moreover, the channels*

$$\Phi^{l,\overline{m}}_{k,O_N^+} \text{ and } \Phi^{\overline{l},m}_{k,O_N^+} \text{ are } O_N^+ - \textit{irreducibly covariant}.$$

Advantages from G-covariance

Suppose that $\Phi : B(H_A) \to B(H_B)$ is irreducibly G-covariant.

- Various averaging techniques, e.g. bi-stochastic property!

$$\Phi\left(\frac{1}{dim(H_A)} Id_A\right) = \Phi\left(\int_G \pi_A(x)\rho\pi_A(x^{-1})dx\right)$$
$$= \int_G \pi_B(x)\Phi(\rho)\pi_B(x^{-1})dx = \frac{1}{dim(H_B)} Id_B.$$

- $C_\chi(\Phi) = \log(dim(H_B)) - H_{min}(\Phi)$. Moreover, any minimizer of H_{min} gives an explicit optimizer of C_χ.

- If Φ is degradable, then $Q(\Phi)$ is attained at $\frac{1}{dim(H_A)} Id_{H_A}$.

We are going to talk about the following properties for Temperley-Lieb channels $\Phi_{k,\mathbb{G}}^{l,\overline{m}}$ and $\Phi_{k,\mathbb{G}}^{\bar{l},m}$:

- Structural properties
  - Entanglement-breaking property
  - Degradability
- Quantitative properties
  - Minimum output entropy
  - One-shot classical/quantum capacities
- Additivity questions and Future works

Entanglement-breaking property

Theorem (BCLY, submitted)

1. *The channel* $\Phi^{l,\overline{m}}_{k,SU(2)}$ *is not entanglement-breaking* $\Leftrightarrow$ *the channel* $\Phi^{l,\overline{m}}_{k,SU(2)}$ *is not PPT* $\Leftrightarrow k > m - l$.
2. *The channel* $\Phi^{\overline{l},m}_{k,SU(2)}$ *is not entanglement-breaking* $\Leftrightarrow$ *the channel* $\Phi^{\overline{l},m}_{k,SU(2)}$ *is not PPT* $\Leftrightarrow k > l - m$.

Example

For $(l, m) = (3, 2)$, we have the following characterization:

$\Phi^{3,\overline{2}}_{5,SU(2)}$	*Not PPT*	$\Phi^{\overline{3},2}_{5,SU(2)}$	*Not PPT*
$\Phi^{3,\overline{2}}_{3,SU(2)}$	*Not PPT*	$\Phi^{\overline{3},2}_{3,SU(2)}$	*Not PPT*
$\Phi^{3,\overline{2}}_{1,SU(2)}$	*Not PPT*	$\Phi^{\overline{3},2}_{1,SU(2)}$	*Entanglement − breaking*!

Entanglement-breaking property

Theorem (BCLY, submitted)

Let $\mathbb{G} = O_N^+$ with $N \geq 3$.

1. *The channel $\Phi^{l,\overline{m}}_{k,O_N^+}$ is* *not entanglement-breaking* *if $k > m - l$.*
2. *The channel $\Phi^{\overline{l},m}_{k,O_N^+}$ is* *not entanglement-breaking* *if $k > l - m$*

   .

Example

For $(l, m) = (3, 2)$, we have the following characterization:

$\Phi^{3,\overline{2}}_{5,O_N^+}$	*Not PPT* if $N >> 1$	$\Phi^{\overline{3},2}_{5,O_N^+}$	*Not PPT* if $N >> 1$
$\Phi^{3,\overline{2}}_{3,O_N^+}$	*Not PPT* if $N >> 1$	$\Phi^{\overline{3},2}_{3,O_N^+}$	*Not PPT* if $N >> 1$
$\Phi^{3,\overline{2}}_{1,O_N^+}$	*Not PPT* if $N >> 1$	$\Phi^{\overline{3},2}_{1,O_N^+}$	???

Degradability

Theorem (BCLY, submitted)

Since $\Phi^{l,\overline{m}}_{k,SU(2)} = \Phi^{\overline{m},l}_{k,SU(2)}$, we may assume $l \geq m$ and we have the following results:

1. $l = m \Rightarrow \Phi^{l,\bar{l}}_{k,SU(2)}$ *is always degradable.*
2. *If $l > m$,*

$\Phi^{l,\overline{m}}_{l+m,SU(2)}$	*degradable*	$\Phi^{\bar{l},m}_{l+m,SU(2)}$	*Not degradable*
	$\vdots$		$\vdots$
$\Phi^{l,\overline{m}}_{k,SU(2)}$	??	$\Phi^{\bar{l},m}_{k,SU(2)}$	*Not degradable*
	$\vdots$		$\vdots$
$\Phi^{l,\overline{m}}_{l-m,SU(2)}$	*degradable*	$\Phi^{\bar{l},m}_{l-m,SU(2)}$	*Not degradable.*

Degradability

Example

1. For $(l, m) = (3, 2)$,

$$\Phi^{3,\overline{2}}_{5,SU(2)} \ \textit{degradable} \qquad \Phi^{\overline{3},2}_{5,SU(2)} \ \textit{Not degradable}$$
$$\Phi^{3,\overline{2}}_{3,SU(2)} \ \textit{Not degradable} \qquad \Phi^{\overline{3},2}_{3,SU(2)} \ \textit{Not degradable}$$
$$\Phi^{3,\overline{2}}_{1,SU(2)} \ \textit{degradable} \qquad \Phi^{\overline{3},2}_{1,SU(2)} \ \textit{Not degradable}.$$

2. For $(l, m) = (4, 3)$,

$$\Phi^{4,\overline{3}}_{7,SU(2)} \ \textit{degradable} \qquad \Phi^{\overline{4},3}_{7,SU(2)} \ \textit{Not degradable}$$
$$\Phi^{4,\overline{3}}_{5,SU(2)} \ \textit{Not degradable} \qquad \Phi^{\overline{4},3}_{5,SU(2)} \ \textit{Not degradable}$$
$$\Phi^{4,\overline{3}}_{3,SU(2)} \ \textit{Not degradable} \qquad \Phi^{\overline{4},3}_{3,SU(2)} \ \textit{Not degradable}$$
$$\Phi^{4,\overline{3}}_{1,SU(2)} \ \textit{degradable} \qquad \Phi^{\overline{4},3}_{1,SU(2)} \ \textit{Not degradable}.$$

Degradability

Theorem (BCLY, submitted)

1. *For $l \geq m$ and sufficiently large $N >> 1$,*

$$\Phi^{l,\overline{m}}_{l+m,O_N^+} \ \textit{Not degradable} \qquad \Phi^{\overline{l},m}_{l+m,O_N^+} \ \textit{Not degradable}$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi^{l,\overline{m}}_{l-m+2,O_N^+} \ \textit{Not degradable} \qquad \Phi^{\overline{l},m}_{l-m+2,O_N^+} \ \textit{Not degradable}$$

$$\Phi^{l,\overline{m}}_{l-m,SU(2)} \ ?? \qquad \Phi^{\overline{l},m}_{l-m,O_N^+} \ \textit{Not degradable}.$$

2. *For $l \leq m$ and sufficiently large $N >> 1$,*

$$\Phi^{l,\overline{m}}_{l+m,O_N^+} \ \textit{Not degradable} \qquad \Phi^{\overline{l},m}_{l+m,O_N^+} \ \textit{Not degradable}$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi^{l,\overline{m}}_{m-l+2,O_N^+} \ \textit{Not degradable} \qquad \Phi^{\overline{l},m}_{m-l+2,O_N^+} \ \textit{Not degradable}$$

$$\Phi^{l,\overline{m}}_{m-l,SU(2)} \ \textit{Not degradable} \qquad \Phi^{\overline{l},m}_{m-l,O_N^+} \ ??.$$

Quantitative aspects of O_N^+-TL channels?

- Minimum output entropy H_{min}
- One-shot classical/quantum capacity $C_\chi/Q^{(1)}$

Minimum output entropies of Temperley-Lieb channels

Definition

The minimum output entropy of a q.channel $\Phi : B(H_A) \to B(H_B)$ is defined by

$$H_{min}(\Phi) = \min_{\rho:\ q.states} H(\Phi(\rho)). \tag{1}$$

Note that $H_{min}(\Phi) = H_{min}(\widetilde{\Phi})$ for any q.channels.

Theorem

1. *(BC18)*

$$0 \leq \lim_{N\to\infty} \left\{ H_{min}(\Phi^{l,\overline{m}}_{k,O_N^+}) - \frac{l+m-k}{2}\log(N) \right\}$$

2. *(BCLY, submitted)*

$$0 = \lim_{N\to\infty} \left\{ H_{min}(\Phi^{l,\overline{m}}_{k,O_N^+}) - \frac{l+m-k}{2}\log(N) \right\}$$

One-shot classical/quantum capacities

Definition

Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel.

1. The Holevo capacity is defined by

$$C_\chi(\Phi) = \sup_{(p_i,\rho_i)_i} \left\{ H(\Phi(\sum_i p_i \rho_i)) - \sum_i p_i H(\Phi(\rho_i)) \right\},$$

where $p_i \geq 0$, $\sum_i p_i = 1$ and ρ_i's are quantum states.

2. The one-shot quantum capacity is defined by

$$Q^{(1)}(\Phi) = \sup_{\rho:\ a\ state} \left\{ H(\Phi(\rho)) - H(\widetilde{\Phi}(\rho)) \right\}.$$

It is known that $Q^{(1)}(\Phi) \leq C_\chi(\Phi)$ for any channel Φ.

Classical/quantum capacities

Definition

Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel. The classical capacity and the quantum capacity are defined by

$$C(\Phi) = \lim_{n\to\infty} \frac{C_\chi(\Phi^{\otimes n})}{n} \textit{ and } Q(\Phi) = \lim_{n\to\infty} \frac{Q^{(1)}(\Phi^{\otimes n})}{n} \textit{ resp.}$$

$$\begin{array}{ccc} Q^{(1)}(\Phi) & \xrightarrow{\leq} & C_\chi(\Phi) \\ \leq \big\downarrow & & \big\downarrow \leq \\ Q(\Phi) & \xrightarrow[\leq]{} & C(\Phi) \end{array}$$

Coding theorems assert that $\left\{ \begin{array}{l} e^{nC(\Phi)} \textit{ codewords} \\ \textit{an } e^{nQ(\Phi)} - \textit{dim'l unit sphere} \end{array} \right.$ are asymptotically communicatable through $\Phi^{\otimes n} : B(H_A^{\otimes n}) \to B(H_B^{\otimes n})$.

Additivity of capacities from structural properties

- Φ: $\left\{ \begin{array}{l} anti - degradable\ or \\ entanglment - breaking \end{array} \right. \Rightarrow Q(\Phi) = Q^{(1)}(\Phi) = 0.$
- Φ: degradable $\Rightarrow Q(\Phi) = Q^{(1)}(\Phi).$
- Φ: entanglement-breaking $\Rightarrow C(\Phi) = C_\chi(\Phi).$

One-shot capacities of Templerley-Lieb channels

Theorem (BCLY, submitted)

For Temperley-Lieb channels of O_N^+, we have

$$(1)\ \lim_{N\to\infty}\left\{Q^{(1)}(\Phi^{l,\overline{m}}_{k,O_N^+}) - \frac{l+k-m}{2}\log(N)\right\} = \lim_{N\to\infty}\left\{C_\chi(\Phi^{l,\overline{m}}_{k,O_N^+}) - \frac{l+k-m}{2}\log(N)\right\} = 0.$$

$$(2)\ \lim_{N\to\infty}\left\{Q^{(1)}(\Phi^{\overline{l},m}_{k,O_N^+}) - \frac{m+k-l}{2}\log(N)\right\} = \lim_{N\to\infty}\left\{C_\chi(\Phi^{\overline{l},m}_{k,O_N^+}) - \frac{m+k-l}{2}\log(N)\right\} = 0.$$

Note that

$$0 = Q(\Phi^{\overline{l},m}_{l+m,SU(2)}) < C(\Phi^{\overline{l},m}_{l+m,SU(2)}) = \log(m+1) \text{ if } l \geq m.$$

Additivity questions on Temperley-Lieb channels

Question (in progress)

For any (l_j, m_j, k_j) *$(1 \le j \le t)$,*

$$Q1. \lim_{N\to\infty}\left\{H_{min}\left(\bigotimes_{j=1}^{t}\Phi_{k_j}^{l_j,\overline{m_j}}\right)-\sum_{j=1}^{t}H_{min}(\Phi_{k_j}^{l_j,\overline{m_j}})\right\}=0?$$

$$Q2. \lim_{N\to\infty}\left\{C_\chi\left(\bigotimes_{j=1}^{t}\Phi_{k_j}^{l_j,\overline{m_j}}\right)-\sum_{j=1}^{t}C_\chi(\Phi_{k_j}^{l_j,\overline{m_j}})\right\}=0?$$

$$Q3. \lim_{N\to\infty}\left\{Q^{(1)}\left(\bigotimes_{j=1}^{t}\Phi_{k_j}^{l_j,\overline{m_j}}\right)-\sum_{j=1}^{t}Q^{(1)}(\Phi_{k_j}^{l_j,\overline{m_j}})\right\}=0?$$

Future works

- Entanglement-assisted classical capacity
- Entanglement of formation of Choi matrices
- Completely bounded minimum output entropies

Thank you for your attention.