On Positive Partial Transpose Squared Conjecture

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NOTATIONS

- Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices over the complex field $\mathbb{C}$.
- A matrix A in $M_n(\mathbb{C})$ is positive semi-definite(PSD), and write $A \geq 0$, if A is hermitian and all eigenvalues of A are non-negative.
- Denote by $M_n^+(\mathbb{C})$ the set of all PSD matrices in $M_n(\mathbb{C})$.
- Denote by $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ the space of all linear maps from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$.

Positive Maps

- A linear map ϕ from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$ is positive if $\phi(M_m^+(\mathbb{C})) \subseteq M_n^+(\mathbb{C})$.
- The identity map on $M_n(\mathbb{C})$ and the transpose map on $M_n(\mathbb{C})$ are denoted by id_n and τ_n respectively.
- A map ϕ is k-positive if the map $id_k \otimes \phi : M_k(M_m(\mathbb{C})) \to M_k(M_n(\mathbb{C}))$ is positive.
- A map ϕ is k-copositive if the map $\tau_k \otimes \phi : M_k(M_m(\mathbb{C})) \to M_k(M_n(\mathbb{C}))$ is positive.

PPT Binding Maps

- A completely (co)positive map is k-(co)positive for every k.
- A map is decomposable if it is the sum of a completely positive map and a completely copositive map.
- A PPT binding map is both completely positive and completely copositive.

Here PPT stands for "positive partial transposition" since the Choi matrix of such a map is positive under partial transpose.

Entanglement Breaking Channels

- A quantum channel is a trace-preserving CP map.
- A linear map ϕ is entanglement breaking if its Choi matrix is separable.

Entanglement breaking quantum channels represent useless processes for any non-classical communication task.

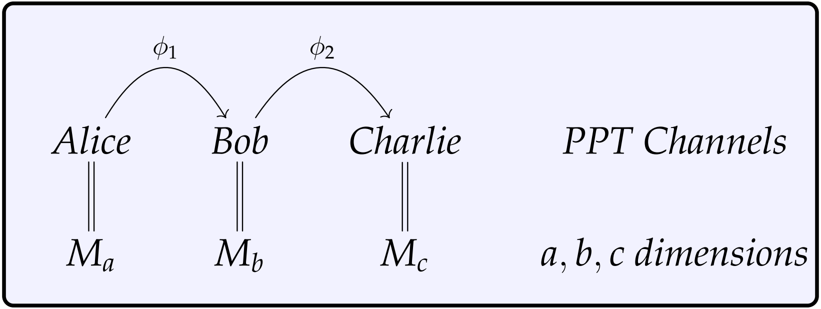
THE PPT SQUARED CONJECTURE

The following conjecture is introduced by M. Christandl.

> For any PPT binding map $\phi \in B(M_n(\mathbb{C}), M_n(\mathbb{C}))$, its square $\phi \circ \phi$ is entanglement breaking.

- Our proof of the conjecture when $n = 3$ is a direct consequence of our result that two-qutrit PPT states have Schmidt number at most two [Chen et al., 2018].
- Our proof is independent from the one found by Müller-Hermes [Christandl et al., 2018].
- Note that the PPT squared conjecture in high dimensional cases are open.

A PICTURE TO ILLUSTRATE



$\Longrightarrow$ Is the composite channel $\phi_2 \circ \phi_1$ entanglement breaking?

Trivial Lifting Zero

Think about how to trivially embed a map ϕ in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$ to $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$.

- The Choi matrix of ϕ resides in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$.
- C_ϕ writes as a block matrix $\begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{pmatrix}$.
- Take it as the Choi matrix $C_\phi \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$:
$$\begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) & \phi(E_{13}) = 0 \\ \phi(E_{21}) & \phi(E_{22}) & \phi(E_{23}) = 0 \\ \phi(E_{31}) = 0 & \phi(E_{32}) = 0 & \phi(E_{33}) = 0 \end{pmatrix} \triangleq C_\phi.$$
- A degenerate $\phi \in B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ admits a similar Choi matrix and reduces to a map in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$

Trivial Lifting One

- Given a linear map $\chi \in B(M_s(\mathbb{C}), M_n(\mathbb{C}))$, fix the canonical matrix unit basis E_{ij}, $i, j = 1, .., s$, in $M_s(\mathbb{C})$, under which the Choi matrix is $C_\chi = [\chi(E_{ij})]_{i,j=1}^{s} \in M_s(M_n(\mathbb{C}))$.
- Given $I = \{n_1, ..., n_p\} \subset \{1, ..., s+p\}$, where $n_1 < \cdots < n_p$, extend the matrix C_χ to a $(s+p) \times (s+p)$ block matrix $C_I^{lift} \in M_{s+p}(M_n(\mathbb{C}))$ by adding one row and one column of $n \times n$ zero matrices at the n_k^{th} level for each $k = 1, ..., p$ as in the next slide:

TRIVIAL LIFTING TWO

$$C_I^{lift} \triangleq \begin{array}{c} \\ 1^{st} \\ \vdots \\ n_k^{th} \\ \vdots \\ (s+p)^{th} \end{array} \begin{array}{c} \begin{array}{ccccc} 1^{st} & \cdots & n_k^{th} & \cdots & (s+p)^{th} \end{array} \\ \begin{pmatrix} \chi(E_{11}) & \cdots & 0 & \cdots & \chi(E_{1,s}) \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \chi(E_{s,1}) & \cdots & 0 & \cdots & \chi(E_{s,s}) \end{pmatrix} \end{array}.$$

TRIVIAL LIFTING THREE

- Denote by $\tilde{\chi}_I$ the map in $B(M_{s+p}(\mathbb{C}), M_n(\mathbb{C}))$ associated with the Choi matrix $C_{\tilde{\chi}_I} = [\tilde{\chi}_p(E_{ij})]_{i,j=1}^{s+p} = C_I^{lift}$.
- The map $\tilde{\chi}_I$ is called a I-trivial lifting of the original map χ. If $I = \{q\}$ is a singleton, simply denote by $\tilde{\chi}_q$ the q-trivial lifting of χ.

Choi's Decomposition

The following result is taken from [Yang et al., 2016].

> Let ϕ be a non-zero k-positive $(2 \leq k < \min\{m, n\})$ map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. There exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p-trivial lifting of a $(k-1)$-positive map in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, for some $p \in \{1, ..., m\}$.

Our proof relies on enhancing a peel-off technique first appeared in [Marciniak, 2010].

When 2-Positivity implies Decomposability

- In $B(M_m(\mathbb{C}), M_n(\mathbb{C})), mn \leq 6$, Woronowicz and Størmer showed that every positive map is decomposable.
- Further we showed that in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, although positive maps may not be decomposable, 2-positive maps are always decomposable.

> Every 2-(co)positive map ϕ in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.

Trade Positivity for Dimension

$2 - positive$	$decomposable$	$decomposable$
$\Vert$	$\uparrow$	$\uparrow$
$2 - positive$	$C.P.$	$1 - positive$
$\uparrow$	$\uparrow$	$\uparrow$
$\phi \in B(M_3, M_3) =$	$\psi \in B(M_3, M_3) +$	$\gamma \in B(M_2, M_3)$

SCHMIDT NUMBER

- The Schmidt rank for a pure state $|\psi\rangle$ is its rank.
- A bipartite density matrix ρ has Schmidt number k if
  1. for any decomposition $\{p_i \geq 0, |\psi_i\rangle\}$ of ρ, at least one of the vectors $|\psi_i\rangle$ has Schmidt rank at least k.
  2. there exists a decomposition of ρ with all vectors $|\psi_i\rangle$ of Schmidt rank at most k.
- Equivalently, $SN(\rho) = \min_{\rho=\sum_i p_i |\psi_i\rangle\langle\psi_i|} \left\{ \max_i SR(|\psi_i\rangle) \right\}$.

The Schmidt number of bipartite states does not increase under LOCC. So the Schmidt number is an entanglement monotone for bipartite states.

Dual cone Correspondence

Translate the aforementioned result into the language of quantum states to answer a question raised in [Kye, 2013]

- $\mathbb{V}_k$ the set of all quantum states of Schmidt number$\leq k$.
- $\mathbb{P}_k$ the set of all k-positive maps.
- $\mathbb{D}$ the cone of all decomposable maps.
- $\mathbb{T}$ the cone of all positive partial transpose states.

Dual Cone Correspondence

$$\begin{array}{ccccccc} \mathbb{V}_1 & \subsetneq & \mathbb{T} & \subsetneq & \mathbb{V}_2 & \subsetneq & \mathbb{V}_3 = (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathbb{P}_1 & \supsetneq & \mathbb{D} & \supsetneq & \mathbb{P}_2 & \supsetneq & \mathbb{P}_3 \cong (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \end{array}$$

A Short Proof

Proof for PPT^2 conjecture when $n = 3$.

- Set ρ as the maximally entangled state in $M_3 \otimes M_3$. So $\sigma := (id_3 \otimes \phi)(\rho)$ is a PPT state in $M_3 \otimes M_3$.
- Let $\sigma = \sum_j p_j |a_j\rangle\langle a_j|$ where each $|a_j\rangle$ has Schmidt rank at most two. That is, $|a_j\rangle \in \mathcal{K}_j \simeq \mathbb{C}^3 \otimes \mathbb{C}^2$.
- Each state $(id_3 \otimes \phi)(|a_j\rangle\langle a_j|)$ is a PPT state in $M_3 \otimes M_2$ up to local equivalence. The Peres-Horodecki criterion says that $(id_3 \otimes \phi)(|a_j\rangle\langle a_j|)$ is separable.
- Using the convex sum of σ we obtain that $(id_3 \otimes (\phi \circ \phi))(\rho) = (id_3 \otimes \phi)(\sigma)$ is separable.

Thank You!

References

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