Quantum f-divergences in von Neumann algebras[1]

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[1] F. Hiai, Quantum f-divergences in von Neumann algebras I. Standard f-divergences; II. Maximal f-divergences, *J. Math. Phys.* **59** (2018), 102202; **60** (2019), 012203.

Plan

- Standard f-divergences
- Maximal f-divergences
- Rény divergences
- Sandwiched Rényi divergences
- Reversibility of quantum operations

Standard f-divergences

- M is a general von Neumann algebra.
- M_*^+ is the set of positive normal linear functionals on M.
- We work in the standard form $(M, L^2(M), J = {}^*, L^2(M)_+)$, where $L^p(M)$ is the Haagerup L^p-space.
- In particular, $\rho \in M_*^+ \longleftrightarrow h_\rho \in L^1(M)_+$ $(h_\rho^{1/2} \in L^2(M)_+)$ gives

$$\rho(x) = \langle h_\rho^{1/2}, x h_\rho^{1/2} \rangle = \mathrm{tr}\,(h_\rho x), \qquad x \in M.$$

- $f : (0, +\infty) \to \mathbb{R}$ is a convex function. Set

$$f(0^+) := \lim_{t \searrow 0} f(t), \quad f'(+\infty) := \lim_{t \to +\infty} \frac{f(t)}{t} \quad \in (-\infty, +\infty].$$

- The transpose of f is

$$\widetilde{f}(t) := t f(t^{-1}), \qquad t \in (0, +\infty).$$

Note that $\widetilde{f}(0^+) = f'(+\infty)$ and $\widetilde{f}\,'(+\infty) = f(0^+)$.

Specializing and modifying the quasi-entropy introduced in [2] [3],

Definition For $\rho, \sigma \in M_*^+$ let $\Delta_{\rho,\sigma}$ be the relative modular operator and

$$\Delta_{\rho,\sigma} = \int_{[0,+\infty)} t\, dE_{\rho,\sigma}(t)$$

be the spectral decomposition. Define the standard f-divergence $S_f(\rho\|\sigma)$ of ρ, σ by

$$\begin{aligned} S_f(\rho\|\sigma) := f(0^+)\sigma(1 - s(\rho)) + \int_{(0,+\infty)} f(t)\, d\big\|E_{\rho,\sigma}(t)h_\sigma^{1/2}\big\|^2 \\ + f'(+\infty)\rho(1 - s(\sigma)) \\ \in (-\infty, +\infty]. \end{aligned}$$

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[2] H. Kosaki, Interpolation theory and the Wigner-Yanase-Dyson-Lieb concavity, *Comm. Math. Phys.* **87** (1982), 315–329.

[3] D. Petz, Quasi-entropies for states of a von Neumann algebra, *Publ. Res. Inst. Math. Sci.* **21** (1985), 787–800.

Proposition

$$S_f(\rho\|\sigma) = S_{\widetilde{f}}(\sigma\|\rho).$$

Example A typical standard f-divergence is the relative entropy $D(\rho\|\sigma)$, due to Umegaki and Araki:

For $f(t) = t\log t$ (hence $\widetilde{f}(t) = -\log t$),

$$S_{t\log t}(\rho\|\sigma) = D(\rho\|\sigma), \qquad S_{-\log t}(\rho\|\sigma) = D(\sigma\|\rho).$$

In particular, when $M = B(\mathcal{H})$, for density (trace-class) operators ρ, σ,

$$D(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma).$$

Operator convex functions In the rest, assume that $f : (0, +\infty) \to \mathbb{R}$ is an operator convex function. Then f admits the unique integral expression (Lesniewski-Ruskai, 1999)

$$
\begin{aligned}
f(t) &= a + b(t-1) + c(t-1)^2 + \int_{[0,+\infty)} \frac{(t-1)^2}{t+s}\, d\mu(s) \\
&= a + b(t-1) + c(t-1)^2 + d\,\frac{(t-1)^2}{t} + \int_{(0,+\infty)} \frac{(t-1)^2}{t+s}\, d\mu(s),
\end{aligned}
$$

where $a, b \in \mathbb{R}$, $c \geq 0$ and μ is a positive measure on $[0, +\infty)$ satisfying $\int_{[0,+\infty)} (1+s)^{-1}\, d\mu(s) < +\infty$.

- For each $n \in \mathbb{N}$, the cut-off operator convex function f_n is

$$f_n(t) := a + b(t-1) + c\,\frac{n(t-1)^2}{t+n} + d\,\frac{(t-1)^2}{t+(1/n)} + \int_{[1/n,n]} \frac{(t-1)^2}{t+s}\,d\mu(s).$$

- Define a finite positive measure ν_n supported on $[1/n, n]$ by

$$d\nu_n(s) := c(1+n)\delta_n + d(1+n)\delta_{1/n} + 1_{[1/n,n]}(s)\frac{1+s}{s}\,d\mu(s).$$

Lemma

$f_n(0^+) < +\infty$, $f_n'(+\infty) < +\infty$, and as $n \to \infty$,

$$f_n(0^+) \nearrow f(0^+),\;\; f_n'(+\infty) \nearrow f'(+\infty),\;\; f_n(t) \nearrow f(t) \quad \text{for } t \in (0, +\infty),$$

$$S_{f_n}(\rho\|\sigma) \nearrow S_f(\rho\|\sigma).$$

Kosaki's[4] variational expression of $D(\rho\|\sigma)$ is extended with a modification as follows:

Theorem (variational expression)

Let L ($\ni \mathbf{1}$) be a subspace of M, dense in M with respect to the strong* operator topology. For every $\rho, \sigma \in M_*^+$,

$$S_f(\rho\|\sigma) = \sup_{n\in\mathbb{N}} \sup_{x(\cdot)} \Big[f_n(0^+)\sigma(1) + f_n'(+\infty)\rho(1)$$

$$- \int_{[1/n,n]} \{\sigma((1 - x(s))^*(1 - x(s))) + s^{-1}\rho(x(s)x(s)^*)\}(1 + s)\, d\nu_n(s) \Big],$$

where the infimum is taken over all L-valued (finitely many values) step functions $x(\cdot)$ on $(0, +\infty)$.

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[4] H. Kosaki, Relative entropy of states: a variational expression, *J. Operator Theory* **16** (1986), 335–348.

Properties of $S_f(\rho\|\sigma)$

Joint lower semicontinuity

$S_f(\rho\|\sigma)$ is jointly lower semicontinuous in $\rho, \sigma \in M_*^+$ in the $\sigma(M_*, M)$-topology.

Joint convexity

$S_f(\rho\|\sigma)$ is jointly convex and jointly subadditive, i.e., for every $\rho_i, \sigma_i \in M_*^+$, $1 \le i \le k$,

$$S_f\Bigg(\sum_{i=1}^k \rho_i \Bigg\| \sum_{i=1}^k \sigma_i\Bigg) \le \sum_{i=1}^k S_f(\rho_i\|\sigma_i).$$

Monotonicity or DPI

If M_0 is another von Neumann algebra and $\Phi : M_0 \to M$ is a unital normal linear Schwarz map, then

$$S_f(\rho \circ \Phi \| \sigma \circ \Phi) \le S_f(\rho \| \sigma).$$

In particular, if M_0 is a unital von Neumann subalgebra of M, then

$$S_f(\rho|_{M_0} \| \sigma|_{M_0}) \le S_f(\rho \| \sigma).$$

Martingale convergence

If $\{M_\alpha\}$ is an increasing net of unital von Neumann subalgebras of M such that $(\bigcup_\alpha M_\alpha)'' = M$, then

$$S_f(\rho|_{M_\alpha} \| \sigma|_{M_\alpha}) \nearrow S_f(\rho \| \sigma).$$

Peierls-Bogolieubov inequality

$$S_f(\rho\|\sigma) \geq \sigma(1) f(\rho(1)/\sigma(1)).$$

Assume that f is non-linear and $\rho, \sigma \neq 0$. Then equality holds in the above if and only if $\rho = (\rho(1)/\sigma(1))\sigma$.

Convergence property

If $\omega_1, \omega_2 \in M_*^+$ and $S_f(\omega_1\|\omega_2) < +\infty$, then for every $\rho, \sigma \in M_*^+$,

$$S_f(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} S_f(\rho + \varepsilon\omega_1\|\sigma + \varepsilon\omega_2).$$

In particular, for every $\rho, \sigma, \omega \in M_*^+$,

$$S_f(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} S_f(\rho + \varepsilon\omega\|\sigma + \varepsilon\omega).$$

Another martingale convergence

Let $\{e_\alpha\}$ be an increasing net of projections in M such that $e_\alpha \nearrow 1$. Then for every $\rho, \sigma \in M_*^+$,

$$\lim_\alpha S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) = S_f(\rho\|\sigma),$$

where $e_\alpha \rho e_\alpha := \rho|_{e_\alpha M e_\alpha}$.

Thus, when $M = B(\mathcal{H})$ with $\dim \mathcal{H} = \infty$, one can define the relative entropy $D(\rho\|\sigma)$ for density operators $\rho, \sigma \geq 0$ as

$$D(\rho\|\sigma) = \lim_\alpha D(E_\alpha \rho E_\alpha \| E_\alpha \sigma E_\alpha),$$

where $\{E_\alpha\}$ is an increasing net of finite rank projections with $E_\alpha \nearrow I$.

Maximal f-divergences

For $\rho, \sigma \in M_*^+$, write $\rho \sim \sigma$ if $\delta\sigma \le \rho \le \delta^{-1}\sigma$ for some $\delta > 0$.

Definition

- Let $\rho, \sigma \in M_*^+$ with $\rho \sim \sigma$. A unique $A_{\rho/\sigma} \in s(\sigma)Ms(\sigma)$ exists so that $h_\rho^{1/2} = A_{\rho/\sigma} h_\sigma^{1/2}$. Set $D_{\rho/\sigma} := A_{\rho/\sigma}^* A_{\rho/\sigma}$, positive invertible in $s(\sigma)Ms(\sigma)$. The maximal f-divergence of ρ, σ is

$$\widehat{S}_f(\rho\|\sigma) := \sigma(f(D_{\rho/\sigma})).$$

With a formal expression,

$$D_{\rho/\sigma} = h_\sigma^{-1/2} h_\rho h_\sigma^{-1/2}, \qquad \widehat{S}_f(\rho\|\sigma) = \mathrm{tr}\, h_\sigma(f(h_\sigma^{-1/2} h_\rho h_\sigma^{-1/2})).$$

Definition (cont.)

- For every $\rho, \sigma \in M_*^+$, the maximal f-divergence of ρ, σ is

$$\widehat{S}_f(\rho\|\sigma) := \lim_{\varepsilon \searrow 0} \widehat{S}_f(\rho + \varepsilon\eta\|\sigma + \varepsilon\eta) \in (-\infty, +\infty],$$

independently of the choice of $\eta \in M_*^+$ with $\eta \sim \rho + \sigma$. This is compatible with the previous definition.

Proposition

$$\widehat{S}_f(\rho\|\sigma) = \widehat{S}_{\widetilde{f}}(\sigma\|\rho).$$

Properties of $\widehat{S}_f(\rho\|\sigma)$

Monotonicity or DPI

Let M, M_0 be a von Neumann algebras and $\Phi : M_0 \to M$ be a unital normal positive map. For every $\rho, \sigma \in M_*^+$,

$$\widehat{S}_f(\rho \circ \Phi\|\sigma \circ \Phi) \le \widehat{S}_f(\rho\|\sigma).$$

Joint convexity

$\widehat{S}_f(\rho\|\sigma)$ is jointly convex in $\rho, \sigma \in M_*^+$.

Integral expression (or 2nd definition)

For every $\rho, \sigma \in M_*^+$, let $D_{\rho/\rho+\sigma} = \int_0^1 t\, dE_{\rho/\rho+\sigma}(t)$ be the spectral decomposition. Then

$$\widehat{S}_f(\rho\|\sigma) = \int_0^1 (1-t) f\left(\frac{t}{1-t}\right) d\left\|E_{\rho/\rho+\sigma}(t) h_{\rho+\sigma}^{1/2}\right\|^2,$$

where $(1-t)f(\frac{t}{1-t})$ is understood as $f(0^+)$ at $t = 0$ and $f'(+\infty)$ at $t = 1$.

Joint lower semicontinuity

$\widehat{S}_f(\rho\|\sigma)$ is jointly lower semicontinuous in the norm topology.

Martingale convergence

If $\{M_\alpha\}$ is an increasing net of unital von Neumann subalgebras of M such that $\left(\bigcup_\alpha M_\alpha\right)'' = M$, then for every $\rho, \sigma \in M_*^+$,

$$\widehat{S}_f(\rho|_{M_\alpha} \| \sigma|_{M_\alpha}) \nearrow \widehat{S}_f(\rho\|\sigma).$$

Variational expression (or 3rd definition)

For every $\rho, \sigma \in M_*^+$,

$$\widehat{S}_f(\rho\|\sigma) = \min\{S_f(p\|q) : (\Psi, p, q) \text{ a reverse test for } \rho, \sigma\},$$

where a reverse test (Ψ, p, q) is a triplet of

$$\Psi : M \to L^\infty(X, \mu), \quad \text{a unital normal positive map,}$$
$$p, q \in L^1(X, \mu)_+, \quad \rho = \Psi_*(p), \quad \sigma = \Psi_*(q).$$

The minimum is attained by a reverse test with $X = [0, 1]$.

Inequality between S_f and $\widehat{S}_f$

For every $\rho, \sigma \in M_*^+$,

$$S_f(\rho\|\sigma) \le \widehat{S}_f(\rho\|\sigma).$$

If $\rho, \sigma \in M_*^+$ commute ($\iff h_\rho h_\sigma = h_\sigma h_\rho$), then $S_f(\rho\|\sigma) = \widehat{S}_f(\rho\|\sigma)$, and if $S_f(\rho\|\sigma) = \widehat{S}_f(\rho\|\sigma) < +\infty$ with some condition of f, then ρ, σ commute. (The proof uses the reversibility theorem.)

Example When $f(t) = t\log t$, $S_{t\log t}(\rho\|\sigma) = D(\rho\|\sigma)$ is the relative entropy, and $\widehat{S}_{t\log t}(\rho\|\sigma) = D_{\mathrm{BS}}(\rho\|\sigma)$ is Belavkin and Staszewski's relative entropy. We have

$$D(\rho\|\sigma) \le D_{\mathrm{BS}}(\rho\|\sigma),$$

and if $D(\rho\|\sigma) = D_{\mathrm{BS}}(\rho\|\sigma) < +\infty$, then ρ, σ commute.

Rényi divergences

Definition

Let $\rho, \sigma \in M_*^+$ with $\rho \neq 0$. When $0 < \alpha \leq 1$, define

$$Q_\alpha(\rho\|\sigma) := \left\|\Delta_{\rho,\sigma}^{\alpha/2} h_\sigma^{1/2}\right\|^2.$$

(Note that $h_\sigma^{1/2} \in D(\Delta_{\rho,\sigma}^{\alpha/2})$ in this case.) When $\alpha > 1$, define

$$Q_\alpha(\rho\|\sigma) := \begin{cases} \left\|\Delta_{\rho,\sigma}^{\alpha/2} h_\sigma^{1/2}\right\|^2 & \text{if } s(\rho) \leq s(\sigma) \text{ and } h_\sigma^{1/2} \in D(\Delta_{\rho,\sigma}^{\alpha/2}), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the α-Rényi divergence of ρ, σ for $\alpha \in (0, \infty) \setminus \{1\}$ is defined as

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log Q_\alpha(\rho\|\sigma).$$

Sandwiched Rényi divergence

The sandwiched Rényi divergences introduced in 2013, due to Müller-Lennert et al,, Wilde-Winter-Yang, ..., has recently been extended to the von Neumann algebra setting by Berta-Scholz-Tomamichel[5] and Jenčová[6] [7]

[5] M. Berta, V. B. Scholz and M. Tomamichel, Rényi divergences as weighted non-commutative vector valued L_p-spaces, *Ann. Henri Poincaré* **19** (2018), 1843–1867.

[6] A. Jenčová, Rényi relative entropies and noncommutative L_p-spaces, *Ann. Henri Poincaré* **19** (2018), 2513–2542.

[7] A. Jenčová, Rényi relative entropies and noncommutative L_p-spaces II, Preprint, arXiv:1707.00047 [quant-ph].

Definition of Berta-Scholz-Tomamichel Let $\rho, \sigma \in M_*^+$. For $\alpha \in [1/2, \infty) \setminus \{1\}$,

$$\widetilde{D}_\alpha^{(\mathrm{BST})}(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \left\| h_\rho^{1/2} \right\|_{2\alpha,\sigma}^{2\alpha},$$

where $\left\| h_\rho^{1/2} \right\|_{p,\sigma}$ is the Araki-Masuda's L^p-norm of the vector representative $h_\rho^{1/2}$ with respect to σ for $1 \le p \le \infty$.

Definition of Jenčová

- For $0 < \alpha < 1$,

$$\widetilde{D}_\alpha^{(J)}(\rho\|\sigma) := \frac{1}{\alpha - 1}\log \mathrm{tr}\left(h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha.$$

- For $1 < \alpha < \infty$,

$$\widetilde{D}_\alpha^{(J)}(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1}\log \|h_\rho\|_{\alpha,\sigma}^\alpha & \text{if } h_\rho \in L^\alpha(M,\sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

where $L^\alpha(M,\sigma)$ is Kosaki's L^α-space.

If $h_\rho \in L^\alpha(M,\sigma)$, then $h_\rho = h_\sigma^{\frac{\alpha-1}{2\alpha}} x h_\sigma^{\frac{\alpha-1}{2\alpha}}$ for some $x \in L^\alpha(M)$, and

$$\|h_\rho\|_{\alpha,\sigma}^\alpha = \|x\|_\alpha^\alpha.$$

With a formal expression,

$$x = h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}}, \quad \|h_\rho\|_{\alpha,\sigma}^\alpha = \left\|h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}}\right\|_\alpha^\alpha = \mathrm{tr}\,(h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha.$$

Reversibility of quantum operations

Reversibility problem

- $\Phi : M_0 \to M$ is a unital normal CP or 2-positive map.
- $\Delta(\rho\|\sigma)$ is a quantum divergence satisfying monotonicity (or DPI)

$$\Delta(\rho \circ \Phi\|\sigma \circ \Phi) \le \Delta(\rho\|\sigma).$$

- If Φ is reversible for $\{\rho, \sigma\}$, i.e., there is a unital normal CP or 2-positive map $\Psi : M \to M_0$ such that $\rho \circ \Phi \circ \Psi = \rho$ and $\sigma \circ \Phi \circ \Psi = \sigma$, then the double use of monotonicity gives $\Delta(\rho \circ \Phi\|\sigma \circ \Phi) = \Delta(\rho\|\sigma)$.
- The problem is whether the equality

$$\Delta(\rho \circ \Phi\|\sigma \circ \Phi) = \Delta(\rho\|\sigma) < +\infty$$

implies reversibility?

- In the von Neumann algebra setting, Petz[8] [9] and Jenčová and Petz[10] formerly studied the reversibility (or sufficiency) via equality in DPI of the relative entropy $D(\rho\|\sigma)$ and the transition probability $P(\rho,\sigma)$, i.e., the standard $\alpha = 1/2$-Rényi divergence.

- More comprehensive results in the finite-dimensional case are in [11] [12] [13].

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[8] D. Petz, Sufficient subalgebras and the relative entropy of states of a von Neumann algebra, *Comm. Math. Phys.* **105** (1986), 123–131.

[9] D. Petz, Sufficient of channels over von Neumann algebras, *Quart. J. Math. Oxford Ser. (2)* **39** (1988), 97–108.

[10] A. Jenčová and D. Petz, Sufficiency in quantum statistical inference, *Comm. Math. Phys.* **263** (2006), 259–276.

[11] F. Hiai, M. Mosonyi, D. Petz and C. Bény, Quantum f-divergences and error correction, *Rev. Math. Phys.* **23** (2011), 691–747.

[12] A. Jenčová, Preservation of a quantum Rényi relative entropy implies existence of a recovery map, *J. Phys. A* **50** (2017), 085303.

[13] F. Hiai and M. Mosonyi, Different quantum f-divergences and the reversibility of quantum operations, *Rev. Math. Phys.* **29** (2017), 1750023.

Petz' recovery map

- $\Phi : M_0 \to M$ is a unital normal positive linear map. Let $\sigma \in M_*^+$ and $e_0 := s_{M_0}(\sigma \circ \Phi)$.
- Petz' recovery map (originally, due to Accardi-Cecchini) with respect to σ is a unital normal positive map $\Psi_\sigma : M \to e_0 M_0 e_0$ defined by

$$\Phi_*(h_\sigma^{1/2} x h_\sigma^{1/2}) = h_{\sigma\circ\Phi}^{1/2} \Psi_\sigma(x) h_{\sigma\circ\Phi}^{1/2}, \qquad x \in M.$$

- Note that $\sigma \circ \Phi \circ \Psi_\sigma = \sigma$ holds automatically. If Φ is 2-positive (resp., CP), then Ψ_σ is 2-positive (resp., CP).
- Ψ_σ can extend to a unital normal positive map $\widetilde{\Psi}_\sigma : M \to M_0$ as

$$\widetilde{\Psi}_\sigma(x) := \Psi_\sigma(x) + \omega(x)(1 - e_0), \qquad y \in M,$$

with a normal state ω of M.

Theorem (case of S_f)

Assume that $\rho, \sigma \in M_*^+$, $s(\rho) \le s(\sigma)$, and that $\Phi : M_0 \to M$ is a unital normal 2-positive map. The following conditions are equivalent:

(i) Φ is reversible for $\{\rho, \sigma\}$;

(ii) $\rho \circ \Phi \circ \Psi_\sigma = \rho$ (also $\sigma \circ \Phi \circ \Psi_\sigma = \sigma$);

(iii) $S_f(\rho \circ \Phi \| \sigma \circ \Phi) = S_f(\rho \| \sigma)$ for any operator convex function on $(0, +\infty)$;

(iv) $S_f(\rho \circ \Phi \| \sigma \circ \Phi) = S_f(\rho \| \sigma) < +\infty$ for some operator convex function on $(0, +\infty)$ such that the support of μ has a limit point in $(0, +\infty)$;

(v) $D_\alpha(\rho \circ \Phi \| \sigma \circ \Phi) = D_\alpha(\rho \| \sigma) < +\infty$ for some $\alpha \in (0, 2) \setminus \{1\}$;

(vi) $P(\rho \circ \Phi, \sigma \circ \Phi) = P(\rho, \sigma)$, i.e., $D_{1/2}(\rho \circ \Phi \| \sigma \circ \Phi) = D_{1/2}(\rho \| \sigma)$;

(vii) $\Phi([D(\rho \circ \Phi) : D(\sigma \circ \Phi)]_t) = [D\rho : D\sigma]_t$ for all $t \in \mathbb{R}$;

(viii) $\Psi_\sigma = \Psi_{\rho+\sigma}$.

Theorem (case of S_f)

Let $\rho, \sigma \in M_*^+$ be arbitrary, and $\Phi : M_0 \to M$ be a unital normal 2-positive map. The following conditions are equivalent:

(i) Φ is reversible for $\{\rho, \sigma\}$;

(ii) $S_f(\rho \circ \Phi \| (\rho + \sigma) \circ \Phi) = S_f(\rho \| \rho + \sigma)$ for any operator convex function on $(0, +\infty)$;

(iii) $S_f(\rho \circ \Phi \| (\rho + \sigma) \circ \Phi) = S_f(\rho \| \rho + \sigma) < +\infty$ for some operator convex function on $[0, +\infty)$ such that the support of μ has a limit point in $(0, +\infty)$;

(iv) $D_\alpha(\rho \circ \Phi \| (\rho + \sigma) \circ \Phi) = D_\alpha(\rho \| \rho + \sigma)$ for some $\alpha \in (0, 2) \setminus \{1\}$;

(v) $P(\rho \circ \Phi, (\rho + \sigma) \circ \Phi) = P(\rho, \rho + \sigma)$;

(vi) $\Phi([D(\rho \circ \Phi) : D((\rho + \sigma) \circ \Phi)]_t) = [D\rho : D(\rho + \sigma)]_t$ for all $t \in \mathbb{R}$.

Question If ρ, σ are arbitrary and $S_f(\rho \circ \Phi \| \sigma \circ \Phi) = S_f(\rho \| \sigma) < +\infty$, then is Φ reversible for $\{\rho, \sigma\}$?

Theorem (case of $\widetilde{D}_\alpha$, $\alpha > 1$) (Jenčová)

$\Phi : M_0 \to M$ is a unital normal 2-positive map. If $\widetilde{D}_\alpha(\rho \circ \Phi \| \sigma \circ \Phi) = \widetilde{D}_\alpha(\rho \| \sigma) < +\infty$, then $\rho \circ \Phi \circ \Psi_\sigma = \rho$.

Theorem (case of $\widetilde{D}_\alpha$, $1/2 < \alpha < 1$) (Jenčová)

Assume that $\rho, \sigma \in M_*^+$, $s(\rho) \le s(\sigma)$, and that $\Phi : M_0 \to M$ is a unital normal CP map. If $\widetilde{D}_\alpha(\rho \circ \Phi \| \sigma \circ \Phi) = \widetilde{D}_\alpha(\rho \| \sigma)$, then $\rho \circ \Phi \circ \Psi_\sigma = \rho$.

Operator connections (means) (Kubo-Ando)

- An operator connection $\tau : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \to B(\mathcal{H})_+$ is represented by an operator monotone function $\phi > 0$ on $(0, +\infty)$ as

$$A \,\tau\, B := A^{1/2}\phi(A^{-1/2}BA^{-1/2})A^{1/2} \quad \text{for invertible } A, B,$$

  and extended to general $A, B \in B(\mathcal{H})_+$ as

$$A \,\tau\, B := \lim_{\varepsilon \searrow 0}(A + \varepsilon I)\,\tau\,(B + \varepsilon I).$$

- τ is well generalized to $\tau : M_*^+ \times M_*^+ \to M_*^+$. Define

$$\sigma \,\tau\, \rho := h_\sigma^{1/2}\phi(D_{\rho/\sigma})h_\sigma^{1/2} \quad \text{if } \rho \sim \sigma,$$

  and extended to general $\rho, \sigma \in M_*^+$ as

$$\sigma \,\tau\, \rho := \lim_{\varepsilon \searrow 0}(\sigma + \varepsilon\eta)\,\tau\,(\rho + \varepsilon\eta),$$

  where $\eta \sim \rho + \sigma$.

Theorem (case of $\widehat{S}_f$)

Let $\rho, \sigma \in M_*^+$ be arbitrary and Φ be a unital normal simply positive map. The following conditions are equivalent:

(i) $\widehat{S}_f(\rho \circ \Phi \| \sigma \circ \Phi) = \widehat{S}_f(\rho \| \sigma)$ for any operator convex function f on $(0, +\infty)$;

(ii) $\widehat{S}_f(\rho \circ \Phi \| (\rho + \sigma) \circ \Phi) = \widehat{S}_f(\rho \| \rho + \sigma)$ for some nonlinear operator convex function f on $[0, +\infty)$;

(iii) $D_2(\rho \circ \Phi \| (\rho + \sigma) \circ \Phi) = D_2(\rho \| \rho + \sigma)$;

(iv) $\Phi_*(\sigma) \, \tau \, \Phi_*(\rho) = \Phi_*(\sigma \, \tau \, \rho)$ for some nonlinear (equivalently, any) operator connection τ;

(v) $\Phi_*(\rho + \sigma) \, \tau \, \Phi_*(\rho) = \Phi_*((\rho + \sigma) \, \tau \, \rho)$ for some nonlinear (equivalently, any) operator connection τ;

(vi) $\Phi_*(h_{\rho+\sigma}^{1/2} D_{\rho/\rho+\sigma} h_{\rho+\sigma}^{1/2}) = h_{\Phi_*(\rho+\sigma)}^{1/2} D_{\Phi_*(\rho)/\Phi_*(\rho+\sigma)} h_{\Phi_*(\rho+\sigma)}^{1/2}$;

(vii) $\Psi_{\rho+\sigma}((D_{\rho/\rho+\sigma})^2) = (\Psi_{\rho+\sigma}(D_{\rho/\rho+\sigma}))^2$. (If Φ is 2-positive, this means that $D_{\rho/\rho+\sigma}$ is in the multiplicative domain of $\Psi_{\rho+\sigma}$.)

Thank you!