The Partial Transpose and its Relation to Freeness

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inspiration: Aubrun (2012), Banica - Nechita (2013)

- $G_1, \ldots, G_{d_1}$ are $d_2 \times p$ random matrices where $G_i = (g_{jk}^{(i)})_{jk}$ and $g_{jk}^{(i)}$ are complex Gaussian $\mathcal{N}(0,1)$ random variables
- the random variables $\{g_{jk}^{(i)}\}_{i,j,k}$ are independent.

- $$W = \frac{1}{d_1 d_2} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right) = \frac{1}{d_1 d_2} (G_i G_j^*)_{ij}$$

  is a $d_1 d_2 \times d_1 d_2$ Wishart matrix, $\frac{p}{d_1 d_2} \longrightarrow c$, $W \xrightarrow{\mathcal{D}} \mathrm{MP}_c$
- $W^{\urcorner} = (d_1 d_2)^{-1} (G_j G_i^*)_{ij}$ partial transpose

[Au] $p, d_1, d_2 \to \infty \Rightarrow W^{\urcorner} \xrightarrow{\mathcal{D}}$ semi-circular: centre c, radius c

[B-N] d_1 fixed, $p, d_2 \to \infty \Rightarrow W^{\urcorner} \xrightarrow{\mathcal{D}} \mathrm{MP}_{\frac{c d_1 (d_1+1)}{2}} \boxplus \left(-\mathrm{MP}_{\frac{c d_1 (d_1-1)}{2}}\right)$

free cumulants

- $W^{\urcorner} = (d_1 d_2)^{-1} (G_j G_i^*)_{ij}$ partial transpose
- $W \xrightarrow{\mathcal{D}} w \in \mathrm{MP}_c$, $W^{\urcorner} \xrightarrow{\mathcal{D}} w^{\urcorner}$
- d_1 fixed, $p, d_2 \to \infty \Rightarrow w^{\urcorner} \in \mathrm{MP}_{\frac{cd_1(d_1+1)}{2}} \boxplus (-\mathrm{MP}_{\frac{cd_1(d_1-1)}{2}})$
- given $\mu \in \mathcal{M}(\mathbb{R})$, a probability measure on $\mathbb{R}$, we let $G(z) = \int (z-t)^{-1} \, d\mu(t)$ be the Cauchy transform of μ and
$$R(z) = G^{\langle -1 \rangle}(z) - z^{-1} = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$$
- $\{\kappa_n\}_n$ are the *free cumulants* of μ
- if $\mu =$ semi-circle with centre c and radius c, then $\kappa_1 = \kappa_2 = c$ and $\kappa_n = 0$ for $n \geqslant 3$
- if $\mu \in \mathrm{MP}_c$ then $\kappa_n = c$ $(\forall n)$
- if $\mu \in \mathrm{MP}_{\frac{cd_1(d_1+1)}{2}} \boxplus (-\mathrm{MP}_{\frac{cd_1(d_1-1)}{2}})$ then for odd n, $\kappa_n = cd_1$ and for even n, $\kappa_n = cd_1^2 = cd_1 + cd_1 + \cdots + cd_1$
- (d_1 odd) can we write $w^{\urcorner}$ as a sum of $(d_1+1)/2$ free operators $w^{\urcorner} = x_0 + x_1 + \cdots + x_{(d_1-1)/2}$ with $\kappa_n^{(x_0)} = cd_1$ $(\forall n)$, and for $i \geqslant 1$ $\kappa_{2n}^{(x_i)} = 2cd_1$ and $\kappa_{2n-1}^{(x_i)} = 0$?

the answer when $d_1 = 5$

$$
\begin{aligned}
w^{\urcorner} = &\begin{pmatrix} w_{11} & 0 & & & \\ 0 & w_{22} & 0 & & \\ & 0 & w_{33} & 0 & \\ & & 0 & w_{44} & 0 \\ & & & 0 & w_{55} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & w_{21} & & & w_{51} \\ w_{12} & 0 & w_{32} & & \\ & w_{23} & 0 & w_{43} & \\ & & w_{34} & 0 & w_{54} \\ w_{15} & & & w_{45} & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & & w_{31} & w_{41} & 0 \\ & 0 & & w_{42} & w_{52} \\ w_{13} & & 0 & & w_{53} \\ w_{34} & w_{24} & & 0 & \\ 0 & w_{15} & w_{35} & & 0 \end{pmatrix}
\end{aligned}
$$

freeness and the computation of cumulants

- $\mathcal{A}$ = unital algebra over $\mathbf{C}$, $\varphi : \mathcal{A} \xrightarrow{linear} \mathbf{C}$, $\varphi(1) = 1$, $(\mathcal{A}, \varphi)$ is a *non-commutative probability space*
- $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$, unital subalgebras, are *freely independent* **if** whenever $a_1, \ldots, a_n \in \mathcal{A}$ with $\varphi(a_i) = 0$ and $a_i \in \mathcal{A}_{j_i}$ with $j_1 \neq j_2 \neq \cdots \neq j_n$, we have $\varphi(a_1 \cdots a_n) = 0$
- if $\mathcal{A}$ is a C*-algebra and $a = a^* \in \mathcal{A}$, a has a distribution, μ_a, with respect to φ: $\int f(t)\mu_a(t) = \varphi(f(a))$ (*usual functional calculus*); such an element has a Cauchy transform, a R-transform: $R(z) = G^{\langle -1 \rangle}(z) - z^{-1} = \kappa_1 + \kappa_2 z + \cdots$, and free cumulants: $\{\kappa_n^{(a)}\}_n$
- if a_1 and a_2 are free then $R_{a_1+a_2} = R_{a_1} + R_{a_2}$, i.e. $\kappa_n^{(a_1+a_2)} = \kappa_n^{(a_1)} + \kappa_n^{(a_2)}$
- conversely: $a_1, \ldots, a_s \in \mathcal{A}$ are free $\Leftrightarrow$ for all $i_1, \ldots, i_n \in [s]$ we have $\kappa_n(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) = 0$ unless $i_1 = i_2 = \cdots = i_n$
- given cumulants $\{\kappa_n\}_n$ we set $R(z) = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$ and then $G^{\langle -1 \rangle}(z) = R(z) + z^{-1}$, and from this a, we obtain a density $\rho(t) = -\pi^{-1}\mathrm{Im}(G(t + 0^+ i))$

freeness and the transpose in the regime: $d_1, d_2 \to \infty$

- Voiculescu (1991) showed that independence and unitary invariance for two ensembles implies asymptotic freeness
- work of Emily Redelmeier on real second order freeness made it clear that the transpose plays an important role in freeness
- Mihai Popa and I (2016) showed that unitarily invariant matrices are asymptotically free from their transposes (& in fact real second order free)
- **Theorem.** In the regime $d_1, d_2 \to \infty$, W, $W^{\urcorner}$, W^{Γ}, and W^{T} are asymptotically free
- **Theorem.** In the regime $d_1, d_2 \to \infty$, U_N, $U_N^{\urcorner}$, U_N^{Γ}, and U_N^{T} are asymptotically $*$-free, for a $N \times N$ Haar distributed random unitary matrix U_N

R-diagonal and even operators: Nica-Speicher (1995)

- in a *-probability space $(\mathcal{A}, \varphi)$ an operator a is *R-diagonal* if $\kappa_n(a^{(\epsilon_1)}, a^{(\epsilon_2)}, \ldots, a^{(\epsilon_n)}) = 0$ unless: n is even and $\epsilon_1 = -\epsilon_2 = \epsilon_3 = \cdots = -\epsilon_n$, where $a^{(-1)} = a^*$ and $a^{(1)} = a$
- if x and y are free and semi-circular then $x + iy$ is R-diagonal
- if U is a Haar distributed orthogonal or unitary matrix and $U \xrightarrow{\mathcal{D}} u$, then u is R-diagonal
- if $x = x^* = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_2(\mathcal{A})$ is free from the matrix units $\{e_{ij}\}_{ij}$ then x_{12} is R-diagonal

[THM] if $x = x^* = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_2(\mathcal{A})$ is free from the matrix units $\{e_{ij}\}_{ij}$ then $\begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}$ and $\begin{pmatrix} 0 & x_{21} \\ x_{12} & 0 \end{pmatrix}$ are free

R-cyclic operators: Nica-Shlyakhtenko-Speicher (1999)

- if $(\mathcal{A}, \varphi)$ is a *-probability space then $(M_n(\mathcal{A}), \varphi)$ is a *-probability space where $\varphi(x) = n^{-1}\varphi(x_{11} + \cdots + x_{nn})$
- if $x = (x_{ij})_{ij} \in M_n(\mathcal{A})$ is such that

  $$\kappa_n(x_{i_1 j_1}, x_{i_2 j_2}, x_{i_3 j_3}, \ldots, x_{i_n j_n}) = 0$$

  unless $j_1 = i_2, j_2 = i_3, \ldots, j_n = i_1$, then we say c is *R-cyclic*
- if $x = M_n(\mathcal{A})$ and x is free from matrix units $\{e_{ij}\}_{ij=1}^{n}$ then x is R-cyclic
- if $\{X_N\}_N$ is a ensemble of random matrices with $X_N \xrightarrow{\mathcal{D}} x$ and which is asymptotically free from deterministic matrices and we write $N = n \times d$ with $d \to \infty$ and n is fixed, then $x \in M_n(\mathcal{A})$ is R-cyclic

diagonal decompositions

- suppose $x \in M_n(\mathcal{A})$, we shall write $x = y_0 + y_1 + \cdots + y_{n-1}$, we call this the *diagonal decomposition* of x, where

$$y_0 = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix} y_1 = \begin{pmatrix} 0 & x_{12} & 0 & \cdots & 0 \\ 0 & 0 & x_{23} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & 0 & x_{n-1,n} \\ x_{n1} & 0 & & 0 & 0 \end{pmatrix}$$

and in general

$$y_k = \begin{pmatrix} 0 & \cdots & x_{1k} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{2,k+1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 \\ 0 & & & & & x_{n-k+1,n} \\ x_{n-k+2,1} & 0 & & & \cdots & 0 \\ 0 & x_{n-k+3,2} & 0 & & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & x_{n,k-1} & 0 \; \cdots & 0 \end{pmatrix}.$$

main theorem

- let $(\mathcal{A}, \varphi)$ be a *-probability space and $(M_n(\mathcal{A}), \varphi)$ the corresponding matrix algebra
- let $\mathcal{D}_n \subseteq M_n(\mathcal{A})$ be the subalgebra of diagonal scalar matrices, and let $\tilde{\varphi} : M_n(\mathcal{A}) \to \mathcal{D}_n$ be the conditional expectation given by $\tilde{\varphi}(A) = \operatorname{diag}(\varphi(a_{11}), \ldots, \varphi(a_{nn}))$
- suppose $x \in M_n(\mathcal{A})$ is R-cyclic, and we write $x^t = y_0 + y_1 + \cdots + y_{n-1}$ in its diagonal decomposition
- then when n is *odd*

  $$y_0,\ \{y_1, y_{n-1}\},\ \{y_2, y_{n-2}\}, \ldots, \{y_{(n-1)/2}, y_{(n+1)/2}\}$$

  are $\tilde{\varphi}$-free
- and when n is *even*

  $$y_0,\ \{y_1, y_{n-1}\},\ \{y_2, y_{n-2}\}, \ldots, \{y_{(n-2)/2}, y_{(n+2)/2}\},\ y_{n/2}$$

  are $\tilde{\varphi}$-free
- if $x = x^*$ then $y_0, y_1, y_2, \ldots, y_{n-1}$ are $\tilde{\varphi}$-$*$-free and $y_1, y_2, \ldots, y_{n-1}$ are $\tilde{\varphi}$-R-diagonal

a word about the proof

- for $d \in \mathcal{D}_n$ with $\operatorname{Im}(d) > 0$, let $\widetilde{G}(d) = \tilde{\varphi}((d-x)^{-1}) \in \mathcal{D}_n$
- from $\widetilde{G}^{\langle -1 \rangle}$ we get the $\mathcal{D}_n$-valued cumulants $\{\tilde{\kappa}_n\}_n$, we shall use these to prove freeness over $\mathcal{D}$
- let s be the matrix that cyclically permutes (backwards) the standard basis of $\mathbf{C}^n$,

$$s = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 \end{pmatrix}.$$

- suppose $x \in M_n(\mathcal{A})$ and $x = y_0 + \cdots + y_{n-1}$ is its diagonal decomposition. If we let $a_k = y_k s^{-k}$ then a_k is the diagonal matrix $\operatorname{diag}(x_{1,k+1}, x_{2,k+2}, \ldots, x_{d,k})$.

$$x \ = \ y_0 + \cdots + y_{n-1} \ = \ a_0 s^0 + a_1 s^1 + a_2 s^2 + \cdots + a_{n-1} s^{n-1}$$

example: when $n = 3$

- $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$y_0 = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}, y_1 = \begin{pmatrix} 0 & x_{12} & 0 \\ 0 & 0 & x_{23} \\ x_{31} & 0 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 0 & x_{13} \\ x_{21} & 0 & \\ 0 & x_{32} & 0 \end{pmatrix}$$

$$a_0 = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}, a_1 = \begin{pmatrix} x_{12} & 0 & 0 \\ 0 & x_{23} & 0 \\ 0 & 0 & x_{31} \end{pmatrix}, a_2 = \begin{pmatrix} x_{13} & 0 & 0 \\ 0 & x_{21} & 0 \\ 0 & 0 & x_{32} \end{pmatrix}$$

$$x = y_0 + y_1 + y_2 = a_0 s^0 + a_1 s^1 + a_2 s^2$$

main technical lemmas

- let $a_{j,k} = s^k a_j s^{-k}$. Then $a_{j,k}$ is the diagonal matrix $a_{j,k} = \mathrm{diag}(x_{j+k+1,k+1}, x_{j+k+2,k+2}, \ldots, x_{j+k+d,k+d})$
- for $i_1, \ldots, i_k \in [n]$ and any $d_1, \ldots, d_{k-1} \in \mathcal{D}_n$ we let $d_{i,l} = s^{-l} d_i s^l$

[LEM. 1] then we have

$$\tilde{\kappa}_k(a_{i_1}s^{i_1}d_1, \ldots, a_{i_{k-1}}s^{i_{k-1}}d_{k-1}, a_{i_k}s^{i_k}) = \tilde{\kappa}_n(a_{i_1}, a_{i_2,i_1}, \ldots, a_{i_n,i_1+\cdots+i_{n-1}})d_{1,i_1}\cdots d_{k-1,i_1+\cdots+i_{k-1}}$$

[LEM. 2] Let $d_1, \ldots, d_{k-1} \in \mathcal{D}_n$ then

$$\tilde{\kappa}_k(y_{i_1}d_1, \ldots, y_{i_{k-1}}d_{k-1}, y_{i_k}) = 0$$

unless either $i_1 \equiv i_2 \equiv \cdots \equiv i_k \equiv 0 \pmod n$ or, n is even and we have $i_l + i_{l+1} \equiv 0 \pmod n$ for $1 \leqslant l \leqslant n-1$.