Chapter 6

Ramifications of the Black-Scholes Model

In this chapter we study various futures issues related to the Black-Scholes model. The reason is that despite its-many some fatal-shortcomings, it still is being used as the standard reference model with which all others are compared and hence serves as a common denominator of communication among traders and that for this reason its deeper knowledge is quit beneficial to understanding the derivative market in general.

6.1 Black-Scholes formula for arbitrary payoff functions

A call or put option has a particular type of payoff function that makes it possible to derive a closed-form pricing formula. If the payoff function, however, is an arbitrarily given function, one cannot have such neat formula. Nonetheless they are not any more difficult to handle.

Suppose $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is a (piecewise C^1) continuous function, and suppose the payoff at the expiry T of the option is given by $\varphi(S_T)$. Then we have learned in the previous chapter that its value $u(t, S_t)$, when the stock price at time t is S_t, can be computed by finding function $u(t, S)$ satisfying the following boundary value problem of PDE:

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - ru = 0$$
$$u(T, S) = \varphi(S).$$

As a practical matter, one usually relies on numerical method to solve this PDE. However when φ is of special form, it can be written as a combination of calls and puts and hence a closed-form formula is available. We list some of the most popular ones.

• **Spread**

Spread is a combination of different series (expiry or strike price) of the same class of options (call or put) on the same stock.

For instance, let X_1 be a call option with strike price K_1 and X_2 a call option with strike price K_2. Assume both are on the same stock and the expiry is also the same. Assume further that $K_1 < K_2$. Then the payoffs of X_1 and X_2 look like as in Figure 6.1.

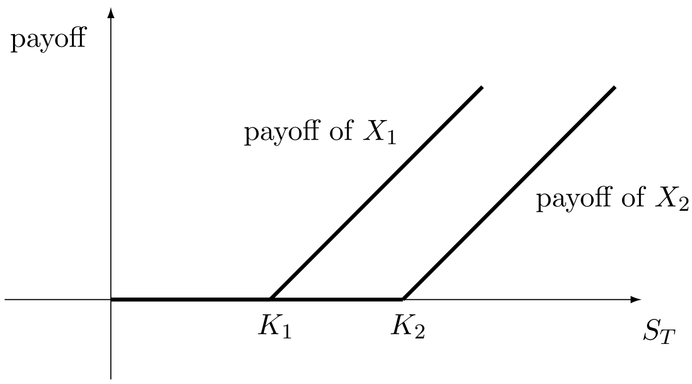


Figure 6.1: Payoffs.

If one combines them by buying X_1 and selling X_2, the resulting combination is called a *bullish spread* whose payoff looks like as in Figure 6.2. If the prices of the constituent options are is taken into consideration the payoff function becomes profit/loss and it looks like as in Figure 6.3.

If on the other hand one buys X_2 and sell X_1, the payoff becomes as in Figure 6.4. This kind of combination is called a *bearish spread.*

Similarly, one can create a payoff function as in Figure 6.5 by buying one call option with strike price K_1 and another with strike price K_3 and selling two call options with strike price K_2 where $K_1 < K_2 < K_3$. This kind combination is called a *butterfly spread.*

Another kind of spread involves two options of the same kind but with different expiry. It is generally called a *calendar spread*, *time spread* or *horizontal spread.* (As a way of nomenclature, the "vertical" spread means spread involving options with the same expiry. So

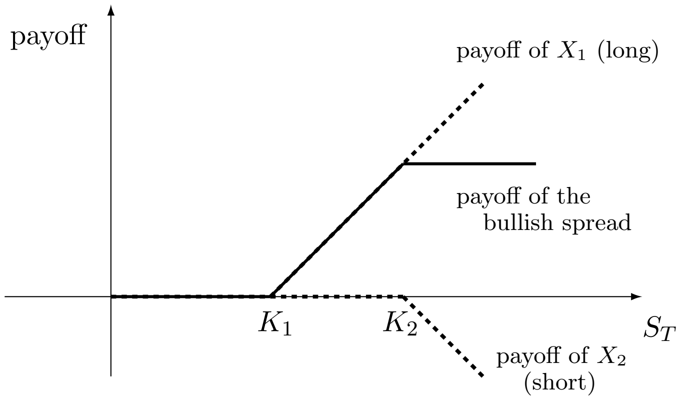


Figure 6.2: Bullish spread.

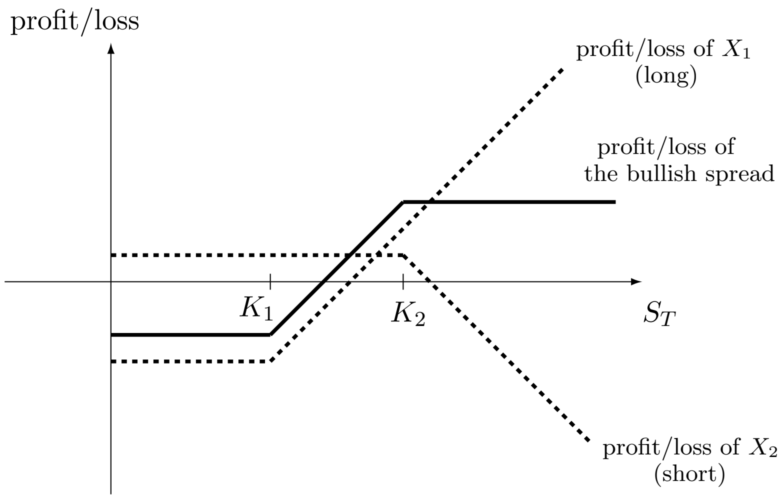


Figure 6.3: Profit/loss of bullish spread.

the options above except the calendar spread may as well be called bullish or bearish vertical spreads.)

- **Straddle and etc.**

One may combine options of different types on the same underlying stocks. A very popular combination is called the *straddle* that combines a put and a call on the same underlying stock with the same expiry and the same strike price. Buying both produces the payoff function as in Figure 6.6.
If the costs of options are taken into account, the profit and loss structure looks like as in Figure 6.7. Any speculator going long this

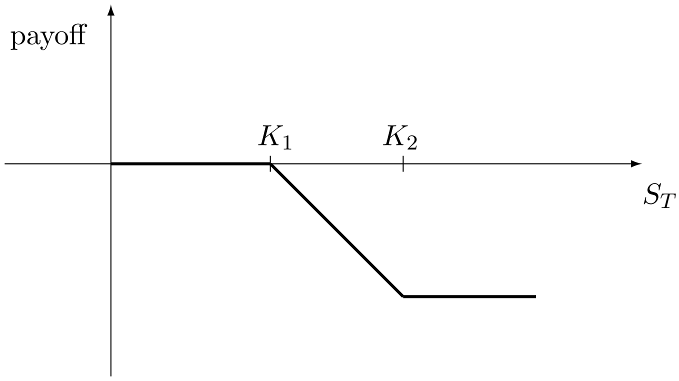


Figure 6.4: Payoff of bearish spread.

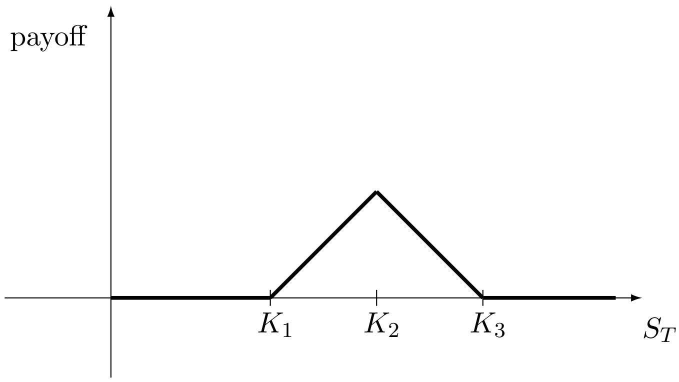


Figure 6.5: Butterfly spread.

kind of straddle is betting that the eventual stock price movement is big enough to net him/her profit. On the other hand any speculator going short this straddle is betting that the stock price does not move much, so that the options writing premium he/she gets would be sufficient to stay in the black.

There are many other combinations called *strip*, *strap*, *strangle*, etc. each of which caters to the whim or market outlook of the traders. Any interested reader may consult many excellent books in this regard.

In general the following lemma shows that any piecewise linear payoff with finitely many break points of its derivative can be written as a combination of puts and calls.

Lemma 6.1. *Let $\varphi(x)$ be a piecewise linear function on $\mathbb{R}^+$ with*

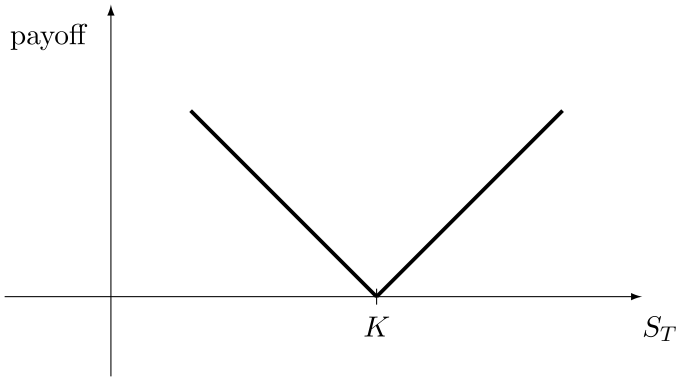


Figure 6.6: Payoff of straddle.

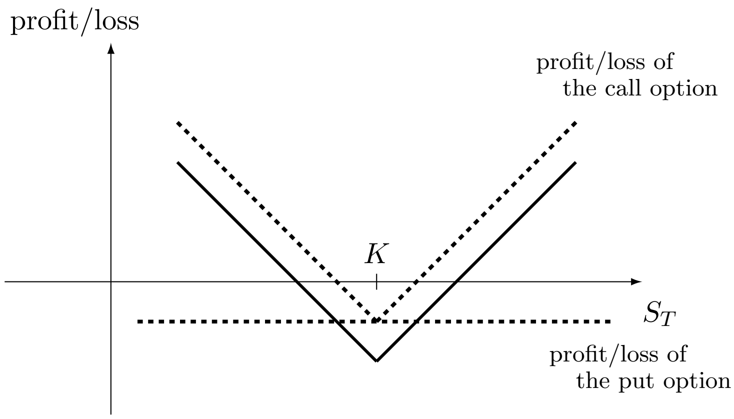


Figure 6.7: Profit/loss of straddle.

finitely many discontinuities of $\varphi'(x)$. *Then* $\varphi(x)$ *can be written as a linear combination of finitely many function of the type* $(x-K)^+$ *or* $(K-x)^+$ *for suitable finitely many constants* K.

We leave the proof of this lemma to the reader. Using it, we immediately have the following important fact.

Theorem 6.2. *Any European contingent claim whose payoff function is a piecewise linear function with finitely many discontinuities of its derivative can be constructed as a portfolio of finitely many European calls and puts, where each constituent option can be held long or short according to the sign of coefficient thereof.*

6.2 Parameter Estimation

In the previous chapter, we learned that the Black-Scholes model assumes that the stock price S_t satisfies

$$\log \frac{S_t}{S_0} = \mu t + \sigma W_t.$$

(In fact $\log S_t/S_0 = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$. But by abuse of notation, we use μ to denote $\mu - \frac{1}{2}\sigma^2$.) In this section, we discuss how one can estimate these parameters μ and σ.

Let S_{t_i} be the stock price at the end of i-th trading day. Then

$$\log \frac{S_{t_i}}{S_{t-1}} = \mu(t_i - t_{i-1}) + \sigma(W_{t_i} - W_{t_{i-1}})$$

where $\Delta t = t_i - t_{i-1} \approx \frac{1}{250}$. (Here we assume there are approximately 250 trading days per year.) Due to the incremental independence of the Brownian motion, we may assume $u_i = \log S_{t_i}/S_{t_{i-1}}$ is a IID Gaussian random variable with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$. By the standard statistical procedure its mean and variance can be estimated as:

$$\mu\Delta t \sim \bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i \tag{6.1}$$

and[1]

$$\sigma\sqrt{\Delta t} \sim \bar{s} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} u_i^2 - (\bar{u})^2}. \tag{6.2}$$

We call $\bar{u}$ the historical daily drift and $\bar{s}$ the historical daily volatility, respectively. We also call

$$\mu = \frac{\bar{u}}{\Delta t} = 250\bar{u}$$

and

$$\sigma = \frac{\bar{s}}{\sqrt{\Delta t}} = \frac{\bar{s}}{\sqrt{1/250}} \approx 15.81\bar{s}$$

the annualized historical draft and volatility, respectively.

Let us now see how reliable these estimates are. Table 6.1 shows various statistical quantities for KOSPI 200 stock price data for every year since 2001 till 2010. In there, one can see that $|\bar{u}|^2$ is two orders of magnitude smaller than $\frac{1}{2}\sum u_i^2$. Therefore in estimating $\bar{s}$, one may use $\sqrt{\frac{1}{n}\sum u_i^2}$ instead of $\sqrt{\frac{1}{n}\sum u_i^2 - |u_i|^2}$. It has an interesting

[1] To get an unbiased estimator, one has to use $\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$. But for large n they are close enough.

	$\bar{u}$	$\lvert\bar{u}\rvert^2$	$\frac{1}{n}\sum u_i^2$	$\sqrt{\frac{1}{n}\sum u_i^2}$	$\bar{s} = \sqrt{\frac{1}{n}\sum u_i^2 - \lvert\bar{u}\rvert^2}$	σ-estimate (1-year)	μ-estimate (1-year)
2001	0.001163	1.352E-06	4.959E-04	0.022270	0.022239	0.351635	0.290702
2002	-0.000349	1.218E-07	4.482E-04	0.021170	0.021167	0.334683	-0.031020
2003	0.001116	1.245E-06	2.890E-04	0.017001	0.016964	0.268228	0.278905
2004	0.000366	1.340E-07	2.393E-04	0.015469	0.015464	0.244514	0.091511
2005	0.001733	3.003E-06	1.185E-04	0.010887	0.010770	0.169950	0.433205
2006	0.000178	3.157E-08	1.357E-04	0.011650	0.011648	0.184178	0.044419
2007	0.001071	1.147E-06	2.175E-04	0.014750	0.014711	0.232595	0.267739
2008	-0.002016	4.063E-06	6.196E-04	0.024892	0.024810	0.392287	-0.503947
2009	0.001644	2.704E-06	2.641E-04	0.016253	0.016169	0.255656	0.411112
2010	0.000800	6.398E-07	9.988E-05	0.009994	0.009962	0.157516	0.199973

Table 6.1: Historical Volatility and Drift.

statistical consequence. First of all, it enables us to pretend that u_i is an IID random variable whose distribution is $N(0, \sigma^2 \Delta t)$. Second, if so, the independent nature of daily events then means that there are roughly 250 independent samples (one per day) to estimate $\sigma\sqrt{\Delta t}$ by $\sqrt{\frac{1}{n}\sum u_i^2}$.

On the other hand, as $|\bar{u}|$ is much smaller than $\sqrt{\frac{1}{n}\sum u_i^2}$, it is nearly impossible to extract meaningful "net" changes contributing solely to $\bar{u}$. The 95% confidence interval of $\bar{u}$ is roughly $(-0.00124 + \bar{u}, 0.00124 + \bar{u})$ when $\mathrm{Var}(u_i) = 0.01$, which also indicates that the estimate of $\bar{u}$ is quite unreliable. So one cannot argue that the independent nature of daily events cannot be utilized in estimating $\bar{u}$ or μ. For that reason, one has to use longer period, say the entire year, to get any number for $\bar{u}$ whose magnitude make sense.

But then as one can see in Table 6.1, μ varies wildly from year to year, whereas σ is more stable. In other words, the estimation of σ is statistically more stable than μ. Fortunately, the Black-Schools model does not depends on μ in any essential way so that the unreliability of μ does not pose any serious impedient to using the model.

In practice, no one believes μ or σ stay constant for a long period of time. Any long term student of market knows that the stock price moves like ebbs and flows. There are seasons when the stock price tends to go up and there are other season when it tanks violently. Accordingly, the volatility for a certain period may be low but it may flare up in time of crisis. Thus it is advised that one should take the Black-Schools model as a sort of shorten-term modeling of the market.

6.3 Implied Volatility and Volatility Smile

In the previous section, we studied how to estimate the volatility σ using the historical stock price data. Although this method of estimating σ is useful in many contexts, it leaves much gap between the model and the actual market price of the options. So many prefer to derive σ directly from the market price of a option, hence the name *implied volatility.*

Recall that the Black-Schools formula gives the valuation of a call or put option as a function of the stock price S_t, the strike price K, the volatility σ, the interest rate r and the time to expiry $T-t$. Namely, C is a function[2] of such quantities:

$$C = C(S_t, K, \sigma, r, t). \tag{6.3}$$

Suppose a market price A of a particular option at some instance of time is given. Treating $S_t, K, r, T-t$ as known quantities, one can find σ that satisfies

$$C(S_t, K, \sigma, r, t) = A. \tag{6.4}$$

To ascertain that such σ exists, one needs to examine the range A can possibly have.

Recall the Black-Schools formula for the call option

$$C_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2).$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_t/Ke^{-r(T-t)}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S_t/Ke^{-r(T-t)}) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Using this formula, one can check how C_t changes as one varies σ while holding all other parameters fixed. Let us first check the how C_t behaves as $\sigma \to 0$. It is trivial to see that if $S_t < Ke^{-r(T-t)}$, d_1 and $d_2 \to -\infty$, hence $N(d_1)$ and $N(d_2) \to 0$. Therefore C_t converges to 0. If, on the other hand, $S_t > Ke^{-r(T-t)}$, d_1 and $d_2 \to +\infty$, hence $N(d_1)$ and $N(d_2) \to 1$. Therefore C_t converges to $S_t - Ke^{-r(T-t)}$. The

[2] In fact, C is a function of the form $C(S_t, K, \sigma, r, T-t)$. But since T is fixed, we express C in the form (6.3).

limit C_t converges to is drawn as Curve ① in Figure 6.8. Let us now check the case where $\sigma \to \infty$. Then it is easy to see that $d_1 \to +\infty$ and $d_2 \to -\infty$, hence $N(d_1) \to 1$ and $N(d_2) \to 0$. Therefore C_t converges to S_t, which is drawn as Curve ② in Figure 6.8. It is clear

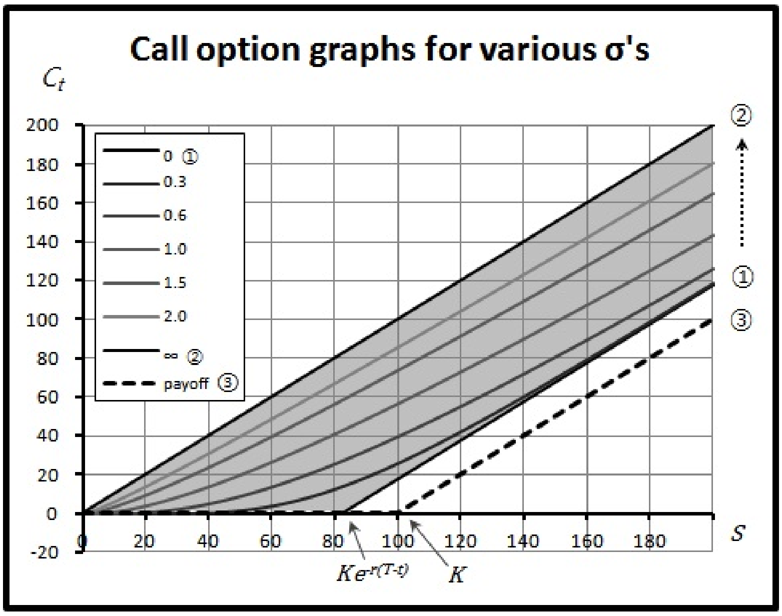


Figure 6.8: Call option graphs for various σ's.

that the graph of the call option lies for any *sigma* between there two curves. Figure 6.8 also shows various curves of C_t for different value of σ. From it, one can observe that as σ increases from 0 to ∞, the corresponding curve increases from ① to ②. Curve ③ is the payoff function of the call option.

Figure 6.9 shows the similar phenomenon for the put option, where one can check the following: first, as $\sigma \to 0$, the graph of the value P_t of the put option converges to Curve ①. Second, as $\sigma \to \infty$, it converges to Curve ②. Third, the graph of the put option for any σ lies between these two curves and as σ increases from 0 to ∞, it increases from Curve ① to Curve ②. Again Curve ③ is the payoff function of the put option. The curious fact that the put option price is less than the payoff for smaller values of S_t is due to the presence of the interest rate.

Therefore, as long as A stays in the shaded region between Curves ① and ②, (6.4) has a solution. In fact, such solution σ is unique as C is a monotonely increasing function of σ. This fact can be verified directly by looking at the sign of $\dfrac{\partial C}{\partial \sigma}$ (see Vega in Section 6.4 below) or can be argued with the principle of finance: namely, the bigger

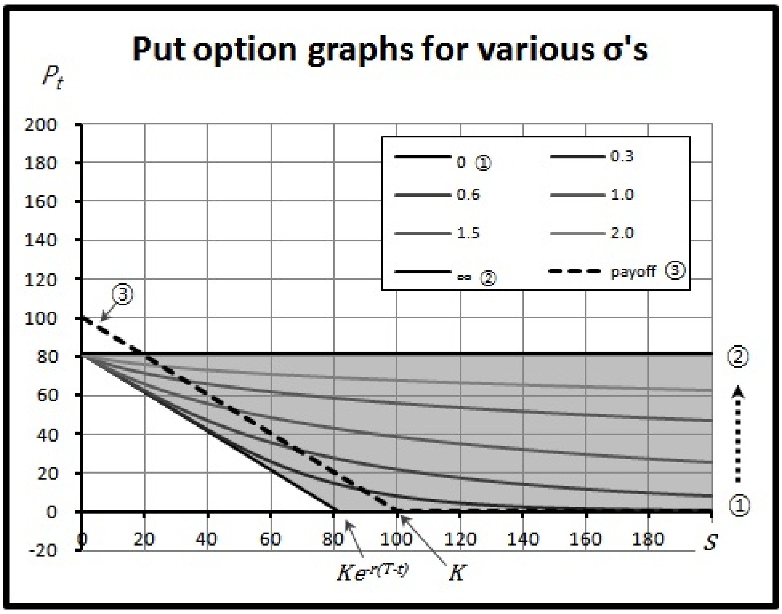


Figure 6.9: Put option graphs for various σ's.

the volatility, the higher the chance for profit for the holder of the option; hence, the higher the option value.

The value of σ calculated this way is called the *impled volatility*, meaning that the volatility is inferred from the market price data of the option via the Black-Schools formula.

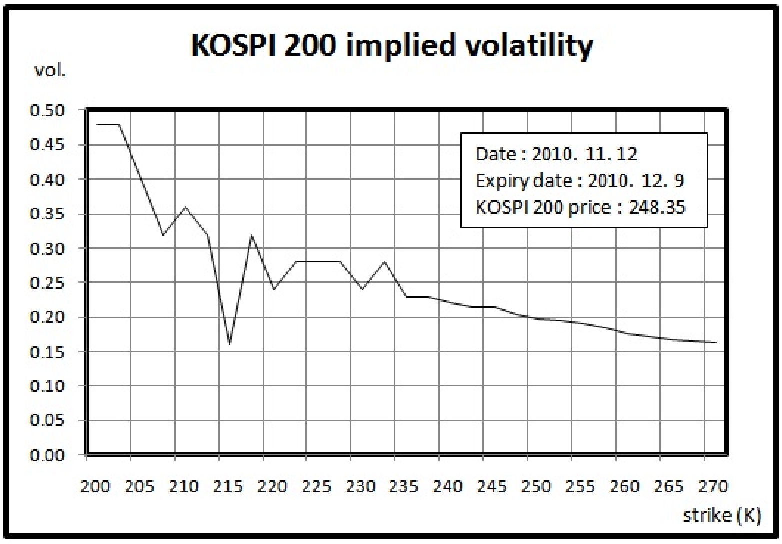


Figure 6.10: Implied volatility graph for KOSPI 200 call options.

Figure 6.10 shows the impled volatility against the various strike prices. According to the assumption of the Black-Schools model, σ is supposed to be a constant. Therefore the implied volatilities must be constant for different Ks. But as one can see in Figure 6.10, it is not the case. Such phenomenon is usually called the *(volatility) smile*. It shows that the assumption of the Black-Schools model does not match what is really going on in the options market.

This volatility smile has many practical implications for traders. To make the Black-Schools model to conform to this phenomenon, one sometimes uses a model for which σ is a stochastic process. (There is a debate as for the utility of such model.) A particular type of such stochastic volatility model is to set σ as a deterministic function of t and S_t, i.e.,

$$\sigma = \sigma(t, S_t).$$

Luckily even if σ is of this form, all the derivations in Chapter 5 leading to the Black-Schools PDE are valid. So we have following boundary value problem.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(t,S)^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$
$$V(T,S) = \varphi(S).$$

This PDE is still linear. But unless $\sigma(t, S_t)$ is of particular form, its solution can only be computed by numerical methods.

There are more general form of stochastic volatility for whose solution one uses a more sophisticated Monte-Carlo method. But we will not go into this in this lecture.

6.4 Sensitivity Analysis: Greeks

It is always a good practice to understand how a formula (for that matter, any recipe or even a theory) changes as the parameter varies. Examining such phenomenon is generally called a sensitive analysis.

In this section, we examine various rates of change of the Black-Schools formula with respect to the change of each important parameter. (In this section, when we draw a specific graph, the option values are computed assuming $K = 100$, $\sigma = 25\%$ and $r = 5\%$.)

- **Delta**

Delta is the rate of change of the option value with respect to the stock price and is denote as:

$$\Delta = \frac{\partial V}{\partial S}$$

Delta of a call option

Let C_t be a call option vale at time t. Recall that the Black-Schools formula for a call option is

$$C_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2). \tag{6.5}$$

Differentiating both sides with respect to S_t, we have

$$\Delta = \frac{\partial C_t}{\partial S_t} = N(d_1) + S_t N'(d_1)\frac{\partial d_1}{\partial S_t} - e^{-r(T-t)} K N'(d_2)\frac{\partial d_2}{\partial S_t}.$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$ and $N'(x)$ is the normal density function, i.e., $(2\pi)^{-\frac{1}{2}}\exp(-x^2/2)$, we get

$$\begin{aligned}
\Delta &= N(d_1) + \frac{\partial d_1}{\partial S_t}\left[S_t N'(d_1) - e^{-r(T-t)} K N'(d_1 - \sigma\sqrt{T-t})\right] \\
&= N(d_1) + \frac{\partial d_1}{\partial S_t}\left[S_t N'(d_1) - e^{-r(T-t)} K \exp\left(d_1\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right) N'(d_1)\right] \\
&= N(d_1) + \frac{\partial d_1}{\partial S_t}\left[S_t N'(d_1) - e^{-r(T-t)} K \exp\Big(\log(S_t/K) + r(T-t)\Big) N'(d_1)\right] \\
&= N(d_1).
\end{aligned}$$

Thus we have

$$\Delta = N(d_1).$$

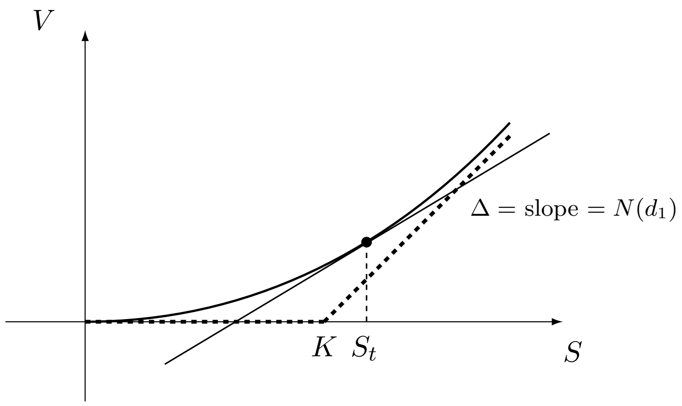


Figure 6.11: Call option graph and slope = Δ

This fact can be seen directly if one recalls that ζ_t is $\frac{\partial C}{\partial S}$ in (5.18) of Chapter 5 and also that ζ_t is $N(d_1)$ as commented in the remark immediately after Theorem 5.8. The significance of Δ is that it represents the number of the underlying stock in the replicating portfolio.

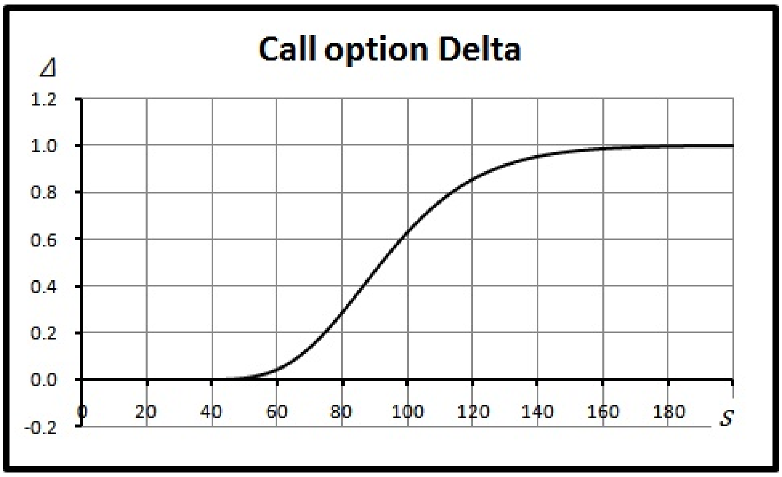


Figure 6.12: Delta graph of a call option.

But the same token, $e^{-rT}KN(d_2)$ is the amount of bank borrowing for the replicating portfolio. (See Theorem 5.8 and the comment thereafter.)

Figure 6.11 shows the graph of the value of a call option together with its Delta for given S_t. It is very important to note that the Delta is the slope of the graph at S_t. It is also important to note that the Delta increases as S_t increases, which means that as S_t increases the replicating portfolio must increase the number of stocks in it.

Figure 6.12 also shows the Delta as a function of S_t. One can see that the Delta changes most rapidly near the strike price.

Delta of a put option

Let P_t be a put option value at time t and Δ_C the Delta of a call option. From the put-call parity, we have

$$P_t = C_t + e^{-r(T-t)}K - S_t$$

Differentiating both sides with respect to S_t, we have

$$\frac{\partial P_t}{\partial S_t} = \Delta_C - 1 = N(d_1) - 1 = -N(-d_1).$$

Thus the Delta of a put option is

$$\Delta = -N(-d_1).$$

This fact can also be seen directly from the Black-Schools formula as in done for the call option.

Figure 6.13 shows the graph of a put option and its Delta as some S_t. The negative slope indicates that the replicating portfolio must have a short position of the underlying stock.

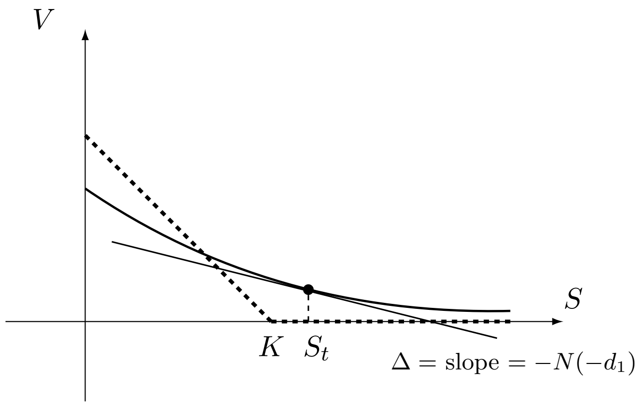


Figure 6.13: Put option graph and slope = Δ

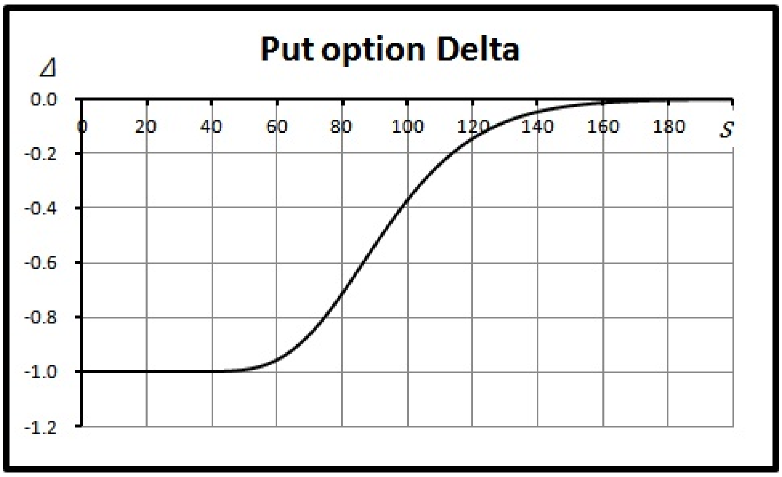


Figure 6.14: Delta graph of a put option.

• **Gamma**

Gamma is the rate of change of Delta with respect to the stock price and is denoted by

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$

Gamma of a call or put option
Let C be a call option's value. Then

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S_t} N(d_1) = N'(d_1)\frac{\partial d_1}{\partial S_t} = \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}.$$

From the put-call parity, we can easily check that the Gamma of a call and the Gamma of a put have the same value. So, we get

$$\Gamma = \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}$$

for both call and put options.

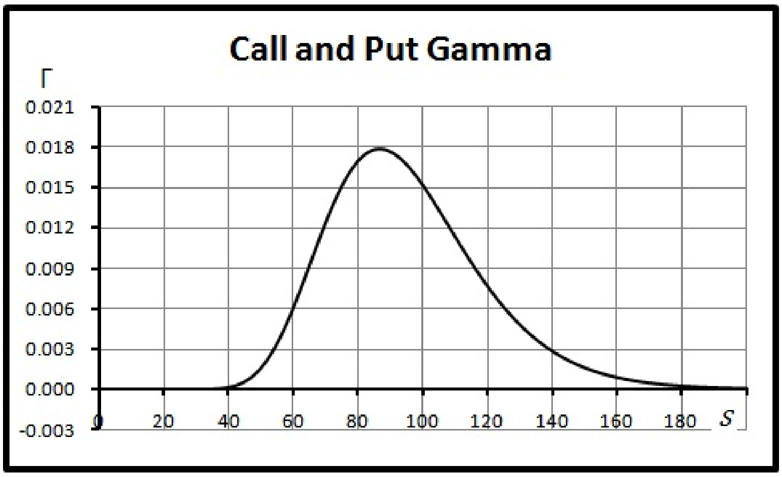


Figure 6.15: Gamma graph of a call/put option.

Note that Γ varies most rapidly near the strike price.

• Rho

Rho is the rate of change of the option value with respect to the interest rate and is denoted by

$$\rho = \frac{\partial V}{\partial r}.$$

Rho of a call option

Let C_t be a call option's value. Differentiating the Black-Schools formula with r, we have

$$\frac{\partial C_t}{\partial r} = S_t N'(d_1)\frac{\partial d_1}{\partial r} + (T-t)e^{-r(T-t)}KN(d_2) - e^{-r(T-t)}KN'(d_2)\frac{\partial d_2}{\partial r}.$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$, $\partial d_1/\partial r = \partial d_2/\partial r$. Thus we have

$$\begin{aligned} \frac{\partial C_t}{\partial r} &= \frac{\partial d_1}{\partial r}\left[S_t N'(d_1) - e^{-r(T-t)}KN'(d_1 - \sigma\sqrt{T-t})\right] \\ &\quad + (T-t)e^{-r(T-t)}KN(d_2). \end{aligned}$$

As in the calculation of the Delta, we already know that

$$S_t N'(d_1) - e^{-r(T-t)} K N'(d_1 - \sigma\sqrt{T-t}) = 0.$$

Thus we have

$$\rho = (T-t)e^{-r(T-t)} K N(d_2).$$

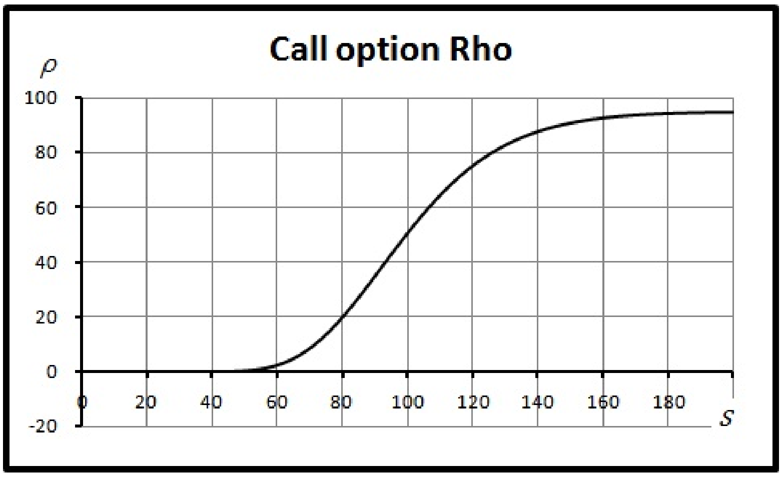


Figure 6.16: Rho graph of a call option.

Figure 6.16 shows the change of ρ with respect to S. In particular, note that since ρ is always positive, the call option's value increases as the interest rate increases.

Rho of a put option

Combining the put-call parity and the Rho value of a call option, we can easily get the Rho of a put option.

$$\rho = -K(T-t)e^{-r(T-t)} N(-d_2).$$

Since ρ is always negative, the option's value decreases as the interest rate increases.

- **Theta**

Θ is the rate of change of the option value with respect to the time and is defined by

$$\Theta = \frac{\partial V}{\partial t}.$$

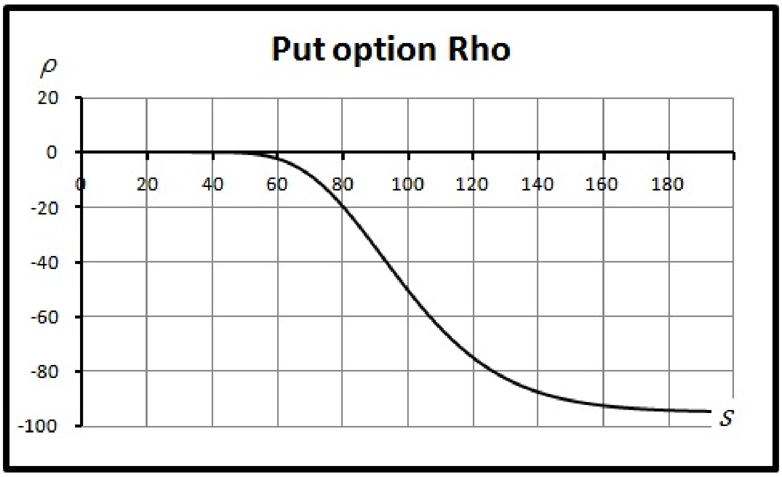


Figure 6.17: Rho graph of a put option.

Theta of a call option

Since $\partial V/\partial t = -\partial V/\partial(T-t)$, we use $\tau = (T-t)$ instead of t in the following calculation. Differentiating a call option value C_t that is given from the Black-Schools formula, we have

$$\frac{\partial C_t}{\partial \tau} = S_t N'(d_1)\frac{\partial d_1}{\partial \tau} + re^{-r\tau}KN(d_2) - e^{-r\tau}KN'(d_2)\frac{\partial d_2}{\partial \tau}.$$

Since $\partial d_1/\partial\tau = \partial d_2/\partial\tau + \sigma/(2\sqrt{\tau})$ and

$$S_t N'(d_1) - e^{-r\tau}KN'(d_2) = 0,$$

replacing $\partial d_1/\partial\tau$ with $\partial d_2/\partial\tau + 1/(2\sqrt{\tau})$, we have

$$\frac{\partial C_t}{\partial \tau} = \frac{S_t\sigma N'(d_1)}{2\sqrt{\tau}} + rKe^{-r\tau}N(d_2).$$

Thus we have

$$\begin{aligned}\Theta &= -\frac{S_t\sigma N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)\\ &= -\frac{S_t\sigma\exp\left(-d_1^2/2\right)}{2\sqrt{2\pi(T-t)}} - rKe^{-r(T-t)}N(d_2).\end{aligned}$$

Since Θ is always negative, the value of the call option decreases as time progresses. Figure 6.19 shows the pattern of the call option's value converging monotonically to the payoff as time passes.

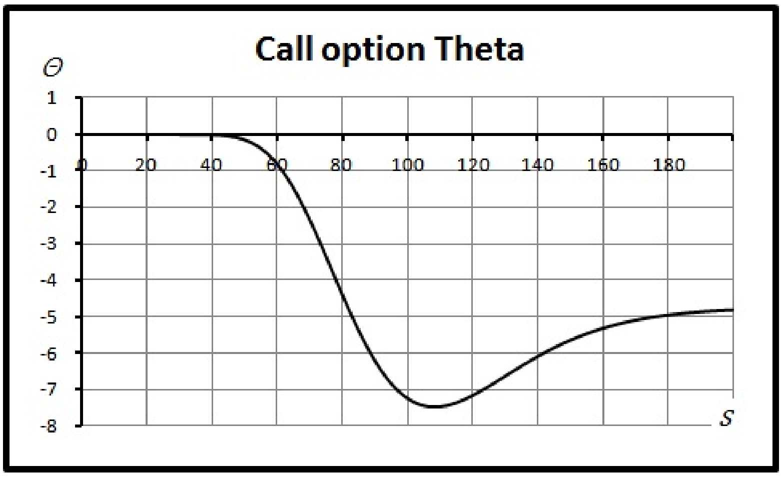


Figure 6.18: Theta graph of a call option.

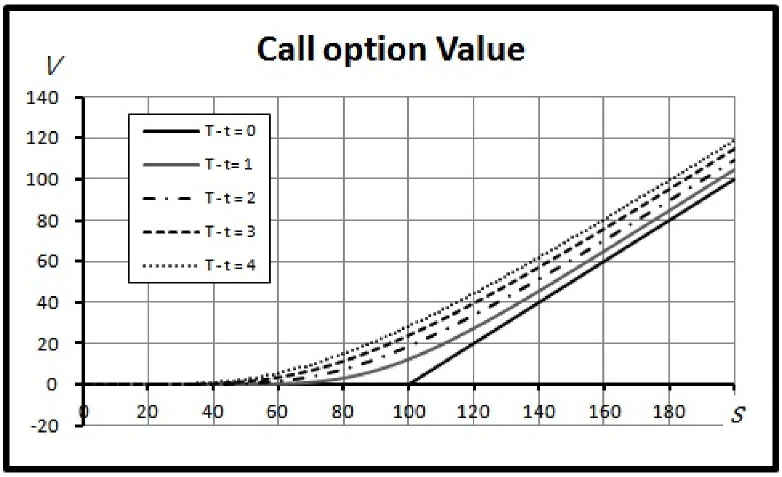


Figure 6.19: Call option graphs for the various $T-t$ values.

Theta of a put option
Differentiating the put-call parity equation with respect to t, we have the following formula for the Theta of the put option.

$$\begin{aligned}\Theta &= -\frac{S_t\sigma N'(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)\\ &= -\frac{S_t\sigma \exp\left(-d_1^2/2\right)}{2\sqrt{2\pi(T-t)}} + rKe^{-r(T-t)}N(-d_2).\end{aligned}$$

Figure 6.21 shows the plot of the Theta against the stock price. One should note that unlike the call option, the put option has mixed sign for the Theta for different values of S. Figure 6.21 shows how the

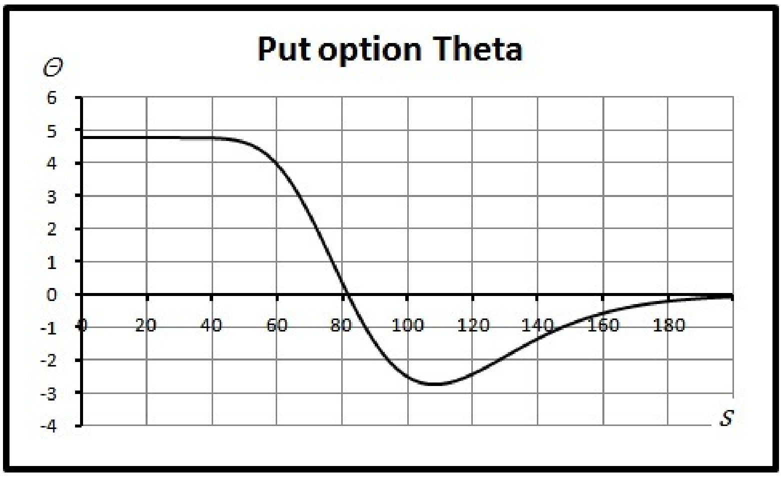


Figure 6.20: Theta graph of a put option.

put option value converges to the payoff function. It is interesting to note that as time progresses for small S_t the put option value increases while for large S_t the put option value decreases.

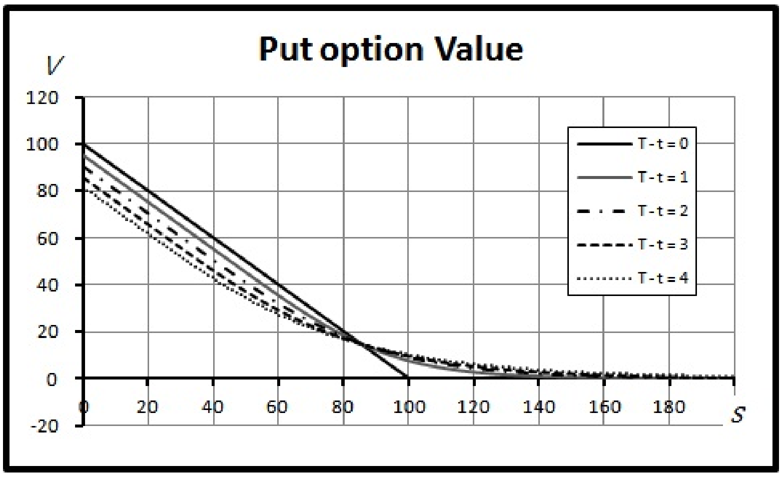


Figure 6.21: Put option graphs for the various $T - t$ values.

6.4.0.3 • Vega

Vega is the rate of change of the option value with respect to the volatility σ and is denoted by

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}.$$

Vega of a call or put option

Since $\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T-t}$, we have

$$\begin{aligned}
\mathcal{V} = \frac{\partial C_t}{\partial \sigma} &= S_t N'(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-r(T-t)}KN'(d_2)\frac{\partial d_2}{\partial \sigma} \\
&= \frac{\partial d_2}{\partial \sigma}\left[S_t N'(d_1) - e^{-r\tau}KN'(d_2)\right] + S_t\sqrt{T-t}N'(d_1) \\
&= S_t\sqrt{T-t}N'(d_1).
\end{aligned}$$

Differentiating the put-call parity equation with respect to σ one can see that the Vega of a call and the Vega of a put are the same. Thus for both call and put options, we have

$$\begin{aligned}
\mathcal{V} &= S_t\sqrt{T-t}\ N'(d_1) \\
&= S_t\sqrt{T-t}\ \frac{\exp\left(-d_1^2/2\right)}{\sqrt{2\pi}}.
\end{aligned}$$

Note that the Vega is always positive, which means that the values of the call and put options increases as the volatility increases. The reader is advised to go back Section 6.3 concerning such changes.

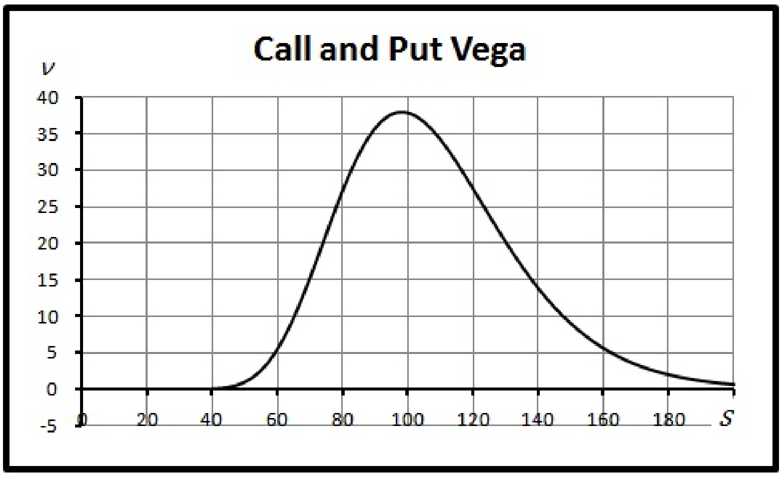


Figure 6.22: Vega graph of a call option.

• **Greeks of general contingent claim**

So far in this section, we have confined ourselves to the Greeks of the call or put options. But the concept of Greeks applies to any contingent claim with arbitrary payoff functions.

To see how it goes, let X be a European option with expiry T with payoff function $\varphi(S_T)$. Then by the verbatim application of the arguments in Section 5.5 the value V_t of X at time t is given by $V_t = V(t,S)$, where $V(t,S)$ is the solution of the following boundary(terminal) value problem:

$$\begin{aligned} &\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} - rV = 0 \\ &V(T,S) = \varphi(S). \end{aligned} \tag{6.6}$$

Then as before the Greeks of X are defined as follows;

- Delta: $\Delta = \frac{\partial V}{\partial S}(t, S_t)$
- Gamma: $\Gamma = \frac{\partial \Delta}{\partial S}(t, S_t) = \frac{\partial^2 V}{\partial S^2}(t, S_t)$
- Rho: $\rho = \frac{\partial V}{\partial r}(t, S_t)$
- Theta: $\Theta = \frac{\partial V}{\partial t}(t, S_t)$
- Vega: $\mathcal{V} = \frac{\partial V}{\partial \sigma}(t, S_t)$.

In particular, if $\varphi(S)$ is a continuous piecewise linear function with finite number of discontinuities of its derivative, Theorem 6.2 implies that X can be constructed as a portfolio (linear combination) of finite number of calls and puts. Any Greek of such X is the same linear combination of the corresponding Greeks of constituent calls or puts. In particular, the Greeks of the spreads, the straddles and the likes can be computed this way.

6.5 Delta and Gamma Hedging

6.5.1 Continuous Delta Hedging

Let X be a European option with payoff function $\varphi(S_T)$, where T is the expiry. Then its value V_t at time t is given by $V_t = V(t,S)$, where $V(t,S)$ is the solution of (6.6).

In Section 5.5, we showed that the replicating portfolio (ξ_t, ζ_t) is given in such a way that

$$\zeta_t = \Delta = \frac{\partial V}{\partial S}(t, S_t),$$

$$\xi_t = e^{-rt}[V_t - \Delta S_t].$$

(See (5.26) of Chapter 5 for further details.)

As time progresses, the delta

$$\Delta = \Delta_t = \frac{\partial V}{\partial S_t}$$

changes continuously. Thus the replicating portfolio has to be readjusted accordingly. In the parlance of finance, this continuous readjustment is called the *continuous hedging* or *continuous delta hedging.* However, this "continuous" hedging is only a theoretical construct; in practice there is no way to do anything continuously (i.e., infinitely often). At best one can do is to trade (hedge) frequently. This gap between theory and practice may be negligible in many cases but when the stock price moves rapidly it could be a cause for serious consequences.

To illustrate, let us draw in Figure 6.23 the graph $y = V(t, S)$ of y as a function of S while holding t fixed in the (S, y) plane. This represents the value of the nonlinear asset (i.e., the option) as a function of possible stock price S. Let us assume that the stock price at time t is S_t. Then the value of the option at time t is V_t and its delta is given by

$$\Delta = \frac{\partial V}{\partial S}(t, S_t),$$

which is the slope of $y = V(t, s)$ at $S = S_t$. Let us also define

$$y = \Delta[S - S_t] + V_t,$$

which is the line passing through (S_t, V_t) with slope Δ. It, in fact, represent the value of the linear asset (stock plus riskless bond) as a function of possible values S of the stock. Suppose now that the

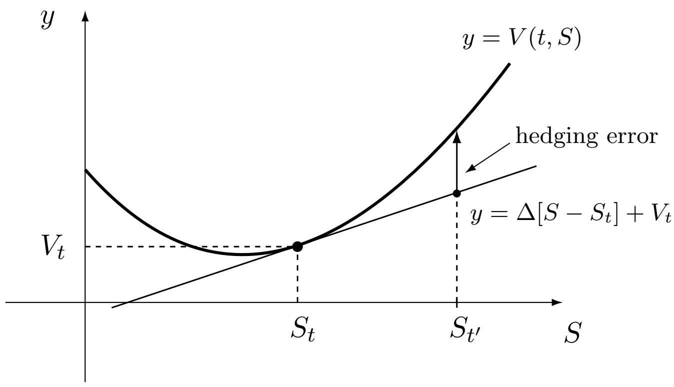


Figure 6.23: Graphs of linear and non-linear assets.

stock price changes to S_t' at time t'. If the time difference $t' - t$ is sufficiently small, $V(t, S)$ and $V(t', S)$ are virtually the same. So let us pretend that the value of the option at time t' can still be computed using $V(t, S)$. Then the value of the option changes from $V(t, S_t)$ to $V(t, S_{t'})$ as a result of stock price change. (Here, we are assuming there is no trade throughout this change.) Similarly, the value of the linear asset (i.e., the replicating portfolio) changes from V_t to $\Delta[S_{t'} - S_t] + V_t$. Then the values of the option and the replicating portfolio do not coincide, which is usually called the *hedging error*. It is depicted in Figure 6.23. Seen from the viewpoint of the option's writer (seller), this hedging error represents the loss. Figure 6.24 shows the hedging error as a function of S $(= S_{t'})$.

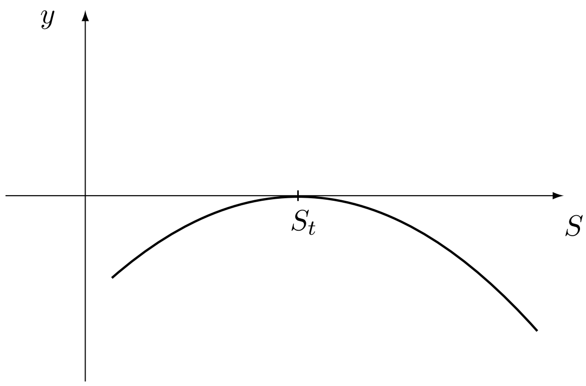


Figure 6.24: Hedging error (loss) of the option's writer.

Figure 6.25 shows the effect of the change of the option's value due to the change in time. Depending on the shape of the graph of $y = V(t, S)$, this time value may help or hurt the option's writer.

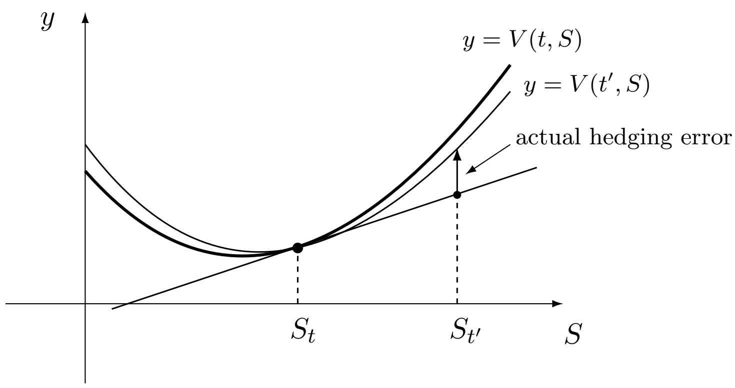


Figure 6.25: Actual hedging error with the time value taken in.

In any case, if the stock price moves rapidly in short time, the change of the option's value due to time value is relatively small, and the dominant factor is the kind of hedging error described in

Figure 6.24. To minimize such hedging error, one is forced to trade frequently, and more importantly, the option's writer should have demanded enough premium over the Black-Schools price to compensate for such contingencies.

6.5.2 Gamma Hedging

The shortcomings of the delta hedging are by now clear. Namely, it is vulnerable to big changes in the stock price in a short time. In order to rectify it, the traders frequently employ the so-called delta-gamma hedging, or in short the gamma hedging. The idea is rather simple. Instead of using the linear assets (stocks and riskless bonds) that have no gamma, the hedging is done by utilizing the plain vanilla (put and call) options that are abundantly available.

Suppose $V = V(t, S)$ is the value of the option to be hedged. Let Δ_V and Γ_V be its delta and gamma, respectively. Construct a portfolio consisting of two other vanilla options: Option 1 and Option 2. Let the delta and gamma of Option 1 be denoted by Δ_1 and Γ_1. Similarly, let Δ_2 and Γ_2 be the delta and gamma of Option 2. Let x be the number of units of Option 1 in the portfolio and y that of Option 2. Then the delta of such portfolio is $\Delta_P = x\Delta_1 + y\Delta_2$ and the gamma of the portfolio is $\Gamma_P = x\Gamma_1 + y\Gamma_2$. We can find the appropriate values of x and y such that

$$\begin{cases} \Delta_P = x\Delta_1 + y\Delta_2 = \Delta_V \\ \Gamma_P = x\Gamma_1 + y\Gamma_2 = \Gamma_V. \end{cases} \tag{6.7}$$

This is a system of linear equation for x and y that can be solved quite easily. The effect of having this portfolio consisting of two options is that its value at S_t matches that of $V = V(t, S)$ up to second order at S_t, which is depicted in Figure 6.26.

By adding more options to the portfolio one can match more Greeks. For instance, let us add another vanilla option, called Option 3. Let Δ_3, Γ_3 be its delta and gamma, respectively. Furthermore, let Λ_1, Λ_2 and Λ_3 be the vegas of Option 1, 2 and 3. Then one can find x, y and z such that

$$\begin{cases} x\Delta_1 + y\Delta_2 + z\Delta_3 = \Delta_V \\ x\Gamma_1 + y\Gamma_2 + z\Gamma_3 = \Gamma_V \\ x\Lambda_1 + y\Lambda_2 + z\Lambda_3 = \Lambda_V \end{cases} \tag{6.8}$$

where Λ_V is the vega of option V in question.

It is obvious that by adding more and more options this way, one can match all the Greeks.

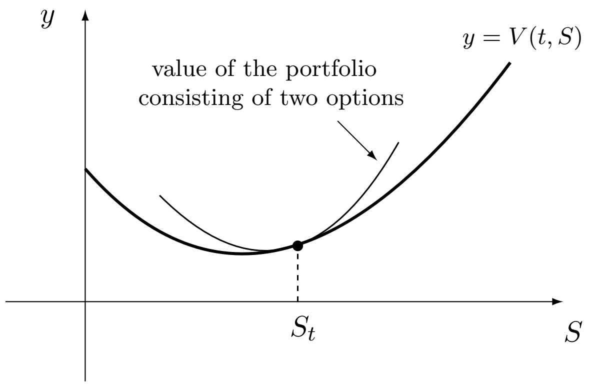


Figure 6.26: Portfolio's value matches V up to second order.

This method of utilizing several non-linear assets (i.e., vanilla call and put options) to match various or all Greeks works even when V is not a single option but itself a portfolio of stocks, options and (riskless) bonds.

However the trouble with this method is that the resulting system of equations may be numerically ill-conditioned, which may yield widely varying, and sometime unrealistic, numerical values for x, y, z, etc. as the parameter changes slightly. So some sort of stabilization scheme must be employed. Furthermore, there are many choices for Option 1, 2, 3, etc. to match the delta, gamma, vega, etc. of V. In fact, which options to utilize is a critical question for which many other aspects must be considered. For instance, the liquidity and the price should be just as important factors.

6.6 Portfolio Insurance

Portfolio insurance is a method/technique developed by Leland and Rubinstein that purports to "insure" a portfolio, if properly constructed and managed according to their recipe, against the downside risk while keeping open the upside potential due to the rise of the stock price. It was conceived in late 1970s and was actively marketed in the 1980s until its shortcomings were revealed in the crash of 1987.

The idea of the portfolio insurance is rather simple. Suppose S_t is a stock price or rather a price of an index like S&P 500 that is hugely liquid. Its put-call parity says that

$$S_t + P_t = C_t + Ke^{-r(T-t)}$$

where P_t and C_t are the prices of the put and the call with expiry T and strike price K, respectively. Using the Black-Scholes formula, (5.43) of Chapter 5, we have

$$S_t + P_t = N(d_1)S_t + Ke^{-r(T-t)}(1 - N(d_2)). \tag{6.9}$$

This formula can be interpreted as saying that the portfolio consisting of one stock and a put option is equivalent to holding $N(d_1)$ shares of stock and $1 - N(d_2)$ units of riskless bond whose face value (i.e., the amount to be paid at the maturity T) is K, or by the same token, $K(1 - N(d_2))$ units of riskless bonds each of which pays 1 at T.

The final payoff $S_T + P_T$ at time T of the portfolio comprising one stock and a put option is depicted in Figure 6.27, whether or

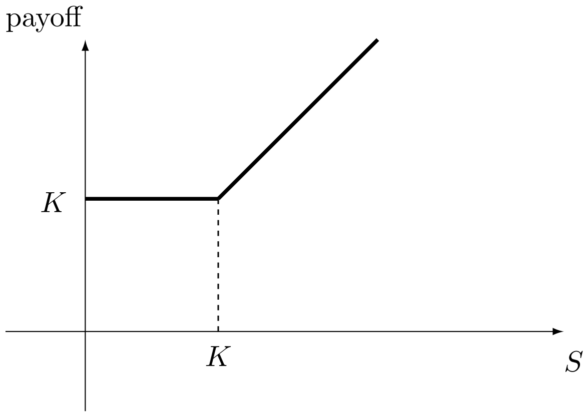


Figure 6.27: Payoff of $S_T + P_T$.

not one engages in dynamic hedging at all. The key observation Leland and Rubinstein made is the following lemma, which becomes the cornerstone of their "portfolio insurance" method.

Lemma 6.3. *There exists K^* such that*

$$S_t + P(S_t, K^*) = K^*$$

where $P(S_t, K)$ is the value of the put option whose strike price of K when the stock price is S_t, while all the other parameters are assumed to be given.

Proof. Define

$$g(K) = S + P(S, K) - K.$$

Then $g(0) = S$, since $P(S, 0) = 0$. On the other hand, as $K \to \infty$, we can check that $d_1 \to -\infty$ and $d_2 \to -\infty$. Therefore by (6.9)

$$\begin{aligned} g(K) &= SN(d_1) + Ke^{-r(T-t)}(1 - N(d_2)) - K \\ &\to -K(1 - e^{-r(T-t)}) < 0. \end{aligned}$$

Thus by the continuity of $g(K)$, there must be K^* such that $g(K^*) = 0$. $\square$

This lemma is remarkable in the sense that it opens up a possibility, albeit theoretical, of minting money. Namely, if one starts with a sum of K^* dollars and use it to buy one share of stock and a put option with the strike price K^*, the payoff structure as in Figure 6.27 guarantees that at time T one cannot lose money but has a possibility of making money in case $S_T > K^*$. (Of course, we have ignored important details like transaction costs, etc.)

Leland and Rubinstein went ahead and tried to exploit this idea. The trouble is that such put option with strike price K^* was not always available in the market place, and even if it was, it may not have been available in sufficient quantity. They therefore decided to synthetically create the portfolio by utilizing (6.9). The left had side of (6.9) is the "magic portfolio" that can "mint" money, which is equal to the right hand side of (6.9). As we said in the comment immediately following (6.9), the right hand side says that as long as one *dynamically* maintains the portfolio consisting of $N(d_1)$ shares of stocks and $1 - N(d_2)$ units of riskless bond with face value K^*, the final payoff will be $S_T + P(S_T, K^*)$ which is always greater than or equal to the initial investment K^*.

However, this portfolio insurance technique did not live up to its promise in time of crisis. In fact, it simply melted down shortly before, during and after the Black Monday of 1987 (October 19, 1987). According to the Brady Commission Report, during the three-day period of October 14 to October 16, 1987, there was a \$12 billion worth of portfolio insurance outstanding and only \$4 billion worth was properly executed according to the plan of portfolio insurance, and at the peak of crash, on Monday, October 19th, 10% of the sales volume of NYSE and 21% of the index futures sales were due to portfolio insurance. What happened was that the sell orders triggered by the dynamic hedging method were left unexcuted while the stock price was dropping precipitously. In short, the *dynamic* hedging scheme envisioned by the designers of portfolio insurance failed to function properly, which resulted in big loss.

There are many reasons for this failure. To list a few:

- As the stock market declines, the delta hedging method employed by the portfolio insurance triggers even more sell orders due to the convex nature of its value function. This helps add greater downward pressure on the stock price, hence creating huge market instability.

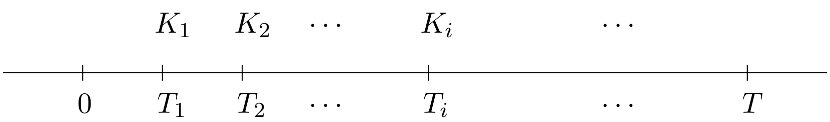
- When there are overwhelmingly more sell orders than buy orders, the sell orders not be properly executed, Thus thereby defeating the purpose of the portfolio insurance. (This is in marked contrast with the usual insurance where the events occur sporadically with very low correlation.)

6.7 Dividend models

Dividend is a stream of cash paid to the holder of a stock. This cash payment occurs at discrete times that are fixed a priori beforehand or they can be ramdom. The amount of dividend at each dividend payment time can be predetermined or it can vary randomly. In this section, we study two models of dividend. The first is the one of deterministic discrete dividend model and the second is the so-called constant dividend model.

6.7.1 Known discrete dividend model

Let $t = 0$ be the present and let T_i be the time of dividend payment with dividend amount K_i and let T be the time at which our option expires.



Suppose the interest rate r is a constant. Then the value $\tilde{I}_t$ at time t of all the dividend paid after t until T is

$$\tilde{I}_t = \sum_i K_i e^{-r(T_i - t)} 1_{(t,T]}(T_i).$$

Similarly the value I_t at time T of all the dividend paid after t until T is

$$I_t = \sum_i K_i e^{r(T - T_i)} 1_{(t,T]}(T_i).$$

When dividend is paid the stock price usually drops by a fixed amount, which is called an ex-dividend drop. Thus it is not a good idea to deal with the stock price process directly. Instead, after

Heath and Jarrow, we primarily deal with the so-called capital gains process G_t, which is assumed to satisfy

$$dG_t = G_t(\mu dt + \sigma dW_t) \tag{6.10}$$

Let D_t be the value at t of the totality of the dividends paid after the present ($t = 0$) until time t. Then obviously

$$D_t = \sum_i K_i e^{r(t-T_i)} 1_{(0,t]}(T_i).$$

Note in particular that

$$D_T = I_0.$$

The net capital gain from time 0 to t is $G_t - G_0$. It is certainly composed of that coming from the appreciation of stock price($S_t - S_0$) and the value at t of the totality of dividends paid after 0 until t. Therefore

$$G_t - G_0 = S_t - S_0 + D_t.$$

Taking differential, we have

$$dG_t = dS_t + dD_t \tag{6.11}$$

We set $G_0 = S_0$, if there is no dividend payment at time $t = 0$; and if there is a dividend payment at $t = 0$, we set S_0 to be the ex-dividend price. Hence in either case we have:

$$\begin{aligned} G_0 &= S_0, \\ G_t &= S_t + D_t. \end{aligned}$$

The trading strategy is as usual composed of ζ_t shares of S_t and ξ_t units of bank account. Then the value process of the portfolio is

$$V_t = \zeta_t S_t + \xi_t B_t.$$

We now assume that the dividend is reinvested in the same stock immediately after the dividend is paid. With this in mind, let us recall the true meaning of "self-financing" condition. A portfolio is self financing if the change of its value is entirely due to the internal market dynamics with no money coming in from or going out to the outside world. Cast in this viewpoint, this portfolio must be self-financing if the change of the portfolio's, value is due to first, the change in stock price; second, the increase in bank account; and third, the dividend. Therefore it is eminently reasonable to *define* (ζ_t, ξ_t) is self-financing if and only if

$$dV_t = \zeta_t dS_t + \xi_t dB_t + \zeta_t dD_t.$$

This and (6.11) then imply that

$$dV_t = \zeta_t dG_t + \xi_t dB_t.$$

This together with (6.10) means that we can work with G_t instead of S_t in deriving the options value formula. Note that

$$G_T = S_T + D_T = S_T + I_0.$$

Let $X = (S_T - K)^+$ be a European call option. Rewritten in terms of G_t, we have

$$X = (G_T - (K + I_0))^+, \tag{6.12}$$

where G_t satisfies the usual geometric Brownian motion (6.10). Therefore one can apply the usual machinery of deriving the Black-Scholes formula with G_t replacing S_t (In here, one has to check that the portfolio is self-financing and it replicates X, etc. But they are all easy exercises and hence are left to the reader.)

The key point in (6.12) is that the exercise price K is replaced with $K + I_0$. Therefore we have the call option value formula

$$C_0 = G_0 N(d_1) - e^{-rT}(K + I_0)N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/(K+I_0)) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\ d_2 &= \frac{\log(S_0/(K+I_0)) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

Similarly, by translating time, we have the following:

Theorem 6.4. *Suppose S_t satisfies the conditions described above with known discrete dividend payments. The value C_t at time t of the call option with strike price K at expiry T is given by*

$$C_t = S_t N(d_1) - e^{-r(T-t)}(K + I_t)N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_t/(K+I_t)) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S_t/(K+I_t)) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Here, S_t is the ex-dividend stock price at time t and I_t is the value at t of the dividends paid after t until T.

6.7.2 Constant dividend yield model

Another simplifying model on dividend is that of constant dividend yield. The gist of this model is that unlike the one in the previous subsection, one assumes that the dividend is paid out at a constant rate like the continuously compounding interest rate model. Thus we assume that if the stock price at t is S_t the dividend paid out during the short time interval $[t, t+dt]$ is proportional to S_t, i.e.

$$dD_t = \delta S_t dt, \tag{6.13}$$

where D_t is the sum($\int$) of dividends paid out from time 0 to t and δ is a positive constant called the dividend payout ratio. We further assume that the dividend is immediately reinvested in the same stock. Thus the dividend $dD_t = \delta S_t dt$ paid out during $[t, t+dt]$ buys δdt shares of stocks. Therefore the holder of the stock actually has increasing number of the stock. he/she holds. Let U_t be the number of the stock due to this reinvestment. Since (6.13) is for a single share, the dividend received by the holder of U_t shares is

$$dD_t = \delta S_t U_t dt,$$

which buys

$$dU_t = \delta U_t dt \tag{6.14}$$

shares of socks. Assuming $U_0 = 1$, (6.14) yields

$$U_t = e^{\delta t}.$$

Assume now that S_t satisfies the usual geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

which gives

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right).$$

Therefore the value $\tilde{S}_t$ of $U_t = e^{\delta t}$ shares of such stock is

$$\begin{aligned}\tilde{S}_t &= S_t e^{\delta t} \\ &= S_0 \exp\left((\mu - \frac{1}{2}\sigma^2 + \delta)t + \sigma W_t\right).\end{aligned}$$

Rewriting this, we can easily see that

$$d\tilde{S}_t = \tilde{S}_t((\mu + \delta)t + \sigma W_t). \tag{6.15}$$

Let (ζ_t, ξ_t) be a portfolio of ζ_t shares of S_t and ξ_t units of bank account. Then its value V_t is given by

$$V_t = \zeta_t S_t + \xi_t B_t.$$

As we did in the previous subsection, the self-financing portfolio's value change comes from the three source: the change in stock price, the change in bank account and the dividend. Therefore the self-financing condition can be phrased as:

$$dV_t = \zeta_t dS_t + \xi_t dB_t + \zeta_t \delta S_t dt.$$

is the self-financing condition. Thus for the self-financing portfolio,

$$\begin{aligned} dV_t &= \zeta_t S_t[(\mu+\delta)dt + \sigma dW_t] + \xi_t dB_t \\ &= e^{-\delta t}\zeta_t \tilde{S}_t[(\mu+\delta)dt + \sigma dW_t] + \xi_t dB_t \\ &= e^{-\delta t}\zeta_t d\tilde{S}_t + \xi_t dB_t, \end{aligned}$$

where in the second equality $\tilde{S}_t = S_t e^{\delta t}$ is used and in the third equality (6.15) is used. Define $\tilde{\zeta}_t = e^{-\delta t} S_t$. Then we have

$$dV_t = \tilde{\zeta}_t d\tilde{S}_t + \xi_t dB_t.$$

From this, we can conclude that

Lemma 6.5. *The portfolio (ζ_t, ξ_t) consisting of ζ_t shares of S_t and ξ_t units of B_t, assuming constant dividend yield, is self-financing if and only if the portfolio $(\tilde{\zeta}_t, \xi_t)$ consisting of $\tilde{\zeta}_t$ shares of(hypothetical) stock $\tilde{S}_t$ and ξ_t units of B_t is self-financing in the usual sense, i.e.,*

$$dV_t = \tilde{\zeta}_t d\tilde{S}_t + \xi_t dB_t.$$

With this lemma the usual machinery of the Black-Scholes model takes over to give us the options formula. In particular, let $X = (S_T - K)^+$ be a European call option with strike price K at the expiry T. Rewriting it, we have

$$X = e^{-\delta T}(\tilde{S}_T - Ke^{\delta T})^+.$$

Thus using the usual Black-Scholes formula, the value C_t of the call option at time t is seen to be

$$C_t = e^{-\delta T}[\tilde{S}_t N(d_1) - e^{-r(T-t)} K e^{\delta T} N(d_2)], \tag{6.16}$$

where

$$\begin{aligned} d_1 &= \frac{\log(\tilde{S}_t/Ke^{\delta T}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(\tilde{S}_t/Ke^{\delta T}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Using $\tilde{S}_t = e^{\delta t} S_t$, one can rewrite (6.16) to get the following

$$C_t = e^{-\delta(T-t)}[S_t N(d_1) - e^{-(r-\delta)(T-t)} K N(d_2)],$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_t/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S_t/K) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

By the put-call parity, the put option price is also easily seen as

$$P_t = e^{-\delta(T-t)}[-S_t N(-d_1) + e^{-(r-\delta)(T-t)} K N(-d_2)].$$

Exercises

6.1.

Answer "yes" or "no" to the following assertion related to the greeks of the Black-Scholes formula. Briefly explain why.

(a) The higher the volatility, the higher the value of a put option.

(b) The higher the stock price, the more shares of stock the hedging portfolio of a call option must have

(c) As the stock price decreases, the seller of a put option must sell more shares short in the hedging portfolio

(d) As the interest rate increases, the value of the put option increases.

(e) All things being equal, the value of a put option increases as time passes if it is deeply in in-the-money territory.

6.2.

(a) Write down the Black-Scholes formula for the call option in the case of constant dividend yield rate δ.

(b) What will C_t converge to as $\sigma \to \infty$?

(c) What will C_t converge to as $\sigma \to 0$?