집합과 수리논리: 기말 고사 2013 년 6 월 12 일 (수) 11:00–12:15 (총점 100 점)

소속: 학번: 이름: 점수:

배점: 문 1, 2, 3, 6 (Need Justifications): 각 12점, 문 4: 4점,
문 5: 48점 (각 3점: 오답 0점, 무답 1점, 정답 3점)

For a set A, let A^* be the set of all finite sequences of elements in A. Two sets are *equipotent* if there exists a bijection between them. We assume ZFC.

1. For $a \neq b$, let $A = \{a, b\}$. Define an ordering relation $<$ on A so that $a < b$. This ordering induces the *dictionary ordering* on A^*. Is A^* a well-ordered set with this induced ordering?

2. Let $H : A^* \to A$ be a function. Suppose f and g are sequences of elements of A such that $f(n) = H(f \upharpoonright n)$ and $g(n) = H(g \upharpoonright n)$ for all $n \in \omega$. Is it true that $f = g$?

3. Does there exist a function $\mu : \mathcal{P}\mathbb{R} \to [0, \infty]$ with the following properties?
   (a) for any interval $[a, b]$ in $\mathbb{R}$, $\mu([a, b]) = b - a$.
   (b) $\mu(S_0 \cup S_1 \cup \cdots) = \mu(S_0) + \mu(S_1) + \cdots$ for any disjoint subsets $S_0, S_1, \ldots$ of $\mathbb{R}$.
   (c) for any subset S of $\mathbb{R}$ and for any $x \in \mathbb{R}$, $\mu(S+x) = \mu(S)$, where $S + x = \{y + x \mid y \in S\}$.

4. Write a natural number in the box below. You will get a better score if your number is closer to 'half of the average' of the natural numbers submitted by the examinees. [ ]

5. In '( )' write T if true, F if false.
   - ( ) Ther exists a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n+1) = f(n)^2 + 1$ for all $n \in \mathbb{Z}$.
   - ( ) Let S be an abelian semi-group. Then there exists an abelian group G and a semi-group homomorphism $i : S \to G$ such that for any abelian group H and any semi-group homomorphism $f : S \to H$ there exists a group homomorphsm $\bar{f} : G \to H$ such that $f = \bar{f} \circ i$.
   - ( ) Any two complete ordered fields are isomorphic.
   - ( ) The set of all functions from an infinite set A into A is equipotent to $\mathcal{P}A$.
   - ( ) The set of all functions from an infinite set A into $\mathcal{P}A$ is equipotent to $\mathcal{P}A$.
   - ( ) If there exist injections from A into B, and from B into A, then there exists a bijection from A onto B.
   - ( ) If there exist surjections from A onto B, and from B onto A, then there exists a bijection from A onto B.
   - ( ) For any infinite cardinal κ, $\kappa^2 = \kappa$.
   - ( ) For any cardinal κ, there exists a cardinal λ such that $\kappa < \lambda$.
   - ( ) The cardinality of the set of ordinals equipotent to ω is the first uncountable cardinal.
   - ( ) Two sets are equipotent if and only if they have the same cardinality.
   - ( ) ZF can not determine whether the continuum hypothesis is true or not.
   - ( ) There exists a statement which can be proved in ZF but not in PA.
   - ( ) Let α be an ordinal and $x \notin \alpha$. Then the *ordered sum* $\alpha \cup \{x\}$ is an ordinal if and only if $x = \alpha$.
   - ( ) The union of any set of ordinals is an ordinal.
   - ( ) Any set of ordinals is an ordinal.

6. Write 3 statements which are equivalent to the Axiom of Choice.

모범답안

[1] No. It has an infinitely decreasing sequence $b, ab, aab, aaab, \ldots$.

[2] Yes. To see this, let $S = \{n \in \omega \mid f(n) = g(n)\}$. Suppose $n \in \omega$ and $k \in S$ for any $k < n$. Then $f(n) = H(f \restriction n) = H(g \restriction n) = g(n)$. Thus $n \in S$. Now the induction principle implies $S = \omega$.

[3] No

[5] All T except the First and the Last.

[6] WO (All set can be well-orderd.), ZL (If any chain in a partially ordered set A has an upper bound, A has a maximal element.), TC (For any sets A and B, there exists an injection either from A into B or from B into A.), $\ldots$