Lecture 1: Historical Remarks

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At ICM(International Congress of Mathematicians) 1900, Paris, David Hilbert(1862–1943) announced 23 Mathematical Problems for the new century. The first one was about the CH(Continuum Hypothesis) and the second one was about the the CA(Consistency of Arithmetic).

CH: Continuum Hypothesis

When he was studying Fourier series, Georg Cantor(1845–1918) investigated various infinite sets of real numbers, and he thought that

> CH: *any infinite set of real numbers is equipotent to either the set* $\mathbb{N}$ *of natural numbers or the set* $\mathbb{R}$ *of real numbers.*

Two sets A and B are *equipotent* or *of the same size* if there is a *bijection* (i.e., one-to-one onto map) between them. In this case, we write

$$A \sim B.$$

For instance,

$$\mathbb{N} \sim 2\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Z} \times \mathbb{Z} \sim \mathbb{Q},$$

where $\mathbb{Z}$ and $\mathbb{Q}$ denote, respectively, the set of integers and rational numbers.

We say that a set is *countable* if there exists an *injection* (i.e., one-to-one map) from from the set *into* $\mathbb{N}$. A countable set is either finte or infinite.

Exercise. Let A be a set of non overlapping disks in the plane. Is A countable?

We also have

$$[0,1] \sim [0,1)$$

between two intervals, since

$$f(x) = \begin{cases} \frac{1}{n+1} & (x = \frac{1}{n} \text{ for some positive integer } n) \\ x & (x \neq \frac{1}{n} \text{ for all } n) \end{cases}$$

gives a bijection.

It is easy to see that

$$[0,1] \sim (0,1) \sim \mathbb{R}.$$

Homework 1: Due Monday, March 11, 2013

1. For sets A, B, C, show that

   (a) $A \sim A$

   (b) $A \sim B \Rightarrow B \sim A$

   (c) $(A \sim B \;\&\; B \sim C) \Rightarrow A \sim C$

2. A complex number z is an *algebraic number* if it is a root of a polynomial with rational coefficients. Show that the set of algebraic numbers is countable.

3. Show that the disk

   $$D := \{z \in \mathbb{C} \mid |z| \leq 1\}$$

   is equipotent to the annulus

   $$A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\},$$

   where $\mathbb{C}$ denotes the set of complex numbers.

CA: Consistency of Arithmetic

Show that $0 \neq 1$.