Lectures 2∼3: Historical Remarks 2

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March 6 & 11, 2013 (v.2.7)

CH and CA were the most important problems in mathematics... in 1900, at least to D. Hilbert.

Infinite Sets

Last time we have discussed various countable sets. For instance the set $\mathbb{A}$ of real algebraic numbers is countable.

If we understand *computable* numbers, then it can be shown that algebraic numbers are also computable. It must be an easy exercise to show that the set $\mathbb{K}$ of computable numbers is also countable.

We have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{E} \subset \mathbb{P} \subset \mathbb{A} \subset \mathbb{K} \subset \cdots \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O},$$

where $\mathbb{E}$ is the set of *Euclidean* or *constructible* numbers (with a straightedge and a compass), $\mathbb{P}$ is the *paper folding* numbers, $\mathbb{H}$ is the set of Hamilton's quaternions, and $\mathbb{O}$ is the set of Cayley's octonions.

Today we will see many *uncountable* sets.

Recall that, for sets A and B, $A \sim B$ if and only if there exists a bijective function from A onto B. For example

$$[0,1] \sim [0,1).$$

Thus

$$(0,1] \sim (0,1) \sim \mathbb{R}.$$

We have more examples.

Theorem 1

$$\mathbb{R} \sim \mathbb{R}^2.$$

To show this, it suffices to show[1]

$$(0,1] \sim (0,1]^2.$$

Theorem 2 (Cantor, 1877) *For any positive integer n*

$$\mathbb{R} \sim \mathbb{R}^n$$

[1] cf. R. Boas, Jr., *A Primer of Real Functions*, The Math. Assoc. America, 1996.

Georg Ferdinand Ludwig Philipp Cantor (March 3, 1845 ∼ Jan. 6, 1918)
Je le vois, mais je ne le crois pas! (I see it, but I don't believe it!),
in a letter to Dedekind, 1877.

Exercise

Let $\mathbb{R}^\infty$ be the set of all finite sequences of $\mathbb{R}$:

$$\mathbb{R}^\infty = \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots .$$

Show that $\mathbb{R} \sim R^\infty$.

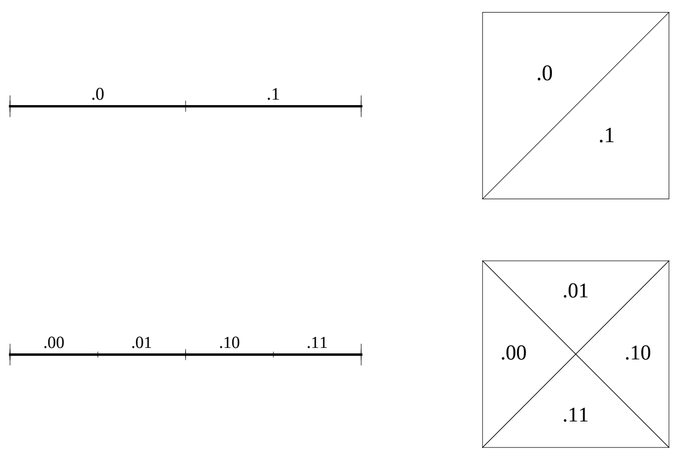
Remark

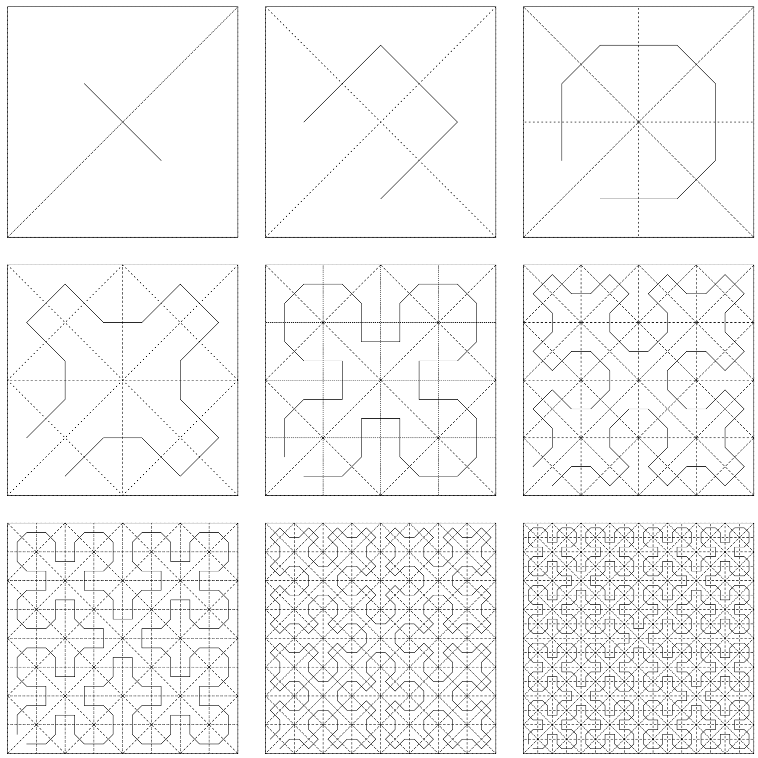
What is the *dimension* of a space?

Space Filling Curve

In 1890, G. Peano found a continuous map from an interval onto a square.

If we use binary system to express real numbers, then it is quite easy to construct such a map.





Exercise. Find a continuous map of $[0,1]$ onto $[0,1]^n$.

L. E. J. Brouwer

A bijection $f : A \to B$ is called a *homeomorphism* if both f and f^{-1} are continuous.

Dedekind conjectured that there is no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$ if $m \neq n$.

In 1911, L. E. J. Brouwer proved the conjecture.

Schröder-Bernstein Theorem

For sets A and B, we write

$$A \preccurlyeq B$$

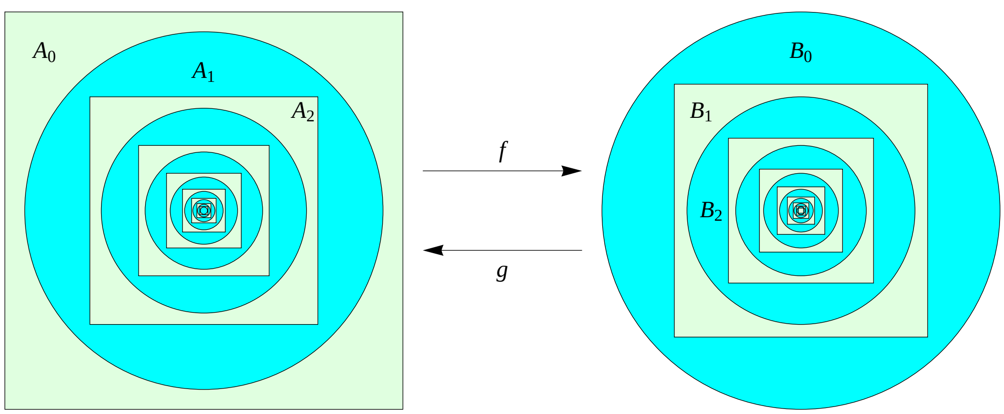
if there exists an injection from A into B, and write

$$A \prec B$$

if $A \preccurlyeq B$ and $A \not\sim B$. It is obvious that $A \preccurlyeq B$ and $B \preccurlyeq C$ implies $A \preccurlyeq C$.

Theorem 3 (Schröder, Bernstein (1897)) [2] *If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \sim B$.*

Proof. We may assume that $A \cap B = \varnothing$. (otherwise we may replace A and B by $A \times \{0\}$ and $B \times \{1\}$, respectively). We are given injections $f : A \to B$ and $g : B \to A$. We will construct a bijection $h : A \to B$.



[2]Proofs were independently discovered. Felix Bernstein(1878–1956) was 19 years old, when he gave a proof at a Cantor's seminar 1897. His proof was published in a book by Borel in 1898. Schröder announced the theorem in an abstract of 1896 paper. His paper published in 1898 was imperfect, and he corrected the proof in1911. Cantor proved this theprem in a 1897 paper using a principle the axiom of choice [Enderton, p.148]. Dedekind proved this theorem in 1887, but he did not publish [Bourbaki, p.325].

Then we have

$$A \supseteq g[B] \supseteq gf[A] \supseteq gfg[B] \supseteq \cdots \supseteq \bigcap_{n=0}^{\infty}(gf)^n[A]$$

$$B \supseteq f[A] \supseteq fg[B] \supseteq fgf[A] \supseteq \cdots \supseteq \bigcap_{n=0}^{\infty}(fg)^n[B].$$

Let

$$A_0 := A - g[B], \quad A_1 := g[B_0] = g[B] - gf[A], \quad A_2 := g[B_1] = gf[A] - gfg[B], \dots$$

$$B_0 := B - f[A], \quad B_1 := f[A_0] = f[A] - fg[B], \quad B_2 := f[A_1] = fg[B] - fgf[A], \dots,$$

i.e., A_n and B_n consists of those elements in A and B, respectively, with n *ancestors* but no more. Clearly, we have

$$f : A_{2n} \sim B_{2n+1}, \quad g : B_{2n} \sim A_{2n+1}.$$

Let

$$A_\infty := \bigcap_{n=0}^{\infty}(gf)^n[A] = A - \bigcup_{n=0}^{\infty} A_n, \quad B_\infty := \bigcap_{n=0}^{\infty}(fg)^n[B] = B - \bigcup_{n=0}^{\infty} B_n.$$

Then

$$f, g^{-1} : A_\infty \sim B_\infty.$$

We have the decompositions

$$A = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_\infty, \quad B = B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_\infty.$$

Now define a new map

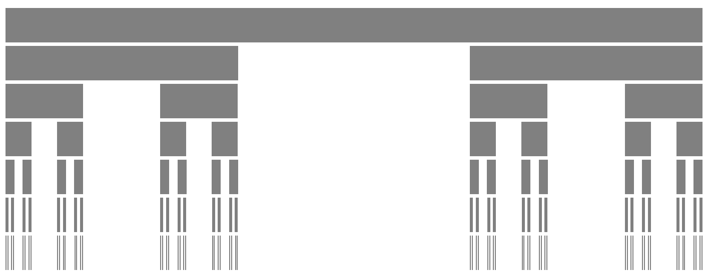
$$h : A \to B, \qquad a \mapsto \begin{cases} f(a) & (a \in A_0 \cup A_2 \cup \cdots \cup A_\infty) \\ g^{-1}(a) & (a \in A_1 \cup A_3 \cup \cdots) \end{cases}$$

Then h is a desired bijection. $\square$

Exercise Show that if $A \prec B$ and $B \prec C$, then $A \prec C$.

Cantor's Dust

In 1874, H. Smith discovered a set. Nowadays this set is usually called the *Cantor set* or the *Cantor's dust*.



In 1884 Cantor showed that the dust D is equipotent to $\mathbb{R}$:

$$D \sim \mathbb{R}.$$

Note that the 'length' of D is 0. In fact the fractal dimension of D is $\frac{\log 2}{\log 3} \approx 0.63$.

Exercises.

1. Any open set in $\mathbb{R}^n$ is equipotent to $\mathbb{R}$.

2. Any closed subset of $\mathbb{R}$ is equipotent to either $\mathbb{N}$ or $\mathbb{R}$.

Power Sets

For any set A, let $\mathcal{P}(A)$ be the set of all subsets of A.

Theorem 4 $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$

Theorem 5 (Cantor, Dec. 7, 1873) $\mathbb{N} \prec \mathbb{R}$

A real number is called a *transcendental* number, if is it not algebraic.

Theorem 6 (Cantor, 1878) *There exists uncountably many transcendental real numbers.*

Theorem 7 *For any set A, $A \prec \mathcal{P}(A)$.*

Proof. The map

$$a \mapsto \{a\}$$

is an injection of A into $\mathcal{P}(A)$. Thus $A \preccurlyeq \mathcal{P}(A)$.

We will show that there exists no surjection of A into $\mathcal{P}(A)$. Let $f : A \to \mathcal{P}(A)$ be given. Consider the subset B of A characterized by the property:

$$\forall a \in A, \qquad (a \in f(a) \Rightarrow a \notin B) \;\&\; (a \notin f(a) \Rightarrow a \in B).$$

In other words, $B = \{a \in A \mid a \notin f(a)\}$. Then for all $a \in A$, $f(a) \neq B$. Thus B is an element in $\mathcal{P}(A)$ not in the range of f. □

Exercise. Show that there exists no injection of $\mathcal{P}(A)$ into A.

Exercise. Let $C^0(\mathbb{R}, \mathbb{R})$ be the set of all continuous real valued functions on $\mathbb{R}$. Show that

$$C^0(\mathbb{R}, \mathbb{R}) \sim \mathbb{R}.$$

We have

$$A \prec \mathcal{P}(A) \prec \mathcal{P}^2(A) \prec \mathcal{P}^3(A) \prec \cdots .$$

If A is the set of all students in our university, then $\mathcal{P}(A)$ is the set of all possible associations of students, and $\mathcal{P}^2(A)$ is the set of all ....

GCH

The *generalized continuum hypothesis* says that for any infinite set A, there exists no set X such that

$$A \prec X \prec \mathcal{P}(A).$$

Question

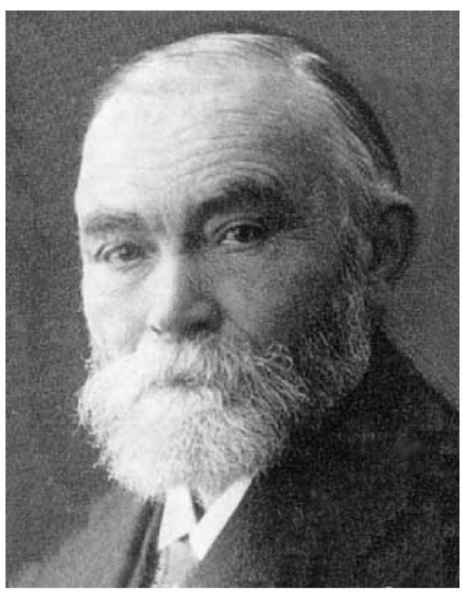
If

$$\mathcal{U} = \{x \mid x = x\}$$

is the collection of all sets, then it contains all subcollections of itself. Is $\mathcal{U}$ still less than $\mathcal{P}(\mathcal{U})$?

Russell's Paradox

G. Frege was one of the founders of modern logic. We owe the quantifying symbols '$\exists$' and '$\forall$' from him.



G. Frege, 1848–1925

Bertrand Russell(1872–1970) said, in his essay[3] in 1901, "*Obviousness is always the enemy of correctness.*" He said many obvious things *just to him.* He said "although Cantor has succeded to introduce the infinities, Cantor's non-existence of the highest infinity is obviously wrong." In 1917 he added a note in this eaasay apologizing for his false comments on Cantor's work [Gardner, Colossal Book]. But there were lots of other obviously wrong judgments in his young essay.

In 1902, Russell wrote a letter to G. Frege, where he asked a question: *Is the set*

$$R := \{x \mid x \notin x\}$$

an element of itself?

Russell's example was also discovered by E. Zermelo independently.

> *A scientist can hardly meet with anything more undesirable than to have the foundations give way just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was nearly through the press.*
>
> G. Frege in *Grundgesetze der Arithmetik* (1903)

내가 아는 어떤 수학자 J는 아주 노래를 잘 합니다. 남은 물론... 그 자신도 그것을 부정하지 않지요. 어느 날 J가 대한수학회 모임에서 다음과 같이 말하였습니다.

[3] First appeared in an American magazine "The International Monthly" under the title "Recent Work in the Philosophy of Mathematics". This essay is contained with the new title "Mathematics and the Metaphysicians" in the late book "Mysticism and Logic and Other Essays" (1918).

“제가 합창단에 가입하여 활동하다 보니... 자기 혼자서만 잘난 독창보다 여러 사람이 어울려 하는 합창이 훨씬 더 훌륭한 일임을 알게 되었습니다...

수학자들이 모여 사는 세상도 마찬가지라고 생각합니다.”

나는 J의 말이 비단 수학자들의 사회뿐 아니라 일반적인 사회에 모두 적용된다고 생각합니다. 역사적으로 인류가 ‘훌륭하다’고 말하는 분들이 가지고 있는 공통적인 성질이 있습니다. 그것은 그들은 대부분 ‘남과 협동하고 자신과는 씨름하며 채찍질하고 꾸준히 노력하는’ 그런 분들이었습니다. 그들은 자신을 바치는... 즉, 헌신하는 분들이었습니다.

그래서 어느 날 나는 다음과 같이 선언하였습니다.

나는 ‘자신과 협동하지 않는 사람’하고만 협동합니다.

내 아내가 그 말을 듣고 나에게 물었습니다: *“자기는 자기하고 협동해?”*

⋮

나는 . . . 사람이 아닙니다...

Poincaré, Weyl, Brouwer

It is a disease from which the human race will soon recover.

H. Poincaré

The truth of $\neg b \vee b$ may not imply either ‘$\neg b$ is ture’ or ‘b is ture’.

L. E. J. Brouwer

To be, or not to be: that is the question:

W. Shakespeare in *Hamlet 3/1*

Fog in the Fog...

H. Weyl

No one shall expel us from the paradise that Cantor has created for us.

D. Hilbert

Classes

There exist *classes* and *atoms* (i.e., *ur-elements* or *individuals*), or ... there exist only *sets*.