Fast Exponentiation Using Split Exponents

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Abstract—We propose a new method to speed up discrete logarithm (DL)-based cryptosystems by considering a new variant of the DL problem, where the exponents are formed as \(e_1 + \alpha e_2\) for some fixed \(\alpha\) and two integers \(e_1, e_2\) with a low weight representation. We call this class of exponents split exponents, and we show that with certain choice of parameters the DL problem on split exponents is essentially as secure as the standard DL problem, while the exponentiation operation using exponents of this class is significantly faster than best exponentiation algorithms given for standard exponents. For example, the speed of scalar multiplication on the standard Koblitz curve K163 is estimated to be accelerated by up to 51.5\% and 23.5\% at the cost of memory for one precomputed point, compared to the TNAF and window TNAF methods, respectively. As for security, we show that the provable security of the DL problem using split exponents is only by a small constant, e.g. 1/4, worse than the security of the standard DL problem. Split exponents can be adopted to speed up various DL-based cryptosystems. We exemplify this on the recent CCA-secure public key encryption of Bellare, Kohno, and Shoup.

Index Terms—Exponentiation, Koblitz Curves, Low Hamming Weight, Discrete Logarithm

I. INTRODUCTION

Exponentiation \(g^e\) on an abelian group (including modular exponentiation in a finite field and scalar multiplication on an elliptic curve) is the most common primitive operation in public-key cryptography. Since exponentiation consists of repeated field multiplications and squarings, the research on speeding up exponentiation strives to reduce the total number of these component operations as well as the complexity of individual operations. However, while the number of squarings necessary for an exponentiation can be reduced by various (off-line) precomputation methods, and the cost of squarings is even more dramatically reduced in binary fields using normal basis or in Koblitz elliptic curves using Frobenius map [1], it remains a challenge to reduce the number of multiplications.

The most well-known methods for reducing the number of multiplications are the sliding window method [2] and the fixed-base comb method [3]. Though from the details of these algorithms they seem to be quite different, they share the property that repeated patterns are computed only once and reused many times. In other words, they can be reformulated in a unified framework as follows: First, convert the exponent \(e\) to a linear combination \(e = \sum_i e_i \alpha_i\) of some fixed exponents \(\alpha_1, \alpha_2, \ldots, \alpha_t\). Then compute \(g^e = \prod_i g_i^{e_i}\) using the precomputed values \(g_i = g^{\alpha_i}\) for each \(i\). In the sliding window method with window size \(w\), the set of \(\alpha_i\)'s consists of all the additive combinations of small powers of 2, i.e. \(2^0, 2^1, \ldots, 2^{w-1}\), whereas the fixed-base comb method uses as the set of \(\alpha_i\)'s all the additive combinations of elements \(2^0, 2^{n/w}, \ldots, 2^{(w-1)n/w}\), where \(n\) is the bit-length of the group order \(p\). (Observe that the fixed-base comb method can be used only with off-line precomputation since many operations are required to compute \(g^{\alpha_i}\).) In both cases, the number of multiplications required for an exponentiation is bounded below by \(\sum_i \omega(e_i) / w - 1\) for relatively small \(w\) where \(\omega(e)\) is the Hamming weight of \(e\) assuming that \(g^{\alpha_i}\)'s are given as precomputed values. (Given \(g^{\alpha_i}\)'s, both methods require \(\sum_{i=1}^{2^w} \omega(e_i) - 1\) multiplications to compute \(g^e\) for \(e = \sum_{i=1}^{2^w} e_i \alpha_i\). Since \(\omega(e) \leq \sum_{i=1}^{2^w} \omega(e_i) \omega(e_i) \leq w \sum_{i=1}^{2^w} \omega(e_i)\), the lower bound is obtained.)

We observe that this lower bound comes from the fact that \(\alpha_i\)'s have a regular form and they have quite small Hamming weights in both methods. Hence, it is an interesting question to look for an exponentiation method to improve this lower bound by using a new form of \(\alpha_i\)'s.

1) Our Contribution: As the first step in this direction of research, we consider the case with \(\alpha_1 = 1\) and \(\alpha_2 = \alpha\) where \(\alpha\) has a large Hamming weight. To be more precise, we consider the set of exponents

\[S_1 + \alpha S_2 := \{e_1 + \alpha e_2 | e_1 \in S_1, e_2 \in S_2\},\]

where \(\alpha \in \mathbb{Z}_p\) and \(S_1\) and \(S_2\) are arbitrary subsets of \(\mathbb{Z}_p\). Let us call an element of \(S_1 + \alpha S_2\) a split exponent. Surprisingly, it turns out that if the product of the cardinalities of \(S_1\) and \(S_2\) is greater than \(p\), the average cardinality of \(S_1 + \alpha S_2\) over all \(\alpha \in \mathbb{Z}_p\) is at least \(p/2\). Furthermore, we show an algorithm for picking a specific good \(\alpha\) such that \(S_1 + \alpha S_2\) covers a significant portion of \(\mathbb{Z}_p\), and we show that the provable security of the discrete logarithm problem using split exponents is only by a small constant, e.g. 1/4, worse than the security of the standard discrete logarithm problem. Our algorithm for picking a good \(\alpha\) is efficient only when the sizes of \(S_1\) and \(S_2\) are unbalanced. We pose an open problem to develop a similar algorithm for the case that \(S_1\) and \(S_2\) are of same size.

To reduce the number of multiplications, \(S_1\) and \(S_2\) should be composed of the elements having small Hamming weights. As a specific instance, we take \(S_1\) and \(S_2\) to be the sets

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of $w$-NAFs with small Hamming weight, where $w$-NAF is a generalization of non-adjacent form (NAF) [4]. Then the number of multiplications required for the exponentiation $g^e$ for $e \in S_1 + \alpha S_2$ given $g$ and $g^\alpha$ is substantially reduced. The advantage of using this new class of exponents is most pronounced in groups where multiplication is much more expensive than squaring, e.g., on binary fields with normal basis representation or on Koblitz curves where we interpret the point addition as a multiplication and the Frobenius map as a squaring, respectively. For example, a scalar multiplication on the standard Koblitz curve K163 [5] requires only 26 point additions, which is a speed-up by 51.5% and 23.5% compared to the TNAF and window TNAF methods [6], [7], respectively. The only overhead to obtain this speedup is memory space for one precomputed point. We remark that to achieve the same speed using precomputation methods (fixed-base window TNAF method), more than seven points should be precomputed and stored.\footnote{For groups in which squaring requires non-negligible cost, existing algorithms may outperform our algorithm. For example, by precomputing a group element $g^{2^n/2}$, the fixed-base comb method [3] can perform an exponentiation faster than our algorithm. Note that our approach does not try to reduce the number of squarings (or point doublings), but it tries to reduce the number of multiplications (or point additions). On the other hand, the most efficient existing precomputation method for Koblitz curves is not the method of [3], but it is the fixed-base window TNAF method [6], [7]. Our method requires significantly smaller amount of memory than the fixed-base window TNAF method to achieve the same speed.}

Finally, we show how split exponents can be used for more efficient implementation of some existing cryptographic schemes based on the discrete logarithm (DL) problem. Namely, we apply the split exponent exponentiation method to the recent encryption scheme of Bellare, Kohno, and Shoup [8] which is the most efficient CCA-secure encryption whose security is provably related to a DL-like problem (the Gap DH problem). Similar adaptations should be possible for long-term secrets in many other DL-based cryptosystems. Also we discuss how to use small Hamming weight exponents to speed up the verification of Schnorr signature.

2) Remark on Applicability: Fast exponentiation is the most required for real-world applications of public key cryptography and such a trend is not changing even in the future ubiquitous computing environments. The notion of split exponent is general in our method, which means that it can be applied to an exponentiation on any abelian group used by public key cryptographic techniques. With regard to usefulness, however, it is more desirable that the group may have fast squaring operation or fast endomorphism for the best performance. Fortunately, there are a number of real-world applications implementing such groups as finite fields of small characteristics and elliptic curves with complex multiplications. They may include not only general applications of public key encryption, digital signature, and authenticated key exchange, but also resource-constrained implementation of them.

It should be noted that our method is the fastest exponentiation algorithm, compared to existing schemes, since we have achieved the fastest exponentiation (scalar multiplication) on Koblitz curves as explained in Section III-B. Our claim is justified easily due to the well-known fact that in general the scalar multiplication on Koblitz curves is always much faster than on other curves. In addition, the memory overhead is only a single elliptic curve point $\alpha P$, where $P$ is the primitive element of the point group. As a result, there should be a great number of promising applications requiring fast exponentiation through our scheme, e.g., an implementation of public key cryptographic techniques in resource-constrained devices such as embedded devices and tiny sensor nodes [9].

3) Related Work: There are many well known general methods for exponentiation including the binary method, the $M$-ary method and the sliding window method [2]. Furthermore, if the base $g$ is fixed, precomputation methods such as BGMW [10], de Rooij [11], and the fixed-base comb method [3] can be used to speed up the computation at the cost of memory.

There are works which consider uniformly generated exponents, but represented in a form which makes exponentiation faster. For example, using the property that a point addition and a point subtraction have almost the same complexity on an elliptic curve, one can speed up scalar multiplication with scalars in a signed-digit form such as NAF [4]. Also, for a special class of curves with fast endomorphism, a scalar can be expanded using a complex radix $\tau$, to reduce the number of point doublings [1], [4], [12], [13]. If we are dealing with a fixed point, we can use precomputation methods such as the fixed-base window TNAF method [6], [7].

Recently, Dimitrov, Imbert and Mishra proposed a double-base expansion method for fast scalar multiplication [14], and there are also various extensions to their work [15], [16], [17], [18], [19], [20], [21]. We are more interested in their Koblitz curve variants [22], [23], [24], [25], because these methods concentrate on reduction in the number of point additions rather than point doublings as our method does. However, the fastest algorithm among them [23] requires $31.09$ additions on average for a scalar multiplication over K163, which is slower than our algorithm by 16.4%. While there is also another interesting approach combining $\tau$-adic expansion and point halving [26], [27], it is slower than the algorithm in [23] as well as ours.

Of a special interest to us are the methods which reduce the number of multiplications (or point additions) by considering exponents of special form. Studies of small Hamming weight exponents [28], [29] and the more recent study of an exponent class of products of random small Hamming weight factors [30] belong to this category. However, none of these schemes is provably as hard as the standard discrete logarithm problem.

4) Organization: In Section 2 we introduce split exponents and show that the discrete logarithm problem on split exponents can be provably almost equivalent to the standard discrete logarithm problem. In Section 3, we give a practical instance of split exponents, analyze the performance of the exponentiations by split exponents and compare it with that of previous algorithms. In Section 4 we show how split exponents can be used to speed up the Bellare-Kohno-Shoup CCA-secure public key encryption and other DL-based schemes. Section 5 devotes to the Schnorr signature scheme with faster signature verification. We conclude in Section 6.
II. DISCRETE LOGARITHM WITH SPLIT EXPONENTS

In this section, we introduce a new class of exponents and show the hardness of the discrete logarithm problem with such exponents.

Let $G$ be an additive abelian group of prime order $p$. All addition and multiplication operations, except when noted otherwise, are in $\mathbb{Z}_p$. If $S_1, S_2 \subseteq \mathbb{Z}_p$ and $\alpha \in \mathbb{Z}_p$, we denote as $S_1 + \alpha S_2$ a set of elements $x \in \mathbb{Z}_p$ such that there exists $x_1, x_2 \in S_1 \times S_2$ s.t. $x = x_1 + \alpha x_2$. We call an element $x \in S_1 + \alpha S_2$, which can be represented as $(x_1, x_2) \in S_1 \times S_2$ s.t. $x = x_1 + \alpha x_2$, a “split exponent”. Let $x \xrightarrow{r} X$ denote assignment of a random uniformly chosen value in set $X$ to variable $x$. The standard Discrete Logarithm (DL) problem is thus the problem of computing $x$ given $P, xP$ where $x \xrightarrow{r} \mathbb{Z}_p$ and $P$ is a generator of $G$. We define the split exponent discrete logarithm problem as the problem of computing $x$ given $P, xP$ for $x \xrightarrow{r} S_1 + \alpha S_2$, where the adversary is additionally given $\alpha P$.

Definition 1: Let $S_1, S_2 \subseteq \mathbb{Z}_p$. Let $P$ be a generator of $G$. Let $\alpha$ be an element in $\mathbb{Z}_p$. Let $A$ be any probabilistic algorithm. We define $A$’s success probability in solving the Split Exponent Discrete Logarithm (SEDL) problem on $(S_1, S_2, \alpha, G)$ as

$$\text{Adv}^{\text{sedl}}_{A, S_1, S_2, \alpha, G} \stackrel{\text{def}}{=} \text{Pr}[A(P, xP) = x | x \xrightarrow{r} S_1 + \alpha S_2].$$

We say that an algorithm $A$ $(t, \epsilon)$-breaks the SEDL on $(S_1, S_2, \alpha, G)$ if $A$ runs in time at most $t$, and $\text{Adv}^{\text{sedl}}_{A, S_1, S_2, \alpha, G}$ is at least $\epsilon$. The $(t, \epsilon)$-SEDL assumption on $(S_1, S_2, \alpha, G)$ is that no adversary $(t, \epsilon)$-breaks the SEDL on $(S_1, S_2, \alpha, G)$.

A. Average Hardness of the Split Exponent DL Problem

The main factor in deciding the hardness of the SEDL problem is the size of set $S_1 + \alpha S_2$. In the following lemma, we give a lower bound on the average size of this set for a random $\alpha$ in $\mathbb{Z}_p$, defined as:

$$C_{S_1, S_2} = \frac{1}{p} \sum_{\alpha \in \mathbb{Z}_p} |S_1 + \alpha S_2|.$$

Lemma 1: For any $S_1, S_2 \subseteq \mathbb{Z}_p$ we have:

$$C_{S_1, S_2} \geq \left( \frac{|S_1||S_2|}{|S_1||S_2| + p - |S_1|} \right) \times p.$$

In particular, if $|S_1||S_2| \geq p$ then $C_{S_1, S_2} \geq p/2$.

In other words, for $|S_1||S_2| \geq p$ the average hardness of the SEDL problem is just factor of 1/2 away from the hardness of the DL problem. Perhaps this fact can be directly used in some DL-based cryptosystem. We use it in section II-B to lower-bound the probability of finding some $\alpha$ for which the set $S_1 + \alpha S_2$ is guaranteed to be large enough so that the hardness of SEDL for this particular $\alpha$ is only a small constant away from the hardness of the DL problem.

Proof: For any $\alpha \in \mathbb{Z}_p$, we define

$$W_\alpha = \{(x, x', y, y') \in S_1^2 \times S_2^2 \mid x + y\alpha = x' + y'\alpha, y \neq y'\}.$$

For any 4-tuple $v = (x, y, x', y') \in S_1^2 \times S_2^2$ with $y \neq y'$, we have $v \in W_\alpha$ for $\alpha = (x'-x)/(y-y')$. Therefore, if $W_\alpha \cap W_{\alpha'}$ is nonempty then $\alpha = \alpha'$. Consequently, sets $W_\alpha$ for $\alpha \in \mathbb{Z}_p$ form a partition of all 4-tuples $(x, y, x', y') \in S_1^2 \times S_2^2$ with $y \neq y'$. In particular we have:

$$\sum_{\alpha \in \mathbb{Z}_p} |W_\alpha| = |S_1|^2|S_2|^2 - |S_1|^2|S_2|.$$

(1)

For each $\alpha \in \mathbb{Z}_p$, we define a relation $\sim_\alpha$ on the set $S_1 \times S_2$:

$$(x, y) \sim_\alpha (x', y') \iff x + y\alpha = x' + y'\alpha.$$

In other words, $(x, y) \sim_\alpha (x', y')$ if $(x, x', y, y') \in W_\alpha$ or $(x, y) = (x', y')$.

It is easy to see that $\sim_\alpha$ is an equivalence relation on $S_1 \times S_2$ and the number of its equivalence classes is $N_\alpha := |S_1 + \alpha S_2|$. Let $V_{\alpha, 1}, \ldots, V_{\alpha, N_\alpha}$ denote the distinct equivalence classes of this relation. Then we have:

$$\sum_{i=1}^{N_\alpha} |V_{\alpha, i}| = |S_1||S_2|, \quad \sum_{i=1}^{N_\alpha} \left( \frac{|V_{\alpha, i}|}{2} \right)^2 = |W_\alpha|. \quad (2)$$

The second equality comes from the fact that an ordered pair of two distinct elements in the same equivalence class corresponds to one element of $W_\alpha$. In other words, we have:

$$\sum_{i=1}^{N_\alpha} |V_{\alpha, i}|^2 = |W_\alpha| + |S_1||S_2|.$$

By applying Cauchy-Schwarz inequality, we have:

$$|S_1|^2|S_2|^2 \leq \left( \sum_{i=1}^{N_\alpha} |V_{\alpha, i}| \right)^2 \leq \left( \sum_{i=1}^{N_\alpha} 1 \right)^2 \left( \sum_{i=1}^{N_\alpha} |V_{\alpha, i}|^2 \right) = N_\alpha (|W_\alpha| + |S_1||S_2|). \quad (3)$$

We use Cauchy-Schwarz inequality again to obtain:

$$\left( \sum_{\alpha \in \mathbb{Z}_p} 1 \right)^2 \leq \left( \sum_{\alpha \in \mathbb{Z}_p} |W_\alpha| + |S_1||S_2| \right) \times \left( \sum_{\alpha \in \mathbb{Z}_p} \frac{1}{|W_\alpha| + |S_1||S_2|} \right). \quad (4)$$

Applying equation (1) we have:

$$\sum_{\alpha \in \mathbb{Z}_p} (|W_\alpha| + |S_1||S_2|) = |S_1|^2|S_2|^2 + (p - |S_1|)|S_1||S_2|. \quad (5)$$

Finally, applying inequalities (3) and (4) we derive:

$$\frac{1}{p} \sum_{\alpha \in \mathbb{Z}_p} N_\alpha \geq \frac{|S_1|^2|S_2|^2}{p} \sum_{\alpha \in \mathbb{Z}_p} \left| \frac{1}{W_\alpha| + |S_1||S_2|} \right| \geq \frac{|p|S_1||S_2|}{|S_1||S_2| + p - |S_1|}.$$

Corollary 1: Let $S_1, S_2$ be two subsets of $\mathbb{Z}_p$.

$$\frac{1}{p} \sum_{\alpha \in \mathbb{Z}_p} (|S_1||S_2| - |S_1 + \alpha S_2|) \leq \frac{|S_1|^2|S_2|^2 - |S_1||S_2|^2}{|S_1||S_2| + p - |S_1|}. \quad (6)$$
In particular, if $|S_1||S_2| \leq \sqrt{p}$, the upper bound is less than 1, and hence the average size of set $S_1 + \alpha S_2$ is between $|S_1||S_2| - 1$ and $|S_1||S_2|$.

B. Hardness of the Split Exponent DL Problem for good $\alpha$'s

In the previous subsection we established that if $|S_1||S_2| \geq p$ the average size of set $S_1 + \alpha S_2$ is at least $1/2$ the size of $\mathbb{Z}_p$ over all $\alpha \in \mathbb{Z}_p$. Trivially, if $S_1 + \alpha S_2 = \mathbb{Z}_p$ for some $S_1, S_2, \alpha$ then SEDL on $S_1, S_2, \alpha$ is at least as hard as the ordinary DL problem. Moreover, if the size of the set $S_1 + \alpha S_2$ is at least a fraction $c$ of the size of the standard DL domain $\mathbb{Z}_p$, then the SEDL problem is related by factor $c$ to the standard DL problem:

**Definition 2**: Let $S_1, S_2 \subseteq \mathbb{Z}_p$. We call $\alpha \in \mathbb{Z}_p$ “$c$-good” for $S_1, S_2$ if $|S_1 + \alpha S_2| \geq c|\mathbb{Z}_p|$.

**Theorem 1**: Let $S_1, S_2 \subseteq \mathbb{Z}_p$. If $\alpha$ is $c$-good for $S_1, S_2$ then the SEDL problem on $S_1, S_2, \alpha$ is at least $(t, \ell, \epsilon/c)$-hard if the DL problem on $G$ is $(t, \epsilon)$-hard.

**Proof**: The proof is immediate: If algorithm $A$ breaks SEDL on $S_1, S_2, \alpha, G$ in time $t$ with probability $\epsilon/c$ then same $A$ breaks DL with probability $\epsilon$ because if $|S_1 + \alpha S_2| \geq c|\mathbb{Z}_p|$ then a random $x \in \mathbb{Z}_p$ is in $S_1 + \alpha S_2$ with probability $c$. Therefore, security of any DL-based cryptosystem degrades only by a factor of $c$ if we replace standard exponents with split exponents for a $c$-good $\alpha$. Two issues remain in order to enable us to utilize this fact in a cryptosystem. We need an efficient procedure to pick $c$-good $\alpha$'s for a good enough constant $c$, e.g. $c = 1/4$, and we need an efficient way to sample the split exponents given $S_1, S_2, \alpha$. (Note that choosing $(x_1, x_2) \leftarrow S_1 \times S_2$ and setting $x = x_1 + \alpha x_2$ is not equivalent to choosing $x \leftarrow S_1 + \alpha S_2$!) These two tasks are handled, respectively, by Algorithms 2 and 1 below. Both algorithms are geared to sets $S_1, S_2$ where $S_2$ is small and $|S_1 + \alpha S_2| \approx p$ as their running times are linear in $|S_2|$ and $p/|S_1 + \alpha S_2|$.

**Algorithm 1** Choosing a Random Split Exponent

On input $S_1, S_2, \alpha$:
1. Randomly select $z \leftarrow \mathbb{Z}_p$.
2. For all $y \in S_2$, check if $x = z - y \alpha \in S_1$. Output $(x, y)$ and stop if it is.
3. If no $(x, y) \in S_1 \times S_2$ is found, go to State 1.

Algorithm 1 outputs a uniformly distributed element $z$ of $S_1 + \alpha S_2$ (represented as pair $(x, y) \in S_1 \times S_2$), with the expected number of $1/c$ iterations provided that $\alpha$ is $c$-good. Hence the expected running time of this algorithm is at most $|S_2|/c$ modular multiplications and checks of membership in $S_1$.

**Algorithm 2** Finding a $c$-good Element $\alpha$

On input $S_1, S_2 \subseteq \mathbb{Z}_p$ and $\tau \in \mathbb{Z}$. Let $c = 2c$.
1. Randomly select $\alpha \leftarrow \mathbb{Z}_p$.
2. Randomly select $x_1, x_2, \ldots, x_\tau \in \mathbb{Z}_p$ and test if each of them belongs to $S_1 + \alpha S_2$ by Step 2 of Algorithm 1.
3. If the number of $x_i$'s in $S_1 + \alpha S_2$ is at least $c\tau$, output $\alpha$. Otherwise go to State 1.

We first estimate the probability $P_{\text{fail}}$ that some inner loop of Algorithm 2 outputs $\alpha$ which is not $c$-good. Suppose that the outer loop picks $\alpha$ which is not $c$-good, i.e. $|S_1 + \alpha S_2| = c'/p < c$. Let $X_1, \ldots, X_\tau$ be random variables s.t. $X_i = 1$ if $x_i \in S_1 + \alpha S_2$, and 0 otherwise. Let $X = X_1 + \cdots + X_\tau$. Using this notation, $P_{\text{fail}} = Pr[X > c\tau]$ for $c = 2c$. If we let $X' = \frac{X}{\tau}$ then the expected value of $X'$ is $\mu = \tau c$. Let $\delta = c/\tau - 1 = 1$. By a Chernoff bound we have:

$$P_{\text{fail}} = \Pr[X' > (c/\tau)\epsilon] = \Pr[X' > \epsilon\tau = (1 + \delta)\mu] < \left(\frac{e^\delta}{(\delta + 1)^{\delta + 1}}\right)^{\epsilon\tau} < 2^{-0.55\tau c}.$$ 

In other words, we can set the probability that Algorithm 2 outputs an incorrect $\alpha$ arbitrarily low by setting a large enough $\tau$.

Now we can estimate the expected running time of Algorithm 2. Let $\theta := |S_1||S_2|/p \geq 1$. Lemma 1 says that $E(X_\alpha) \leq \frac{p}{\theta + 1}$ for a random variable $X_\alpha := p - |S_1 + \alpha S_2|$. Applying Markov inequality for $X_\alpha$, we obtain:

$$\Pr[X_\alpha \geq (1 - \epsilon)p] \leq \frac{E(X_\alpha)}{(1 - \epsilon)p} \leq \frac{1/(\theta + 1)}{1 - \epsilon},$$

which implies that a random $\alpha$ in $\mathbb{Z}_p$ is $\epsilon$-good with probability

$$P_1 \geq \frac{\theta/(\theta + 1) - \epsilon}{1 - \epsilon}.$$

If $\alpha$ is $\epsilon$-good, at least $\epsilon\tau$ elements from randomly selected $\tau$ elements from $\mathbb{Z}_p$ belong to $S_1 + \alpha S_2$ with probability

$$P_2 = \sum_{i \geq \epsilon\tau} \left(\frac{\tau}{i}\right) c'(1 - \epsilon)^{i-1} \approx 1/2.$$ 

If $|S_1||S_2| \geq p$ and we use Algorithm 1 to test if $x_i$'s belong to $S_1 + \alpha S_2$, the expected running time of Algorithm 2 is at most

$$T = (P_1 P_2)^{-1} \times \tau \times |S_2|.$$

**Example**: If $\theta \geq 1$, we may take $c = 1/4$ and $\tau = 576$. Then $P_{\text{fail}} \leq 2^{-80}$. We also have $P_1 \geq (\theta - 1)/(\theta + 1)$ and $P_2 \approx 1/2$, which gives the running time of Algorithm 2:

$$T = 1152|S_2| \cdot (\theta - 1)/(\theta + 1),$$

which is less than $212^2|S_2|$ for $\theta \geq 2$.

III. IMPLEMENTATION OF_SPLIT_EXONENTS

In this section, we propose to use small Hamming weight $w$-NAFs as a possible instance of the split sets $S_1$ and $S_2$. We show that split exponents implemented in this form substantially accelerate scalar multiplication on Koblitz curves, where a $w$-NAF is interpreted as a $\tau$-adic representation for a complex number $\tau$. We also show that $w$-NAF based split exponents give similar speed-up for exponentiation on binary fields represented in normal bases.
A. Low Hamming Weight Exponents

We define \( w \)-NAFs following the definitions given in [4], [31], [32]:

**Definition 3:** Let \( w \) be an integer \( \geq 2 \) and \( D \) a subset of \( \mathbb{Z} \) with \( 0 \notin D \). A \( w \)-NAF with the digit set \( D \) is a sequence of digits satisfying the following two conditions:

1) Each non-zero digit belongs to \( D \).
2) Among any \( w \) consecutive digits, at most one is non-zero.

We denote a \( w \)-NAF as a string \( a = (a_{m-1} \cdots a_0) \), where \( m \) is the length of \( a \). The length of \( a \) is defined to be the smallest \( i \) such that \( a_{i-1} \neq 0 \). The (Hamming) weight of \( a \) is defined to be the number of non-zero digits in its \( w \)-NAF representation. In this paper, we are especially interested in \( w \)-NAFs of small Hamming weight.

We will consider the set of split exponents \( S_1 + \alpha S_2 \) for \( \alpha \in \mathbb{Z}_p \) where \( S_1 \) and \( S_2 \) are the sets of \( w \)-NAFs of fixed Hamming weight. The number of \( w \)-NAFs of length \( \leq m \) and weight \( t \) with a coefficient set \( D \) is given [31] by

\[
\left( m - (w - 1)(t - 1) \right) |D|^t.
\]

Given a base \( k \), a \( w \)-NAF form can be naturally mapped into the \( k \)-adic representation of an integer via \( (a_{m-1} \cdots a_0) \mapsto \sum_{i=0}^{m-1} a_i k^i \). Therefore we can identify a \( w \)-NAF with its conversion to an integer, and we can use a \( w \)-NAF as an exponent of modular exponentiations in finite fields and scalar multiplications in elliptic curves.

B. Scalar Multiplication on Koblitz Curves by Split Exponents

In this subsection, we introduce a fast scalar multiplication algorithm by split exponents based on small Hamming weight \( w \)-NAFs and analyze its efficiency.

Consider an ordinary elliptic curve \( E \) defined over \( \mathbb{F}_q \) with \( \#E(\mathbb{F}_q) = q + 1 - t \) and gcd\((q, t) = 1 \). The Frobenius map \( \tau \) is defined as follows:

\[
\tau : E(\mathbb{F}_q) \to E(\mathbb{F}_q); (x, y) \mapsto (x^{q^t}, y^{q^t}),
\]

where \( \mathbb{F}_q \) is the algebraic closure of \( \mathbb{F}_q \). The Frobenius map \( \tau \) is a root of the characteristic equation \( \chi(\tau) = \tau^2 - \tau + q \) in \( \text{End}(E) \). We denote \( E(\mathbb{F}_{q^n}) \) by the subgroup of \( E(\mathbb{F}_q) \) consisting of \( \mathbb{F}_q \)-rational points. Let \( G \) be the subgroup of \( E(\mathbb{F}_{q^n}) \) generated by \( P \) with a prime order \( \ell \) satisfying \( \ell \nmid \#E(\mathbb{F}_q) \) and \( \ell \nmid \#E(\mathbb{F}_q) \).

We may consider a \( w \)-NAF as a \( \tau \)-adic representation of an integer with the nonzero digit set \( D = \{ \pm 1, \ldots, \pm (2^{w-1} - 1) \} \). It is called a \( \tau \)-adic \( w \)-NAF and denoted by \( a = \langle a_{m-1} \cdots a_0 \rangle_\tau \) or \( a = \sum_{i=0}^{m-1} a_i \tau^i \). Note that given a \( \tau \)-adic \( w \)-NAF \( a = \langle a_{m-1} \cdots a_0 \rangle_\tau \) and a point \( Q \) in \( G \), \( aQ \) is defined as \( aQ := \sum_{i=0}^{m-1} a_i \tau^i(Q) \). The following theorem [31], [33] shows that different \( \tau \)-adic \( w \)-NAFs act as different scalars for scalar multiplication if their lengths are bounded.

**Theorem 2:** [31], [33] \( a = \langle a_{m-1} \cdots a_0 \rangle_\tau \) and \( b = \langle b_{m'-1} \cdots b_0 \rangle_\tau \) be two \( \tau \)-adic \( w \)-NAFs. Then \( aQ = bQ \) for some nonzero \( Q \in G \) implies that \( m = m' \) and \( a_i = b_i \) for all \( i \) if both of \( m \) and \( m' \) are equal to or less than

\[
M_{\ell, w} = \log_q \left( \ell/(q^{w/2} + 1)^2 \right) - (w - 1).
\]

A scalar multiplication by a \( \tau \)-adic \( w \)-NAF can be done similarly to the window TNAF method [6], [7]. The only difference is to use \( P_{j} = iP \) instead of \( P_{j} = (i \mod \tau^w)P \). (Refer to Algorithm 4 in Appendix A.) If a scalar is of the form \( k = k_1 + \alpha k_2 \) and \( Q = \alpha P \) is given together with \( P \), then \( kP \) can be computed as \( k_1 P + k_2 Q \) using simultaneous scalar multiplication sharing the \( \tau \) operations. It requires \( (2^w - 1) + (\text{wt}(k_1) + \text{wt}(k_2) - 1) \) point additions, two point doublings and \((m-1)\) \( \tau \) operations. (For \( w = 2 \), no doublings are required.)

We can further reduce the number of point additions by sharing the point additions in the table construction stage. More precisely, given \( k_1 = \sum_{j=0}^{m-1} k_{1,j} \tau^j \) and \( k_2 = \sum_{j=0}^{m-1} k_{2,j} \tau^j \), we first compute \( R_i = \sum_{k_{1,j} = \pm 1} \text{sign}(k_{1,j}) \tau^j(P) + \sum_{k_{2,j} = \pm 1} \text{sign}(k_{2,j}) \tau^j(Q) \) for each \( i \) and then compute

\[
kP = k_1 P + k_2 Q = R_1 + 3R_3 + \cdots + (2^{w-1} - 1)R_{2^{w-1}-1}
\]

using the BGMW technique [10]. The detailed procedure is given in Algorithm 3. It requires \( \text{wt}(k_1) + \text{wt}(k_2) + 2^{w-2} - 2 \) point additions, one point doubling, and \( 2(m-1) \) \( \tau \) operations. (For \( w = 2 \), no doubling is required, since Step 3 is not executed.) Thus it significantly reduces the number of point additions at the cost of additional \((m-1)\) \( \tau \) operations.

**Algorithm 3** Scalar multiplication by a split scalar

1. Input \( P, Q, k_1 \), and \( k_2 \).
2. Scanning stage:
   1. Set \( R_i \leftarrow O \) for \( i = 1, 3, 5, \ldots, 2^{w-1} - 1 \).
   2. For \( j = 0 \) up to \( m - 1 \)
      1. Set \( R_{k_{1,j}} \leftarrow R_{k_{1,j}} + \text{sign}(k_{1,j}) \tau^j(P) \).
      2. Set \( R_{k_{2,j}} \leftarrow R_{k_{2,j}} + \text{sign}(k_{2,j}) \tau^j(Q) \).
3. Computation stage for
   1. \( k_1 P + k_2 Q = R_1 + 3R_3 + \cdots + (2^{w-1} - 1)R_{2^{w-1}-1} \):
      1. Set \( S \leftarrow R_{2^{w-1}-1} \); Set \( T \leftarrow R_{2^{w-1}-1} \).
      2. For \( i = 2^{w-1} - 3, 2^{w-1} - 5, \ldots, 5, 3 \)
         1. Set \( S \leftarrow S + R_i \).
         2. Set \( T \leftarrow T + S \).
      3. Set \( T \leftarrow 2T \).
      3.4 Set \( T \leftarrow T + S + R_1 \).
4. Output \( T \).

Now compare the performances of these algorithms with those of existing scalar multiplication algorithms. We consider the standard methods such as the TNAF and window TNAF methods [6], [7] as well as more recently proposed methods using double bases [23], [25], since to our knowledge, they are the fastest algorithms for scalar multiplication with a non-fixed point over a Koblitz curve.

Table 1 shows appropriate parameters \( m \) and \( t_1, t_2 \) for various \( w \)'s. \( m \) is chosen to guarantee the uniqueness of \( \tau \)-adic \( w \)-NAFs according to Theorem 2 and \( t_1, t_2 \) are chosen so that \( |S_1| \approx 2^{140} \) and \( |S_2| \approx 2^{2162} \) where \( S_i (i = 1, 2) \) is the set of \( \tau \)-adic \( w \)-NAFs of length \( m \) and weight \( t_i \). The last
three columns show the numbers of point operations required for the window TNAF algorithm and two proposed algorithms over the Koblitz curve K163. The number of point operations for the window TNAF algorithm is \(2^{w-2} - 1 + \frac{162}{w+1}\) additions and 162 + \(\kappa\) applications of \(\tau\) where \(\kappa\) comes from the table construction [6].

If we use a normal basis to represent the underlying field elements of an elliptic curve, then the computation of a Frobenius map is almost free, and we can ignore the ‘T’ terms in the table. In this case, the optimal value of \(w\) for the window TNAF method is \(w = 5\), and its cost is 34A. On the other hand, for the split scalars, the optimal choice is Algorithm 3 with \(w = 4\), which requires 25A+1D. Thus assuming the cost for a doubling is approximately the same as that of an addition, the expected speed-up is 23.5%. Note that the methods in [23] and [25] require 31.09 and 36.37 point additions over K163, respectively, which implies that our method is faster than them by 16.4% and 28.5%.

On the other hand, if a polynomial basis is used, the performance analysis becomes a tedious task because the cost for squaring operations cannot be completely ignored. According to our precise analysis given in Appendix B, the speed-ups over the TNAF and window TNAF methods are expected to be 36–40% and 10–15%, respectively.

### C. Binary Fields with Split Exponents

In binary fields, we consider the set of split exponents \(S_1 + \alpha S_2 \subseteq \mathbb{Z}_p\) where \(S_1\) and \(S_2\) are the sets of \(w\)-NAFs of fixed Hamming weights with the nonzero digit set \(D = \{1, 3, 5, \ldots, 2^w - 1\}\). They are called unsigned \(w\)-NAFs. If we use normal basis representation and replace by a squaring operation, we cannot obtain a similar speed-up to the Koblitz curve case.

Given a split exponent \(x = x_1 + \alpha x_2\) for unsigned \(w\)-NAFs \(x_1\) and \(x_2\) with length \(m\), each of which has weight \(t_1\) and \(t_2\) respectively, we can compute \(g^x\) for an arbitrary field element \(g\) using Algorithm 5 in Appendix C.) It requires \(t_1 + t_2 + 2^w - 3\) multiplications.

The second algorithm is to reduce the number of multiplications using the BGMW technique [10]. Its complexity is \(t_1 + t_2 + 2^{w-1} - 2\). (For the detailed procedure, refer to Algorithm 6 in Appendix C.)

Now we need to consider how to generate unsigned \(w\)-NAFs uniformly. One can easily show that every positive integer has exactly one unsigned \(w\)-NAF [29]. Moreover, all unsigned \(w\)-NAFs of length at most \(\lfloor \log p \rfloor - w + 1\) are distinct.

Table II compares the performance when \(m\) is the largest integer less than or equal to \(\lfloor \log p \rfloor - w + 1\). Note that typical cryptographic applications use short exponents of 160 bits over finite fields of order \(2^{1024}\) to guarantee the \(2^{80}\) security. Thus we assume that the exponentiation algorithms use 160-bit exponents and \(|S_1||S_2| \approx 2^{160}\). According to this table, the expected speed-up is 23.7%, which is similar to the case of Koblitz curves.

We remark that as \(m\) gets larger, the performance gain is expected to be better since we can reduce \(t_1\) and \(t_2\). But when \(m\) is larger than \(\lfloor \log p \rfloor - w + 1\), distinct unsigned \(w\)-NAFs of length \(m\) can be congruent modulo \(p\). It would be an interesting problem to devise an algorithm to pick \(w\)-NAFs almost uniformly from \(\mathbb{Z}_p\) for larger \(m\).

### IV. APPLICATIONS OF EXPONENTIATION BY SPLIT EXPONENTS

In this section, we show how split exponents can be adopted in cryptographic schemes based on the discrete logarithm problem, and we show the efficiency gains resulting from these modifications.

#### A. Public Key Encryption

Bellare, Kohno, and Shoup [8] proposed a CCA-secure version of ElGamal encryption that achieves the fastest encryption and decryption among ElGamal-based schemes by re-using the ephemeral ElGamal key. We show that usage of split exponents can speed up both the encryption and the decryption operations even further at the cost of doubling the size of public keys and increasing the setup cost.

**Setup:** Let \(G\) be a prime-order subgroup of \(\mathbb{Z}_p\) where \(p\) is its order and \(P\) is a generator. Let \(S_1\) and \(S_2\) be two subsets of \(\mathbb{Z}_p\) such that \(|S_1||S_2| \geq 2^{160}\), \(|S_1| \approx 2^{120}\), and \(|S_2| \approx 2^{40}\). A 1/4-good element \(\alpha \in \mathbb{Z}_p\) for \(S_1, S_2\) is chosen using Algorithm 2, and the public parameters are \(\text{Param} = \langle G, S_1, S_2, P, P_1 := \alpha P, P_2 := \alpha^2 P, E, H \rangle\), where \(E\) is a CCA-secure symmetric key encryption scheme and \(H\) is a collision resistant hash function from bit strings onto group \(G\), modeled as a random oracle in the security analysis.

**Key Generation:** Each user \(U_i\) chooses a secret key \(x_i + \alpha y_i\) uniformly from \(S_1 + \alpha S_2\), and publishes a public key \((X_i, Y_i)\) such that \(X_i = x_iP + y_iP_1\) and \(Y_i = x_iP_1 + y_iP_2\). Similarly \(s\) chooses an ephemeral secret key \(u_1 + \alpha v_1\) uniformly from \(S_1 + \alpha S_2\), and computes \(R_i = u_iP + v_iP_1\) and \(S_i = u_iP_1 + v_iP_2\).

**Encryption:** The encryption of a message \(m\) of the user \(U_j\) to the user \(U_i\) is \(\langle R_j, S_j, C \rangle\) where \(C = E_K(m)\) and \(K = H(R_j, S_j, X_i, Y_i, u_jX_i + v_jY_i)\).

**Decryption:** Given \(\langle R_j, S_j, C \rangle\), compute \(K = H(R_j, S_j, X_i, Y_i, x_iR_j + y_iS_j)\) and decrypt the ciphertext \(C = E_K(m)\) using \(K\).

A corresponding argument to Theorem 1 shows that if \(\alpha\) is 1/4-good then the hardness of the Gap Diffie Hellman Problem on group \(G\) on split exponents is related by factor 1/16 to the hardness of the Gap Diffie Hellman Problem (see e.g. [8]). Therefore the “chosen sender and receiver” security of the BKS encryption scheme (where the adversary attacks a fixed sender/receiver pair of players) is provably related by factor 1/16 to the Gap Diffie Hellman Problem. The reduction that shows this follows the reduction in [8]. Since the reduction is for “chosen sender chosen receiver” adversary, the reduction does not have to sample the split exponent keys of other players in the network. Also, note that the encryption and
decryption procedures can avoid having to verify whether the receiver’s or sender’s keys are elements in group $G$ because points on this Koblitz curve form a group $G'$ of size $2p$, and existence of the DDH oracle on $G'$ is therefore implied by the existence of the DDH oracle on $G$. (The first oracle can be implemented with a single call to the second one together and a single test of membership in $G$.)

Compared with the original scheme, we could see from Table I that both encryption and decryption are accelerated by 23.5% with normal basis (and 10–15% with polynomial basis) by virtue of split exponents. Though the size of public key is doubled and the setup cost is increased to select a 1/4-good element $\alpha$, the computational cost of both encryption and decryption are significantly reduced by using split exponents.

B. Other Diffie-Hellman and ElGamal Variants

The same speed-up can be achieved in any Diffie-Hellman system which uses fixed exponents, where the relatively high cost of uniform sampling of set $S_1 + \alpha S_2$ can be amortized. For example, the Diffie-Hellman Key Agreement with fixed exponents can be made more efficient at the cost of doubling the public key size. If the public key is $\langle X_i, Y_i \rangle$ where $X_i = (x_i + \alpha y_i)P$ and $Y_i = \alpha X_i$, we could speed up the shared key computation by 23.5% again, since $x_jX_i + y_jY_i = (x_i + \alpha y_i)(x_j + \alpha y_j)P = x_iX_j + y_iY_j$.

In versions of ElGamal encryption where the ephemeral key is refreshed for each encryption the split exponents can only be used for the public keys, i.e. $U_i$’s public key is a pair $X_i, Y_i$, but the ephemeral key is just $R_i = uP$ for $u \sim \mathbb{Z}_p$. This would slow down encryption process, but it would accelerate the decryption operation by 23.5%.

V. SCHNORR SIGNATURES WITH w-NAF

Note that the advantage of split exponents comes from the condition that each of $S_i$ consists of $\sqrt{p}$ elements of small Hamming weight chosen from $p$ elements. Thus if the set of challenges of the Schnorr signature consists of $\sqrt{p}$ elements, we can enjoy the advantage of small Hamming weight exponents without using split exponents.

In Asiacrypt 2000, Schnorr and Jacobson analyzed [34] the security of Schnorr signatures in the generic group model and the random oracle model for the hash function, and showed that the scheme has $2^{80}$ security level in the model as long as the challenge $c$ is chosen uniformly in any set with $2^{80}$ elements.

We describe a variant of the Schnorr signature scheme on Koblitz curves where the challenge $c$ in the NIZK for the discrete logarithm on which the Schnorr signature scheme is based is not a random element in $\mathbb{Z}_p$ but a value $c$ uniformly chosen from the set $S$ of $w$-NAFs of fixed weight. Since there is an efficient algorithm to pick up a $w$-NAF integer of fixed weight $t$ [31], we can easily obtain an efficient full-domain hash function to $S$ by the standard technique.

**Setup:** Let $G$ be a prime-order subgroup of the elliptic curve points from a Koblitz curve defined over $\mathbb{F}_{2^{163}}$, where $p$ is its order and $P$ is a generator. Let $S$ be a subset of $\mathbb{Z}_p$. The public parameters are $\text{Param} = \langle G, S, P, H \rangle$, where $H$ is a collision resistant full domain hash function $H : \{0,1\}^* \rightarrow S$.

**Key Generation:** Each user $U_i$ chooses a secret key $x_i \sim \mathbb{Z}_p$ and publishes a public key $X_i = x_iP$.

**Signing:** A signature on message $m$ is a tuple $\langle m, c, s \rangle$ where $R = -kP$, $c = H(m, R)$ and $s = k + cx_i$, for $k \sim \mathbb{Z}_p$.

**Verification:** Signature $\langle c, s \rangle$ on $m$ is accepted if

$$c = H(m, -sP + cx_i).$$

The verification of Schnorr signature mainly consists of two scalar multiplications, that is, $-sP$ and $cx_i$. The above variant improves the speed of $cX_i$ computation. If we use $w = 4$ and $t = 11$, Algorithm 4 requires only 14 elliptic curve additions, so the speed-ups using a normal basis is $(19 - 14)/19 = 26.3\%$ over the window TNAF. For the number of elliptic curve additions required to compute $cX_i$ for other parameters, refer to Table III. Note that the first scalar multiplication is a fixed point multiplication, so it can be easily sped up using various fixed-base precomputation methods.

We may consider split exponents for challenges of Schnorr

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**Table I**

| Parameters | $|S_1|$ | $|S_2|$ | Number of multiplications |
|-----------|--------|--------|--------------------------|
| $w$ | $m$ | $t_1$ | $t_2$ | $\text{Win.TNAF}$ | $\times \text{Alg. 4}$ | $\text{Alg. 3}$ |
| 2 | 157 | 28 | 6 | $1.1 \times 2^{42}$ | $1.8 \times 2^{20}$ | 54A+162T | 33A+156T | 33A+312T |
| 3 | 156 | 32 | 5 | $1.5 \times 2^{42}$ | $1.0 \times 2^{31}$ | 42A+165T | 29A+2D+155T | 28A+1D+310T |
| 4 | 154 | 30 | 5 | $1.6 \times 2^{42}$ | $1.7 \times 2^{29}$ | 36A+165T | 28A+2D+151T | 25A+1D+306T |
| 5 | 152 | 17 | 4 | $1.1 \times 2^{42}$ | $1.8 \times 2^{31}$ | 34A+168T | 34A+2D+151T | 27A+1D+302T |
| 6 | 150 | 14 | 4 | $1.7 \times 2^{42}$ | $1.6 \times 2^{34}$ | 38A+174T | 47A+2D+149T | 32A+1D+298T |

**Table II**

| Parameters | $|S_1|$ | $|S_2|$ | Number of multiplications |
|-----------|--------|--------|--------------------------|
| $w$ | $m$ | $t_1$ | $t_2$ | $\text{Alg. 5}$ | $\times \text{Alg. 3}$ | $\text{Alg. 6}$ |
| 2 | 159 | 27 | 6 | $1.2 \times 2^{44}$ | $1.0 \times 2^{32}$ | 53 | 33 |
| 3 | 158 | 22 | 5 | $1.8 \times 2^{42}$ | $1.1 \times 2^{29}$ | 42 | 29 |
| 4 | 157 | 19 | 4 | $1.7 \times 2^{44}$ | $1.1 \times 2^{30}$ | 38 | 29 |
| 5 | 156 | 16 | 4 | $1.1 \times 2^{42}$ | $1.0 \times 2^{41}$ | 41 | 34 |
| 6 | 155 | 13 | 4 | $1.5 \times 2^{44}$ | $1.8 \times 2^{34}$ | 53 | 47 |
signatures. That is, the challenge value $c = (c_1, c_2)$ is chosen uniformly from $S_1 \times S_2$ where each of $S_i$ is a set of $w$-NAF of fixed Hamming weight. Then we need to publish one more public parameter $\alpha P$ for $\alpha \in \mathbb{Z}_p$ and one more public key $Y_i = \alpha X_i$ for each user $U_i$. Then the signature is a tuple $(m, c_1, c_2, s)$ for a message $m$ where $R = -kP$, $(c_1, c_2) = H(m, R)$, and $s = k + (c_1 + \alpha c_2)z_i$ for $k \in \mathbb{Z}_p$. The signature is verified by checking $(c_1, c_2) = H(m, -sP + c_1 X_1 + c_2 Y_i)$.

By Corollary 1, if $|S_1| = |S_2| = 2^{10}$ then the average size of $|S_1 + \alpha S_2|$, for random $\alpha$, is at least $2^{80} - 1$. Hence we may assume that for an overwhelming fraction of $\alpha$’s the size of $|S_1 + \alpha S_2|$ is indeed very close to $2^{80}$, which would imply almost the same lower bound on the complexity of forging a signature as the argument in [34]. If we use 3-NAF of weight 5 for $(c_1, c_2)$, the $c_i X_i + c_2 Y_i$ takes only 11 elliptic curve additions which is an improvement of $42.1\%$ over the window TNAF in normal basis representation. For a specific $\alpha$, however, we need to investigate more the distribution of $|S_1 + \alpha S_2|$.

VI. CONCLUSION AND FURTHER STUDY

In this paper, we proposed a new variant of the discrete logarithm problem, called SEDL, which is the discrete logarithm on a class of exponents we call split exponents, and we showed how to adopt these exponents to speed up encryption and decryption operations in a BKS encryption scheme [8].

An interesting open problem is to show hardness of the SEDL problem on all but negligible fraction of $\alpha$’s. In this paper, we showed that the SEDL problem is hard for a random $\alpha$, and hence we also showed that the hardness of the SEDL problem is a small constant away from the hardness of the standard DL problem for certain “good” $\alpha$’s. However, we showed an efficient generation of such $\alpha$’s only if $S_1$ and $S_2$ are unbalanced, e.g. $|S_1| = 2^{120}$ and $|S_2| = 2^{40}$. Furthermore, we showed an efficient sampling algorithm from the set of split exponents for such a fixed good $\alpha$, also only in the case when set $S_2$ is much smaller than $S_1$. Finally, both algorithms, the one for finding good $\alpha$’s and the sampling algorithm, are time consuming, which limits their applicability. If the SEDL problem was hard for all but negligible $\alpha$’s then split exponents could be used to speed up a much larger class of DL-based cryptosystems, because they could then be applied to ephemeral values in these cryptosystems, and not just to the long-term private keys, as in our variant of the BKS encryption scheme. Similarly, extending our results to sets $S_1$ and $S_2$ of the same size would result in further speeding up of exponentiation operations.

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In this paper, we compared the performance of split exponent-based exponentiation with those of existing exponentiation algorithms only in the 80-bit security setting. But it will also be interesting to analyze the significance of split exponents from an asymptotic point of view as the security parameter grows. In addition, incorporating split exponents in the double-base setting such as [22], [23], [25] will be a promising research direction.

### Table III

| Parameters | $w$ | $m$ | $t$ | $|S|$ | Number of point operations |
|-----------|-----|-----|-----|-----|---------------------------|
|           | 2   | 157 | 15  | $1.0 \times 2^{11}$ | 2FA+79T, 1FA+1D+156T |
|           | 3   | 156 | 13  | $1.1 \times 2^{85}$ | 2FA+80T, 1FA+1D+155T |
|           | 4   | 154 | 11  | $1.5 \times 2^{85}$ | 1FA+82T, 1FA+1D+153T |
|           | 5   | 152 | 9   | $1.2 \times 2^{84}$ | 2FA+85T, 1FA+1D+151T |
|           | 6   | 150 | 8   | $1.1 \times 2^{84}$ | 2FA+91T, 2FA+1D+149T |
Algorithm 4 Scalar Multiplication by a $\tau$-adic $w$-NAF

1. Input $P$ and $k = (k_{m-1}k_{m-2}\ldots k_0)_\tau$, where $k$ is given as a $\tau$-adic $w$-NAF.

2. Table construction stage:
   2.1 Set $P_0 \leftarrow O$, $P_1 \leftarrow P$ and $P_2 \leftarrow 2P$.
   2.2 For $j = 1$ up to $2^{w-2} - 1$, set $P_{2j+1} \leftarrow P_{2j-1} + P_2$.

3. Scalar multiplication stage:
   3.1 Find the largest $i$ s.t. $k_i \neq 0$. Set $Q \leftarrow \text{sign}(k_i)P_{|k_i|}$.
   3.2 For $j = i - 1$ down to 0
       3.2.1 Set $Q \leftarrow \tau Q$.
       3.2.2 if $k_j \neq 0$ then set $Q \leftarrow Q + \text{sign}(k_j)P_{|k_j|}$.

4. Output $Q$.

Algorithm 4 and Algorithm 3 when the underlying binary field is represented as a polynomial basis. First, we revise Table 1 to discriminate the computational requirements for each part of the algorithms, and construct Table IV. That is, we denoted the cost for table construction or scanning and the cost for other parts as separate terms in the new table.

If the point operations are done in affine coordinates, we can obtain the relations $1A = 1F + 2M + 1S$, $1D = 1F + 2M + 2S$, and $1T = 2S$, where $I, M, S$ represent the computational costs for a field inversion, a field multiplication, and a field squaring, respectively. On the other hand, if we consider projective coordinates, it is desirable to use a mixed coordinate system where a doubling is performed using two points in López-Dahab (LD) projective coordinates and an addition is performed with one point represented in LD projective coordinates and the other in affine coordinates [35]. In this case, the costs for a doubling and an addition are $3M + 5S$ and $8M + 5S$, respectively [7], [35]. However, in order to use this coordinate system, some part of the scalar multiplication should still be done in affine coordinates. For example, for the window TNAF method, table construction is done in affine coordinates and the remaining part is done in mixed coordinates. The final result of a scalar multiplication is obtained by converting the result of the last point operation into affine coordinates, which requires $1+2M+1S$ since an LD projective point $(X : Y : Z)$, $Z \neq 0$ corresponds to the affine point $(X/Z, Y/Z^2)$.

For $2\times$ Algorithm 4, we have two options to use mixed coordinates. The first choice is to apply a similar approach to the window TNAF method. That is, table construction is done in affine coordinates and the remaining part is computed using mixed-coordinate additions and LD Frobenius maps. Another choice is to use mixed coordinates in table construction and LD projective additions and LD Frobenius maps in the remaining computation. Note that the costs of an LD projective addition and an LD Frobenius map are $13M + 6S$ and $3S$, respectively. According to our analysis, the first option is preferred in most cases. Therefore we consider only the first option.

We also have two options for Algorithm 3. The first choice is to use affine coordinates in Step 2. Then in Step 3, the computation of $S$’s is done in mixed coordinates, and $T$’s are computed using LD projective additions. The second choice is to use affine Frobenius maps and mixed additions in Step
Tables V shows the number of field operations for the three scalar multiplication algorithms using the above coordinate systems. Tables VI, VII and VIII present the total costs represented in the number of squarings using the estimation $I/M = 5, M/S = 7 [7]$. The minimum values over possible choices of $w$ are highlighted. We can summarize the three tables as follows:

- If $I/M = 5$, then affine coordinates are preferable. In this case, the parallel execution of Algorithm 4 with window size $w = 4$ is the best choice for the computation of $k_1P + k_2(\alpha P)$, and it is faster than the TNAF algorithm ($w = 2$) and the window TNAF algorithm for $kP$ by 40.21% and 11.20%, respectively.

- If $I/M = 8$, then mixed coordinates are better than affine coordinates for the window TNAF algorithm. But these two coordinates provide similar performances for the other two algorithms. Consequently, the parallel execution of Algorithm 4 with window size $w = 4$ is the best choice, and the speed-ups over the TNAF and window TNAF methods are 36.69% and 10.14%, respectively.

- If $I/M = 10$, mixed coordinates are preferable in most cases. Algorithm 3 ($w = 4$) with m-2 coordinates is the best choice, and it is faster than the TNAF and the window TNAF algorithms by 38.45% and 14.94%, respectively.

APPENDIX C

**EXponentiation Using Sparse UnSigned $w$-NAFs**

In this section, we present exponentiation algorithms using split sparse exponents. First, we start with a simple algorithm (Algorithm 5) that uses free squaring over a normal basis, which can be seen as an improved version of [36]. It requires $2(w−1) = wt(x) − 1 = 2w−1 + wt(x) − 2$ multiplications, where $wt(x)$ denotes the weight of $x$.

Algorithm 5 Exponentiation $g^x$ using normal basis representation

1. Input $g$ and $x = (x_{m−1}, x_{m−2}, \ldots, x_0)$. where $x_0$ is given as an unsigned $w$-NAF.
2. Table construction stage:
   1. Set $g_0 ← 1$, $g_1 ← g$ and $g_2 ← g^2$.
   2. For $j = 1$ up to $2^w − 1$, set $g_{j+1} ← g_{j} \times g_2$.
3. Exponentiation stage:
   1. Find the largest $i$ such that $x_i ≠ 0$. Then set $y ← g_i$.
   2. For $j = i − 1$ downto 0
      1. Set $y ← y^2$.
      2. If $x_j ≠ 0$ then set $y ← y \times g_j$.
4. Output $y$.

If an exponent is of the form $x = x_1 + \alpha x_2$ and $h = g^\alpha$ is given as input together with $g$, then $g^r$ can be computed as $g^{x_2} h^{x_1}$. A naive algorithm to compute $g^r h^{x_2}$ is to use Algorithm 5 twice, and it requires $2(2w−1) + (wt(x)−1) + wt(x) + wt(x) − 3$ multiplications. However, a modification of the BGMW method [10] gives a significant speed-up: the number of required multiplications for Algorithm 6 is $wt(x_1) + wt(x_2) + 2w−1 − 2$. 
TABLE V

COMPARISON OF THE NUMBER OF FIELD OPERATIONS FOR THREE SCALAR MULTIPLICATION ALGORITHMS

<table>
<thead>
<tr>
<th>w</th>
<th>Win. TNAF [6]</th>
<th>2×Alg. 4</th>
<th>Alg. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>affine</td>
<td>mixed</td>
<td>affine</td>
</tr>
<tr>
<td>1</td>
<td>I</td>
<td>M</td>
<td>S</td>
</tr>
<tr>
<td>2</td>
<td>54</td>
<td>108</td>
<td>378</td>
</tr>
<tr>
<td>3</td>
<td>42</td>
<td>84</td>
<td>368</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
<td>72</td>
<td>366</td>
</tr>
<tr>
<td>5</td>
<td>34</td>
<td>68</td>
<td>370</td>
</tr>
<tr>
<td>6</td>
<td>38</td>
<td>76</td>
<td>386</td>
</tr>
</tbody>
</table>

Algorithm 6 Exponentiation by a split exponent

1. Input \(g, h, x_1\) and \(x_2\).
2. Set \(s ← 1\), \(t ← 1\).
3. For \(i = 2^w - 1, 2^w - 3, \ldots, 3, 1\)
   3.1 For each \(j\) such that \(x_{1,j} = i\), set \(s ← s \times g^{2^j}\).
   3.2 For each \(j\) such that \(x_{2,j} = i\), set \(s ← s \times h^{2^j}\).
   3.3 if \(i = 1\) then set \(t ← t \times s\); else set \(t ← t \times s^2\).
4. Output \(t\).

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