

HÖLDER REGULARITY OF SOLUTIONS TO DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS

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ABSTRACT. We establish the Alexandroff-Bakelman-Pucci estimate, the Harnack inequality, and the Hölder continuity of solutions to degenerate elliptic equations of the non-divergence form

$$(0.1) \quad Lu := x a_{11} u_{xx} + 2\sqrt{x} a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y = g$$

on $x \geq 0$, with bounded measurable coefficients. We also establish similar regularity results in the parabolic case.

1. INTRODUCTION

This paper concerns with the regularity of solutions to degenerate parabolic equations of the non-divergence form

$$(1.1) \quad Lu := x a_{11} u_{xx} + 2\sqrt{x} a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y - u_t = g$$

on $x \geq 0$, with bounded measurable coefficients which satisfy the weak ellipticity condition

$$(1.2) \quad a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$$

and the lower bound $b_1 \geq c > 0$. More precisely, we will establish the Alexandroff-Bakelman-Pucci estimate, the Harnack inequality, and the Hölder continuity of solutions to equation (1.1), generalizing the classical by now result of Krylov and Safonov [KS] and Tso [T], for the strictly parabolic case.

The existence of regular solutions to the Dirichlet problem of (1.1) has been shown by Kohn and Nirenberg in [KN] and, for a more general class of equations with smooth coefficients, by Lin and Tso in [LT]. In both [KN] and [LT] their authors also established global L^2 -estimates of solutions of (1.1) in suitable weighted Sobolev norms. The applications of such degenerate problems to probability theory [F1][F2] was commented in [KN].

Our motivation for the study of (1.1), besides its own interest, arises from the regularity question of the free-boundary problem associated with the Gauss Curvature flow with flat sides. This is the flow describing the deformation of a weakly convex compact surface Σ in $\mathbb{R}^3$ by its Gaussian Curvature [H], [DH1]. If the initial surface Σ has flat sides, then the parabolic equation describing the motion of the hypersurface becomes degenerate where the curvature becomes zero. Hence, the junction Γ between each flat side and the strictly convex part of the surface, where the equation becomes degenerate, behaves like a free-boundary propagating with finite speed. Assuming that the surface Σ near the interface is represented by a graph $z = f(x, y, t)$, the function f evolves by the fully nonlinear equation

$$(1.3) \quad f_t = \frac{\det D^2 f}{(1 + |Df|^2)^{3/2}}$$

with the flat side $\Sigma_1(t) = \{(x, y, t) | f(x, y, t) = 0\}$. Daskalopoulos and Hamilton [DH1], showed the existence of a C^∞ -smooth up to the interface solution of (1.3), under the initial assumption that $g = \sqrt{2f}$ vanishes linearly at the interface and hence the equation for $g(x, y, t) = \sqrt{2f(x, y, t)}$ has a linear degeneracy. A simple local coordinate change from $(x, y, g(x, y, t))$ to $(h(z, y, t), t, z)$ transforms the free-boundary $g = 0$ into the fixed hyperplane $z = 0$. Moreover, h satisfies the fully-nonlinear equation of

$$(1.4) \quad h_t = \frac{z(h_{zy}^2 - h_{zz}h_{yy}) + h_z h_{yy}}{(z^2 + h_z^2 z^2 h_y^2)^{3/2}}$$

and its linearized equation satisfies a degenerate equation of type (1.1), under suitable conditions. The short time existence of a smooth up to the interface solution $z = g(x, y, t)$ in [DH1] is based on $C^{2,\alpha}$ a-priori Schauder estimates for solutions of (1.1) with C^α -coefficients.

In [DL2], the authors have recently shown that the function $z = g(x, y, t)$ will remain smooth up to the interface, for all time $0 < t < T_c$, with T_c denoting the vanishing time of the flat side. By means of first and second a-priori derivative bounds it is shown in [DL2] that each first order derivative of $z = h(x, y, t)$ satisfies an equation of the form (1.1). Therefore, the Hölder continuity Theorem 3.1 in Section 3, implies that h is of class $C^{1,\alpha}$, which constitutes the basic regularity estimate in [DL2].

Similar regularity questions arise in the free-boundary problem associated to the Porous medium equation [DH2], [K]

$$(1.5) \quad f_t = f \Delta f + \nu |Df|^2, \quad \nu > 0$$

satisfied by the pressure f of a gas through a porous medium. Indeed, Daskalopoulos, Hamilton and Lee [DHL] showed the all-time C^∞ regularity of solutions to (1.5) with root concave initial data, based on the Hölder a priori estimate of solutions to degenerate equations of the divergence form

$$(1.6) \quad x_n \Delta_{\mathbb{R}^{n-1}} u - x_n^{-\sigma} \partial_{x_n} (x_n^{1+\sigma} a^j \partial_j u) - u_t = g.$$

Such an estimate was shown by Koch in [K], by a Moser's iteration argument, appropriately scaled according to a singular metric. Local a priori $C^{2,\alpha}$ -estimates for degenerate equations of the form

$$(1.7) \quad Lu := x (a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy}) + b_1 u_x + b_2 u_y - u_t = g$$

with C^α -coefficients satisfying the ellipticity condition (1.2) and the lower bound $b_1 \geq c > 0$, was shown in [DH2], as the main step on establishing the short time existence of a smooth up to the interface solution of (1.5) with suitable $C^{2,\alpha}$ initial data. Because of the degeneracy of the equation, all the estimates are scaled according to the an appropriate singular metric.

All the above results generalize in dimensions $n > 2$. The question of C^α -regularity of solutions to (1.7) with bounded measurable coefficients satisfying (1.2) and $b_1 \geq c > 0$ is still an open problem. One also may ask similar questions on various types of degeneracies of the type

$$(1.8) \quad Lu := \sum_{i=1}^n x^{\alpha_i} x^{\alpha_j} a_{ij} u_{ii} + \sum_{i=1}^n b_i u_i + cu - u_t = g.$$

Let us also mention that the C^α , $C^{1+\alpha}$ and $C^{2+\alpha}$ regularity of solutions to degenerate elliptic equation of the type of (1.7) in the case that $b_1 \leq 0$ has been established by Lin and Wang in [LW].

We will assume throughout this paper that the coefficients of the operator L in (1.1) satisfy the bounds

$$(1.9) \quad a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}$$

and

$$(1.10) \quad |a_{ij}|, |b_i| \leq \lambda^{-1}$$

and

$$(1.11) \quad \frac{2b_1}{a_{11}} \geq \nu > 0$$

for some constants $0 < \lambda < 1$ and $0 < \nu < 1$.

In Section 2 we will establish the Alexandroff-Bakelman-Pucci estimate, the Harnack estimate and the Hölder continuity of solutions to the corresponding elliptic equations

$$(1.12) \quad Lu := x a_{11} u_{xx} + 2\sqrt{x} a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y = g$$

under the same assumptions (1.9)-(1.11) on its coefficients. In Section 3 we will show how one can generalize these results to the parabolic case. Since most of the proofs will be similar to the elliptic case, we will only draft the proofs of the parabolic results.

Let us also emphasize that all our proofs generalize to higher dimensions $n \geq 3$.

2. THE ELLIPTIC CASE.

Let (x_0, y_0) be a point in $\mathbb{R}^2$, with $x_0 \geq 0$. For any number $r > 0$, let us denote by $\mathcal{C}_r(x_0, y_0)$ the cube

$$\mathcal{C}_r(x_0, y_0) = \{(x, y) : x \geq 0, |x - x_0| \leq r, |y - y_0| \leq r\}.$$

Let us also denote by μ the measure

$$(2.1) \quad d\mu = x^{\frac{\nu}{2}-1} dx dy.$$

Our goal is to prove the following result:

Theorem 2.1. *Assume that the coefficients of the operator L are smooth on $\mathcal{C}_\rho(x_0, y_0)$, $\rho > 0$, and satisfy the bounds (1.9) and (1.11). Then, there exist a number $0 < \alpha < 1$ so that, for any $r < \rho$*

$$\|u\|_{C_s^\alpha(\mathcal{C}_r(x_0, y_0))} \leq C(r, \rho) \left(\|u\|_{C^\circ(\mathcal{C}_\rho(x_0, y_0))} + \left(\int_{\mathcal{C}_\rho} g^2 d\mu \right)^{1/2} \right)$$

for all smooth functions u on $\mathcal{C}_\rho(x_0, y_0)$ for which $Lu = g$.

From now on we will assume that the operator L satisfies conditions (1.9) and (1.11). Throughout this section we will denote by s the variable

$$s = \sqrt{x}.$$

The operator L can be expressed in the (s, y) variables as

$$L_s u := \frac{a_{11}}{4} u_{ss} + a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{4s} \left[\frac{2b_1}{a_{11}} - 1 \right] u_s + b_2 u_y$$

and hence introducing the new elliptic coefficient matrix

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{12} & \tilde{a}_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}}{4} & \frac{a_{12}}{2} \\ \frac{a_{12}}{2} & a_{22} \end{pmatrix}$$

the operator L_s takes the form

$$(2.2) \quad L_s u = \tilde{a}_{11} u_{ss} + 2\tilde{a}_{12} u_{sy} + \tilde{a}_{22} u_{yy} + \frac{\tilde{a}_{11}}{s} \left[\frac{b_1}{2\tilde{a}_{11}} - 1 \right] u_s + b_2 u_y.$$

The matrix $\tilde{a}_{ij}$ satisfies

$$(2.3) \quad \tilde{\lambda} |\xi|^2 \leq \tilde{a}_{ij} \xi_i \xi_j \leq \tilde{\lambda}^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}$$

with $\tilde{\lambda} = \lambda/4$ and

$$(2.4) \quad |b_i| \leq \lambda^{-1} \quad \text{and} \quad \frac{b_1}{2\tilde{a}_{11}} \geq \nu > 0$$

with $0 < \nu < 1$. We will also denote by $\bar{L}_s$ our model operator

$$\bar{L}_s u = u_{ss} + u_{yy} + (\nu - 1) \frac{u_s}{s}$$

which may also be expressed in the form

$$\bar{L}_s u = s^{1-\nu} [s^{\nu-1} u_s]_s + u_{yy}.$$

2.1. Alexandrov-Bakelman-Pucci Estimate. Let us consider the new variable

$z = \frac{s^{2-\nu}}{2-\nu}$. Then

$$\frac{dz}{ds} = s^{1-\nu}$$

implying that

$$\bar{L}_s u = s^{2(1-\nu)} u_{zz} + u_{yy}.$$

Pick a point (s_0, y_0) such that $s_0 \geq 0$ and for $r > 0$ we define the cube

$$\mathcal{C}_r(s_0, y_0) = \{(s, y) : s \geq 0, |s - s_0| \leq r, |y - y_0| \leq r\}.$$

Consider the gradient map $Z = (u_z, u_y)$ in the (z, y) variables, and define the set

$$(2.5) \quad \Gamma^+ = \left\{ (s, y) \in \mathcal{B}_\rho : \frac{\partial(u_z, u_y)}{\partial(z, y)} \leq 0, u_z \leq 0 \right\}.$$

We will show the following Alexandrov-Bakelman-Pucci maximum principle for solutions of the equation (1.12). Our arguments follow the ideas in the proof of

Theorem 9.1 in [GT]. However, because of the degeneracy of equation (1.12) we need to scale the estimates differently. To simplify the notation, we will denote in the next two Theorems by (a_{ij}) the matrix $(\tilde{a}_{ij})$ and by λ the number $\tilde{\lambda}$.

Theorem 2.2. *Let u be a classical subsolution of equation*

$$(2.6) \quad L_s u := a_{11} u_{ss} + 2a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{s} \left[\frac{b_1}{2a_{11}} - 1 \right] u_s + b_2 u_y \geq g$$

on $\mathcal{C}_\rho = \mathcal{C}_\rho(s_0, y_0)$, $\rho < 1$, with coefficients satisfying conditions (2.3) and (2.4). Assume in addition that $u \leq 0$ on $\{|s - s_0| = \rho, |y - y_0| = \rho\} \cap \mathcal{C}_\rho(s_0, y_0)$. Then,

$$\sup_{\mathcal{C}_\rho} u^+ \leq C(\lambda, \nu) \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \left(\int_{\Gamma^+} (g^-)^2(s, y) s^{\nu-1} ds dy \right)^{1/2}$$

with

$$(2.7) \quad \rho_\nu(s_0) = (s_0 + \rho)^{2-\nu} - s_0^{2-\nu}.$$

Proof. Assume that u^+ takes a positive maximum

$$M = \max_{\mathcal{C}_\rho} u^+$$

at the point (s, y) and let ρ_ν be the distance defined by (2.7). Consider the set Γ^+ defined by (2.5). Let us observe that since u is a classical subsolution of (2.6), and therefore at least C^2 -smooth up to $x = 0$, we have $u_s = 2s u_x = 0$ at $s = \sqrt{x} = 0$ and in addition $u_z = s^{\nu-1} u_s = 2s^\nu u_x = 0$ at $s = z = 0$. In particular, this implies that $\{u_s \leq 0\} = \{u_z \leq 0\}$. Then, a simple geometric argument shows that

$$D = \left[-\frac{cM}{\rho_\nu(s_0)}, 0 \right] \times \left[-\frac{cM}{\rho}, \frac{cM}{\rho} \right] \subset Z(\Gamma^+)$$

for some uniform constant c , where $Z(\Gamma^+)$ denotes the image of Γ^+ under the gradient map $Z = (u_z, u_y)$. Hence

$$(2.8) \quad |D| \leq |Z(\Gamma^+)| = \int_{\Gamma^+} \left| \det \left(\frac{\partial Z}{\partial(s, y)} \right) \right| ds dy.$$

On the other hand

$$\begin{aligned} |Z(\Gamma^+)| &= \int_{\Gamma^+} \left| \det \left(\frac{\partial Z}{\partial(s, y)} \right) \right| ds dy = \int_{\Gamma^+} \left| \det \left(\frac{\partial Z}{\partial(z, y)} \right) \frac{dz}{ds} \right| ds dy \\ &= \int_{\Gamma^+} \left| \det \left(\frac{\partial Z}{\partial(z, y)} \right) \right| s^{1-\nu} ds dy = \int_{\Gamma^+} |u_{zz} u_{yy} - u_{zy}^2| s^{1-\nu} ds dy \\ &= \int_{\Gamma^+} |s^{2(1-\nu)} u_{zz} u_{yy} - (s^{1-\nu})^2 u_{zy}^2| s^{\nu-1} ds dy \\ &= \int_{\Gamma^+} |\det E| d\mu \end{aligned} \quad (2.9)$$

with $d\mu = s^{\nu-1} ds dy$ and

$$E = \begin{pmatrix} s^{2(1-\nu)} u_{zz} & s^{1-\nu} u_{zy} \\ s^{1-\nu} u_{zy} & u_{yy} \end{pmatrix} = \begin{pmatrix} u_{ss} + \frac{(\nu-1)u_s}{s} & u_{sy} \\ u_{sy} & u_{yy} \end{pmatrix}.$$

Since, $\frac{\partial(u_z, u_y)}{\partial(z, y)} \leq 0$ on Γ^+ , $-E \geq 0$, i.e., $|\det E| = \det(-E)$. Hence, by formula (9.10) in [GT] and (2.6), we conclude

$$\begin{aligned} 2 [\det(a_{ij}) \cdot \det(-E)]^{\frac{1}{2}} &\leq \left(a_{11} \left[u_{ss} + \frac{(\nu-1)u_s}{s} \right] + 2a_{12}u_{sy} + a_{22}u_{yy} \right)^- \\ &\leq \left(a_{11}u_{ss} + 2a_{12}u_{sy} + a_{22}u_{yy} + \frac{a_{11}[\frac{b_1}{2a_{11}} - 1]}{s}u_s + \frac{a_{11}[\nu - \frac{b_1}{2a_{11}}]}{s}u_s \right)^- \\ &\leq g^- + |b_2||u_y| + \left(\frac{a_{11}[\nu - \frac{b_1}{2a_{11}}]}{s}u_s \right)^- \end{aligned}$$

The last term in the above estimate is actually equal to zero, since $u_z = u_s/s \leq 0$ on Γ^+ and $\nu - \frac{b_1}{2a_{11}} \leq 0$ by condition (2.4). Hence

$$2 [\det(a_{ij}) \cdot |\det E|]^{\frac{1}{2}} \leq g^- + |b_2||u_y|.$$

Hölder's inequality then implies the estimate

$$2 [\det(a_{ij}) \cdot |\det E|]^{\frac{1}{2}} \leq (k^2(g^-)^2 + |b_2|^2)^{\frac{1}{2}} \cdot (k^{-2} + |u_y|^2)^{\frac{1}{2}}$$

for all numbers $k > 0$. Using the bound $\det(a_{ij}) \geq \lambda^2$ we then conclude the bound

$$(2.10) \quad |\det E|^{\frac{1}{2}} \cdot (k^{-2} + |u_y|^2)^{-\frac{1}{2}} \leq \frac{1}{2} \lambda^{-1} (k^2(g^-)^2 + |b_2|)^{\frac{1}{2}}.$$

Considering the function G on $\mathbb{R}^2$ defined by

$$G(\xi, \zeta) = (k^{-2} + \xi^2)^{-1},$$

instead of (3.4) we have the formula

$$(2.11) \quad \int_D G \leq \int_{\Gamma^+} G(Z) \left| \frac{\partial Z}{\partial(s, y)} \right| ds dy = \int_{\Gamma^+} (k^{-2} + u_y^2)^{-1} |\det E| d\mu.$$

Combining (2.10) and (2.11) and using the bound $|b_2| \leq \lambda^{-1}$, we obtain the estimate

$$(2.12) \quad \int_D G \leq \frac{1}{4\lambda^2} \int_{\Gamma^+} (k^2(g^-)^2 + \lambda^{-2}) d\mu.$$

To compute the integral $\int_D G$, let us recall that $D = [-\frac{cM}{\rho_\nu(s_0)}, 0] \times [-\frac{cM}{\rho}, \frac{cM}{\rho}]$, so that

$$\begin{aligned}
 \int_D G &\geq \int_{-\frac{cM}{\rho_\nu(s_0)}}^0 \int_{-\frac{cM}{\rho}}^{\frac{cM}{\rho}} (k^{-2} + \xi^2)^{-1} d\xi d\zeta \\
 (2.13) \quad &\geq \frac{c\rho}{\rho_\nu(s_0)} \int_{B_{\frac{cM}{\rho}}} (k^{-2} + \xi^2 + \zeta^2)^{-1} d\xi d\zeta \\
 &= \frac{c\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^2 k^2 M^2}{\rho^2})
 \end{aligned}$$

for some small constant $c = c(\lambda, \nu) > 0$. From (2.12) and (2.13) we obtain

$$\frac{c\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^2 k^2 M^2}{\rho^2}) \leq \frac{1}{4\lambda^2} \int_{\Gamma^+} (k^2 (g^-)^2 + \lambda^{-2}) d\mu.$$

Let us set k by $k^{-2} = \lambda^2 \int_{\Gamma^+} (g^-)^2 d\mu$ in the above estimate so that

$$\begin{aligned}
 \frac{1}{4\lambda^2} \int_{\Gamma^+} (k^2 (g^-)^2 + \lambda^{-2}) d\mu &= \frac{1}{4\lambda^4} \left(1 + \int_{\Gamma^+} d\mu \right) \\
 &\leq C(\lambda) \left(1 + \int_{\mathcal{C}_\rho} s^{\nu-1} ds dy \right) \leq C(\lambda, \nu)
 \end{aligned}$$

for some constant $C = C(\lambda, \nu)$. Combining the above we conclude that

$$\frac{\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^2 k^2 M^2}{\rho^2}) \leq C(\lambda, \nu).$$

Since $\alpha = \frac{\rho}{\rho_\nu(s_0)} \geq 1$, when $s_0 < 1$ and $\rho < 1$, the estimate $\alpha \log(1+x) \geq \log(1+\alpha x)$ then implies that

$$\log(1 + \frac{c^2 k^2 M^2}{\rho \rho_\nu(s_0)}) \leq C(\lambda, \nu).$$

Exponentiating, we finally obtain the estimate

$$M \leq C(\lambda, \nu) \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \left(\int_{\Gamma^+} (g^-)^2 d\mu \right)^{\frac{1}{2}}$$

finishing the proof of the Theorem.

Replacing u by $-u$ in the above Theorem and defining the set

$$\Gamma^- = \left\{ (s, y) \in \mathcal{C}_\rho : \frac{\partial(u_z, u_y)}{\partial(z, y)} \geq 0, u_z \geq 0 \right\}$$

we obtain:

Theorem 2.3. *Let u be a classical supersolution of equation*

$$(2.14) \quad L_s := a_{11} u_{ss} + 2a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{s} \left[\frac{b_1}{2a_{11}} - 1 \right] u_s + b_2 u_y \leq g$$

on $\mathcal{C}_\rho = \mathcal{C}_\rho(s_0, y_0)$, with coefficients satisfying conditions (2.3) and (2.4). Assume in addition that $u \geq 0$ on $\{|s - s_0| = \rho, |y - y_0| = \rho\} \cap \mathcal{C}_\rho(s_0, y_0)$. Then,

$$\sup_{\mathcal{C}_\rho} u^- \leq C(\lambda, \nu) \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \left(\int_{\Gamma^-} (g^+)^2(s, y) s^{\nu-1} ds dy \right)^{1/2}$$

with $\rho_\nu(s_0)$ as in (2.7).

2.2. The Barrier Function. We will construct, in this paragraph, an important for our purposes barrier function. A similar function was introduced by Caffarelli in [C]. To simplify the computations in this paragraph we will go back to the original (x, y) variables, assuming that L satisfies conditions (1.9) - (1.11). We begin by introducing a new distance function. For a point $(x_0, y_0) \in \mathbb{R}^2$, with $0 \leq x_0 \leq 1$, let us define the distance function d_γ by

$$(2.15) \quad d_\gamma^2((x, y), (x_0, y_0)) = (\sqrt{x} - \sqrt{x_0})^2 + \gamma^2 (y - y_0)^2$$

with

$$\gamma^2 = \frac{\nu\lambda}{10}.$$

Recall that $0 < \lambda < 1$ is the ellipticity constant and $0 < \nu < 1$ the positive constant so that (1.11) holds. Notice that in the (s, y) variables, with $s = \sqrt{x}$ the distance function d_γ^2 may be expressed as

$$d_\gamma^2((s, y), (s_0, y_0)) = (s - s_0)^2 + \gamma^2 (y - y_0)^2.$$

For $r > 0$, let $Q_r(x_0, y_0)$ denote the cube

$$Q_r(x_0, y_0) = \{ (x, y) : x \geq 0, |\sqrt{x} - \sqrt{x_0}| \leq r, \gamma |y - y_0| \leq r \}$$

and let $\mathcal{B}_\rho(x_0, y_0)$ denote the ball

$$\mathcal{B}_r(x_0, y_0) = \{ (x, y) : x \geq 0, d_\gamma((x, y), (x_0, y_0)) \leq r \}.$$

Lemma 2.4. *There exists a smooth function ϕ on the half space $\mathbb{R}_+^2$ and positive constants C and $K > 1$ depending only on the constants λ and ν , such that*

$$(2.16) \quad \begin{cases} \phi & \geq 0 & \text{on } \mathbb{R}_+^2 \setminus \mathcal{B}_{3\sqrt{2}}(x_0, y_0) \\ \phi & \geq -2 & \text{in } Q_{\frac{3}{2}}(x_0, y_0) \end{cases}$$

and

$$(2.17) \quad L\phi \leq C\xi, \quad \text{on } \mathbb{R}_+^2$$

where $\xi = \bar{\xi}(d^2)$ is a continuous function on $\mathbb{R}^n$ with $0 \leq \xi \leq 1$ and $\text{supp } \xi \subset Q_{\frac{1}{2}}(x_0, y_0)$. Moreover, $\phi \geq -K$ on $\mathbb{R}_+^2$.

Proof. To simplify the notation, let us set for any $r > 0$, $\mathcal{B}_r = \mathcal{B}_r(x_0, y_0)$ and $Q_r = Q_r(x_0, y_0)$. Introducing the new distance function

$$\bar{d}^2 = \frac{(x - x_0)^2}{x + x_0} + \gamma^2 (y - y_0)^2.$$

one can easily see that

$$(2.18) \quad d_\gamma \leq \bar{d} \leq \sqrt{2} d_\gamma$$

since

$$|\sqrt{x} - \sqrt{x_0}| \leq \frac{|x - x_0|}{\sqrt{x + x_0}} \leq \sqrt{2} |\sqrt{x} - \sqrt{x_0}|.$$

Define the function

$$\phi = M_1 - \frac{M_2}{(\bar{d}^2)^\alpha}, \quad \text{on } \mathcal{B}_4 \setminus \mathcal{B}_{\frac{1}{4}}$$

with $\alpha > 0$ a sufficiently large constant, depending only on λ and ν , to be determined in the sequel. One can choose M_1 and M_2 , depending on λ, ν and α , so that

$$\phi \equiv 0, \quad \text{on } \bar{d} = 3\sqrt{2} \quad \text{and} \quad \phi = -2, \quad \text{on } \bar{d} = 3.$$

Hence, by (2.18)

$$\phi \leq 0, \quad \text{on } \mathcal{B}_4 \setminus \mathcal{B}_{3\sqrt{2}} \quad \text{and} \quad \phi = -2, \quad \text{on } \mathcal{B}_{3\sqrt{2}/2} \setminus \mathcal{B}_{\frac{1}{4}}.$$

It is possible to extend ϕ as a smooth function $\phi = \bar{\phi}(\bar{d})$ on $\mathbb{R}_+^2$ in such a way that (2.16) holds and also $L\phi \leq 0$ on $\mathbb{R}_+^2 \setminus \mathcal{B}_4$. This, in particular, will imply that

$$L\phi \leq C(\nu, \lambda) \xi, \quad \text{on } Q_{\frac{3}{2}} \cup (\mathbb{R}_+^2 \setminus \mathcal{B}_3).$$

Hence, it remains to show that $L\phi \leq C(\nu, \lambda) \xi$ on $\mathcal{B}_4 \setminus Q_{\frac{3}{2}}$. Since $\mathcal{B}_{\frac{1}{4}} \subset Q_{\frac{3}{2}}$, it is enough to show that

$$(2.19) \quad L\phi \leq 0 \quad \text{on } \mathcal{B}_4 \setminus \mathcal{B}_{\frac{1}{4}}.$$

To simplify the notation, let us set $\theta = \bar{d}^2$, so that

$$\phi = M_1 - \frac{M_2}{\theta^\alpha}.$$

A direct computation shows that

$$\begin{aligned} L\phi = & x a_{11} \left[\frac{\alpha M_2}{\theta^{\alpha+1}} \theta_{xx} - \frac{\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} \theta_x^2 \right] + a_{22} \left[\frac{\alpha M_2}{\theta^{\alpha+1}} \theta_{yy} - \frac{\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} \theta_y^2 \right] \\ & - 2\sqrt{x} a_{12} \left[\frac{\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} \theta_x \theta_y \right] + b_1 \left[\frac{\alpha M_2}{\theta^{\alpha+1}} \theta_x \right] + b_2 \left[\frac{\alpha M_2}{\theta^{\alpha+1}} \theta_y \right] \end{aligned}$$

i.e.,

$$\begin{aligned} L\phi &= \frac{\alpha M_2}{\theta^{\alpha+1}} [x a_{11} \theta_{xx} + a_{22} \theta_{yy} + b_2 \theta_y] + b_1 \left[\frac{\alpha M_2}{\theta^{\alpha+1}} \theta_x \right] \\ &\quad - \frac{\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} [x a_{11} \theta_x^2 + 2\sqrt{x} a_{12} \theta_x \theta_y + a_{22} \theta_y^2] \end{aligned}$$

Notice first that by the ellipticity condition (1.9) we have

$$x a_{11} \theta_x^2 + 2\sqrt{x} a_{12} \theta_x \theta_y + a_{22} \theta_y^2 \geq \lambda [x \theta_x^2 + \theta_y^2].$$

Also, by direct calculation

$$\theta_x = \frac{(x+3x_0)(x-x_0)}{(x+x_0)^2} \quad \text{and} \quad \theta_{xx} = \frac{8x_0^2}{(x+x_0)^3}$$

while

$$\theta_y = \frac{2\nu\lambda}{10} (y-y_0) \quad \text{and} \quad \theta_{yy} = \frac{2\nu\lambda}{10}.$$

Hence, using again the bounds (1.9) - (1.11), we obtain

$$\begin{aligned} (2.20) \quad L\phi &\leq \frac{\alpha M_2 \lambda^{-1}}{\theta^{\alpha+1}} \left[\frac{8x x_0^2}{(x+x_0)^3} + \frac{2\nu\lambda}{10} + \frac{(x+3x_0)(x-x_0)^+}{(x+x_0)^2} + \frac{2\nu\lambda}{10} |y-y_0| \right] \\ &\quad - \frac{\nu\alpha M_2}{\theta^{\alpha+1}} \left[\frac{(x+3x_0)(x-x_0)^-}{(x+x_0)^2} \right] \\ &\quad - \frac{2\lambda\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} \left[\frac{x(x+3x_0)^2(x-x_0)^2}{(x+x_0)^4} + \frac{4\nu^2\lambda^2}{100} (y-y_0)^2 \right] \end{aligned}$$

Let us consider a point $P = (x, y) \in \mathcal{B}_4 \setminus \mathcal{B}_{\frac{1}{4}}$. We will show that there exists a constant $\alpha = \alpha(\nu, \lambda)$, sufficiently large, for which $L\phi \leq 0$ at P . We separate the two cases:

Case 1: $x \leq \frac{1}{2}x_0$. In this case, (2.20) implies that

$$\begin{aligned} (2.21) \quad L\phi &\leq \frac{\alpha M_2 \lambda^{-1}}{\theta^{\alpha+1}} \left[\frac{8x}{x_0} + \frac{2\nu\lambda}{10} |y-y_0| \right] - \frac{\nu\alpha M_2}{\theta^{\alpha+1}} \left(\frac{3}{8} - \frac{2}{10} \right) \\ &\quad - \frac{2\lambda\alpha(\alpha+1)M_2}{\theta^{\alpha+2}} \left[\frac{9x}{x_0} \cdot \frac{(x-x_0)^2}{x+x_0} + \frac{4\nu^2\lambda^2}{100} (y-y_0)^2 \right] \end{aligned}$$

Since $d_\gamma((x, y), (x_0, y_0)) \geq \frac{1}{4}$ we have $\theta \geq c(\lambda, \nu) > 0$. In addition

$$|\sqrt{x} - \sqrt{x_0}|^2 \geq \frac{1}{32} \quad \text{or} \quad \frac{\lambda\nu}{10} |y-y_0|^2 \geq \frac{1}{32}.$$

When $|\sqrt{x} - \sqrt{x_0}|^2 \geq \frac{1}{32}$, one can deduce from (2.21) that

$$\begin{aligned} L\phi &\leq \frac{\alpha M_2}{\theta^{\alpha+1}} \left[\frac{8\lambda^{-1}x}{x_0} + \frac{2\nu}{10} |y-y_0| - \frac{\nu}{10} \right] \\ &\quad - \frac{\alpha(\alpha+1)M_2}{\theta^{\alpha+1}} \left[c_1(\nu, \lambda) \frac{x}{x_0} + c_2(\nu, \lambda) (y-y_0)^2 \right] \leq 0 \end{aligned}$$

for α sufficiently large, depending only on λ and ν . On the other hand, when

The negativity come from the condition like $x \leq \frac{\lambda\nu}{100}$ so that $\frac{\alpha M_2}{\theta^{\alpha+1}} \left[\frac{8\lambda^{-1}x}{x_0} + \frac{2\nu}{10} |y-y_0| - \frac{\nu}{10} \right] \leq 0$.

$\frac{\lambda\nu}{10} |y - y_0|^2 \geq \frac{1}{32}$ the estimate (2.21) implies that

$$L\phi \leq \frac{\alpha M_2}{\theta^{\alpha+1}} [C(\nu, \lambda) - c(\nu, \lambda)(\alpha + 1)] \leq 0$$

again for $\alpha = \alpha(\lambda, \nu)$ sufficiently large.

Case 2: $x \geq \frac{1}{2}x_0$. Then for a point $P = (x, y) \in \mathcal{B}_4 \setminus \mathcal{B}_{\frac{1}{4}}$ where

$$\frac{1}{4} \leq |\sqrt{x} - \sqrt{x_0}|^2 + \frac{\lambda\nu}{10} |y - y_0|^2 \leq 4$$

and with $x_0 \leq 1$, (2.20) implies the estimate

$$L\phi \leq \frac{\alpha M_2}{\theta^{\alpha+1}} [C(\lambda, \nu) - (\alpha + 1)c(\lambda, \nu)] \leq 0$$

for $\alpha = \alpha(\lambda, \nu)$ sufficiently large.

The following Lemma follows by simply rescaling the function ϕ .

Lemma 2.5. *Given $\rho > 0$, there exists a smooth function ϕ_ρ on the half space $\mathbb{R}_+^2$ and positive constants C and $K > 1$ such that*

$$(2.22) \quad \begin{cases} \phi_\rho & \geq 0 & \text{on } \mathbb{R}_+^2 \setminus \mathcal{B}_{3\sqrt{2}\rho}(x_0, y_0) \\ \phi_\rho & \geq -2 & \text{in } Q_{\frac{3\rho}{2}}(x_0, y_0) \end{cases}$$

$$(2.23) \quad L\phi_\rho \leq \frac{C}{\rho^2} \xi_\rho, \quad \text{on } \mathbb{R}_+^2$$

where $0 \leq \xi_\rho \leq 1$ is a continuous function on $\mathbb{R}^n$ with $\text{supp} \xi_\rho \subset Q_{\frac{\rho}{2}}(x_0, y_0)$. Moreover, $\phi_\rho \geq -K$ on $\mathbb{R}_+^2$.

Proof. Let $\phi = \bar{\phi}(\bar{d}^2)$ be the function constructed in the previous Lemma. Define the function ϕ_ρ by

$$\phi_\rho = \bar{\phi}\left(\frac{\bar{d}}{\rho}\right).$$

Then, clearly ϕ_ρ satisfies conditions (2.22). Moreover,

$$L\phi_\rho(d) = \frac{1}{\rho^2} L\bar{\phi}\left(\frac{d}{\rho}\right)$$

implying condition (2.23).

2.3. The Harnack Inequality. Fix a point $(x_0, y_0) \in \mathbb{R}_+^2$ and set $s_0 = \sqrt{x_0}$. Let us now go back to the (s, y) variables (with $s = \sqrt{x}$) assuming, throughout this section, that the operator $L_s u$ is defined as

$$(2.24) \quad L_s u := a_{11} u_{ss} + 2a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{s} \left[\frac{b_1}{2a_{11}} - 1 \right] u_s + b_2 u_y$$

with L_s satisfying conditions (2.3) and (2.4). Denoting, for any $r > 0$, by $Q_r(s_0, y_0)$ the cube

$$Q_r(s_0, y_0) = \{ (s, y) : s \geq 0, |s - s_0| \leq r, \gamma |y - y_0| \leq r \}$$

we will show the following Harnack inequality for solutions to equation $L_s u = g$.

Theorem 2.6. *Let $u \geq 0$ be a classical solution of equation $L_s u = g$ in $Q_\rho(s_0, y_0)$, where g is a bounded and continuous function on $Q_\rho(s_0, y_0)$. Then,*

$$(2.25) \quad \sup_{Q_{\frac{\rho}{2}}(s_0, y_0)} u \leq C \left(\inf_{Q_{\frac{\rho}{2}}(s_0, y_0)} u + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right)$$

with $d\mu = s^{\nu-1} ds dy$ and $\rho_\nu(s_0)$ given by (2.7).

Theorem 2.6 follows as a direct consequence of the next basic for our purposes Lemma.

Lemma 2.7. *Let $u \geq 0$ be a classical solution of equation $L_s u = g$ in $Q_{3\sqrt{2}\rho}(s_0, y_0)$, where g is a bounded and continuous function on $Q_{3\sqrt{2}\rho}(s_0, y_0)$. Then, there exists constants ϵ_0 and C depending only on λ and ν , such that whenever $\inf_{Q_{\frac{\rho}{8}}(s_0, y_0)} u \leq 1$ and*

$$\rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(s_0, y_0), d\mu)} \leq \epsilon_0,$$

then $\sup_{Q_{\frac{\rho}{8}}(s_0, y_0)} u \leq C$.

Let us begin the proof of Lemma 2.7 by showing the following Corollary of Theorem 2.3 and Lemma 2.5. In the sequel we will denote by $|\mathcal{A}|_\mu$ the *normalized* measure of a set $\mathcal{A}$ with respect to $d\mu = s^{\nu-1} ds dy$, namely

$$|\mathcal{A}|_\mu = \frac{\gamma^\nu}{2} \int_{\mathcal{A}} s^{\nu-1} ds dy.$$

For future reference, let us notice that the measure $|Q_\rho(s_0, y_0)|_\mu$ of the cube $Q_\rho(s_0, y_0)$ is equal to

$$(2.26) \quad |Q_\rho(s_0, y_0)|_\mu = \frac{\gamma^\nu}{2} \int_{y_0 - \frac{\rho}{\gamma}}^{y_0 + \frac{\rho}{\gamma}} \int_{\bar{s}}^{s_0 + \rho} s^{\nu-1} ds dy = [(s_0 + \rho)^\nu - \bar{s}_0^\nu] \rho$$

with $\bar{s}_0 = \max(s_0 - \rho, 0)$.

Lemma 2.8. *Let u be a classical supersolution of equation $L_s u \leq g$ in $Q_{3\sqrt{2}\rho}(s_0, y_0)$. Then, there exist constants $\epsilon_0 > 0$, $0 < k < 1$ and $K > 1$ so that if $u \geq 0$ in $Q_{3\sqrt{2}\rho}(s_0, y_0)$ with $\inf_{Q_{\frac{3\rho}{2}}(s_0, y_0)} u \leq 1$ and*

$$(2.27) \quad \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(s_0, y_0), d\mu)} \leq \epsilon_0,$$

then

$$(2.28) \quad |\{u \leq K\} \cap Q_\rho(s_0, y_0)|_\mu \geq k |Q_\rho(s_0, y_0)|_\mu.$$

Proof. To simplify the notation, we will denote for any $r > 0$, $Q_r = Q_r(s_0, y_0)$ and $\mathcal{B}_r = \mathcal{B}_r(s_0, y_0)$, where

$$\mathcal{B}_r(s_0, y_0) = \{(s, y) : \bar{d}_\gamma((s, y), (s_0, y_0)) \leq r\}.$$

Set $w = u + \phi_\rho$, where ϕ_ρ is the barrier function of Lemma 2.5, expressed in the (s, y) variables. Then,

$$L_s w \leq g + \frac{C}{\rho^2} \xi_\rho \quad \text{on } \mathcal{B}_{3\sqrt{2}\rho}$$

In addition, $w \geq 0$ on $\partial\mathcal{B}_{3\sqrt{2}\rho}$, since $u \geq 0$ on $Q_{3\sqrt{2}\rho}$ and $\phi_\rho \geq 0$ on $\mathbb{R}^2 \setminus \mathcal{B}_{3\sqrt{2}\rho}$. Also, $\inf_{Q_{\frac{3\rho}{2}}} w \leq -1$, since $\inf_{Q_{\frac{3\rho}{2}}} u \leq 1$ and $w \leq -2$ on $Q_{\frac{3\rho}{2}}$. Hence, $\inf_{\mathcal{B}_{3\sqrt{2}\rho}} w \leq -1$. We therefore can apply the ABP estimate, Theorem 2.3, to conclude that

$$1 \leq \inf_{\mathcal{B}_{3\sqrt{2}\rho}} w^- \leq C(\lambda, \nu) \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \left(\int_{\Gamma^-} (g + C\xi)^2(s, y) s^{\nu-1} ds dy \right)^{1/2}$$

with $\rho_\nu(s_0)$ given by (2.7) and

$$\Gamma^- = \left\{ (s, y) \in \mathcal{B}_{3\sqrt{2}\rho} : \frac{\partial(u_z, u_y)}{\partial(z, y)} \geq 0, u_z \geq 0 \right\}, \quad z = \frac{s^{2-\nu}}{2-\nu}.$$

Using that $0 \leq \xi_\rho \leq 1$ and $\text{supp}\xi \subset Q_{\frac{2}{\rho}}$, we conclude the estimate

$$1 \leq C \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(x_0, y_0), d\mu)} + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \frac{C}{\rho^2} |\Gamma^- \cap Q_{\frac{\rho}{2}}|_\mu^{\frac{1}{2}}.$$

Choosing ϵ_0 sufficiently small so that $C \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(x_0, y_0), d\mu)} \leq \frac{1}{2}$, the previous estimate implies the lower bound

$$\frac{1}{2} \leq C \rho^{-\frac{3}{2}} \rho_\nu(s_0)^{\frac{1}{2}} |\Gamma^- \cap Q_{\frac{\rho}{2}}|_\mu^{\frac{1}{2}}.$$

Observing also that $w \leq 0$ on Γ^- so that $u(x) \leq -\phi(x) \leq K$, we finally conclude the estimate

$$c \frac{\rho^3}{\rho_\nu(s_0)} \leq |\{u \leq K\} \cap Q_{\frac{\rho}{2}}|_\mu.$$

Since $|\{u \leq K\} \cap Q_\rho|_\mu \geq |\{u \leq K\} \cap Q_{\frac{\rho}{2}}|_\mu$, to finish the proof of (2.28), it is enough to show that for ρ sufficiently small

$$(2.29) \quad |Q_\rho(s_0, y_0)|_\mu \leq \frac{C \rho^3}{\rho_\nu(s_0)}.$$

Indeed, using (2.26) we have

$$\delta(\rho) := \frac{\rho_\nu(s_0)}{\rho^3} \cdot |Q_\rho(s_0, y_0)| = \frac{2[(s_0 + \rho)^{2-\nu} - s_0^{2-\nu}] \cdot [(s_0 + \rho)^\nu - \bar{s}_0^\nu]}{\rho^2}.$$

When $s_0 \leq 2\rho$, then

$$\delta(\rho) \leq \frac{2(3\rho)^\nu \cdot (3\rho)^{2-\nu}}{\rho^2} \leq C(\nu).$$

On the other hand, when $s_0 > 2\rho$, then $s_0 - \rho \geq \rho \geq s_0/2$, implying that

$$\delta(\rho) \leq \frac{2[(s_0 + \rho)^{2-\nu} - s_0^{2-\nu}] \cdot [(s_0 + \rho)^\nu - (s_0 - \rho)^\nu]}{\rho^2} \leq C(\nu) s_0^{1-\nu} s_0^{\nu-1} \leq C(\nu)$$

proving (2.29), therefore finishing the proof of the Lemma.

Before we proceed with the continuation of the proof of Lemma 2.7, we will state the following Corollary of the well known Calderón-Zygmund decomposition. Starting with the cube $Q_\rho(s_0, y_0)$, we split it into four cubes of half size and we split each one of these four cubes into four other cubes of half the size. Iterating this process we obtain cubes called dyadic cubes. If Q is a dyadic cube different than $Q_\rho(s_0, y_0)$, we say that $\tilde{Q}$ is the predecessor of Q , if Q is one of the four cubes obtained from dividing $\tilde{Q}$. Recalling that $|A|_\mu = \frac{\gamma\nu}{2} \int_A s^{\nu-1} ds dy$, we have the following Lemma:

Lemma 2.9. *Let $A \subset B \subset Q_\rho(s_0, y_0)$ be measurable sets and $0 < \delta < 1$ such that*

- (a) $|A|_\mu \leq \delta |Q_\rho(s_0, y_0)|$, and
- (b) *If Q is a dyadic cube such that $|A \cap Q|_\mu > \delta |Q|_\mu$, then $\tilde{Q} \subset B$.*

Then, $|A|_\mu \leq \delta |B|_\mu$.

Proof. The proof of this Lemma is very similar to the standard case (see in [CC], Lemma 4.2). We use the Calderón-Zygmund technique, following the lines of the proof of lemma 4.2 in [CC]. By assumption we have that

$$\frac{|A \cap Q_\rho(s_0, y_0)|_\mu}{|Q_\rho(s_0, y_0)|_\mu} = \frac{|A|_\mu}{|Q_\rho(s_0, y_0)|_\mu} \leq \delta.$$

We subdivide Q_ρ into four dyadic cubes. If one of these cubes, Q , satisfies $|A \cap Q|_\mu / |Q|_\mu \leq \delta$, we then split Q into four dyadic cubes and we iterate this process.

In this way we find a family of dyadic cubes, $Q^1, Q^2, \dots$ (different from $Q_\rho(s_0, y_0)$) satisfying

$$\frac{|A \cap Q^i|_\mu}{|Q^i|_\mu} > \delta, \quad \forall i$$

and such that if $x \notin \cup Q^i$, then x belongs to a infinite number of closed dyadic cubes Q with diameters tending to zero and $|A \cap Q|_\mu / |Q|_\mu \leq \delta < 1$. Applying the Lebesgue differentiation theorem to χ_A with respect to the measure $d\mu$, and using that $d\mu$ is absolutely continuous with respect to the Lebesgue measure, we deduce that $\chi_A \leq \delta < 1$ for a.e. $x \notin \cup Q^i$. Hence, $A \subset \cup Q^i$ except of a set of measure zero.

Consider the family of predecessors of the cubes Q^i , and relabel them so that $\{\tilde{Q}_i\}_{i \geq 1}$ are pairwise disjoint. Then, $A \subset \cup \tilde{Q}_i$ and from the way we chose the cubes Q^i , we have

$$\frac{|A \cap \tilde{Q}_i|_\mu}{|\tilde{Q}_i|_\mu} \leq \delta, \quad \forall i.$$

Since $|A \cap Q^i|_\mu / |Q^i|_\mu > \delta$ and (b) holds, we have that $\tilde{Q}_i \subset B$, for every $i \geq 1$. Hence

$$A \subset \bigcup_{i \geq 1} \tilde{Q}_i \subset B.$$

We conclude that

$$|A|_\mu \leq \sum_{i \geq 1} |A \cap \tilde{Q}_i|_\mu \leq \delta \sum_{i \geq 1} |\tilde{Q}_i|_\mu = \delta |\cup \tilde{Q}_i|_\mu \leq \delta |B|_\mu,$$

finishing the proof of the Lemma.

Lemma 2.10. *There exist universal constants $\epsilon_0 > 0$, $0 < k < 1$ and $K > 1$ so that if $u \geq 0$ is a supersolution of equation $L_s u \leq g$ in $Q_{3\sqrt{2}\rho}(s_0, y_0)$ with $\inf_{Q_{\frac{3\rho}{2}}(s_0, y_0)} u \leq 1$ and g satisfies (2.27), then*

$$(2.30) \quad |\{u \geq K^j\} \cap Q_\rho(s_0, y_0)|_\mu \leq (1 - k)^j |Q_\rho(s_0, y_0)|_\mu$$

for $j = 1, 2, 3, \dots$

As a consequence, we have that

$$(2.31) \quad |\{u \geq t\} \cap Q_\rho(s_0, y_0)|_\mu \leq d t^{-\epsilon} |Q_\rho(s_0, y_0)|_\mu, \quad \forall t > 0$$

where d and ϵ are positive universal constants.

Proof. To simplify the notation, let us denote for any $r > 0$ by $Q_r = Q_\rho(s_0, y_0)$. We will proceed by induction. For $j = 1$, (2.30) follows from (2.28). Suppose that (2.30) holds for $j - 1$ and set

$$A = |\{u \geq K^j\} \cap Q_\rho|_\mu \quad \text{and} \quad B = |\{u \geq K^{j-1}\} \cap Q_\rho|_\mu.$$

We will apply Lemma 2.9. Clearly $A \subset B \subset Q_\rho$ and

$$|A|_\mu \leq |\{u > K\} \cap Q_\rho|_\mu \leq (1-k)|Q_\rho|_\mu$$

by Lemma 2.8. It remains to prove condition (b) in Lemma 2.9, that is we need to show that if $Q = Q_{\frac{\rho}{2^i}}(\bar{s}, \bar{y})$ is a dyadic cube such that

$$(2.32) \quad |A \cap Q|_\mu > (1-k)|Q|_\mu$$

then $\tilde{Q} \subset B$. Assume the opposite, namely that there exists a point P such that

$$(2.33) \quad P \in \tilde{Q} \quad \text{and} \quad u(P) < K^{j-1}.$$

Consider the function

$$\tilde{u} = \frac{u}{K^{j-1}}.$$

Then $\tilde{u}$ satisfies

$$L\tilde{u} \leq \tilde{g}, \quad \text{on } Q_{3\sqrt{2}l}(\bar{s}, \bar{y})$$

with $\tilde{g} = g/K^{j-1}$ and $l = 1/2^i$. Also, notice that since $P \in \tilde{Q} \subset Q_{\frac{3l}{2}}(\bar{s}, \bar{y})$, we have

$$\inf_{Q_{\frac{3l}{2}}(\bar{s}, \bar{y})} \tilde{u} \leq \frac{u(P)}{K^{j-1}} \leq 2.$$

It is easy to check that $\tilde{u}$ satisfies all the other hypotheses of lemma 2.8, implying that

$$|\{\tilde{u} \leq K\} \cap Q|_\mu \geq k|Q|_\mu$$

or equivalently

$$|\{u \leq K^j\} \cap Q|_\mu \geq k|Q|_\mu.$$

Hence

$$|Q \cap A|_\mu = |\{u > K^j\} \cap Q|_\mu \leq (1-k)|Q|_\mu$$

contradicting (2.32). This finishes the proof of (2.30). The proof of (2.31) follows immediately from (2.30) taking $d = (1-k)^{-1}$ and ϵ such that $1-k = K^{-\epsilon}$.

Lemma 2.11. *Let u be a classical subsolution of equation $L_s u \geq g$ in $Q_{3\sqrt{2}\rho}(s_0, y_0)$. Assume that g satisfies (2.27) and u satisfies (2.31). Then, there exist constants $K_0 > 1$ and $\sigma > 1$ such that for ϵ as in (2.31) and $\theta = K_0/(K_0 - 1) > 1$, the following holds: if $i \geq 1$ is an integer and $P = (s_1, y_1)$ is a point such that*

$$(2.34) \quad P \in Q_{\frac{\rho}{4}}(s_0, y_0)$$

and

$$(2.35) \quad u(P) \geq \theta^{i-1} K_0,$$

then

$$Q^i := Q_{l_i \rho}(P) \subset Q_\rho(s_0, y_0) \quad \text{and} \quad \sup_{Q^i} u \geq \theta^i K_0$$

where $l_i = \sigma K_0^{-\epsilon/2} \theta^{-\epsilon i/2}$.

Proof. We follow the lines of the proof of Lemma 4.7 in [CC]. Take $\sigma > 0$ and $K_0 > 0$ such that

$$(2.36) \quad (i) \quad \frac{1}{2} \sigma^2 > \frac{12^2 d 2^\epsilon}{\nu} \quad \text{and} \quad (ii) \quad \sigma K_0^{-\epsilon/2} + d K_0^{-\epsilon} \leq \frac{1}{6}$$

with d and ϵ as in (2.27). Assuming that $\sup_{Q^i} u < \theta^j K_0$, we will derive a contradiction. By (2.34) and (2.36) (ii), we have

$$Q_{l_i \rho/(3\sqrt{2})}(P) \subset Q_{l_i \rho}(P) \subset Q_\rho(s_0, y_0).$$

Hence (2.31) implies

$$(2.37) \quad \begin{aligned} |\{u \geq \theta^i \frac{K_0}{2}\} \cap Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu &\leq \\ &\leq |\{u \geq \theta^i \frac{K_0}{2}\} \cap Q_\rho(s_0, y_0)|_\mu \leq d \theta^{-i\epsilon} \left(\frac{K_0}{2}\right)^{-\epsilon} |Q_\rho(s_0, y_0)|. \end{aligned}$$

Consider now the function

$$v = [\theta K_0 - \frac{u}{\theta^i - 1}] / [(\theta - 1) K_0].$$

We claim that v satisfies the assumptions of Lemma 2.10 on $Q_{l_i \rho/(3\sqrt{2})}(P)$. Hence, by (2.31) we conclude that

$$|\{v \geq K_0\} \cap Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu \leq d K_0^{-\epsilon} |Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu.$$

Since $u \leq \theta^j K_0/2$ if and only if $v \geq K_0$, we conclude that

$$(2.38) \quad |\{u \leq \theta^j \frac{K_0}{2}\} \cap Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu \leq d K_0^{-\epsilon} |Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu.$$

Combining (2.37) and (2.38) we obtain

$$(2.39) \quad |Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu \leq d \theta^{-i\epsilon} \left(\frac{K_0}{2}\right)^{-\epsilon} |Q_\rho(s_0, y_0)|_\mu + d K_0^{-\epsilon} |Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu.$$

To estimate the ratio

$$R = \frac{|Q_\rho(s_0, y_0)|_\mu}{|Q_{l_i \rho/(3\sqrt{2})}(P)|_\mu}$$

from above, we apply formula (2.26) to show the estimate

$$|Q_\rho(s_0, y_0)|_\mu \leq [(s_0 + \rho)^\nu - (s_0 - \rho)^\nu] \rho$$

and

$$|Q_{l_i\rho/(3\sqrt{2})}(P)|_\mu \geq [(s_1 + \frac{l_i\rho}{3\sqrt{2}})^\nu - s_1^\nu] \frac{l_i\rho}{3\sqrt{2}} \geq \nu (s_1 + \frac{l_i\rho}{3\sqrt{2}})^{\nu-1} \left(\frac{l_i\rho}{3\sqrt{2}} \right)^2$$

when $P = (s_1, y_1)$. Combining the above we find that

$$(2.40) \quad R \leq \frac{1}{\nu\rho} [(s_0 + \rho)^\nu - (s_0 - \rho)^\nu] (s_1 + \frac{l_i\rho}{3\sqrt{2}})^{1-\nu} \left(\frac{3\sqrt{2}}{l_i} \right)^2.$$

When $s_0 \leq 2\rho$, then $(s_0 + \rho)^\nu - (s_0 - \rho)^\nu \leq (3\rho)^\nu$ and $s_1 \leq 9\rho/4$ (since $P \in Q_{\frac{\rho}{4}}(s_0, y_0)$) in (2.40). Hence

$$R \leq \frac{3^\nu}{\nu} \left(\frac{9}{4} + \frac{l_i}{3\sqrt{2}} \right)^{1-\nu} \left(\frac{3\sqrt{2}}{l_i} \right)^2$$

Using the bound $\frac{l_i}{3\sqrt{2}} \leq 3/4$ we conclude that

$$R \leq \frac{3}{\nu} \left(\frac{3\sqrt{2}}{l_i} \right)^2, \quad \text{if } s_0 \leq \rho.$$

On the other hand, when $s_0 \geq 2\rho$, the estimates

$$(s_0 + \rho)^\nu - (s_0 - \rho)^\nu \leq \nu\rho(s_0 - \rho)^{\nu-1} \leq \nu\rho \left(\frac{s_0}{2} \right)^{\nu-1}$$

and

$$s_1 + \frac{l_i\rho}{3\sqrt{2}} \leq s_0 + \frac{\rho}{4} + \frac{l_i\rho}{3\sqrt{2}} \leq s_0 + \rho \leq 2s_0$$

in (2.40), imply

$$R \leq 4^{1-\nu} \left(\frac{3\sqrt{2}}{l_i} \right)^2, \quad \text{if } s_0 \geq \rho.$$

Combining both cases, and using that $\nu < 1$ we finally obtain the bound

$$R = \frac{|Q_\rho(s_0, y_0)|_\mu}{|Q_{l_i\rho/(3\sqrt{2})}(P)|_\mu} \leq \frac{4}{\nu} \left(\frac{6}{l_i} \right)^2 = \frac{1}{\nu} \left(\frac{12}{l_i} \right)^2$$

which in combination with (2.39) gives

$$\frac{l_i^2}{12^2} \leq \frac{d}{\nu} \theta^{-i\epsilon} \left(\frac{K_0}{2} \right)^{-\epsilon} + d K_0^{-\epsilon} \frac{l_i^2}{12^2}$$

Using (2.36)(ii) we conclude

$$\frac{1}{2} \frac{l_i^2}{12^2} \leq \frac{d}{\nu} \theta^{-i\epsilon} \left(\frac{K_0}{2} \right)^{-\epsilon}.$$

The definition of l_i in the above estimate gives

$$\frac{\sigma^2}{2} \leq \frac{12^2 d 2^\epsilon}{\nu}$$

a contradiction to (2.36)(i).

It remains to verify that v satisfies the assumptions of Lemma 2.10 on $Q_{\tilde{\rho}}(P)$, with $\tilde{\rho} = l_i \rho / (3\sqrt{2})$. Clearly, the function v satisfies the equation $Lv \leq \tilde{g}$ on $Q_{\tilde{\rho}}(P)$, with

$$\tilde{g} = -\frac{g}{\theta^{i-1}(\theta-1)K_0}.$$

In addition $v > 0$ on $Q_{\tilde{\rho}}(P)$, since $\sup_{Q_{l_i \rho}(P)} < \theta^i K_0$, by assumption. Also, (2.35) implies that $\inf_{Q_{\tilde{\rho}}(P)} \leq 1$. It remains to verify that

$$\tilde{\rho}^{\frac{1}{2}} \tilde{\rho}_\nu(s_1)^{\frac{1}{2}} \|\tilde{g}\|_{L^2(Q_{\tilde{\rho}}(P), d\mu)} \leq \epsilon_0$$

with $\tilde{\rho}_\nu(s_1) = (s_1 + \tilde{\rho})^{2-\nu} - s_1^{2-\nu}$. Since

$$\|\tilde{g}\|_{L^2(Q_{\tilde{\rho}}(P), d\mu)} = \frac{1}{\theta^{i-1}(\theta-1)K_0} \|g\|_{L^2(Q_{\tilde{\rho}}(P), d\mu)},$$

$Q_{\tilde{\rho}}(P) \subset Q_\rho(s_0, y_0)$ and g satisfies (2.27), it is enough to show that

$$\frac{\tilde{\rho}^{\frac{1}{2}} \tilde{\rho}_\nu(s_1)^{\frac{1}{2}}}{\theta^{i-1}(\theta-1)K_0} \leq \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}}.$$

Let us first estimate from above the ratio

$$\eta = \frac{\tilde{\rho}_\nu(s_1)}{\rho_\nu(s_0)} = \frac{(s_1 + \tilde{\rho})^{2-\nu} - s_1^{2-\nu}}{(s_0 + \rho)^{2-\nu} - s_0^{2-\nu}}.$$

When $s_0 \leq \rho/2$, then $s_1 \leq 3\rho/4$. Using also that $\tilde{\rho} = l_i \rho / (3\sqrt{2}) \leq 3\rho/4$, we obtain

$$\eta \leq \frac{(s_1 + \tilde{\rho})^{1-\nu} \tilde{\rho}}{\rho^{2-\nu} - (\frac{\rho}{2})^{2-\nu}} \leq \frac{(2\rho)^{1-\nu} \frac{l_i \rho}{3\sqrt{2}}}{(2^{2-\nu} - 1)(\frac{\rho}{2})^{2-\nu}} \leq \frac{8l_i}{3\sqrt{2}}.$$

When $s_0 \geq \rho/2$, then $s_1 + \tilde{\rho} \leq s_0 + \rho \leq 3s_0$ implying the estimate

$$\eta \leq \frac{(3s_0)^{1-\nu} \tilde{\rho}}{s_0^{1-\nu} \rho} \leq \frac{l_i}{\sqrt{2}} \leq \frac{8l_i}{3\sqrt{2}}.$$

In both cases $\eta \leq 8l_i / (3\sqrt{2}) \leq 3l_i$. Therefore, if

$$\zeta = \frac{3l_i}{\theta^{i-1}(\theta-1)K_0} \leq 1$$

the desired estimate holds. To show the last inequality, let us use that $\theta > 1$, $\theta = 2(\theta-1)K_0$ and $l_i = \sigma K_0^{-\epsilon/2} \theta^{-\epsilon i/2}$ to find that

$$\zeta = \frac{6\sigma K_0^{-\epsilon/2} \theta^{-\epsilon i/2}}{\theta^i} \leq 6\sigma K_0^{-\epsilon/2}.$$

Hence, $\zeta \leq 1$, by (2.36)(ii), therefore finishing the proof of the Lemma.

We are now in position to give the proof of Lemma 2.7.

Proof of Lemma 2.7. By the assumptions of Lemma 2.7, and using Lemmas 2.8 and 2.10, one can easily show that u satisfies the hypotheses of Lemma 2.12.

Since $l_i = \sigma K_0^{-\epsilon/2} \theta^{-\epsilon i/2}$, with $K_0 > 1$ and $\theta > 1$, there exists a large integer i_0 , depending only on universal constants, such that

$$(2.41) \quad \sum_{i \geq i_0} l_i \leq \frac{1}{8}.$$

We claim that

$$\sup_{Q_{\frac{\rho}{8}}(s_0, y_0)} u \leq \theta^{i_0-1} K_0$$

therefore finishing the proof of the lemma. To show this claim, we proceed by contradiction. If the claim is not true, then there exists a point P_{i_0} with

$$P_{i_0} \in Q_{\frac{\rho}{8}}(s_0, y_0) \quad \text{and} \quad u(P_{i_0}) \geq \theta^{i_0-1} K_0.$$

In particular $P_{i_0} \in Q_{\frac{\rho}{4}}(s_0, y_0)$. Hence, by lemma 2.11, there exists a point P_{i_0+1} such that

$$P_{i_0+1} \in Q_{l_{i_0}\rho}(P_{i_0}) \quad \text{and} \quad u(P_{i_0+1}) \geq \theta^{i_0} K_0.$$

We can repeat this process, to obtain a sequence of points P_i , $i \geq i_0$, such that

$$P_{i+1} \in Q_{l_i\rho}(P_i) \quad \text{and} \quad u(P_{i+1}) \geq \theta^i K_0 \quad \forall i \geq i_0$$

if we can actually show that each such point P_i satisfies

$$P_i \in Q_{\frac{\rho}{4}}(s_0, y_0).$$

To this end, let $P_i = (s_i, y_i)$. Then, by (2.41) we have

$$|s_i - s_0| \leq |s_{i_0} - s_0| + \sum_{k=i_0}^{i-1} |s_{k+1} - s_k| \leq \frac{\rho}{8} + \sum_{k \geq i_0} l_k \rho \leq \frac{\rho}{4}$$

and also

$$\gamma |y_i - y_0| \leq \gamma |y_{i_0} - y_0| + \sum_{k=i_0}^{i-1} \gamma |y_{k+1} - y_k| \leq \frac{\rho}{8} + \sum_{k \geq i_0} l_k \rho \leq \frac{\rho}{4}$$

implying that $P_i \in Q_{\frac{\rho}{4}}(s_0, y_0)$, therefore finishing the proof of Lemma 2.7.

Proof of Theorem 2.6. Let $(\bar{s}, \bar{y})$ be a point in $Q_{\frac{\rho}{2}}(s_0, y_0)$ and set $\tilde{\rho} = \rho/100$ so that $Q_{3\sqrt{2}\tilde{\rho}}(\bar{s}, \bar{y}) \subset Q_{\rho}(s_0, y_0)$. One can easily check that, for any $\delta > 0$, the function

$$u_{\delta} = \left(\inf_{Q_{\frac{\rho}{8}}(\bar{s}, \bar{y})} u + \delta + \epsilon_0^{-1} \tilde{\rho}^{\frac{1}{2}} \tilde{\rho}_{\nu}(\bar{s})^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\tilde{\rho}}(\bar{s}, \bar{y}), d\mu)} \right)^{-1}$$

satisfies the hypotheses of Lemma 2.7 on $Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y})$. Hence by Lemma 2.7 we conclude that $\sup_{Q_{\frac{\rho}{8}}(\bar{s}, \bar{y})} u_\delta \leq C$, implying, after letting $\delta \rightarrow 0$, that

$$(2.42) \quad \sup_{Q_{\frac{\rho}{8}}(\bar{s}, \bar{y})} u \leq C \left(\inf_{Q_{\frac{\rho}{8}}(\bar{s}, \bar{y})} u + \tilde{\rho}^{\frac{1}{2}} \tilde{\rho}(\bar{s})^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y}), d\mu)} \right)$$

for a universal constant C . One can easily show, using the same arguments as in the proof of Lemma 2.11, that

$$\eta = \frac{\tilde{\rho}_\nu(\bar{s})}{\rho_\nu(s_0)} \leq \eta_0$$

for some universal constant η_0 . Hence, (2.25) follows from (2.42) via a standard covering argument.

We finish this section with two important Theorems (see also [GT] and [CC]). The first Theorem is a weak Harnack estimate for nonnegative supersolutions u of equation $L_s u \leq g$.

Theorem 2.12. *Let $u \geq 0$ be a supersolution of equation $L_s u \leq g$ in $Q_\rho(s_0, y_0)$, where g is a bounded and continuous function on $Q_\rho(s_0, y_0)$. Then, there exist universal constants $p_0 > 0$ and C such that*

$$(2.43) \quad \left(\int_{Q_{\frac{\rho}{4}}(s_0, y_0)} u^{p_0} d\mu \right)^{\frac{1}{p_0}} \leq C \left\{ \inf_{Q_{\frac{\rho}{2}}(s_0, y_0)} u + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right\}$$

with $d\mu = s^{\nu-1} ds dy$ and $\rho_\nu(s_0)$ given by (2.7).

Proof. Let $u \geq 0$ be a supersolution of equation $L_s u \leq g$ in $Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y})$ such that $\inf_{Q_{\frac{3\rho}{2}}(\bar{s}, \bar{y})} u \leq 1$ and $\rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y}), d\mu)} \leq \epsilon_0$, with ϵ_0 as in Lemma 2.8. Then, by Lemma 2.10, we have

$$|\{u \geq t\} \cap Q_\rho(\bar{s}, \bar{y})|_\mu \leq dt^{-\epsilon} |Q_\rho(\bar{s}, \bar{y})|_\mu$$

. As a consequence, for $p_0 = \frac{\epsilon}{2}$, we obtain

$$(2.44) \quad \begin{aligned} \int_{Q_\rho(\bar{s}, \bar{y})} u^{p_0} d\mu &= p_0 \int_0^\infty t^{p_0-1} |\{u \geq t\} \cap Q_\rho(\bar{s}, \bar{y})|_\mu dt \\ &\leq p_0 \left(\int_0^1 t^{p_0-1} dt + \int_1^\infty t^{p_0-1} t^{-\epsilon} dt \right) |Q_\rho(\bar{s}, \bar{y})|_\mu = C(\epsilon) |Q_\rho(\bar{s}, \bar{y})|_\mu. \end{aligned}$$

Let $(\bar{s}, \bar{y}) \in Q_{\frac{\rho}{4}}(s_0, y_0)$ and $\tilde{\rho} = \frac{\rho}{100}$ sufficiently small so that $Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y}) \subset Q_{\rho}(s_0, y_0)$. Set

$$u_{\delta} = u \left(\delta + \inf_{Q_{\frac{3\tilde{\rho}}{2}}(\bar{s}, \bar{y})} u + \epsilon_0^{-1} \tilde{\rho}^{\frac{1}{2}} \rho_{\nu}(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\tilde{\rho}}(\bar{s}, \bar{y}), d\mu)} \right)$$

so that u_{δ} satisfies all the assumptions of Lemma 2.10 on $Q_{3\sqrt{2}\tilde{\rho}}(\bar{s}, \bar{y})$. Hence

$$\left(\int_{Q_{\rho}(\bar{s}, \bar{y})} u_{\delta}^{p_0} d\mu \right)^{\frac{1}{p_0}} \leq C |Q_{\rho}(\bar{s}, \bar{y})|_{\mu}^{\frac{1}{p_0}}.$$

The desired inequality (2.43) now follows via a standard covering argument.

The last Theorem in this section is a local maximum principle for subsolutions u of equation $L_s u \geq g$.

Theorem 2.13. *Let u be a subsolution of equation $L_s u \geq g$ in $Q_{\rho}(s_0, y_0)$, where g is a bounded and continuous function on Q_{ρ} . Then, for any $p > 0$, we have*

(2.45)

$$\sup_{Q_{\frac{\rho}{2}}(s_0, y_0)} u \leq C(p) \left\{ \left(\int_{Q_{\frac{3\rho}{4}}(s_0, y_0)} u^p d\mu \right)^{\frac{1}{p}} + \rho^{\frac{1}{2}} \rho_{\nu}(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{\rho}(s_0, y_0), d\mu)} \right\}$$

with $d\mu = s^{\nu-1} ds dy$, $\rho_{\nu}(s_0)$ given by (2.7), and $C(p)$ a constant depending only on λ, ν and p .

Proof. Let u be subsolution of equation $L_s u \geq g$ in $Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y})$, where $\rho^{\frac{1}{2}} \rho_{\nu}(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{3\sqrt{2}\rho}(\bar{s}, \bar{y}), d\mu)} \leq \epsilon_0$, with ϵ_0 as in Lemma 2.8. If, in addition, $u^+ \in L^{\epsilon}(Q_{\rho}(\bar{s}, \bar{y}))$ with

$$\|u^+\|_{L^{\epsilon}(Q_{\rho}(\bar{s}, \bar{y}), d\mu)} \leq d^{\frac{1}{\epsilon}} |Q_{\rho}(\bar{s}, \bar{y})|_{\mu}^{\frac{1}{\epsilon}}$$

then

$$|\{u \geq t\} \cap Q_{\rho}(\bar{s}, \bar{y})|_{\mu} \leq t^{-\epsilon} \int_{Q_{\rho}(\bar{s}, \bar{y})} (u^+)^{\epsilon} d\mu \leq d t^{-\epsilon} |Q_{\rho}(\bar{s}, \bar{y})|_{\mu}$$

. It follows that (2.31) holds for u and hence the proof of Lemma 2.7, which only uses (2.31), implies that

$$\sup_{Q_{\frac{\tilde{\rho}}{8}}(\bar{s}, \bar{y})} u \leq C.$$

Rescaling, as in Theorem 2.12 we obtain (2.45) with $p = \epsilon$. To obtain (2.45) for all $p > 0$ we use interpolation.

2.4. Hölder Continuity. In this section we will present the proof of Theorem 2.2. First, under the same notation as in the previous section, we will show the following continuity result:

Lemma 2.14. *Let u be a classical solution of equation $L_s u = g$ in $Q_\rho(s_0, y_0)$, where g is a bounded and continuous function. Then, for a universal constant $\theta < 1$, and a universal constant C , we have*

$$(2.46) \quad \text{osc}_{Q_{\frac{\rho}{2}}(s_0, y_0)} u \leq \theta \text{osc}_{Q_\rho(s_0, y_0)} u + C \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)}.$$

Proof. For any $r > 0$, set $m_r := \inf_{Q_r(s_0, y_0)} u$, $M_r := \sup_{Q_r(s_0, y_0)} u$ and $\omega_r := \text{osc}_{Q_r(s_0, y_0)} u$. Applying the Harnack inequality (2.25) to the nonnegative functions $u - m_\rho$ and $M_\rho - u$ on $Q_\rho(s_0, y_0)$ we obtain

$$M_{\frac{\rho}{2}} - m_\rho \leq C \left(m_{\frac{\rho}{2}} - m_\rho + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right)$$

and

$$M_{\frac{\rho}{2}} - m_{\frac{\rho}{2}} \leq C \left(M_\rho - M_{\frac{\rho}{2}} + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right).$$

Adding both inequalities we get

$$\omega_{\frac{\rho}{2}} + \omega_\rho \leq C \left(\omega_\rho - \omega_{\frac{\rho}{2}} + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right)$$

which implies that

$$\omega_{\frac{\rho}{2}} \leq \frac{C-1}{C+1} \omega_\rho + \frac{2C}{C+1} \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)}.$$

We are now in position to prove our Hölder continuity result. Theorem 2.1 is a direct consequence of the next Theorem.

Theorem 2.15. *Let u be a classical solution of equation (2.24) in $Q_{\rho_0}(s_0, y_0)$, where g is a bounded and continuous function. Then, there exist positive constants C and $\alpha < \frac{1}{2}$, depending only on λ and ν , such that*

$$(2.47) \quad \text{osc}_{Q_\rho(s_0, y_0)} u \leq C \rho^\alpha \left(\rho_0^{-\alpha} \sup_{Q_\rho(s_0, y_0)} |u| + \rho_0^{\frac{1}{2}-\alpha} (s_0 + \rho_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho(s_0, y_0), d\mu)} \right).$$

Proof. Set $\omega(\rho) = \text{osc}_{Q_\rho(s_0, y_0)} u$. By Lemma 2.14 we have

$$\omega(\rho/2) \leq \theta \omega(\rho) + k(\rho)$$

with $\theta < 1$ an absolute constant and

$$k(\rho) = \rho^{\frac{1}{2}} (s_0 + \rho_0)^{\frac{1}{2}} \|g\|_{L^2(Q_{\rho_0}(s_0, y_0), d\mu)}.$$

Both functions ω and k are non-decreasing. Hence, (2.47) follows by Lemma 8.23 in [GT].

3. THE PARABOLIC CASE

We will now extend the results of the previous section to the parabolic case. We will consider degenerate equations of the form

$$Lu - u_t = g$$

where L is the operator defined given by (1.1) and satisfying conditions (1.9) - (1.11).

Denoting, for any number $\rho > 0$ and any point (x_0, y_0, t_0) , $x_0 \geq 0$, by $\mathcal{C}_\rho = \mathcal{C}_\rho(x_0, y_0, t_0)$ the parabolic cube

$$\mathcal{C}_\rho = \{(x, y, t) : x \geq 0, |x - x_0| \leq \rho^2, |y - y_0| \leq \rho, t_0 - \rho^2 \leq t \leq t_0\}$$

and by μ the measure $d\mu = x^{\frac{\nu}{2}-1} dx dy$, we will show the following analogue of Theorem 2.1.

Theorem 3.1. *Assume that the coefficients of the operator L are smooth on $\mathcal{C}_\rho$, $\rho > 0$, and satisfy the bounds (1.9) - (1.11). Then, there exist a number $0 < \alpha < 1$ so that, for any $r < \rho$*

$$\|u\|_{C_s^\alpha(\mathcal{C}_r)} \leq C(r, \rho) \left(\|u\|_{C^\circ(\mathcal{C}_1)} + \left(\int_{\mathcal{C}_\rho} g^3(x, t) d\mu dt \right)^{1/3} \right)$$

for all smooth functions u on $\mathcal{C}_\rho$ for which $Lu - u_t = g$.

The proof of Theorem 3.1 follows the lines of the proof of the corresponding elliptic result, Theorem 2.1. We will only present the proof of Alexandroff-Bakelman-Pucci estimate, Theorems 3.2 and 3.3, and the proof of the existence of the barrier function, Lemma 3.4, which differs from the elliptic case. The rest of the results follow from the elliptic analogies in a standard manner, as in [W1], [W2].

3.1. Alexandrov-Bakelman-Pucci Estimate. In this section we will show the parabolic version of the Alexandrov-Bakelman-Pucci Estimate, following the lines of the proof elliptic result, Theorem 2.2. The proof of the ABP estimate in the strictly parabolic case was given by Tso in [T]. As in paragraph 2.1, because of the degeneracy of the equation, we introduce the new variable $z = \frac{s^{2-\nu}}{2-\nu}$, so that $\frac{dz}{ds} = s^{1-\nu}$. Consider this time the gradient map

$$(3.1) \quad Z(z, y, t) = (u_z, u_y, u - (z u_z + y u_y))$$

so that

$$(3.2) \quad \det \left(\frac{\partial Z}{\partial(z, y, t)} \right) = u_t \left[\det \left(\frac{\partial \bar{Z}}{\partial(z, y)} \right) \right], \quad \bar{Z}(z, y) = (u_z, u_y)$$

and set

$$\Gamma^+ = \left\{ (s, y, t) \in \mathcal{C}_\rho : \frac{\partial(u_z, u_y)}{\partial(z, y)} \leq 0, u_z \leq 0, u_t \geq 0 \right\}.$$

Denoting by $\mathcal{C}_r(s_0, y_0, t_0)$ the cube

$$\mathcal{C}_r(s_0, y_0, t_0) = \{(s, y) : s \geq 0, |s - s_0| \leq r, |y - y_0| \leq r, t_0 - r^2 \leq t \leq t_0\}$$

for any point (s_0, y_0, t_0) with $s_0 \geq 0$ and any $r > 0$, we will show the following parabolic analogue of the Alexandrov-Bakel'man-Pucci maximum principle (Theorems 2.2 and 2.3 of paragraph 2.1).

Theorem 3.2. *Let u be a classical solution of equation $L_s u - u_t = g$ on $\mathcal{C}_\rho = \mathcal{C}_\rho(s_0, y_0, t_0)$, with coefficients satisfying conditions (2.3) - (2.4). Assume in addition that $u \leq 0$ on $\{|s - s_0| = \rho, |y - y_0| = \rho, t - t_0 = \rho^2\} \cap \mathcal{C}_\rho$. Then,*

$$\sup_{\mathcal{C}_\rho} u^+ \leq C(\lambda, \nu) \rho^{\frac{2}{3}} \rho_\nu(s_0)^{\frac{1}{3}} \left(\int_{\Gamma^-} (g^-)^3(s, y, t) s^{\nu-1} ds dy dt \right)^{1/3}$$

with

$$(3.3) \quad \rho_\nu(s_0) = (s_0 + \rho)^{2-\nu} - s_0^{2-\nu}.$$

Proof. We will only give an outline of the proof, pointing out the differences from the elliptic case. Let us suppose that u^+ takes a positive maximum

$$M = \max_{\mathcal{C}_\rho} u^+$$

at the point (s, y, t) and let ρ_ν be the distance defined by (3.3) Then

$$D = \left[-\frac{cM}{\rho_\nu(s_0)}, 0\right] \times \left[-\frac{cM}{\rho}, \frac{cM}{\rho}\right] \times \left[-\frac{cM}{\rho}, \frac{cM}{\rho}\right] \subset Z(\Gamma^+)$$

for some uniform constant c , where $Z(\Gamma^+)$ denotes the image of Γ^+ under the gradient map Z given by (3.1). Hence

$$(3.4) \quad |D| \leq |Z(\Gamma^+)| = \int_{\Gamma^+} \left| \det \left(\frac{\partial Z}{\partial(s, y, t)} \right) \right| ds dy dt.$$

On the other hand, (3.2) and the computations leading to formula (2.9), imply that

$$(3.5) \quad |Z(\Gamma^+)| = \int_{\Gamma^+} |u_t \det E| d\mu dt$$

with $d\mu = s^{\nu-1} ds dy$ and

$$E = \begin{pmatrix} s^{2(1-\nu)} u_{zz} & s^{1-\nu} u_{zy} \\ s^{1-\nu} u_{zy} & u_{yy} \end{pmatrix} = \begin{pmatrix} u_{ss} + \frac{(\nu-1)u_s}{s} & u_{sy} \\ u_{sy} & u_{yy} \end{pmatrix}.$$

Since, $u_t \geq 0$ and $\frac{\partial(u_z, u_y)}{\partial(z, y)} \leq 0$ on Γ^+ , $|u_t \det E| = u_t (-\det E)$. Hence the estimate

$$3[u_t \det(a_{ij}) \cdot (-\det E)]^{\frac{1}{3}} \leq \left(a_{11} \left[u_{ss} + \frac{(\nu-1)u_s}{s} \right] + 2a_{12} u_{sy} + a_{22} u_{yy} - u_t \right)^{-}$$

implies the bound

$$3[u_t \det(a_{ij}) \cdot |\det E|]^{\frac{1}{3}} \leq g^- + |b_2| |u_y|$$

and by Hölder's inequality

$$3[u_t \det(a_{ij}) \cdot |\det E|]^{\frac{1}{3}} \leq (k^3 (g^-)^3 + |b_2|^3)^{\frac{1}{3}} \cdot (k^{-\frac{3}{2}} + |u_y|^{\frac{3}{2}})^{\frac{2}{3}}$$

for all numbers $k > 0$. Using the bound $\det(a_{ij}) \geq \lambda^2$ we then conclude the estimate

$$(3.6) \quad (u_t |\det E|)^{\frac{1}{3}} \cdot (k^{-\frac{3}{2}} + |u_y|^{\frac{3}{2}})^{-\frac{2}{3}} \leq \frac{1}{3} \lambda^{-1} (k^3 (g^-)^3 + |b_2|^3)^{\frac{1}{3}}.$$

Hence, considering the function G on $\mathbb{R}^3$ defined by

$$G(\xi, \zeta, \tau) = (k^{-\frac{3}{2}} + \xi^{\frac{3}{2}})^{-2}$$

we have the formula

$$(3.7) \quad \int_D G \leq \int_{\Gamma^+} G(Z) \left| \det \left(\frac{\partial Z}{\partial(s, y, t)} \right) \right| ds dy dt = \int_{\Gamma^+} (k^{-\frac{3}{2}} + u_y^{\frac{3}{2}})^{-2} |u_t \det E| d\mu dt.$$

Combining (3.6) and (3.7) and using the bound $|b_2| \leq \lambda^{-1}$, we obtain the estimate

$$(3.8) \quad \int_D G \leq \frac{1}{27\lambda^3} \int_{\Gamma^+} (k^3 (g^-)^3 + \lambda^{-3}) d\mu dt.$$

To compute the integral $\int_D G$, let us recall that $D = [-\frac{cM}{\rho_\nu(s_0)}, 0] \times [-\frac{cM}{\rho}, \frac{cM}{\rho}] \times [-\frac{cM}{\rho}, \frac{cM}{\rho}]$, so that, similarly to (2.13) we obtain

$$(3.9) \quad \begin{aligned} \int_D G &= \frac{c\rho}{\rho_\nu(s_0)} \int_{B_{\frac{cM}{\rho}}} (k^{-\frac{3}{2}} + \xi^{\frac{3}{2}})^{-2} d\xi d\zeta d\tau \\ &\geq \frac{c\rho}{\rho_\nu(s_0)} \int_{B_{\frac{cM}{\rho}}} (k^{-3} + \xi^3)^{-1} d\xi d\zeta d\tau \geq \frac{c\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^3 k^3 M^3}{\rho^3}) \end{aligned}$$

From (3.8) and (3.9) we obtain

$$\frac{c\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^3 k^3 M^3}{\rho^3}) \leq \frac{1}{27\lambda^3} \int_{\Gamma^+} (k^3 (g^-)^3 + \lambda^{-3}) d\mu dt.$$

Let us set k by $k^{-3} = \lambda^3 \int_{\Gamma^+} (g^-)^3 d\mu dt$ to finally conclude (after some calculations) that

$$\frac{\rho}{\rho_\nu(s_0)} \log(1 + \frac{c^3 k^3 M^3}{\rho^3}) \leq C(\lambda, \nu).$$

Since $\alpha = \frac{\rho}{\rho_\nu(s_0)} \geq 1$, when $s_0 < 1$ and $\rho < 1$, the estimate $\alpha \log(1+x) \geq \log(1+\alpha x)$ then implies that

$$\log(1 + \frac{c^3 k^3 M^3}{\rho^2 \rho_\nu(s_0)}) \leq C(\lambda, \nu).$$

Exponentiating, we finally conclude the estimate

$$M \leq C(\lambda, \nu) \rho^{\frac{2}{3}} \rho_\nu(s_0)^{\frac{1}{3}} \left(\int_{\Gamma^+} (g^-)^3 d\mu dt \right)^{\frac{1}{3}}$$

finishing the proof of the Theorem.

Replacing u by $-u$ in the above Theorem and defining the set

$$\Gamma^- = \left\{ (s, y) \in \mathcal{C}_\rho : \frac{\partial(u_z, u_y)}{\partial(z, y)} \geq 0, u_z \geq 0, u_t \geq 0 \right\}$$

we obtain:

Theorem 3.3. *Let u be a classical solution of equation*

$$L_s := a_{11} u_{ss} + 2a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{s} \left[\frac{b_1}{2a_{11}} - 1 \right] u_s + b_2 u_y = g$$

on $\mathcal{C}_\rho = \mathcal{C}_\rho(s_0, y_0)$, with coefficients satisfying conditions (2.3) - (2.4). Assume in addition that $u \geq 0$ on $\{ |s - s_0| = \rho, |y - y_0| = \rho, t - t_0 = \rho^2 \} \cap \mathcal{C}_\rho$. Then,

$$\sup_{\mathcal{C}_\rho} u^- \leq C(\lambda, \nu) \rho^{\frac{2}{3}} \rho_\nu(s_0)^{\frac{1}{3}} \left(\int_{\Gamma^-} (g^+)^3(s, y, t) s^{\nu-1} ds dy dt \right)^{1/3}$$

with $\rho_\nu(s_0)$ given by (3.3).

3.2. The Barrier Function. As in the elliptic case, for the proof of the Harnack estimate will need to construct a barrier function, similar to the barrier function introduced by Wang in [W1]. To simplify the computations in this paragraph we will go back to the original (x, y, t) variables, assuming that L satisfies conditions (1.9) and (1.11). Similarly to paragraph 2.2, for any two points (x, y) and (x_0, y_0) in $\mathbb{R}_+^2$, we introduce the distance function d_γ defined by

$$(3.10) \quad d_\gamma^2((x, y), (x_0, y_0)) = (\sqrt{x} - \sqrt{x_0})^2 + \gamma^2(y - y_0)^2$$

with $\gamma > 0$ a sufficiently small constant depending on λ, ν , to be determined in the sequel. Recall that $0 < \lambda, \nu < 1$ the positive constant so that (1.11) holds. Notice

that in the (s, y) variables, with $s = \sqrt{x}$ the distance function d_γ^2 can be expressed as

$$d_\gamma^2((s, y), (s_0, y_0)) = (s - s_0)^2 + \gamma^2 (y - y_0)^2.$$

For $r > 0$, let $Q_r(x_0, y_0, t_0)$ denote the cube

$$Q_r(x_0, y_0, t_0) = \{ (x, y) : x \geq 0, |\sqrt{x} - \sqrt{x_0}| \leq r, \gamma |y - y_0| \leq r, t_0 - r^2 \leq t \leq t_0 \}.$$

Also let us denote by $\mathcal{B}_r(x_0, y_0)$ the ball

$$\mathcal{B}_r(x_0, y_0) = \{ (x, y) : x \geq 0, d_\gamma((x, y), (x_0, y_0)) \leq r \}$$

and by $K_r(x_0, t_0, y_0)$ the parabolic cylinder

$$K_r(x_0, t_0, y_0) = \mathcal{B}_r(x_0, y_0) \times (t_0 - r^2, t_0].$$

We will show the following analogue of Lemma 2.2 in [W1].

Lemma 3.4. *For any point $(x_0, y_0) \in \mathbb{R}^2$ with $0 \leq x_0 \leq 1$ and any number $0 < \rho \leq 1$ let us set $K_{3\sqrt{2}\rho} = \mathcal{B}_{3\sqrt{2}\rho}(x_0, y_0) \times (0, 18\rho^2)$, $Q_{\frac{\rho}{2}}^1 = Q_{\frac{\rho}{2}}(x_0, y_0, \frac{\rho^2}{4})$ and $Q_{\frac{3\rho}{2}}^2 = Q_{\frac{3\rho}{2}}(x_0, y_0, \frac{10\rho^2}{4})$. Then, there exists a function ϕ_ρ on $K_{3\sqrt{2}\rho}$, such that*

$$(3.11) \quad \begin{cases} \phi_\rho \geq 1 & \text{in } Q_{\frac{3\rho}{2}}^2 \\ \phi_\rho \leq 0 & \text{on } \partial_p K_{3\sqrt{2}\rho} \end{cases}$$

and

$$(3.12) \quad L\phi_\rho - (\phi_\rho)_t \geq 0 \quad \text{on } K_{3\sqrt{2}\rho} \setminus Q_{\frac{\rho}{2}}^1.$$

Moreover, we have

$$\|\phi_\rho\|_{C^{1,1}(K_{3\sqrt{2}\rho})} \leq \frac{C(\lambda, \nu)}{\rho^2}.$$

Proof. This Lemma is the parabolic analogue of Lemma 2.5. As in the elliptic case, we will first show the Lemma in the case that $\rho = 1$. The general case will follow by an appropriate dilation. Similarly to Lemma 2.4 we introduce the new distance function

$$\bar{d}_\gamma^2 = \frac{(x - x_0)^2}{x + x_0} + \gamma^2 (y - y_0)^2.$$

which is equivalent to d_γ since

$$(3.13) \quad d_\gamma \leq \bar{d} \leq \sqrt{2} d_\gamma.$$

Let us consider the function

$$\omega(x, y, t) = [18 - \bar{d}^2((x, y), (x_0, y_0))] \Lambda(x, y, t)$$

with

$$\Lambda(x, y, t) = \frac{1}{4\pi t} e^{-\frac{\bar{d}^2((x, y), (x_0, y_0))}{t}}.$$

For numbers $0 < \tau_0 < 1$, $m > 1$ and $l > 1$, to be determined in the sequel, set

$$u(x, y, t) = e^{-mt} \omega^l(x, y, t + \tau_0) - M(\tau_0)$$

with

$$M(\tau_0) = \sup\{\omega^l(x, y, \tau_0) : \bar{d}((x, y), (x_0, y_0)) \geq \frac{1}{2}\}.$$

Then, it follows by (3.13) that $u \leq 0$ on $\partial_p K_{3\sqrt{2}} \setminus Q_{\frac{1}{2}}^1$. Moreover, we can choose τ_0 sufficiently close to zero, depending only on γ , such that we still have $u > 0$ on $Q_{\frac{3}{2}}^2$.

To simplify the notation let us set $\theta(x, y) = \bar{d}^2((x, y), (x_0, y_0))$, so that

$$\omega = (18 - \theta) \Lambda \quad \text{and} \quad \Lambda = \frac{1}{4\pi t} e^{-\frac{\theta}{t}}.$$

Also, let us set

$$\mathcal{L}u := u_t - Lu = u_t - (\tilde{a}_{ij} u_{ij} + b_i u_i)$$

with

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} = \begin{pmatrix} x a_{11} & \sqrt{x} a_{12} \\ \sqrt{x} a_{21} & a_{22} \end{pmatrix}.$$

A direct computation shows that

$$\mathcal{L}u = e^{-mt} \omega^{l-2} \{ l [\omega (\omega_t - \tilde{a}_{ij} \omega_{ij} - b_i \omega_i) - (l-1) \tilde{a}_{ij} \omega_i \omega_j] - m \omega^2 \}$$

with

$$\omega_i = -\left[\frac{1}{t + \tau_0} (18 - \theta) + 1\right] \Lambda \theta_i \quad \text{and} \quad \omega_t = (18 - \theta) \Lambda \left[\frac{\theta}{(t + \tau_0)^2} - \frac{1}{t + \tau_0}\right]$$

and

$$\omega_{ij} = -\left[\frac{1}{t + \tau_0} (18 - \theta) + 1\right] \Lambda \theta_{ij} + \frac{1}{t + \tau_0} \left[\frac{1}{t + \tau_0} (18 - \theta) + 2\right] \Lambda \theta_i \theta_j.$$

Combining the above we find that

$$\begin{aligned} \mathcal{L}u &= l e^{-mt} \omega^{l-2} \Lambda^2 \left\{ (18 - \theta)^2 \left[\frac{\theta}{(t + \tau_0)^2} - \frac{1}{t + \tau_0} + \frac{1}{t + \tau_0} \tilde{a}_{ij} \theta_{ij} \right. \right. \\ &\quad \left. \left. - \frac{1}{(t + \tau_0)^2} \tilde{a}_{ij} \theta_i \theta_j + \frac{1}{t + \tau_0} b_i \theta_i - \frac{m}{l} \right] \right. \\ &\quad \left. + (18 - \theta) \left[\tilde{a}_{ij} \left(\theta_{ij} - \frac{2}{t + \tau_0} \theta_i \theta_j \right) + b_i \theta_i \right] \right. \\ &\quad \left. - (l-1) \tilde{a}_{ij} \left[1 + \frac{1}{t + \tau_0} (18 - \theta) \right]^2 \theta_i \theta_j \right\}. \end{aligned} \tag{3.14}$$

Hence, using that $l > 1$, we obtain

$$\mathcal{L}u \leq l e^{-mt} \omega^{l-2} \Lambda^2 [(18 - \theta)^2 I + (18 - \theta) II]$$

with

$$I = \frac{\theta}{(t + \tau_0)^2} - \frac{1}{t + \tau_0} + \frac{1}{t + \tau_0} \tilde{a}_{ij} \theta_{ij} - \frac{1}{(t + \tau_0)^2} \tilde{a}_{ij} \theta_i \theta_j + \frac{1}{t + \tau_0} b_i \theta_i - \frac{m}{l}$$

and

$$II = \tilde{a}_{ij} \theta_{ij} - 2(l + 1) \tilde{a}_{ij} \frac{1}{t + \tau_0} \theta_i \theta_j + b_i \theta_i.$$

By assumptions (1.9) and (1.11) we have

$$\tilde{a}_{ij} \theta_{ij} \leq \lambda^{-1} [x \theta_{xx} + \theta_{yy}]$$

and

$$\tilde{a}_{ij} \theta_i \theta_j \geq \lambda [x \theta_x^2 + \theta_y^2]$$

while

$$|b_i| \leq \lambda \quad \text{and} \quad b_1 \geq \frac{\nu \lambda}{2}.$$

Also, by direct computation

$$\theta_x = \frac{(x + 3x_0)(x - x_0)}{(x + x_0)^2} \quad \text{and} \quad \theta_{xx} = \frac{8x_0^2}{(x + x_0)^3}$$

while

$$\theta_y = 2\gamma^2(y - y_0) \quad \text{and} \quad \theta_{yy} = 2\gamma^2$$

and $\theta_{xy} = 0$. In particular one can observe that

$$|\theta|, |\theta_x|, |\theta_y|, |x \theta_{xx}|, |\theta_{yy}| \leq C(\gamma) \quad \text{on } K_{3\sqrt{2}}(P_0)$$

when $x_0, |y_0| \leq 1$. Therefore, the term I can be easily estimated as

$$I \leq \frac{C(\gamma, \lambda, \nu)}{\tau_0^2} - \frac{m}{l} \leq -\frac{m}{2l}$$

for $\frac{m}{l}$ sufficiently large, depending only on γ, λ and ν (since τ_0 depends only on γ).

The term II can be estimated as

$$II \leq \lambda^{-1} [x \theta_{xx} + \theta_{yy} + |\theta_y| + \theta_x^+] - \frac{\nu \lambda}{2} \theta_x^- - c(\gamma, \lambda) (l + 1) [x \theta_x^2 + \theta_y^2]$$

where θ_{ij} and θ_i are given above. When $d_\gamma((x, y), (x_0, y_0)) \geq \frac{1}{4}$, then one may use the same arguments as in the proof of lemma 2.4 to deduce that $II \leq 0$, when $\gamma = \gamma(\lambda, \nu)$ and $l = l(\lambda, \nu)$ are chosen sufficiently large. In the case where $d_\gamma((x, y), (x_0, y_0)) < \frac{1}{4}$ we have $(18 - \theta) \geq c(\nu, \lambda) > 0$ and hence

$$II \leq C(\nu, \lambda) \leq C(\nu, \lambda) (18 - \theta)$$

so that we still have

$$(18 - \theta)^2 I + (18 - \theta) II \leq (18 - \theta)^2 \left[-\frac{m}{2l} + C(\nu, \lambda) \right] \leq 0$$

by choosing m sufficiently large.

Summarizing the above, we have constructed a function u satisfying $\mathcal{L}u \leq 0$ in $K_{3\sqrt{2}}$ and also such that $u \leq 0$ on $\partial_p K_{3\sqrt{2}} \setminus Q_{\frac{1}{2}}^1$, $u > c(\nu, \lambda) > 0$ on $Q_{\frac{3}{2}}^2$. Moreover, it is easy to observe that

$$(3.15) \quad \|u\|_{C^{1,1}} \leq C(\nu, \lambda).$$

We can modify u in such a way that (3.15) still holds, $\mathcal{L}u \leq 0$ on $K_{3\sqrt{2}} \setminus Q_{\frac{1}{2}}^1$, $u \leq 0$ at $\partial K_{3\sqrt{2}}$ and $u > 0$ in $Q_{\frac{3}{2}}^2$. Finally, setting

$$\phi = \frac{u}{\inf_{Q_{\frac{3}{2}}^2} u}$$

so that $\phi \geq 1$ in $Q_{\frac{3}{2}}^2$, we conclude that ϕ is the desired barrier function.

We have constructed above the barrier function $\phi = \bar{\phi}(\bar{d}, t)$ on $K_{3\sqrt{2}}$. To construct the barrier function ϕ_ρ on $K_{3\sqrt{2}\rho}$, for any $0 < \rho < 1$, we set

$$\phi_\rho = \bar{\phi}_\rho(\bar{d}, t) = \bar{\phi}\left(\frac{\bar{d}}{\rho}, \frac{t}{\rho^2}\right).$$

Clearly

$$L\phi_\rho - (\phi_\rho)_t = \frac{1}{\rho^2} (L\phi - \phi_t) \geq 0, \quad \text{on } K_{3\sqrt{2}\rho} \setminus Q_{\frac{1}{2}}^1$$

and it also satisfies (3.11). Moreover, we have

$$\|\phi_\rho\|_{C^{1,1}(K_{3\sqrt{2}\rho})} = \frac{1}{\rho^2} \|\phi\|_{C^{1,1}(K_{3\sqrt{2}})} \leq \frac{C(\nu, \lambda)}{\rho^2}$$

concluding that ϕ_ρ satisfies all the required conditions.

3.3. The Harnack Inequality. Fix a point (x_0, y_0, t_0) with $x_0 \geq 0$, and set $s_0 = \sqrt{x_0}$. Let us now go back to the (s, y) variables (with $s = \sqrt{x}$) assuming, throughout this section, that u is a solution of the equation

$$(3.16) \quad L_s u := a_{11} u_{ss} + 2a_{12} u_{sy} + a_{22} u_{yy} + \frac{a_{11}}{s} \left[\frac{b_1}{2a_{11}} - 1 \right] u_s + b_2 u_y - u_t = g$$

with L_s satisfying conditions (2.3) and (2.4). Denoting, for any $r > 0$, by $Q_r(s_0, y_0, t_0)$ the cube

$$Q_r(s_0, y_0, t_0) = \{ (s, y) : s \geq 0, |s - s_0| \leq r, \gamma |y - y_0| \leq r t_0 - r \leq t \leq t_0 \}$$

we have the following Harnack inequality for solutions to (3.16).

Theorem 3.5. *Let $u \geq 0$ be a classical solution of equation (3.16) in $Q_\rho(s_0, y_0, t_0)$, where g is a bounded and continuous function on $Q_\rho(s_0, y_0, t_0)$. Then,*

$$(3.17) \quad \sup_{Q_{\frac{\rho}{2}}(s_0, y_0, t_0 - \frac{3\rho^2}{4})} u \leq C \left(\inf_{Q_{\frac{\rho}{2}}(s_0, y_0, t_0)} u + \rho^{\frac{3}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^3(Q_\rho(s_0, y_0, t_0), d\mu)} \right)$$

with $d\mu = s^{\nu-1} ds dy dt$ and $\rho_\nu(s_0)$ given by (3.3).

The proof of Theorem 3.5, based upon the A-B-P estimate, Theorem 3.3, and the barrier function given in Lemma 3.4, follows along the lines of the proof of the corresponding elliptic Theorem 2.6. One may now follow the proof of Theorem 2.1, with the standard adaptations to the parabolic case to show Theorem 3.1.

We finish by stating the parabolic analogies of the weak Harnack estimate, Theorem 2.12 and the local maximum principle Theorem 2.13.

To simplify the notation, let us set, for any $r > 0$, $Q_r := Q_r(s_0, y_0, t_0)$ and $Q_r^- := Q_r(s_0, y_0, t_0 - \frac{3\rho^2}{4})$.

The first Theorem is a weak Harnack estimate for nonnegative supersolutions u of equation $L_s u \leq g$.

Theorem 3.6. *Let $u \geq 0$ be a supersolution of equation $L_s u \leq g$ in $Q_\rho := Q_\rho(s_0, y_0, t_0)$, where g is a bounded and continuous function on Q_ρ . Then, there exist universal constants $p_0 > 0$ and C such that*

$$(3.18) \quad \left(\int_{Q_{\frac{\rho}{4}}^-} u^{p_0} d\mu \right)^{\frac{1}{p_0}} \leq C \left(\inf_{Q_{\frac{\rho}{2}}} u + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^2(Q_\rho, d\mu)} \right)$$

with $d\mu = s^{\nu-1} ds dy dt$ and $\rho_\nu(s_0)$ given by (3.3).

The last Theorem in this section is a local maximum principle for subsolutions u of equation $L_s u \geq g$.

Theorem 3.7. *Let u be a subsolution of equation $L_s u \geq g$ in $Q_\rho := Q_\rho(s_0, y_0, t_0)$, where g is a bounded and continuous function on Q_ρ . Then, for any $p > 0$, we have*

$$(3.19) \quad \sup_{Q_{\frac{\rho}{2}}} u \leq C(p) \left\{ \left(\int_{Q_{\frac{3\rho}{4}}} u^p d\mu \right)^{\frac{1}{p}} + \rho^{\frac{1}{2}} \rho_\nu(s_0)^{\frac{1}{2}} \|g\|_{L^3(Q_\rho, d\mu)} \right\}$$

with $d\mu = s^{\nu-1} ds dy dt$, $\rho_\nu(s_0)$ given by (3.3), and $C(p)$ a constant depending only on λ, ν and p .

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