OBSTACLE PROBLEM FOR NONLINEAR 2^{nd}-ORDER ELLIPTIC OPERATOR

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Abstract.
We study the obstacle problem for fully nonlinear second-order uniformly elliptic operators. We can show the existence of a continuous viscosity solution in the general setting,and we can show $C^{1,1}$-regularity of the viscosity solution when the operator is convex or concave. $C^{1,\alpha}$-regularity of the free boundary is established when the operator is convex in any dimension or concave in two dimensions.

When F is a linear operator,many authors studied the existence and the regularity of the weak solution, and the regularity of the free boundary. Hans Lewy and Guido Stampacchia considered the least superharmonic function in their paper[LG]. L. Caffarelli, and Kinderleher showed the gradient of the solution has the same modulus of continuity as the gradient of the obstacle. The penalized problem was widely pursued by many authors. Hans Brezis and David Kinderleher proved the $C^{1,1}$-regularity of the solution in [Br].

In [Ca2], L. Caffarelli used the $C^{1,1}$-regularity and the homogeneity of the operator to get the regularity of the free boundary by blowing up the solution. Much understanding of the viscosity solution of the nonlinear elliptic operator was improved in [Ish],[Ev],[Kry], and [Ca3].

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Our principle results are the existence of the viscosity solution, the $C^{1,1}$-regularity of the solution, and the $C^{1,\alpha}$-regularity of the free boundary.

In the first section, we consider the least super-solution of (0.1),and the corresponding penalized problem. The least super solution will give us the viscosity solution by Perron's method [Ish]. We show the continuity of the solution by the Harnack inequality.

In the second section the $C^{1,1}$-regularity of the solution is proved by extending the argument in [Ca1]. By means of the Harnack inequality, the question can be interpreted in terms of the boundary behavior of the solution, and by the comparison theorem, the solution is also controllable around the boundary.

In the third section, we show the C^1-regularity of the free boundary by the compactness method which was developed in [Ca2]. Our approach consists in blowing up our solution, characterizing the limiting "cone" solution and deducing the regularity of the original solution. The critical estimate in this approach is the lower bound for the second derivative of the solution by the Harnark type estimate. It is proved for the convex operator in any dimension, but for the concave operator, it is proved only in two dimensions by the relation between two eigenvalues of the solution.

In the fourth section, $C^{1,\alpha}$-regularity of the free boundary is established when the operator is homogeneous of degree one. First, we found a cone of direction where the solution is monotone. Then the potential theory in [Fa] implies the $C^{1,\alpha}$-regularity of the all level surface of the solution including the free boundary.

In the last section, we show the $C^{1,\alpha}$-regularity of the free boundary of the general case by the renormalization and approximation argument.

I wish to express my thanks to Luis Caffarelli for his suggestion of this problem and advice.

1. THE EXISTENCE AND THE CONTINUITY THEORY

At this section, we are going to study the existence of the lower semicontinuous viscosity solution and the continuity of the solution.

Theorem 1.1.

There exists a lowersemicontinuous viscosity supersolution u *which satisfies (0.1).*

Proof It is proved by Perron's method in H. Ishii [Ish]

Theorem 1.2 (a generalization of Evans theorem)**.**

If u *is continuous in* $\operatorname{supp}(F(D^2u))$*, then* u *is continuous in* Ω

Proof The only possible problem is on the free boundary. Assume u is discontinuous at some point x_o in $\operatorname{supp}((F(D^2u))$. There exists a sequence x_k in the complement of $\operatorname{supp}((F(D^2u))$ converging to x_o s.t. $u(x_k)$ converges to μ(possibly ∞) with $\mu > \liminf_{x\to x_o} u + K = u(x_o) + K$
Without loss of generality, we can assume $\liminf_{x\to x_o} u = u(x_o) = 0$.
So for any $\delta > 0$, there is a small neighborhood of x_o, with $u(x) \geq -\delta$.

Now choose r_k as large as possible such that $B_{r_k}(x_k)$ is in the complement of $\mathrm{supp}(F(D^2u))$.

So for x in our neighborhood, $u(x)+\delta \geq 0$ and $u(x_k)+\delta \geq \mu > 0$ for large k.

Now choose k large enough to guarantee that $B_{r_k}(x_k)$ is contained in our neighborhood. By the Harnack inequality,$u(x)+\delta \geq C\mu$ in $B_{r_k/2}(x_k)$.

C is universal,in particular, independent of r_k.

Choose small $\delta > 0$ s.t. $u(x) \geq C\mu - \delta \geq \frac{C}{2}\mu$ in $B_{r_k/2}(x_k)$.

Let $y_k \in \mathrm{supp}(F(D^2u)) \cap \partial B_{r_k}(x_k)$.

Now by the weak Harnark inequality,

$$\begin{aligned} u(y_k)+\delta &\geq C\Big(\frac{1}{B_{2r_k}(y_k)}\int_{B_{2r_k}(y_k)}(u+\delta)^p\Big)^{\frac{1}{p}} \\ &\geq C\Big(\frac{1}{B_{r_k/2}(y_k)}\int_{B_{r_k/2}(y_k)}(u+\delta)^p\Big)^{\frac{1}{p}} \\ &\geq C\mu+\delta \end{aligned}$$

(for small C). So $u(y_k) \geq C\mu > 0$,independent of K, and since the y_k converge to x_o, we have a contradiction!

Q.E.D.

Theorem 1.3.

u is continuous on $\overline{\Omega}$.

Proof We know u is continuous in the interior of Ω from the previous theorem. Now we are going to study the continuity on the boundary. If $\partial\Omega$ is smooth, it will satisfy the exterior sphere condition at each point on the boundary.

So we can find a barrier on the boundary [GT]

The comparison principle says that the solution is trapped between upper and lower barriers which are continuous and coincide with the

boundary data at the given point on the boundary. As x approaches to the given point, the solution assumes the boundary data. Therefore u is continuous on $\overline{\Omega}$.
Q.E.D.

Now we are going to give another proof of the existence of the continuous viscosity solution by using the standard penalized problem.By the uniqueness argument, it turns out to be the least supper solution given in the theorem above. We can improve the regularity slightly better which is not necessary in the next arguments, but we will include it here because of the technical interest.

Theorem 1.4 (the existence)**.**

There is a u in $W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for all $1 \leq p < \infty$ s.t.

$$\begin{aligned} -F(D^2u) &\geq 0 \\ u &\geq \phi \ in \ \Omega \\ (-F(D^2u))(u-\phi) &= 0 \\ u &= 0 \ on \ \partial\Omega \end{aligned} \tag{1.1}$$

Proof There is a continuous family of functions $\beta_\epsilon(t) \in C^\infty$ when $\epsilon < 1$ s.t.

$$\begin{array}{ll} \beta'_\epsilon(t) & \geq 0 \\ \beta_\epsilon(t) & \rightarrow -\infty \text{ if } t < 0, \epsilon \rightarrow 0 \\ \beta_\epsilon(t) & \rightarrow 0 \text{ if } t > 0, \epsilon \rightarrow 0 \\ \beta_\epsilon(t) & \leq 0 \\ \beta_\epsilon(0) & \geq -C \end{array}$$

where C is a constant independent of ϵ. Let's consider the penalized problem

$$
\begin{aligned}
-F(D^2u) + \beta_\epsilon(u-\phi) &= 0 \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
\tag{1.2}
$$

For the technical reason, let's truncate the graph of β_ϵ at the level $\pm N$.

$$
\beta_{\epsilon,N} = \begin{cases} N & \text{if } \beta_{\epsilon,N} \geq N \\ \beta_{\epsilon,N} & \text{if } -N \leq \beta_{\epsilon,N} \leq N \\ -N & \text{if } \beta_{\epsilon,N} \geq N \end{cases}
$$

Consider the problem:

$$
\begin{aligned}
-F(D^2u) + \beta_{\epsilon,N}(u-\phi) &= 0 \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
\tag{1.3}
$$

For each $v \in L^p_{loc}(\Omega) \cap C^o(\overline{\Omega})$ By [Ca2], there is $u \in W^{2,p}_{loc}(\Omega) \cap C^o(\overline{\Omega})$ satisfying

$$
\begin{aligned}
-F(D^2u) + \beta_{\epsilon,N}(v-\phi) &= 0 \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

From $|\beta_{\epsilon,N}(v-\phi)| \leq N$, we cab deduce

$$
||u||_{W^{2,p}_{loc}(\Omega)} \leq R
$$

where R is independent of v, and u is ρ_*-modulus continuous for some ρ_* independent of v by the theorem (4.14) in [Ca3]

On the other hand, $W^{2,p}_{loc}(\Omega)$ is compact subset of L^p_{loc}, and ρ_* continuous functions make a compact subset of $C^o(\overline{\Omega})$. So $v \rightarrow u = Tv$ is a compact operator.

By the Schauder fixed point theorem, there is u s.t.

$$
\exists u = Tu
$$

So there is $u_{\epsilon,N} \in W^{2,p}_{loc}(\Omega) \cap C^o(\overline{\Omega})$ s.t. $u_{\epsilon,N}$ is a solution of (1.3)

We estimate $w = \beta_{\epsilon,N}(u_{\epsilon,N} - \phi)$ at first, $w \leq C$ by the definition of $\beta_\epsilon(t)$

There is a $x_o \in \overline{\Omega}$ where w assumes its minimum on $\overline{\Omega}$ since w is continuous on $\overline{\Omega}$

Without loss of generality we assume

$$w(x_o) \leq 0, w(x_o) < \beta_\epsilon(0)$$

and x_o is not in $\partial\Omega$ since $u_{\epsilon,N}(x_o) - \phi(x_o) > 0$ for $x_o \in \partial\Omega$

The monotonicity of β_ϵ implies $w(x_o) \geq \beta_\epsilon(0)$ It's a contradiction.

So x_o is on Ω, and $u_{\epsilon,N} - \phi$ has the minimum at x_o since $\beta_{\epsilon,N}$ is monotone.

We can think $(u_{\epsilon,N} - \phi)(x_o) < 0$ and

$$-F(D^2u_{\epsilon,N} - D^2\phi) \leq 0 \text{ at } x_o$$

since $D^2(u_{\epsilon,N} - \phi) \geq 0$ at the minimum point x_o.

At x_o, by the uniform ellipticity,

$$\begin{aligned} \beta_{\epsilon,N}(u - \phi)(x_o) \quad &= F(D^2u_{\epsilon,N}) = F(D^2u_{\epsilon,N} - D^2\phi + D^2\phi) \\ &\geq F(D^2u_{\epsilon,N} - D^2\phi) + \lambda||D^2\phi^+|| - \Lambda||D^2\phi^-|| \\ &\geq \lambda||D^2\phi^+|| - \Lambda||D^2\phi^-|| \geq -C \end{aligned}$$

for some constant C. Therefore $|\beta_{\epsilon,N}(u - \phi)| \leq C$ where C is independent of ϵ and N

$$|F(D^2u_{\epsilon,N})| \leq C$$

By the standard estimate in the uniform elliptic equation,

$$||u_{\epsilon,N}||_{W^{2,p}_{loc}(\Omega)\cap C(\Omega)} \leq C \tag{1.4}$$

If N is large, $u_{\epsilon,N}$ is the solution of the penalized problem(1.2).

So we can drop N in $u_{\epsilon,N}$

On the other hand $|\beta_{\epsilon,N}(u_\epsilon - \phi)| < C$ gives

$$|(u_{\epsilon,N} - \phi)(x) - (u_{\epsilon,N} - \phi)(y)| \leq \rho^*(|x - y|) \tag{1.5}$$

where ρ^* is independent of ϵ, N.

By (1.4)(1.5),there are $\epsilon = \epsilon_m \to 0$ s.t.

$$u_{\epsilon,m} \to u \qquad \text{weakly in } W^{2,p}_{loc}(\Omega)$$

$$u_{\epsilon,m} \to u \qquad \text{uniformly in } \overline{\Omega}$$

$|\beta_\epsilon(u_\epsilon - \phi)| < C$ gives $u \geq \phi$, and

$$\beta_\epsilon(u_{\epsilon_m} - \phi) \to 0 \qquad \text{on } u > \phi$$

$$\overline{\lim}_{\epsilon_m \to 0} \beta_{\epsilon_m}(u_{\epsilon_m} - \phi) \leq 0$$

By the standard argument in the viscosity solution,

$$-F(D^2 u) \geq 0 \qquad \text{a.e. in } \Omega$$

$$F(D^2 u) = 0 \qquad \text{a.e. on } \{u > \phi\}$$

Q.E.D.

Theorem 1.5 (The uniqueness)**.**

u in theorem (1.4) is the least super-solution of the problem(0.1)

Proof Let v be the least super-solution of (0.1).
By the definition of $v, v \leq u$. Let Assume $D = \{x | u > v\} \neq \phi$
$F(D^2 u) = 0 \geq F(D^2 v_*)$ on D in the viscosity sense. On $\partial D, u = v$.
So $v \geq u$ in D. It is a contradiction. Q.E.D.

2. $C^{1,1}$-REGULARITY

We are going to show the solution has uniform $C^{1,1}$ estimate across the free boundary.

Lemma 2.1.

u is the viscosity solution of the (0.1)
$u(x_o) = \phi(x_o), L_{x_o}(x) = \phi(x_o) + D\phi(x_o)(x - x_o)$

$\sup_{B_r(x_o)} |\phi(x) - Ł_{x_o}(x)| \leq \lambda$

Then

$$\sup_{B_{r/2}(x_o)} |u(x) - Ł_{x_o}(x)| \leq C\lambda$$

for some universal constant C

Proof In $B_r(x_o)$,

$$L_{x_o}(x) - \lambda \leq \phi(x) \leq u(x)$$

We need to show $u(x) \leq L_{x_o}(x) + C\lambda$ in $B_{r/2}(x_o)$.

In $B_r(x_o)$,let

$$v(x) = u(x) - L_{x_o}(x) - \lambda$$

$$\Lambda(u) = \{u = \phi\}, N(u) = \{u > \phi\}, \Gamma(u) = \partial\Lambda(u) \cap \partial N(u).$$

Then

$$v(x) \geq 0, v(x) \leq 2\lambda \qquad \text{in } B_r(x_o) \cap \Lambda(u)$$

,and

$$F(D^2 v(x)) \leq 0$$

Let w be a viscosity solution s.t. $F(D^2 w(x)) = 0$ and $w = v$ on ∂B_r.

Then $w \leq v$.

On $\partial B_r, v = w$ implies $v < w + 2\lambda$.

On $\Lambda(u), v < 2\lambda$ implies $v < w + 2\lambda$.

Therefore $v \leq w + 2\lambda$ in $B_r(x_o) \cap N(u)$ which means $v \leq w + 2\lambda$ in $B_r(x_o)$, and $0 \leq w(x_o) \leq v(x_o) \leq 2\lambda$.

By the Harnack inequality, on $B_{r/2}(x_o), w \leq C\lambda$,.

Therefore $u(x) - L_{x_o}(x) \leq C\lambda$ for some $C > 0$

Q.E.D.

Theorem 2.2.

If u is the viscosity solution of (0.1), u is $C^{1,1}_{loc}(\Omega) \cap C(\overline{\Omega})$

Proof Since $\Gamma(u)$ is compact in Ω, there is a $\delta > 0$ s.t. $d(\partial\Omega, \Gamma(u)) \geq 2\delta > 0$
For any $y_o \in \Gamma(u)$, $B_\delta(y_o) \subset \Omega$.
For any $x \in B_\delta(y_o) \cap N(u)$, there is x_o which is the closest point of x to $\Gamma(u)$.
Since $\phi \in C^2$,

$$|\phi(x) - L_{x_o}(x)| \leq |D^2\phi||x - x_o|^2$$

By the lemma(2.1),

$$|u(x) - L_{x_o}(x)| \leq C|x - x_o|^2 \qquad \text{if } |x - x_o| < \frac{\rho}{2}$$

When $r < R < \frac{\rho}{2}$,

$$\begin{aligned}\sup_{B_r} |D(u(x) - L_{x_o}(x))| &\leq \frac{C}{R} \sup_{B_R} |u(x) - L_{x_o}(x)| \\ &\leq \frac{C}{R} R^2 \leq CR\end{aligned}$$

where $L_{x_o}(x) = \phi(x_o) + D\phi(x_o)(x - x_o)$. Since x_o is on $\Lambda(u)$, $L_{x_o}(x) = D\phi(x_o) = Du(x_o)$. Therefore when $r < \frac{\rho}{2}$,

$$|Du(x) - Du(x_o)| \leq Cd(x, x_o) \tag{2.1}$$

Now we are going to combine the estimate on the free boundary (2.1) and the following interior estimate. Since $F(D^2u) = 0$ in $N(u)$,

$$F_{ij}(D^2u)D_{ij}u_k = 0 \qquad \text{for any } 1 \leq k \leq n$$

$a_{ij}(x) = F_{ij}(D^2u)$ is the measurable coefficient.

$$a_{ij}(x)D_{ij}u_k = 0 \qquad \text{in } N(u)$$

For any two points $x, x' \in B_{\frac{\rho}{2}}(y_o)$, we want to show $|u_k(x) - u_k(x')| \leq C|x - x'|$. Let's take $x_o, x'_o \in \Gamma(u)$ s.t.
$d(x, \Gamma(u)) = d(x, x_o), d(x', \Gamma(u)) = d(x', x'_o)$

If $d(x,x') \le \frac{1}{2}\max(d(x,x_o),d(x',x'_o))$,
then the interior estimate of the uniform elliptic equation says

$$|u_k(x) - u_k(x')| \le C|x - x'|$$

If $d(x,x') \ge \frac{1}{2}\max(d(x,x_o),d(x',x'_o))$,
then

$$\begin{aligned} d(x_o,x'_o) &\le d(x,x_o) + d(x,x') + d(x',x'_o) \\ &\le Cd(x,x') \end{aligned}$$

$$\begin{aligned} |u_k(x) - u_k(x')| &= |u_k(x) - u_k(x_o)| + |u_k(x_o) - u_k(x'_o)| \\ &\quad + |u_k(x'_o) - u_k(x')| \\ &\le Cd(x,x_o) + |\phi_k(x_o) - \phi_k(x'_o)| + Cd(x',x'_o) \\ &\le Cd(x,x') \end{aligned}$$

Q.E.D.

3. THE REGULARITY OF THE FREE BOUNDARY

We would like to reduce the regularity problem of the free boundary to a standard form for using the compactness method. We need the following lemma to $F(D^2\phi) > 0$ in a neighborhood which will imply the nondegeneracy of the solution.

Lemma 3.1.

$\Lambda(u) = \{x|u(x) = \phi(x)\} \subset \{x|F(D^2\phi) < 0\}$

Proof $V_1 = \{x|F(D^2\phi) > 0\}$
Let $x \in \partial V_1$. By our assumption ($F(D^2\phi)$ and $DF(D^2\phi)$ do not vanish simultaneously),the cone

$$C_\epsilon = \{z| < z - x, D(F(D^2\phi)) > \epsilon|z - x|\} \cap \{z||z - x| < \delta\epsilon\}$$

satisfies $C_\epsilon \subset V_1$.

$$0 \geq F(D^2u) - F(D^2\phi) = a_{ij}(x)D_{ij}(u - \phi)$$

where $a_{ij}(x) = \int_0^1 F(\theta D^2u(x) - (1-\theta)D^2\phi(x))d\theta$, and $u - \phi > 0$ in C_ϵ. If ϵ is very small, we can find positive subsolution of the type $r^{1,\alpha}f(\theta)$ like Lemma(3.10). We will discuss how to find the subsolution later in Lemma(3.10). On the other hand, the solution u is $C^{1,1}$ across the free boundary. Therefore x is not in $\Lambda(u)$.

Q.E.D.

Since Λ is compact,there is $\nu > 0$ s.t. $\Lambda \subset \{x|F(D^2\phi) < -\nu\}$.
Then for all $x_o \in \partial\Lambda$, there is a small $\epsilon > 0$ s.t.

$$B_\epsilon(x_o) \subset \{x|F(D^2\phi) < -\nu\}$$

By rescaling the solution, we can think,
$v = u - \phi \geq 0$ in B_1. $F(D^2v + D^2\phi) = 0$ in $N(u) = \{u(x) > 0\}$
$|D^2v|_{L^\infty(B_1)} \leq M$,$v \in C^{1,1}(B_1)$,$0 \in \Gamma(u) = \partial\Lambda(u) \cap \partial N(u)$.
We would like to study the simple case at first, and to extend the argument in general case by modifying it a little bit.

The simplified case is the following.
$X = (x_1, \cdots, x_n) \in R^n$
$B_r(x) = \{y \in R^n| \parallel x - y \parallel < r\}$
u is the nonnegative function on some domain $D \subset R^n$
$\Lambda(u) = \{x \in D|u(x) = 0\}$
$N(u) = \{x \in D|u(x) > 0\}$
$\Gamma(u) = \partial\Lambda(u) \cap \partial N(u)$

- $F(u)$ is the uniformly elliptic operator and $F \in C^2$ i.e.
  $\lambda \parallel N \parallel \leq F(M+N) - F(M) \leq \Lambda \parallel N \parallel$
  where $N \geq 0$ and M, N : n x n symmetric matrix
- $u \in P_r(0 < r \leq \infty)$
  $\Leftrightarrow$
  (1) $u \in C^{1,1}(B_r), \sup_{B_r} |D_{ij}u| \leq M$
  (2) $u \geq 0$ and $0 \in \Gamma(u)$
  (3) $F(D^2u) = 1$ in $N(u)$

Remark We are going to use the case

$$3') \qquad F(D^2u - I) = 0 \qquad \text{in } N(u)$$

instead of 3). There is no difference in the argument by the lemmas in Section 4. We would like to think 3) for the technical simplicity

We are going to use the notion of thinness of $\Lambda(u)$ in B_r by the quantity

$$\delta_r(\Lambda) = \frac{m.d.(\Lambda \cap B_r)}{r}$$

where $m.d.(\Lambda \cap B_r)$ is the infimum of the distance between two pairs of parallel hyperplans such that $\Lambda \cap B_r$ is contained in the strip determined by them.
For example, if Λ is a ellipsoid, $m.d.(\Lambda)$ is the twice of the length of the shortest axis.

Our theorem says that if we have more than a critical amount $\sigma(r)$ of zero set $\Lambda(u)$ in the notion of $\delta_r(\Lambda)$, the free boundary in a neighborhood of zero is C^1. In addition, the critical amount $\sigma(r)$ goes to zero with r. We can state the same theorem 1 in [Ca2] for the nonlinear elliptic case.

Theorem 3.2.

Let u be in P_1.
Then there exists a positive, non decreasing function $\sigma(r)$, with $\sigma(0^+) = 0$ s.t.
If for some $r \in (0,1), \delta_r(\lambda(u)) > \sigma(r)$
Then the free boundary $\Gamma(u)$, in a neighborhood $B_{\tilde{r}}(0)$, is the graph of a C^1.

In addition, we can get the $C^{1,\alpha}$-regularity of the free boundary when the operator is homogeneous of degree one.

Theorem 3.3.

When F is homogeneous of degree one, the same assumption with Theorem (3.2) implies that the level surface of u,i.e. $\{u = \epsilon\}$ are uniformly $C^{1,\alpha}$ graphs and so is the free boundary $\Gamma(u)$.

3.1. **Convexity and Non Degeneracy.**

Lemma 3.4.

If $u \in P_1, x \in \overline{N(u)}$, then

$$\sup_{y \in B_\rho(x)} [u(y) - u(x)] \geq \frac{\rho^2}{2n\overline{\Lambda}}$$

Proof Let $x \in N(u)$

$$g(y) = u(y) - u(x) - \frac{|x-y|^2}{2n\overline{\Lambda}}$$

on $N(u) \cap B_\rho(x)$
$F(D^2 g) = F(D^2 u - \frac{I}{\Lambda}) \geq F(D^2 u) - \Lambda \frac{I}{n\overline{\Lambda}} \geq 0$
So by the maximum principle, the maximum is at $\partial(N(u) \cap B_\rho(x))$ and

$g(x) = 0$

On the other hand, $g < 0$ on $\partial\Omega$, so

$$\sup_{\partial B_\rho(x)} g \geq 0$$

$$\sup_{y \in \partial B_\rho(x)} [u(y) - u(x) - \frac{|x-y|^2}{2n\overline{\Lambda}}] \geq 0$$

$$\sup_{y \in B_\rho(x)} [u(y) - u(x)] \geq \sup_{y \in \partial B_\rho(x)} [u(y) - u(x)] \geq \frac{\rho^2}{2n\overline{\Lambda}}$$

Q.E.D.

Corollary 3.5.

If $u^{(m)} \in P_r$ and $u^{(m)} \to u_o$ uniformly in the compact subsets,then $u_o \in P_r$ and $\overline{N(u_o)} \supset \overline{\lim} N(u^{(m)})$

Proof We would like to use the contradiction argument.
We assume $\overline{N(u_o)}$ doesn't contain $\overline{\lim} N(u^{(m)})$
Then there is a y^o in $\overline{\lim} N(u^{(m)}) \setminus \overline{N(u_o)}$
We can find a small ϵ neighborhood of y_o s.t. $B_\epsilon(y^o) \cap \overline{N(u_o)}$ is empty.
Let's choose $\{y_m\}$ s.t. $y_m \in N(u^{(m)})$ and $y_m \to y^o$
By the lemma(3.2),

$$\sup_{y \in B_{\epsilon/2}(y_m)} u^{(m)}(y) \geq \sup_{y \in B_{\epsilon/2}(y_m)} [u^{(m)}(y) - u^{(m)}(y_m)] \geq \frac{1}{2n}\frac{\epsilon^2}{2} = \frac{\epsilon^2}{8n}$$

For a large m,

$$\sup_{y \in B_\epsilon(y^o)} u^{(m)}(y) \geq \sup_{y \in B_{\epsilon/2}(y^o)} u^{(m)}(y) \geq \frac{\epsilon^2}{8n} > 0$$

But $u^{(m)} \to 0$ uniformly on $B_\epsilon(y^o)$. It is a contradiction. Q.E.D.

Lemma 3.6.

If u is a nonnegative $C^{1,1}(\overline{B_\rho(x_o)}))$ function with a norm $||u||_{C^{1,1}} \leq M$ and for some point $Y_o \in \partial B(x_o), u(Y_o) = 0$ and $Du(Y_o) = 0$

,then for given $\delta, 0 < \delta < \frac{1}{2}$, and a pure second derivative u_{ii}, there is $\Gamma \subset B_{\delta^m \rho}(x_1)$ and x_1 s.t. $d(x_1, \partial B(x_o)) \geq \sqrt{\delta}\rho$ and a large $m > 0$ s.t.

$$|\Gamma| \geq C\delta^m |B_\rho|$$

$$\min_{\Gamma} u_{ii} \geq -CM\delta^{1/2}$$

for some universal constant C.

Proof $y_1 = (1-\delta)y_o + \delta x_o$.
For any $y_1 + z \in B_{\delta'}(y_1)$ s.t. $\delta' = \frac{\delta\rho}{4}$
, $|y_1 + z - y_o| \leq 2\delta\rho$, $|Du(y_1+z)| \leq M(2\delta\rho)$, $|u(y_1+z)| \leq \frac{M}{2}(2\delta\rho)^2$.
Let's choose a interval I_z from $y_1 + z$ along $\pm i^{th}$-direction which means $I_z = [y_1 + z, y_2 + z] = z + I$
s.t. $|I_z| = \frac{1}{2}\delta^{1/2}\rho$ and $I = [y_1, y_2]$.

$$0 \leq u(y_2+z) = u(y_1+z) \pm u_i(y_1+z)|I| + \int_I \int_I u_{ii}$$

$$0 \leq \int_{B_{\delta'}} u(y_2+z)dz = \int_{B_{\delta'}} u(y_1+z)dz \pm \int_{B_{\delta'}} u_i(y_1+z)dz|I| + \int_I \int_I \int_{B_{\delta'}} u_{ii}dz$$

$$\leq \{\frac{M}{2}(2\delta\rho)^2 \pm M(2\delta\rho)|I|\}|B_{\delta'}| + \int_I \int_I \int_{B_{\delta'}} u_{ii}dz$$

Let's choose the length of I_z as $\frac{1}{2}\delta^{1/2}\rho$

$$\frac{1}{|B_{\delta'}|}\int_I \int_I \int_{B_{\delta'}} u_{ii}dz \geq -CM\rho^2\delta^{3/2}$$

$$\max_{y \in I} \frac{1}{|B_{\delta'}|}\int_{B_{\delta'}} u_{ii}dz \geq -CM\delta^{1/2}$$

$$\exists \tilde{y} \in I \qquad s.t. \qquad \frac{1}{|B_{\delta'}|}\int_{B_{\delta'}} u_{ii}(\tilde{y}+z)dz \geq -CM\delta^{1/2}$$

Let $\tilde{\Gamma} = \{u_{ii}(\tilde{y}+z) \geq -\eta\}$ where $\eta = 2CM\delta^{1/2}$

$$\frac{1}{|B_{\delta'}|}\int_{B_{\delta'} \cap \tilde{\Gamma}} u_{ii} + \frac{1}{|B_{\delta'}|}\int_{B_{\delta'} \cap \tilde{\Gamma}^c} u_{ii} \geq -CM\delta^{1/2}$$

$$\frac{1}{|B_{\delta'}|}\int_{B_{\delta'} \cap \tilde{\Gamma}} u_{ii} \geq -CM\delta^{1/2} - \frac{1}{|B_{\delta'}|}\int_{B_{\delta'} \cap \tilde{\Gamma}^c} u_{ii}$$

$$u_{ii} \leq M \qquad \text{on } B_{\delta'} \cap \tilde{\Gamma}$$

$$u_{ii} \leq -\eta \qquad \text{on } B_{\delta'} \cap \tilde{\Gamma}^c$$

$$\begin{aligned} M\frac{|B_{\delta'} \cap \tilde{\Gamma}|}{|B_{\delta'}|} &\geq \frac{1}{|B_{\delta'}|}\int_{B_{\delta'}\cap\tilde{\Gamma}} u_{ii} \\ &\geq -CM\delta^{1/2} + \eta\frac{|B_{\delta'}| - |B_{\delta'} \cap \tilde{\Gamma}|}{|B_{\delta'}|} \\ &\geq -CM\delta^{1/2} + \eta - \eta\frac{|B_{\delta'} \cap \tilde{\Gamma}|}{|B_{\delta'}|} \end{aligned}$$

$$(M+\eta)\frac{|B_{\delta'} \cap \tilde{\Gamma}|}{|B_{\delta'}|} \geq \eta - CM\delta^{1/2}$$

Since $\eta = 2CM\delta^{1/2}$

$$\frac{|B_{\delta'} \cap \tilde{\Gamma}|}{|B_{\delta'}|} \geq \frac{CM\delta^{1/2}}{M+\eta} \geq \frac{CM\delta^{1/2}}{2M} \geq C\delta^{1/2}$$

$$|B_{\delta'} \cap \tilde{\Gamma}| \geq C\delta^{1/2}|B_{\delta'}| \geq C\delta^{n+\frac{1}{2}}\rho^n$$

since $\delta' = \frac{\delta\rho}{4}$. Let $\Gamma = B_{\delta'} \cap \tilde{\Gamma}$. Then $u_{ii} \geq -CM\delta^{1/2}$ on Γ and $|\Gamma| \geq C\delta^{n+\frac{1}{2}}\rho^n$ Q.E.D.

Now we are going to get lower estimates for the second derivative of the solution. At first we consider the convex operator F where $D_{ii}u(x)$ is a subsolution of the linearized equation.

Lemma 3.7.

F is convex, for any directional derivative

$$D_{ii}u(x) \geq -C|\log|x||^{-\epsilon} \qquad \textit{for small } |x|$$

Proof We will use the Harnack inequality inductively by shrinking the radius of the ball centered at 0. Let

$$D_{ii}u(x) \geq -M_k \qquad \text{on } B_{(\frac{1}{2})^k}$$

Let's choose any x_o s.t. $|x_o| \leq (\frac{1}{2})^{k+1}$.
We can find the biggest ball $B_\rho(x_o) \subset N(u)$,contacting the free boundary at one point y_o. Since $y_o \in \partial B_\rho(u) \cap \Lambda(u), u(y_o) = 0, Du(y_o) = 0.$

By the Lemma(3.6), for given $\delta > 0$ s.t. $0 < \delta < \frac{1}{2}$,
There are Γ, x_1,and m corresponding to Lemma(3.6). So

$$\min_{\Gamma} u_{ii} \geq -C_M \delta^{\frac{1}{2}}$$

If we use the lemma(5.1) in the Appendix,

$$u_{ii}(x_o) \geq -M_k + \gamma \delta^N (M_k - C\delta^{\frac{1}{2}})$$

Let's choose $C\delta^{\frac{1}{2}} = \epsilon M_k$ for some $\epsilon > 0$.
Then

$$u_{ii}(x_o) \geq -M_k + CM^{\tilde{N}} \qquad \text{for some } \tilde{N}$$

$$-M_{k+1} \geq -M_k + CM_k^{\tilde{N}}$$

A standard argument shows that

$$M_k \sim k^{-\epsilon} \sim |\log |x||^{-\epsilon}$$

Q.E.D.

The second case is the concave F which is depending on eigenvalues of u in R^2. The largest eigenvalue behaves like a supersolution, and two eigenvalues are balanced though the uniform elliptic operator F. Thanks to this balance, the smallest eigenvalue behaves like a subsolution.

Lemma 3.8.

If F is concave in R^2
F is depending only on eigenvalues,
then $D_{ll}u(x) \geq -C|\log |x||^{-\epsilon}$ for small $|x|$.

Proof$F(\lambda_1, \lambda_2) = 1$ where $\lambda_1 \leq \lambda_2$ and $F(0, \alpha) = 1$.
Let $\lambda_1 \geq -M_k$ on $B_{(\frac{1}{2})^k}$.

$\lambda_2 - \alpha \leq N_k$ for some N_k s.t. $F(-M_k, \alpha + N_k) = 1$ and $N_k > 0$

By the similar argument, there are Γ, x_1, m s.t. $\min_\Gamma(\lambda_1(x)) \geq -C_m \delta^{\frac{1}{2}}$

So $\max_\Gamma(\lambda_2(x) - \alpha) \leq \tilde{C_M} d\delta^{\frac{1}{2}}$ for some directional derivative l.

On the other hand, $D_{ll}u(x)$'s are sub-solutions of $Lw = 0$ where $Lw = F_{ij}(D^2u)D_{ij}w$

By the lemma(5.1) in the appendix,

$$(N_k - (D_{ll}u(x_o) - \alpha)) \geq \lambda\delta^N(N_k - \tilde{C}_M\delta^{\frac{1}{2}})$$

Let's choose $C\delta^{\frac{1}{2}} = \epsilon M_k$ for small $\epsilon > 0$

Then

$$D_{ll}u(x_o) - \alpha \leq N_k - CN_k^{\tilde{N}}$$

$$\lambda_2(x)|_{B_{\frac{1}{2}}^{k+1}} - \alpha \leq N_k - CN_k^{\tilde{N}}$$

By the uniform ellipticity of F,

$$-M_{k+1} \geq -M_k + \tilde{C}M_k^{\tilde{N}}$$

So $M_k \sim k^{-\epsilon} \sim C|\log|x||^{-\epsilon}$

Therefore

$$D_{ll}u(x) \geq \lambda_1(x) \geq -C|\log|x||^{-\epsilon}$$

for small $|x|$

Q.E.D.

Lemma 3.9.

If $u_{\epsilon_m}^{(m)} = \frac{1}{\epsilon^m}u^{(m)}(\epsilon_m x)$ *converges to* u_o *as* ϵ_n *converges to* 0

Proof

$$\begin{aligned} D_{ll}u_{\epsilon_m}^{(m)}(x) = D_{ll}u^{(m)}(\epsilon_m X) &\geq -C|\log|\epsilon_m x||^{-\epsilon} \\ &\geq -C|\log|\epsilon_m R||^{-\epsilon} \end{aligned}$$

if $|x| \leq R$

So $u_{\epsilon_m}^{(m)}(x) + \frac{C}{2}|\log|\epsilon_m R||^{-\epsilon}|x|^2$ is convex.

If ϵ_m goes to 0, it converges uniformly to $u_o(x)$ in $|x| \leq R$. Therefore

$u_o(x)$ is convex.

Q.E.D.

3.2. **the regularity of convex solution.** Let's define $P^* = \{u \in P_1 | u$ is convex $\}$.

Lemma 3.10.

If $u \in P_1^*$ *and* $\delta_1(\Lambda(u)) \geq \epsilon > 0$

$$\frac{1}{\epsilon_m^2} u(\epsilon_m x) \to u_o(x)$$

,then $\Lambda(u_o(x)) = \{x | < x, e_n > \leq 0\}$ *for some coordinate system.*

Proof $\Lambda(u_o(x))$ is a convex cone which is generated by $\Lambda(u)$
If $\Lambda(u_o(x))$ is not a half plane, for some polar coordinate,

$$\Lambda(u_o(x)) \subset \{x | x = (\rho\cos\theta, \rho\sin\theta, x_3, \cdots, x_n), \theta_o \leq |\theta| \leq \pi\}$$

where $\theta_o > \frac{\pi}{2}$
We choose θ_1 where $\frac{\pi}{2} < \theta_1 < \theta_0$, and α so that $\alpha\theta_1 = \pi$.
Let $w = D_l u_o$. Then $w = 0$ on $\partial\Lambda(u_o(x))$,and w is a Lipschitz function in virtue of $C^{1,1}$-regularity.

$$F(D^2 u_o) = 1$$

$$F_{ij}(D^2 u_o) D_{ij}(u_o)_l = 0$$

Let $a_{ij}(x) = F_{ij}(D^2 u_o)$ be a bounded measurable elliptic coefficients. We use a barrier function to estimate the w from below.

$$v = r^\alpha (e^{-\beta \sin\alpha\theta} - e^{-\beta})$$

Then, in the polar coordinate

$$Lv = a_{ij} D_{ij} v = a_{rr} D_{rr} v + \frac{a_{r\theta}}{r} D_{r\theta} v + \frac{a_{\theta\theta}}{r^2} D_{\theta\theta} v + b_r D_r v$$

$$\begin{aligned}
D_{rr}v &= \alpha(\alpha-1)(e^{-\beta sin\alpha\theta}-e^{-\beta}) \\
\frac{1}{r}D_{r\theta}v &= \alpha r^{\alpha-2}(-\beta\alpha\cos\alpha\theta)e^{-\beta\sin\theta} \\
\frac{1}{r^2}D_{\theta\theta}v &= r^{\alpha-2}((\beta\alpha\cos\alpha\theta)^2+\beta\alpha^2\sin\alpha\theta)e^{-\beta\sin\theta} \\
\frac{1}{r}D_{r}v &= \alpha r^{\alpha-2}(e^{-\beta\sin\alpha\theta}-e^{-\beta})
\end{aligned}$$

For large β, the term $\frac{1}{r^2}D_{\theta\theta}v$ will dominate and it is positive.

On the other hand, $a_{\theta\theta}\geq\overline{\lambda}>0$ where $\overline{\lambda}$ is a elliptic coefficient. If we choose large β, $Lv\geq 0$. Therefore v is a sub-solution and zero on $\partial\Lambda(u_o)$.

By the comparison theorem,$0<v<w$ in $\{x||\theta|<\theta_1\}$.

On the other hand, w is $C^{0,1}$. It means that w will decay to 0 faster than any r^α where $0<\alpha<1$. It contradicts against the decay rate of v, when r goes to 0.

Therefore $\partial\Lambda(u_o)$ is flat.

Q.E.D.

Lemma 3.11.

For any $\epsilon>0,\delta>0$, there exists a $\lambda=\lambda(\epsilon,\delta)$ such that if $u\in P_1^*(M)$ and $\delta_1(\Lambda(u))>\epsilon$, then in an appropriate system of coordinates

$$\Lambda(u)\supset B_\lambda\cap\{x:\alpha(x,-e_n)<\frac{\phi}{2}-\delta\}$$

$$N(u)\supset B_\lambda\cap\{x:\alpha(x,e_n)<\frac{\phi}{2}-\delta\}$$

Proof Look Lemma4.4 in [Fr]

Q.E.D.

We are going to write Lemma(4.5) in Chapter 2 of [Fr] for the completeness of the proof.

Lemma 3.12.

If $u \in P_1^*$ *and* $\delta_{\frac{1}{4}}(\Lambda(u)) \geq \epsilon > 0$,
then there is an appropriate system of coordinates, $\mu > 0$, *a* C^1*-function s.t.*

$$\Lambda(u) \cap B_\mu = \{x | x_n \leq g(x')\}$$

where $x' = (x_1, \cdots, x_{n-1})$

Proof Since $\Lambda(u) \cap B_{1/2}(0) \supset \Lambda(u) \cap B_{\frac{1}{4}}(y)$ if $|y| < \frac{1}{4}$,

$$\text{mindiam}(\Lambda(u) \cap B_{\frac{1}{2}}(u)) \geq \text{mindiam}(\Lambda(u) \cap B_{\frac{1}{4}})$$

$$= \frac{1}{4}\delta_{1/4}(\Lambda(u)) \geq \frac{\epsilon}{4}$$

So we can use the previous lemma w.r.t. any point $y \in \Gamma, |y| < \frac{1}{4}$.
By the previous lemma $y = 0 \in \Gamma$,for any $\delta = \frac{1}{m}$,there is a system of coordinate$(e_i^m)(i = 1, \cdots, n)$ s.t.

$$\begin{aligned} \Lambda(u) &\supset B_\lambda \cap \{x | \alpha(x, -e_n) < \frac{\pi}{2} - \delta\} \\ N(u) &\supset B_\lambda \cap \{x | \alpha(x, -e_n) < \frac{\pi}{2} - \delta\} \end{aligned} \tag{3.1}$$

holds with $e_n = e_n^m$.
$\Lambda(u)$ is convex. So

$$\begin{aligned} \{x^o + te_n^m\} \cap \Lambda(u) =& \{x^o + te_n^m | t < t^o\} \\ \{x^o + te_n^m\} \cap N(u) =& \{x^o + te_n^m | t > t^o\} \end{aligned} \tag{3.2}$$

If $x = \sum_i x_i e_i^m$, then $x^o + t^o e_n^m$ can be represented by $x_n = g^m(x')$ where $x' = (x_2, \cdots, x_n)$, and $\Lambda(u) = \{x_n \leq g^m(x')\}$. Since (3.2) hold for (e_i^m) in B_{λ_m} (Λ_m depends on $\delta = \frac{1}{m}$ and $\epsilon; \lambda_m \to 0$), it follows that for a suitable choice of the $e_i^m, 1 \leq i \leq m-1$,and for a subsequence,

$$e_i^m \to e_i^o$$

$$(e_i^o) = T^m(e_i^m), \qquad \text{where } T^m \text{ is an orthogonal matrix}$$

Thus $T^m \to I$ where I is an identity matrix.

From(3.1),

$$\frac{|g^m(x') - g^m(0)|}{|x'|} \le \frac{C}{m} \qquad |x'| \le \lambda_m$$

Therefore $x_n = g^o(x)$ is a representation of $\Lambda(u)$ and g^o is differentiable at 0 with zero gradient.

We can do the same argument about $y \in \Lambda(u)$ with $|y|$ small enough.

Thus there is a system of coordinates $(e_i^{m,y})$,and $\Lambda(u)$ can be a limiting on $(e_i^{o,y})$ as $(m \to \infty)$,and $\Gamma(u)$ can be represented in a ρ_o-neighborhood of y where ρ_o is independent of y.

$$x_n \le g^{m,y}(x') \qquad \text{or} \qquad x_n \le g^{o,y}(x')$$

where $x = \sum x - ie_i^{m,y}$ or $x = \sum x_i E_i^{0,y}$,respectively.

Therefore

$$\frac{|g^{o,y}(x') - g^o(0)|}{|x'|} \le \beta_m \qquad \text{if } |x'| < \lambda_m, \beta_m \to 0 \tag{3.3}$$

The system of coordinates $e_i^{m,y}$ is related to $e_i^{0,y}$ by

$$e_i^{m,y} = T^{m,y}(e_i^{o,y})$$

where $T^{m,y}$ converges uniformly to I with respect to y.

$$e_i^{o,y} = T^{o,y}(e_i^o)$$

and

$$T^{o,y} \to I \qquad \text{if } y \to 0 \tag{3.4}$$

by the uniform size of the cones.

We can rewrite (3.3) in terms of the system of coordinates (e_i^o).

$$g^o(y+h) - g^0(y) = hc(y) + ho(1)$$

where $0(1) \to 0$ as $|h| \to 0$,uniformly w.r.t. y, and $|c(y)| \le C$.

Therefore $g^o \in C^1$. Q.E.D.

3.3. **Improvement of Convexity for Γ of large minimum diameter.**

Lemma 3.13.

Given ϵ, δ, there exits $\rho_o(\epsilon,\delta)$ s.t.
If $u \in P_1$, and $\delta_\rho(\Lambda) \geq \epsilon$ for $\rho \leq \rho_o$,
then in an appropriate system of coordinates,

$$\Lambda(u) \supset B_{\rho\frac{\lambda}{2}} \cap \{x| - x_n \geq 2\rho\gamma\delta\}$$
$$N(u) \supset B_{\rho\frac{\lambda}{2}} \cap \{x|x_n \geq 2\rho\lambda\delta\}$$

Proof Otherwise, there exists $u_m \in P_1$, and ρ_m , where $\rho_m \to 0$, contradicting the lemma(3.10).

$$u^{(m)}_{\rho_m} \to u_o \in P^*$$
$$\delta_{\rho_o}(\Gamma(u_o)) \geq \overline{\lim}\, \delta_{\rho_o}(\Lambda(u^{(m)}_{\rho_m})) \geq \epsilon$$

We obtain a contradiction to Lemma(3.11). Q.E.D.

Remark

$$\Gamma(u) \supset B_{\rho\frac{\lambda}{2}} \cap \{x| - x_n \geq 2\rho\gamma\delta\}$$
$$\delta_{\rho\frac{\lambda}{2}}(\Gamma(u)) \geq \frac{\frac{\rho\lambda}{2} - 2\rho\lambda\delta}{\frac{\rho\lambda}{2}} = 1 - 4\delta \geq \frac{1}{2} \geq \epsilon$$

for

$$\epsilon, \delta < \frac{1}{8}$$

Corollary 3.14.

Given $\epsilon, \delta < \frac{1}{8}$,
there is an appropriate system of coordinates with $n-$vector $e^{(k)}_n$

$$\Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^k} \cap \{x| - x^{(k)}_n \geq \rho(\frac{\lambda}{2})^k 4\delta\}$$
$$N(u) \supset B_{\rho(\frac{\lambda}{2})^k} \cap \{x|x^{(k)}_n \geq \rho(\frac{\lambda}{2})^k 4\delta\}$$

Lemma 3.15.

Given $\epsilon, \delta < \frac{1}{8}$,
then, there is a $\mu < 1$ *s.t.*

$$D_{ii}u|_{N(u)\cap B_{\rho(\frac{\lambda}{2})^k}} \geq -M\mu^{k-1}$$

Proof We would like to consider two cases. The fist one is when the operator F is convex,and the second,when F is concave in R^2.
Case 1) If F is convex ,the any second derivative $D_{ii}u$ is a supper solution of a linear operator $L = a_{ij}(x)D_{ij}$ where $a_{ij}(x) = F_{ij}(D^2u)$.
There is

$$\phi(x) \qquad \text{s.t.} \qquad \begin{cases} \phi(x) = 0 & \text{when } x_n = -\frac{1}{2} - \delta \text{ where } \delta > 0 \\ \phi(x) = -1 & \text{on } \partial B_1 \cap \{x_n > -\frac{1}{2}\} \\ -1 < \phi < 0 & \text{on } \partial B_1 \cap \{-\frac{1}{2} - \delta < x_n < -\frac{1}{2} \end{cases}$$

Let

$$h \qquad \text{s.t.} \qquad \begin{cases} Lh &= 0 \qquad \text{on } B_1 \cap \{x_n < -\frac{1}{2} - \delta\} \\ h &= \phi \qquad \text{on } \partial B_1 \cap \{x_n < -\frac{1}{2} - \delta\} \end{cases}$$

Therefore

$$-\mu = \inf_{B_{\frac{1}{2}}} h > -1$$

By the induction,

$$D_{ii}u|_{N(u)\cap B_{\rho(\frac{\lambda}{2})^k}} \geq -M\mu^{k-1}$$

Case 2)$F(\lambda_1, \lambda_2) = 1$ and F is concave.
Claim: If $\lambda_1 \geq -M \qquad$ in $N(u) \cap B_{\rho(\frac{\lambda}{2})}$
,then $\lambda_1 \geq -M\mu \qquad$ in $N(u) \cap B_{\rho(\frac{\lambda}{2})^2}$
If this claim is true, we can use it inductively to get

$$\lambda_1 \geq -M\mu^{k-1} \qquad \text{in } N(u) \cap B_{\rho(\frac{\lambda}{2})^K}$$

Now we are going to prove the claim. By the lemma(3.10)

$$\Lambda(u) \supset B_{\rho\frac{\lambda}{2}} \cap \{x| - x_n \geq 2\rho\lambda\delta\}$$

$$N(u) \supset B_{\rho \frac{\lambda}{2}} \cap \{x | x_n \geq 2\rho\lambda\delta\}$$

Let $F(0, \alpha) = 1$.

$\lambda_1 \geq -M$ if and only if $\lambda_2 - \alpha \leq N$ for some $N > 0$, since F is elliptic. There is

$$\phi(x) \begin{cases} = 0 & \text{when } x_n = -\frac{1}{2} - \delta (\delta > 0) \\ = N & \text{on } \partial B_1 \cap \{x_n > -\frac{1}{2}\} \\ 0 < \phi < N & \text{on } \partial B_1 \cap \{-\frac{1}{2} - \delta < x_n < -\frac{1}{2}\} \end{cases}$$

Let $Lw = F_{ij}(D^2 u) D_{ij} w = a_{ij}(x) D_{ij} w$ where $a_{ij}(x) = F_{ij}(D^2 u)$. Then there is

$$w \qquad \text{s.t.} \qquad \begin{cases} Lw & = 0 \qquad \text{on } B_1 \cap \{x_n < -\frac{1}{2} - \delta\} \\ w & = \phi \qquad \text{on } \partial B_1 \cap \{x_n < -\frac{1}{2} - \delta\} \end{cases}$$

For all direction l, $D_{ll}u - \alpha$ is a subsolution of L , which is less than N on $\partial B_1 \cap \partial N(u)$ since $\lambda_2 - \alpha \leq N$,and less than 0 on $B_1 \cap \partial N(u)$. Thus $D_{ll}u - \alpha \leq w$ in $B_1 \cap N(u)$. By Harnack inequality,

$$\sup_{B_{1/2} \cap N(u)} w \leq \gamma N$$

implies

$$\sup_{B_{1/2} \cap N(u)} (D_{ll}u - \alpha) \leq \gamma N$$

for all l.

$$\sup_{B_{1/2} \cap N(u)} (\lambda_2 - \alpha) \leq \lambda N$$

$$\inf_{B_{1/2} \cap N(u)} \lambda_1 \geq -\mu M$$

Therefore the claim is true. By the remark after the claim, the lemma is true. Q.E.D.

3.4. **Regularity of the Free Boundary.**

Lemma 3.16.

If $u \in C^{1,1}, u(0) = 0, D_{ii}u > -\tau$, *then for* $0 < t < 1$,

$$u(tx) \leq |x|^2 \tau + u(x)$$

Proof Let $\max_{0\le t\le 1} u(tx) = u(t_0x)$.

If $t_o < 1$, then $0 = u(0) = u(t_ox) + \int\int D_{ii}u \ge u(t_ox) - \tau|x|^2$

So $u(tx) \le u(t_ox) \le \tau|x|^2$

If $t_o = 1$, then $u(tx) \le u(x)$ for $0 \le t \le 1$.

So $u(t,x) \le u(x) + \tau|x|^2$ for $0 \le t \le 1$. Q.E.D.

Corollary 3.17.

If $x_o \in N(u) \cap [B_{\rho(\frac{\lambda}{2})^k} \backslash B_{\rho(\frac{\lambda}{2})^{k+1}}]$

then there is y_o s.t.

$$|x_o - y_o| \le \theta_k = \sqrt{Mn}\mu^{(k-2)/2}\rho(\frac{\lambda}{2})^{k-2}$$

$$\{sy_o\} \subset N(u)$$

if $|sy_o| < \rho(\frac{\lambda}{2})^{k-1}$, and $s > 1$.

Proof By the lemma (3.2)

$$\sup_{\partial B_{\theta_k(x)}} u \ge \frac{\theta_k^2}{2n}$$

There is $y \in \partial B_{\theta_k}(x)$ s.t. $u(y) \ge \frac{\theta_k^2}{2n}$

By the lemma (3.14) and $D_{ii}u|_{N(u)\cap B_{\rho(\frac{\lambda}{2})^{k-1}}} \ge -M\mu^{k-2}$,

if $|sy| < \rho(\frac{\lambda}{2})^{k-1}$,where $t = \frac{1}{s}, tx = y$,then

$$\begin{aligned}\frac{1}{2n\overline{\lambda}}(Mn\mu^{k-2}\rho^2(\frac{\lambda}{2})^{2(k-2)}) \le u(y) \ &\le (\rho(\frac{\lambda}{2})^{k-1})^2M\mu^{k-2} + u(sy)\\ &= M\rho^2(\frac{\lambda}{2})^{2(k-1)}\mu^{k-2} + u(sy)\end{aligned}$$

$$0 < \frac{1}{2}M\rho^2(\frac{\lambda}{2})^{2(k-1)}\mu^{k-2} \le u(sy)$$

Therefore $sy \in N(u)$

Q.E.D.

Remark At the previous lemma (3.15),the angle between x and y,

$$\alpha(x,y) \le C\frac{\mu^{\frac{k-2}{2}}}{\lambda}$$

Lemma 3.18.

There is $k_o(\epsilon,\delta) > 0$ s.t. for suitable constant C,in an appropriate system of coordinates

(1) $\Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^{k_o}} \cap \{x|\alpha(x,-e_n) \leq \frac{\pi}{2} - C\delta\}$
(2) $N(u) \supset B_{\rho(\frac{\lambda}{2})^{k_o}} \cap \{x|\alpha(x,+e_n) \leq \frac{\pi}{2} - C\delta\}$

Proof About (1), if (1) is not true, there is

$$x \in (B_{\rho(\frac{\lambda}{2})^l} \backslash B_{\rho(\frac{\lambda}{2})^{l+1}}) \cap N(u)$$

for some large $l \geq k_o$.
By the lemma (3.15)
there is y s.t. $|x-y| \leq \theta_l = \sqrt{Mn}\mu^{(l-2)/2}\rho(\frac{l}{2})^{l-2}$

$$sy \in N(u) \qquad \text{s.t. } |sy| \leq \rho(\frac{\lambda}{2})^{l-1}$$

There is $s_1 > 1$, s.t. $|s_1 y| < \rho(\frac{\lambda}{2})^{l-1}$
Let $x_1 = s_1 y \in B_{\rho(\frac{\lambda}{2})^{l-1}} \backslash B_{\rho(\frac{\lambda}{2})^l}$
Then

$$\alpha(x,x_1) \leq C\frac{\mu^{\frac{l-2}{2}}}{\lambda}$$

$$x_1 \in B_{\rho(\frac{\lambda}{2})^{l-1}} \backslash B_{\rho(\frac{\lambda}{2})^l}$$

By the induction,

$$\alpha(x_k,x_{k+1}) \leq C\frac{\mu^{\frac{l-2-k}{2}}}{\lambda}$$

$$x_{k+1} \in B_{\rho(\frac{\lambda}{2})^{l-(k+1)}} \backslash B_{\rho(\frac{\lambda}{2})^{l-k}}$$

For some $\tilde{k}, x_{\tilde{k}} \in \partial B_{\rho(\frac{\lambda}{2})^{k_o}}$

$$\begin{aligned}
\alpha(x, x_{\tilde{k}}) &= \sum_{i=1}^{\tilde{k}} \alpha(x_{i-1}, x_i) \\
&\leq \sum_{i=1}^{\tilde{k}} C \frac{\mu^{\frac{l-1-i}{2}}}{\lambda} \\
&= C \frac{\mu^{\frac{l-2}{2}}}{\lambda} \frac{(\frac{1}{\mu})^{\frac{\tilde{k}}{2}}) - 1}{\frac{1}{\mu^{\frac{1}{2}}} - 1} \\
&= C \frac{\mu^{\frac{l-2}{2}} (\frac{1}{\mu})^{\frac{\tilde{k}-1}{2}}}{\lambda(1 - \mu^{\frac{1}{2}})} \\
&= C \frac{\mu^{\frac{l-1}{2} - \frac{\tilde{k}-\frac{1}{2}}{2}} - \mu^{\frac{l}{2}}}{\lambda(-\mu^{\frac{1}{2}} + 1)} \\
&\leq C \frac{\mu^{\frac{l-2-\tilde{k}}{2}}}{\lambda(1-\mu)} = \frac{C}{\lambda(1-\mu)} \mu^{k_o - 2} \leq \frac{\delta}{4}
\end{aligned}$$

for large k_o

$C > 5$ in (1)

$$\begin{aligned}
\alpha(x, -e_n) &< \frac{\pi}{2} - 5\delta \\
\alpha(x_{\tilde{k}}, -e_n) &< \frac{\pi}{2} - 4\delta
\end{aligned}$$

$x_{\tilde{k}} \in \partial B_{\rho(\frac{\lambda}{2})^{k_o}}$ and $x_{\tilde{k}} \in N(u)$. It is a contradiction against the lemma(3.16)

About (2)

We assume (2) is not true for any positive constant C and integer $k_o > 0$, then for any large constant C^*,there is

$$x_o \in \Gamma(u) \cap B_{(\frac{1}{2})\rho(\frac{\lambda}{2})^{k_o}} \cap \{x | \alpha(x, e_n) \leq \frac{\pi}{2} - C^* \delta\}$$

By the previous argument,$\Gamma(u)$ contains a cone with vertex x_o opening $\frac{\pi}{2} - C\delta$.Then the axis of the cone must be away from $-e_n$ by $C_o\delta$ by Corollary(3.14).

If C^* is large,0 is an interior point of the cone. It is a contradiction. Q.E.D.

Proof of Theorem(3.2) We can use the same argument which is used in the convex case.
To find $\sigma(r)$, we are going to choose r corresponding to σ.
If $\delta_r > \sigma$, then for any point a neighborhood taken as a new center we will have
$\delta_s > \frac{\sigma}{2}$ for some $s < r = \rho(\frac{\sigma}{2}, \frac{1}{8})$ from Lemma(3.13). Then for each r, there is a $\sigma(r)$ through the relation $r = \frac{1}{2}\rho(\frac{\sigma}{2}, \frac{1}{8})$. Q.E.D.

4. $C^{1,\alpha}$-regularity of the free boundary

Now we would like to show the $C^{1,\alpha}$ regularity of the free boundary in any dimension for the convex operator , and in two dimensions for the concave operator.

Lemma 4.1.

If u is convex , in P_1, and $\delta(\Lambda(u) \cap B - 1) \geq \delta_o > 0$,
then there are a small $r_o(\delta_o)$, and ϵ_o s.t. for some system of coordinates if $\epsilon < \epsilon_o$, then
$[D_{e_n} + \epsilon D_{e_j}]u(x) \geq \epsilon d(x, \Gamma(u))$

Proof

By the scalling, it is enough to prove the lemma for $d(x, \partial\Omega) = 1$.
The nondegeneracy, Lemma(3.4) says $\sup_{B_1(x_o)} u \geq \frac{1}{2n\bar{\Lambda}}$.
So we can find a point y s.t. $u(y) \geq \frac{1}{2n\bar{\lambda}}$.
Let's join y and a point on the free boundary in B_1 by a line segment.

Since u is convex, the line segment and the normal direction to the free boundary at x_o have an angle which is smaller than ϵ_o.
In addition, the length of this line is of order 1.
Since u is bigger than $\frac{1}{2n\bar{\lambda}}$ at one end of the line segment and 0 at the other end of it,
we can say $D_{e_n+\epsilon e_\tau}u \geq c_o > 0$ at a point y_1 on the line segment for an uniform constant c_o.
$C^{1,1}$-estimate implies that y_1 is far from the free boundary uniformly.
On the other hand, $D_{e_n+\epsilon e_\tau}u$ is a solution of the linearized equation.
By the Harnack inequality on the chains of balls connecting y_1 and x, $D_{e_n+\epsilon e_\tau}u \geq c_1 > 0$ for a uniform constant c_1.
Q.E.D

Remark
Let's define $u_k = \frac{1}{\epsilon_o^{2k}}u(\epsilon^k x)$.
By $C^{1,1}$-estimate of u_k and Lemma(4.1), u_k converges to a convex function uniformly in C^1.
From the Lemma(4.1), we can say that ,for a given $\lambda > 0$, there is a k s.t. $D_{e_n+\epsilon e_\tau}u_k \geq [d(x, \partial\Omega) - C\lambda]$
where $u_k = \frac{1}{\epsilon_o^{2k}}u(\epsilon^k x)$.

Now we are going to show the gradient of the solution is increasing in a cone of direction even though it is not convex

Lemma 4.2.

If u is in P_1 and $\delta(\Lambda(u) \cap B-1) \geq \delta_o > 0$,
then there are a small $r_o(\delta_o)$, and ϵ_o

s.t. for some system of coordinates
if $\epsilon < \epsilon_o$, *then*
$[D_{e_n} + \epsilon D_{e_j}]u(x) \geq \epsilon d(x, \Gamma(u))$

We know by using the argument in Lemma(4.1) that the result of this Lemma is true as soon as we get $[D_{e_n} + \epsilon D_{e_j}]u(x)$ is nonnegative From the remark above, we know $D_{e_n+\epsilon e_\tau} u_k$ is nonnegative at the point outside of γ-neighborhood of the free boundary.
Now we are going to show $D_{e_n+\epsilon e_\tau} u_k \geq 0$
in the region $B_{r_o/2} \cap N_\gamma(\Gamma(u_k))$.
Let's use contradiction argument.
Suppose that for $x_o \in N_\gamma(\Gamma(u_k))$ s.t.

$$D_{e_n+\epsilon e_\tau} u_k(x_o) < 0$$

Consider the auxiliary function

$$h(x) = u_k(x) - \frac{1}{2n\overline{\Lambda}}|x - x_o|^2 - D_{e_n+\epsilon_\tau} u_k(x)$$

in $B_\rho(x_o) \cap N(u_k)$.
Since the operator F is homogeneous of degree one, $D_{e_n+\epsilon e_\tau} u_k$ is a solution of the linearized equation.
$Lh = F_{ij}(D^2u)D_{ij}h = 0$ in a domain, and $h(x_o) > 0$.
Then $h > 0$ on $\partial(B_\rho(x_o) \cap N(u_k))$, since $h \leq 0$ on $\partial N(u_k)$.
Therefore $h(x) > 0$ for some x along ∂B_ρ.
At x, we get

$$u_k(x) - D_{e_n+\epsilon e_\tau} u_k \geq C\rho^2$$

On the other hand,

$$u_k(x) - D_{e_n+\epsilon e_\tau} u_k \leq C_1 d^2(x, \partial\Omega) - C_2 d(x, \partial\Omega) + C_3\gamma$$

That is

$$\rho^2 \leq C_1 d^2 - C_2 d + C_3\gamma$$

,and $d \leq \rho + \gamma = d(x, x_0) + d(x_o, \partial N(u_k))$
Hence

$$\rho^2 \leq C_1(\rho+\gamma)^2 - C_2(\rho+\gamma) + C_3\gamma$$

This is a contradiction for small enough $\rho \leq \frac{r_o}{2}$, and much smaller γ
Q.E.D.

Proof of Theorem(3.3) We are going to use the potential theory in [Fa]. At this point we have a smooth solution in $N(u)$ by the interior estimate of the solution and the C^1 regularity of the free boundary,Theorem(3.2). For any direction l,

$$Lu_l = F_{ij}(D^2u)D_{ij}u_l = 0 \qquad \text{in } N(u)$$

Let $a_{ij}(x) = F_{ij}(D^2u)$. Then $a_{ij}(x)$ is smooth in $N(u)$.
From Lemma(4.2), $D_{e_n+\epsilon_o e_\tau}u$ and $D_{e_n}u$ are positive. Theorem (1.2.2) in [Fa] implies the C^α regularity of

$$\frac{u_\nu + \epsilon u_\tau}{u_\nu}$$

,where ν is the normal direction,and τ is the tangential direction.
In addition, the Hölder norms are uniform and defined all the way to the free boundary .
We can send ϵ to 0. Therefore so is $\frac{u_\tau}{u_\nu}$.
On the other hand,

$$\frac{u_\tau}{u_\nu} = D_\tau f_\epsilon(x_1, \cdots, x_{n-1})$$

along any level surface $\{x_n = f_\epsilon\} = \{u = \epsilon\}$.
Let's send ϵ to 0.
Then the free boundary will be $C^{1,\alpha}$.
Q.E.D.

5. THE GENERAL CASE

We would like to consider the general $C^{2,\alpha}$ obstacle.
Due to the remark in the beginning of the Section 3, we can consider the following standard case without the loss of the generality. $X = (x_1, \cdots, x_n) \in R^n$
$B_r(x) = \{y \in R^n | \parallel x - y \parallel < r\}$
u is the nonnegative function on some domain$D \subset R^n$
$\Gamma(u) = \{x \in D | u(x) = 0\}$
$N(u) = \{x \in D | u(x) > 0\}$
$\Lambda(u) = \partial\Gamma(u) \cap \partial N(u)$

- $F(u)$ is the uniformly elliptic operator and $F \in C^2$ i.e.
  $\overline{\lambda} \parallel N \parallel \leq F(M+N) - F(M) \leq \overline{\Lambda} \parallel N \parallel$
  where $N \geq 0$ and $M, N :$ n x n symmetric matrix
- $u \in P_r(0 < r \leq \infty)$
  $\Leftrightarrow$
  (1) $u \in C^{1,1}(B_r), \sup_{B_r} |D_{ij}u| \leq M$
  (2) $u \geq 0$ and $u \in \Gamma(u)$
  (3) $F(D^2u + D^2\phi) = 0$ in $N(u)$ where $f(x) \geq \nu > 0$
  $F(D^2\phi) \leq -\nu < 0$ and $||\phi||_{C^{2,\frac{1}{2}}} < \infty$

Many of the arguments are similar with the special case, Section 3, except for the $C^{1,\alpha}$-regularity of the free boundary. We would like to point out only different part in the similar case.

Theorem 5.1.

Let u be in P_1.
Then there exists a positive, non decreasing function $\sigma(r)$, with $\sigma(0^+) = 0$ s.t.

If for some $r \in (0,1), \delta_r(\lambda(u)) > \sigma(r)$

Then the free boundary $\Gamma(u)$*, in a neighborhood* $B_{\tilde{r}}(0)$*, is the graph of a* C^1*.*

Theorem 5.2.

When F *is homogeneous of degree one, the same condition with Theorem(5.1) implies that for some* $r* > 0$*,*

the free boundary $\Gamma(u)$*, in a neighborhood* $B_{r*}(0)$*, is the graph of a* $C^{1,\alpha}$*.*

Lemma 5.3.

If $u \in P_1, x \in N(u)$*,then*

$$\sup_{y \in B_\rho(x)} [u(y) - u(x)] \geq \frac{\rho^2}{2n\overline{\Lambda}}$$

Proof Let $\tilde{F}(D^2u, x) = F(D^2u + D^2\phi) - F(D^2\phi) = f(x) \geq \nu > 0$

$\tilde{F}$ is uniformly elliptic with the elliptic constant $\overline{\lambda}, \overline{\Lambda} > 0$

Then we can use the same argument in the proof of lemma(3.2). Q.E.D.

Lemma 5.4.

F *is convex, for any directional derivative*

$$D_{ll}u(x) \geq -C|\log|x||^{-\epsilon} \qquad \textit{for small } |x|$$

Proof We will use the Harnack inequality inductively by shrinking the radius of the ball centered at 0. Let

$$D_{ll}u(x) \geq -M_k \qquad \text{on } B_{(\frac{1}{2})^k}$$

Let's choose any x_o s.t. $|x_o| \leq (\frac{1}{2})^{k+1}$.

We can find the biggest ball $B_\rho(x_o) \subset N(u)$, contacting the free boundary at one point y_o. Since $y_o \in \partial B_\rho(u) \cap \Lambda(u), u(y_o) = 0, Du(y_o) = o$.

By the lemma(3.6), for given $\delta > 0$ s.t. $0 < \delta < \frac{1}{2}$, there are Γ, x_1,and m corresponding to the lemma(3.6). So

$$\min_{\Gamma} u_{ii} \geq -C_M \delta^{\frac{1}{2}}$$

Since $F(D^2u + D^2\phi) = 0$ and F is convex,

$$L(u_{ll} + \phi_{ll}) \leq 0$$

where $Lw = a_{ij}(x)D_{ij}w$ and $a_{ij}(x) = F_{ij}(D^2u + D^2\phi)$
If we use the lemma(5.1) in the Appendix,

$$u_{ii}(x_o) + \phi_{ll}(x_o) \geq \delta^n \min_{\Gamma}(u_{ll} + \phi_{ll}) + (1 - \gamma\delta^N) \min_{B_\rho(x_o)}(u_{ll} + \phi_{ll})$$

$$\begin{aligned} u_{ll}(x_o) &\geq \min_{B_\rho(x_o)} \phi_{ll} - \phi_{ll}(x_o) - M_k + \alpha\delta^N(M_k - C\delta^{1/2}) \\ &\geq -M_k + \alpha\delta^N(M_k - C\delta^{1/2}) - ||\phi_{ll}||_{C^\alpha}\rho^\alpha \end{aligned}$$

Let's choose $C\delta^{\frac{1}{2}} = \epsilon M_k$ for some $\epsilon > 0$.

Then

$$u_{ii}(x_o) \geq -M_k + CM^{\tilde{N}} \qquad \text{for some } \tilde{N}$$

$$-M_{k+1} \geq -M_k + CM_k^{\tilde{N}} - C2^{-\beta k}$$

Therefore

$$M_k \sim k^{-\epsilon} \sim |\log|x||^{-\epsilon}$$

Q.E.D.

Lemma 5.5.

If F is concave in R^2

F is depending only on eigenvalues,

then $D_{ll}u(x) \geq -C|\log|x||^{-\epsilon}$ for small $|x|$.

Proof $D^2\phi$ is continuous at 0.Let $\lambda_1(D^2\phi(0)) = \beta$

There is α s.t. $F(\beta, \alpha) = 0$

$F(\lambda_1, \lambda_2) = 0$ where $\lambda_1 \leq \lambda_2$

and $\lambda_1 = \lambda_1(D^2u + D^2\phi), \lambda_2 = \lambda_2(D^2u + D^2\phi)$

Let $\lambda_1 - \beta \geq -M_k$ on $B_{(\frac{1}{2})^k}$.

$\lambda_2 - \alpha \leq N_k$ for some N_k s.t. $F(\beta - M_k, \alpha + N_k) = 0$ and $N_k > 0$

By the similar argument with Lemma(3.6) by using the geodesics on the graph of u, there are Γ, x_1, m s.t.

$$\min_{\Gamma}(\lambda_1(D^2u)) \geq -C_m\delta^{\frac{1}{2}}$$

$$\min_{\Gamma}(\lambda_1(D^2u + D^2\phi)) \geq -C_m\delta^{\frac{1}{2}} + \min_{\Gamma}\lambda_1(D^2u)$$

$$\min_{\Gamma}(\lambda_1 - \beta) \geq -C_M\delta^{\frac{1}{2}} + \min_{\Gamma}\lambda_1(D^2\phi) - \beta \geq -C_M\delta^{\frac{1}{2}} - C(\frac{1}{2})^{\frac{k}{2}}$$

since $D^2\phi$ is Hölder continuous.

Therefore

$$\max_{\Gamma}(\lambda_2(x) - \alpha) \leq \tilde{C_M}\delta^{\frac{1}{2}} + \tilde{C}(\frac{1}{2})^{\frac{k}{2}}$$

$$\max_{\Gamma}(D_{ll}u(x) - \alpha) \leq \tilde{C_M}\delta^{\frac{1}{2}} + \tilde{C}(\frac{1}{2})^{\frac{k}{2}}$$

for all directional derivative l. $D_{ll}u(x)$'s are sub-solutions of $Lw = 0$ where $Lw = F_{ij}(D^2u)D_{ij}w$

By the lemma(6.1) in the appendix,

$$(N_k - (D_{ll}u(x_o) - \alpha)) \geq \lambda\delta^N(N_k - \tilde{C}_M\delta^{\frac{1}{2}} - \tilde{C}(\frac{1}{2})^{\frac{k}{2}})$$

We choose $C\delta^{\frac{1}{2}} = \epsilon M_k$ for small $\epsilon > 0$

Then

$$D_{ll}u(x_o) - \alpha \leq N_k - CN_k^{\tilde{N}} + \tilde{C}(\frac{1}{2})^{\frac{k}{2}}$$

$$\lambda_2(x)|_{B^{k+1}_{\frac{1}{2}}} - \alpha \le N_k - CN_k^{\tilde{N}}\tilde{C}(\frac{1}{2})^{\frac{k}{2}}$$

By the uniform ellipticity of F,

$$-M_{k+1} \ge -M_k + \tilde{C}M_k^{\tilde{N}} - C2^{-\frac{k}{2}}$$

So $M_k \sim k^{-\epsilon} \sim C|\log|x||^{-\epsilon}$

Therefore

$$D_{ll}u(x) \ge \lambda_1(x) \ge -C|\log|x||^{-\epsilon}$$

for small $|x|$

Q.E.D.

Proof of Theorem(5.1) By means of Lemma (5.3)(5.4)(5.5), the similar argument with Theorem(3.2) implies Theorem (5.1). Q.E.D.

Lemma 5.6.

Let w_1, w_2 be two solutions of the same problem in B_1, with

(1) $F(D^2w_i + D^2\phi_i) = 0$ *in* $N(w_i) = \{w_i > 0\}$.
(2) $w_1 = w_2$ *on* ∂B_1
(3) $|D^2\phi_1 - D^2\phi_2|_{L^\infty} < \epsilon$

Then $|w_1 - w_2| < C\epsilon$.

Proof Let's compare $w_1 + \epsilon(1-|x|^2) + \phi_2$ with $w_2 + \phi_2$. On ∂B_1,

$$w_1 + \epsilon(1-|x|^2) + \phi_2 \ge w_2 + \phi_2$$

The ellipticity and

$$D^2\phi_1 - D^2\phi_2 + 2n\epsilon I > 0$$

implies

$$F(D^2w_1 - 2n\epsilon + D^2\phi_2) \leq F(D^2w_1 + D^2\phi_1) = F(D^2w_2 + D^2\phi_2) = 0$$

By the comparison principle,

$$w_1 + \epsilon(1 - |x|^2) + \phi_2 \geq w_2 + \phi_2$$

Therefore

$$w_1 + \epsilon(1 - |x|^2) \geq w_2$$

Q.E.D.

Consider the two sets

$$S^+(\nu, 0) = \{x| < x, \nu > \leq A|x|^{1+\epsilon}\}$$

$$S^-(\nu, 0) = \{x| < x, \nu > \leq -A|x|^{1+\epsilon}\}$$

Lemma 5.7.

Suppose that we are given a set Ω *such that* $0 \in \partial\Omega$*, and for any* $x_o \in \partial\Omega \cap B_{1/2}$*,*
there exists a $\nu(x_o)$ *such that*

$$B_1 \cap S^-(\nu, x_o) \subset \Omega \cap B_1 \subset S^+(\nu, x_o) \cap B_1$$

Then, in a ball $B_r(0)$*,* $\partial\Omega$ *is a* $C^{1,\alpha}$*-graph in the direction of* $\nu(0)$*.*

Proof The angle between $\nu(x_o)$ and $\nu(x_1)$ can be estimated as

$$|\nu(x_o) - \nu(x_1)| \leq C|x_o - x_1|^\alpha$$

by the fact that $\partial\Omega$ is at the same time in $S^+\backslash S^-(\nu(x_o))$ and in $S^+\backslash S^-(\nu(x_1))$
Q.E.D.

Now we would like to state an inductive argument to show $C^{1,\alpha}$-regularity of the free boundary.
Let $T^{\pm}(\epsilon, \nu) = \{x| < x, \nu > \leq \pm\epsilon\}$

Proof of Theorem(5.2) We would like to show the inductive expression of Lemma(4.7) without losing the generality. We claim that for some $\lambda < 1$,$\Gamma(u) \cap B_{2^{-k}}$ is in a $\lambda^k 2^{-k}$-strip i.e. $T^-(\nu, \lambda^k 2^{-k}) \cap B_{2^{-k}} \subset \Gamma(u) \cap B_{2^{-k}} \subset T^+(\nu, \lambda^k 2^{-k}) \cap B_{2^{-k}}$.
We might approximate u in B_{r_k} by a solution w of the free boundary problem s.t.

$$\begin{aligned} F(D^2 w - I) &= 0 \qquad B_{r_k} \cap N(w) \\ w &= u \qquad \text{on } \partial B_{r_k} \\ r_k &= 2^{-(1-\theta)k} \end{aligned}$$

Without the loss of generality, we can assume that $D^2\phi(0) = -I$ by the assumption on ϕ.
The Hölder continuity of $D^2\phi$ says

$$|D^2\phi + I|_{B_{r_k}} \leq \epsilon_o r_k^\alpha = \epsilon_o 2^{-(1-\theta)k\alpha}$$

We normalize B_{r_k} to B_1. Then $B_{2^{-k}}$ will be normalized to $B_{2^{-\theta k}}$.

On the normalized picture, by Lemma(4.6),

$$\overline{u}_k = \frac{1}{r_k^2} u(r_k x)$$

$$\overline{w}_k = \frac{1}{r_k^2} w(r_k x)$$

$$|\overline{u}_k - \overline{w}_k| \leq \epsilon_o 2^{-(1-\theta)k\alpha}$$

By Theorem(3.18), the free boundary of $\overline{w}_k$ is $C^{1,\alpha}$.

The quadratic growth, and Lemma(5.3),says $\Gamma(u_k)$ is in $C2^{\frac{-(1-\theta)k\alpha}{2}}$-neighborhood of $\Gamma(w_k)$.

Since $\Gamma(w_k)$ is $C^{1,\alpha}$, $\Gamma(w_k) \cap B_{2^{-\theta k}}$ is contained in a $2^{(-\theta k)(1+\alpha)}$-strip.

If θ is small, $\frac{(1-\theta)k\alpha}{2} > \theta k(1+\alpha)$. So $\Gamma(u_k)$ is contained in a strip of width $2(2^{-\theta k(1+\alpha)})$.

If we renormalize it back, $\Gamma(u) \cap B_{2^{-k}}$ is in a strip of width $(2^{-\theta\alpha})^k 2^{-k}$.

Let's choose $\lambda = 2^{-\theta\alpha}$. Q.E.D.

6. Appendix

$\{a_{ij}\}$ is the uniformly elliptic coefficient, i.e.

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$

Lemma 6.1.

w is a smooth function satisfying $-a_{ij}w_{ij} \geq 0$ in $B(R)$. for some $0 < R < R_0$,Let Γ be a closed subset of $B(R)$ s.t. $\Gamma \subset B_{\delta^m R}(x), dist(\Gamma, \partial B(R)) \geq \delta R, |\Gamma| \geq C\delta^m|B(R)|$ for some large $m > 0$, ,for some $C > 0$,and for

some x satisfying $d(x, \partial B_\rho) \geq \sqrt{\delta} R$

Then there exists a constant $\gamma = \gamma(n, \lambda, \Lambda) > 0$,and $N > 1$ s.t.

$$\min_{B(R/2)} w \geq \gamma \delta^N \min_{\Gamma} w + (1 - \gamma \delta^N) \min_{B(R)} w$$

ProofLet $u = w - \min_{B(R)} w \geq 0$ The the weak Harnack inequality says that if $B_{2r}(y) \subset \Omega$,then

$$(\frac{1}{|B_r|} \int_{B_r} u^p)^{\frac{1}{p}} \leq C \inf_{B_r} u \tag{6.1}$$

where p and C are positive constants depending n, λ, Λ, and R.

We can get the universal constants by the rescaling. We would use the covering argument.

$$\Gamma \subset B_{\delta^m R} \qquad \text{s.t. } d(x, \partial B_\rho) \geq \sqrt{\delta} R$$

There is $\{B_k\}_{k=1}^{l}$ s.t. the center of B_{k+1} is on B_k, $B_l - B_{\delta^m R}(x_1)$, and the radius of B_k is $R(\frac{1}{2})^k$.

If we use (6.1),we can get two facts.

(1)

$$(\frac{1}{|B_k|} \int_{B_k} u^p)^{\frac{1}{p}} \leq C \inf_{B_k} u$$

(2)

$$\inf_{B_{k+1}} u \leq \inf_{B_k \cap B_{k+1}} u \leq (\frac{1}{|B_k \cap B_{k+1}|} \inf_{B_k \cap B_{k+1}} u^p)^{\frac{1}{p}} \leq C (\frac{1}{|B_k|} \int_{B_k} u^p)^{\frac{1}{p}}$$

If we use (1)(2), $\inf_{B_k} u \geq C \inf_{B_{k+1}} u$ for some universal constant C.So

$$\begin{aligned} \inf_{B_{\frac{R}{2}}} u &\geq C^l \inf_{B_l} u \\ &\geq C^{l+1} \delta^{\frac{m}{p}} \inf_{\Gamma} u \end{aligned}$$

And

$$(1-\delta)R = R(1-\sum_{i=1}^{l}(\frac{1}{2})^i)$$

$$1-\delta = \frac{1-(\frac{1}{2})^l}{1-\frac{1}{2}}$$

Therefore $\delta = (\frac{1}{2})^{l-1}$. Finally,we could have

$$\inf_{B_{\frac{R}{2}}} u \geq \delta^N \inf_{\Gamma} u$$

for some γ, N which is universal.

Q.E.D.

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