# All time $C^{\infty}$-Regularity <br> of interface in degenerated diffusion: a geometric approach 

P. DASKALOPOULOS R. Hamilton AND K. LEE

University of California, Irvine
Department of Mathematics
Irvine, CA 92697

## 1. Introduction

We study the in this work the geometry of the interface in the porous medium equation

$$
\frac{\partial u}{\partial t}=\Delta u^{m}, \quad m>1
$$

with initial data nonnegative, integrable and compactly supported. It is well known that this equation describes the evolution in time of various diffusion processes, in particular the diffusion of biological species and the flow of a gas through a porous medium. In the last case $u$ represents the density, while $f=m u^{m-1}$ represents the pressure of the gas and satisfies the equation

$$
f_{t}=f \Delta f+\frac{1}{m-1}|D f|^{2}
$$

When $u=0$, then $f=0$ and both of the above equations become degenerate. This degeneracy results into the interesting phenomenon of the finite speed of propagation: If the initial data $u^{0}$ is compactly supported in $\mathbb{R}^{n}$, the solution $u(\cdot, t)$ will remain compactly supported for all time $t$. In [8] Daskalopoulos and Hamilton showed that under certain assumptions on the initial data the free boundary $\Gamma=\partial \overline{\operatorname{supp} u}$ is a smooth surface when $0<t<T$, for some $T>0$. It is well known [1] that, in general, the free-boundary will not remain smooth for all time: advancing free boundaries may hit each other, creating singularities.

In this paper we address the question: under what geometric assumptions on the initial data, the free-boundary will remain smooth at all time?

Let us consider the initial value problem for the pressure $f$, namely the problem

$$
\begin{cases}\frac{\partial f}{\partial t}=f \Delta f+r|D f|^{2} & (x, t) \in \mathbb{R}^{n} \times[0, \infty)  \tag{1.1}\\ f(x, 0)=f^{0} & x \in \mathbb{R}^{n}\end{cases}
$$

with $r=1 /(m-1)$, where $f^{0}$ is non-negative and supported on the compact set

$$
\Omega=\left\{x \in \mathbb{R}^{n}: f^{0}(x)>0\right\} .
$$

It is well known that for any integrable initial data $f^{0} \geq 0$ the initial value problem (1.1) admits a unique weak solution $f$ on $\mathbb{R}^{n} \times(0, \infty)$. Moreover, it follows by the results in [4] that the pressure $f$ as well as the interface is Hölder continuous

If the initial interface $\partial \Omega$ is convex, then it will not necessarily remain convex, since its shape at time $t>0$ will depend on the speed of the free-boundary, namely the gradient $D f$ of the pressure, near the interface. However, we will show in this work that, if the pressure $f$ is initially a concave function, which in particular implies that its interface is convex, then the support of $f(\cdot, t)$ will remain convex for all time $0 \leq t<\infty$. In particular, under certain regularity initial assumptions, the free-boundary will be a smooth surface.

One may ask: is the matrix inequality

$$
D_{i j}^{2} f \leq 0
$$

preserved under the flow ? In other words, if the initial pressure $f^{0}$ is weakly concave, will $f$ remain weakly concave for all time ? Surprisingly, this is not the case. Instead, we will show in Section 2 that the matrix inequality

$$
D_{i j}^{2} \sqrt{f} \leq 0
$$

is preserved under the flow: if the initial pressure $f^{0}$ is root- concave, then $f$ will remain root-concave for all time. Hence, the interface

$$
\Gamma=\partial\left\{x \in \mathbb{R}^{n}: f(x, t)>0\right\}
$$

will be convex for all $t>0$. Using the geometry of the level sets of $f$ we will show that the pressure $f$ is $C^{\infty}$-smooth up to the interface, for all $t>0$. In particular the interface will be smooth.

The above discussion is summarized in the following result, which will be shown in Section 4.

Theorem 1.1. Assume that the function $f^{0}$ is smooth up to the boundary of $\Omega$, and in addition it is root-concave in $\Omega$ and satisfies the non-degeneracy condition

$$
\begin{equation*}
f^{0}+\left|D f^{0}\right|^{2} \geq c>0 \tag{1.2}
\end{equation*}
$$

for some $c>0$. Then, the solution $f$ of the initial value problem (1.1) is a smooth function smooth up to the interface $\Gamma$ and $f(\cdot, t)$ is root-concave, for all $0 \leq t<\infty$. In particular, the free boundary $\Gamma$ is a smooth surface.

Remark. If the initial pressure $f^{0}$ is concave on its support, then it is also rootconcave. Hence, in this case also solution to (1.1) is smooth up to the interface for all time.

The $C^{\infty}$ regularity assumption on the initial data $f^{0}$ in Theorem 1.1 can be weakend to only assume that $f \in C^{1}(\bar{\Omega})$ :
Theorem 1.2 Assume that $f^{0} \in C^{1}(\bar{\Omega})$ is root-concave in $\Omega$ and it satisfies the non-degeneracy condition (1.2) and the lower bound on the Laplacian

$$
\begin{equation*}
\Delta f^{0} \geq-K \quad \text { in } \quad \mathbb{R}^{n} \tag{1.3}
\end{equation*}
$$

in the distributional sense, for some constant $K>0$. Then, the solution $f$ of the initial value problem (1.1) is a smooth function smooth up to the interface $\Gamma$ and $f(\cdot, t)$ is root-concave, for all $0<t<\infty$. In particular, the free boundary $\Gamma$ is a smooth surface.

## 2. The Root-Concavity Estimate

Let $A$ be a compact subset of $\mathbf{R}^{n} \times[0, T], T>0$ with smooth lateral boundary and let $f$ be a smooth solution of equation:

$$
\frac{\partial f}{\partial t}=f \Delta f+r|D f|^{2}, \quad \text { on } \quad A
$$

for some $0<r<\infty$, with $f=0, D f \neq 0$ on the lateral boundary of $A$ and $f>0$ inside $A$.

Theorem 2.1. If $\sqrt{f}$ is weakly concave at $t=0$, it remains so for all $t$.

Proof. We must show that the matrix inequality

$$
D_{i j}^{2}(\sqrt{f}) \leq 0
$$

is preserved in time.
To simplify the notation we denote $\partial / \partial t$ by as subscript $t$ and $\partial / \partial x^{k}$ by a subscript $k$. Then we can write the evolution of $f$ as

$$
\begin{equation*}
f_{t}=f f_{k k}+r f_{k}^{2} \tag{2.1}
\end{equation*}
$$

where the summation convention is used. Since

$$
D_{i j}^{2}(\sqrt{f})=\frac{1}{2 \sqrt{f}}\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}\right)
$$

it is enough to show that the matrix inequality

$$
A_{i j}=f_{i j}-\frac{f_{i} f_{j}}{2 f} \leq 0
$$

is preserved. What we will show is that $A_{i j} \leq \delta I_{i j}$ for all $\delta>0$. Of course this implies that $A_{i j} \leq 0$. To do this we choose a positive function $\psi=\psi(t)$ with

$$
\begin{equation*}
\psi_{t}>c\left(\psi+\psi^{2}\right) \tag{2.2}
\end{equation*}
$$

for a suitable constant $c$ which will depend on bounds for $f, f_{i}$ and $f_{i j}$ and which is sufficiently small at $t=0$. Consider the quadratic

$$
Z=\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right) V^{i} V^{i}
$$

in the vector $V$.
We will show that the inequality

$$
Z<0
$$

is preserved for all $V^{i}$ provided (1) holds. Since $Z<0$ at $t=0$, by compactness there will be a first time $t_{0}>0$ when $Z=0$ at a point $x_{0} \in \bar{A}$ and at vector $V_{0}$ with $\left|V_{0}\right|=1$, while $Z<0$ for all $t<t_{0}$ at all $x$ and in all directions $V$.

Interior Estimate. Assume that $x_{0}$ belongs to the interior of the set $A$. We extend $V_{0}$ to be a smooth vector field $V$ in a neighborhood of $\left(x_{0}, t_{0}\right)$ in space time so that:

$$
\begin{equation*}
V_{j}^{i}=\frac{1}{2 f}\left(f_{k} V^{k}\right) I_{i j} \tag{2.3}
\end{equation*}
$$

at the point $\left(x_{0}, t_{0}\right)$. (Note there may be an obstruction to (2.3) holding this in the full neighborhood, but we only need it to hold at the point $\left(x_{0}, t_{0}\right)$ ). Also (to simplify the notation) we choose the extension so that

$$
V_{t}^{j}=0 \quad \text { and } \quad V_{k k}^{j}=0
$$

at $\left(x_{0}, t_{0}\right)$. Now

$$
Z=\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right) V^{i} V^{j}
$$

is a function of $x$ and $t$ only. Differentiating equation (1) we compute

$$
f_{i t}=f f_{i k k}+f_{i} f_{k k}+2 r f_{k} f_{i k}
$$

and

$$
f_{i j t}=f f_{i j k k}+2 f_{i} f_{j k k}+2 r f_{k} f_{i j k}+f_{i j} f_{k k}+2 r f_{i k} f_{j k}
$$

Thus

$$
\left(f_{i j} V^{i} V^{j}\right)_{t}=\left\{f f_{i j k k}+2 f_{i} f_{j k k}+2 r f_{k} f_{i j k}+f_{i j} f_{k k}+2 r f_{i k} f_{j k}\right\} V^{i} V^{j}
$$

at $\left(x_{0}, t_{0}\right)$. Also, using (2.3) we compute

$$
\left(f_{i j} V^{i} V^{j}\right)_{k}=\left\{f_{i j k}+\frac{f_{j k} f_{i}}{f}\right\} V^{i} V^{j}
$$

and

$$
f\left(f_{i j} V^{i} V^{j}\right)_{k k}=\left\{f f_{i j k k}+2 f_{k k j} f_{i}+f_{j k} f_{i k}-\frac{f_{j k} f_{i} f_{k}}{2 f}+\frac{f_{k k} f_{i} f_{j}}{2 f}\right\} V^{i} V^{j}
$$

at $\left(x_{0}, t_{0}\right)$. Hence

$$
\begin{align*}
\left(f_{i j} V^{i} V^{j}\right)_{t} & =f\left(f_{i j} V^{i} V^{j}\right)_{k k}+2 r f_{k}\left(f_{i j} V^{i} V^{j}\right)_{k}  \tag{2.4}\\
& +f_{k k}\left\{\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}\right) V^{i} V^{j}\right\}-2 r \frac{f_{j k} f_{i} f_{k}}{f} V^{i} V^{j} \\
& +2 r f_{i k} f_{j k} V^{i} V^{j}-f_{j k}\left\{\left(f_{i k}-\frac{f_{i} f_{k}}{2 f}\right) V^{i}\right\} V^{j}
\end{align*}
$$

of $\left(x_{0}, t_{0}\right)$.
On the other hand

$$
\left(\frac{f_{i} f_{j}}{f} V^{i} V^{j}\right)_{t}=\left\{2 f_{j} f_{i k k}-\frac{f_{i} f_{j}}{f} f_{k k}+\frac{4 r f_{j} f_{k} f_{i k}}{f}-\frac{r f_{i} f_{j} f_{k}^{2}}{f^{2}}\right\} V^{i} V^{j}
$$

while

$$
\left(\frac{f_{i} f_{j}}{f} V^{i} V^{j}\right)_{k}=\frac{2 f_{i k} f_{j}}{f} V^{i} V^{j}
$$

and

$$
\left(\frac{f_{i} f_{j}}{f} V^{i} V^{j}\right)_{k k}=\left\{\frac{2 f_{i k k} f_{j}}{f}-\frac{f_{i k} f_{j} f_{k}}{f^{2}}+\frac{2 f_{i k} f_{j k}}{f}+\frac{f_{k k} f_{i} f_{j}}{f^{2}}\right\} V^{i} V^{j}
$$

Thus

$$
\begin{align*}
\left(\frac{f_{i} f_{j}}{2 f} V^{i} V^{j}\right)_{t} & =f\left(\frac{f_{i} f_{j}}{2 f} V^{i} V^{j}\right)_{k k}+2 r f_{k}\left(\frac{f_{i} f_{j}}{2 f} V^{i} V^{j}\right)_{k}  \tag{2.5}\\
& -f_{i k}\left\{\left(f_{j k}-\frac{f_{j} f_{k}}{2 f}\right) V^{j}\right\} V^{i}-\frac{r f_{i} f_{j} f_{k}^{2}}{2 f^{2}} V^{i} V^{j}
\end{align*}
$$

Since the vector $V_{0}$ must be a null eigenvector for the matrix

$$
f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}
$$

at $\left(x_{0}, t_{0}\right)$, we have:

$$
A_{i j} V^{i} \equiv\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}\right) V^{i}=\psi I_{i j} V^{i}
$$

at $\left(x_{0}, t_{0}\right)$.
If $\psi=0$ all terms involving $A_{i j} V^{i}$ in (2.4) and (2.5) drop out; if not at least they are bounded by

$$
c \psi|V|^{2} .
$$

for some constant $c$. Therefore from (2.4) and (2.5) we compute:

$$
Z_{t} \leq Z_{k k}+2 r f_{k} Z_{k}+\left[c \psi-\psi_{t}\right]|V|^{2}+R
$$

with

$$
\begin{aligned}
R & =2 r f_{i k} f_{j k} V^{i} V^{j}-2 r \frac{f_{j k} f_{i} f_{k}}{f} V^{i} V^{j}+\frac{r f_{I} f_{j} f_{k}^{2}}{2 f^{2}} V^{i} V^{j} \\
& =2 r f_{j k}\left\{\left(f_{i k}-\frac{f_{i} f_{k}}{2 f}\right) V^{i}\right\} V^{j}-\frac{r f_{i} f_{k}}{f}\left\{\left(f_{j k}-\frac{f_{j} f_{k}}{2 f}\right) V^{j}\right\} V^{i}
\end{aligned}
$$

which again can be estimated as

$$
R \leq c \psi|V|^{2}
$$

We conclude that at $\left(x_{0}, t_{0}\right)$

$$
\begin{equation*}
Z_{t} \leq f Z_{k k}+2 r f_{k} Z_{k}+\left(c \psi-\psi_{t}\right)|V|^{2} \tag{6}
\end{equation*}
$$

and by choosing $\psi$ satisfying (2) we can make

$$
Z_{t}<f Z_{k k}+2 r f_{k} Z_{k}
$$

at $\left(x_{0}, t_{0}\right)$.
Regardless of the way we extended $V_{0}$ to $V$ we still have $Z=0$ at $\left(x_{0}, t_{0}\right)$ and $Z \leq 0$ for all $t \leq t_{0}$ At all $x$ in a neighborhood of $x_{0}$. Thus $Z_{t} \geq 0, Z_{k}=0$ and $\Delta Z \leq 0$ at $\left(x_{0}, t_{0}\right)$. Now if we choose

$$
\psi_{t}>c\left(\psi+\psi^{2}\right)
$$

then since $\left|V_{0}\right|=1$ we get that $0<0$ in (2.6), which is a contradiction. Hence

$$
Z<0
$$

in the interior of $A$.
Estimate at the boundary. Suppose now that the inequality $Z<0$ fails at a point $x_{0}$ on the boundary in a direction $V_{0} \neq 0$. Assume $\left|V_{0}\right|=1$. Clearly $V_{0}$ is tangent to the boundary at $x_{0}$, so

$$
f_{i} V^{i}=0
$$

at $x_{0}$ (since $f \equiv 0$ at the boundary).

Lemma 2.2. At $x_{0}$ at time $t_{0}$ in the direction $V_{0}$ we have

$$
f_{k} f_{i j k} V^{i} V^{j} \leq 0
$$

Proof: Choose a path $x(s)$ parameterized by $s$ with $x=x_{0}$ at $s=0$ and

$$
\frac{d x^{k}}{d s}=f_{k}
$$

so that the path lies in the interior region $f>0$ for small $s>0$. Then choose a vector field $V(s)$ along the path with $V=V_{0}$ and

$$
\frac{d V^{k}}{d s}=\frac{1}{\left|f_{l}\right|^{2}}\left(f_{i} f_{i j} V^{j}\right) f_{k}
$$

at $s=0$. Along this path the functions $f$ and $f_{i} V^{i}$ are both smooth and both zero at $s=0$. Then by L'Hospital's rule

$$
\lim _{s \rightarrow 0} \frac{f_{i} V^{i}}{f}=\lim _{s \rightarrow 0} \frac{d\left(f_{i} V^{i}\right) / d s}{d f / d s}
$$

Now

$$
\frac{d}{d s} f=f_{k} \frac{d x^{k}}{d s}=\left|f_{k}\right|^{2}
$$

while

$$
\frac{d}{d s}\left(f_{i} V^{i}\right)=f_{i j} V^{i} \frac{d x^{j}}{d s}+f_{i} \frac{d V^{i}}{d s}=2 f_{j} f_{i j} V^{i}
$$

which gives

$$
\lim _{s \rightarrow 0} \frac{f_{i} V^{i}}{f}=\frac{2 f_{i} f_{i j} V^{j}}{\left|f_{k}\right|^{2}}
$$

Consider the function

$$
Q=\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right) V^{i} V^{j}
$$

along the path $x(s)$ in the direction $V(s)$. Therefor for $s>0$

$$
\begin{aligned}
\frac{d Q}{d s} & =\left(f_{i j} \frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right)_{k} V^{i} V^{j} \frac{d x^{k}}{d s}+2\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right) \frac{d V^{i}}{d s} V^{j} \\
& =r f_{k}\left(f_{i j k}-\frac{f_{i k} f_{j}}{f}+\frac{f_{i} f_{j} f_{k}}{2 f^{2}}\right) V^{i} V^{j}+2\left(f_{i j}-\frac{f_{i} f_{j}}{2 f}-\psi I_{i j}\right) \frac{d V^{i}}{d s} V^{j}
\end{aligned}
$$

In fact the function $Q$ extends to be smooth at $s=0$, because $f_{i} V^{i} / f$ does (since $f$ only vanishes at first order). Therefore we can evaluate $d Q / d s$ at $s=0$ by taking
the limit. Note that $d Q / d s \leq 0$ at $s=0$ since $Q=0$ at $s=0$, while $Q \leq 0$ for $s \geq 0$. Rewrite

$$
\begin{aligned}
\frac{d Q}{d s}= & r f_{k} f_{i j k} V^{i} V^{j}-r\left(f_{k} f_{i k} V^{i}\right)\left(\frac{f_{j} V^{j}}{f}\right)+\frac{r}{2}\left|f_{k}\right|^{2}\left(\frac{f_{i} V^{i}}{f}\right)\left(\frac{f_{j} V^{j}}{f}\right) \\
& +2 f_{i j} \frac{d V^{i}}{d s} V^{j}-\left(f_{i} \frac{d V^{i}}{d s}\right)\left(\frac{f_{j} V^{j}}{f}\right)-\psi V^{i} \frac{d V^{j}}{d s}
\end{aligned}
$$

Now we can take the limit as $s \rightarrow 0$. Using our chosen value of $d V^{i} / d s$ and our limit of $f_{i} V^{i} / f$ we get

$$
\frac{d Q}{d s}=r f_{k} f_{i j k} V^{i} V^{j}
$$

at $s=0$ (after several cancellations!) This proves the Lemma, since $d Q / d s \leq 0$.
Now we study the time evolution. Pick a path $x(t)$ for $t \leq t_{0}$ with $x(t)$ always in the boundary, and with $x=x_{0}$ at $t=t_{0}$. We also pick a path $V(t)$ for $t \leq t_{0}$ with $V(t)$ always tangent to the boundary at the point $x(t)$ at time $t$ for $t \leq T_{0}$, and with $V=V_{0}$ at $t=t_{0}$.

The variation $\frac{d x^{k}}{d t}$ is constrained by the equation that $f=0$ on the boundary, so

$$
\frac{d}{d t} f=0
$$

along the path $x(t)$, which makes

$$
f_{t}+f_{k} \frac{d x^{k}}{d t}=0
$$

¿From the equation

$$
f_{t}=f f_{k k}+r f_{k}^{2}=r f_{k}^{2}
$$

on the boundary where $f=0$. Thus we need to have

$$
f_{k}\left(\frac{d x^{k}}{d t}+r f_{k}\right)=0
$$

and this is the only constraint on $\frac{d x^{k}}{d t}$. Therefore we chose

$$
\frac{d x^{k}}{d t}=-r f_{k}
$$

Wherein the variation $\frac{d V^{k}}{d t}$ is constrained by the equation $f_{k} V^{k}=0$, so

$$
\frac{d}{d t}\left(f_{k} V^{k}\right)=0
$$

In the vector $V(t)$ along the path $x(t)$, which makes

$$
f_{k} \frac{d V^{k}}{d t}+f_{k t} V^{k}+f_{j k} \frac{d x^{j}}{d t} V^{k}=0
$$

¿From the equation

$$
f_{T}=f f_{k k}+r f_{k}^{2}
$$

we get

$$
f_{t i}=f f_{i k k}+f_{i} f_{k k}+2 r f_{k} f_{i k}
$$

and on the boundary where $f=0$ in the direction $V^{i}$ where $f_{i} V^{i}=0$ we get

$$
f_{t i} V^{i}=2 r f_{k} f_{i k} V^{i}
$$

Using $d x^{k} / d t=-r f_{k}$, we see that $d V^{k} / d t$ is constrained by

$$
f_{k}\left\{\frac{d V^{k}}{d t}+r f_{i k} V^{i}\right\}=0
$$

and this is the only constraint. Therefore we choose

$$
\frac{d V^{k}}{d t}=-r f_{i k} V^{i}
$$

Now consider the function

$$
Q=\left\{f_{i j}-\psi I_{i j}\right\} V^{i} V^{j}
$$

along the path $x(t)$ in the direction $V(t)$. Since $f_{i} V^{i}=0$ along this path, this agrees with our previous quadratic, and we have $Q \leq 0$ for $t \leq t_{0}$ while $Q=0$ at $t=t_{0}$. Therefore $\frac{d Q}{d t} \geq 0$ at $t=t_{0}$. We compute $d Q / d t$ along the path.

$$
\frac{d Q}{d t}=\left\{f_{i j t}-\psi^{\prime} I_{i j}\right\} V^{i} V^{j}+f_{i j k} \frac{d x^{k}}{d t} V^{i} V^{j}+2\left\{f_{i j}-\psi I_{i j}\right\} \frac{d V^{i}}{d t} V^{j}
$$

¿From the equation

$$
f_{i j t}=f f_{i j k k}+f_{i} f_{j k k}+f_{j} f_{i k k}+f_{i j} f_{k k}+2 r f_{k} f_{i j k}+2 r f_{i k} f_{j k}
$$

and since $f=0$ and $f_{i} V^{i}=0$ on our path, we get

$$
f_{i j t} V^{i} V^{j}=2 r f_{k} f_{i j k} V^{i} V^{j}+f_{k k}\left(f_{i j} V^{i} V^{j}\right)+2 r\left(f_{i k} V^{i}\right)\left(f_{j k} V^{j}\right)
$$

Now use $f_{i j} V^{i} V^{j}=\psi|V|^{2}$ at $t=t_{0}$ and $d x^{k} / d t=-r f_{k}$ and $d V^{k} / d t=-r f_{i k} V^{i}$ to compute

$$
\frac{d Q}{d t}=\left\{-\psi^{\prime}+f_{k k} \psi+2 r \psi^{2}\right\}|V|^{2}
$$

(after some cancellation!) If $f_{k k} \leq c$ and $\psi^{\prime}>c \psi+2 r \psi^{2}$ we must have $d Q / d t<0$, contradicting $d Q / d t \geq 0$. The contradiction shows the quadratic

$$
\left(f_{i j}-\frac{f f_{j}}{2 f}-\psi I_{i j}\right) V^{i} V^{j}
$$

must stay strictly $<0$. This proves our claim that $\sqrt{f}$ is concave, since $\psi$ can be as small as we like.

## 3. Gradient Estimates

We will assume, throughout this section, that $f$ is a solution of the initial value problem

$$
\begin{cases}\frac{\partial f}{\partial t}=f \Delta f+r|D f|^{2} & \text { on } \mathbb{R}^{n} \times[0, T] \\ f(x, 0)=f^{0} & \text { in } \mathbb{R}^{n}\end{cases}
$$

which is smooth up to the interface for $0 \leq t \leq T$.

Lemma 3.1. If $f$ satisfies

$$
\Delta f \geq-K
$$

in the distributional sense at $t=0$, then

$$
\Delta f \geq-\frac{K \sigma}{K t+\sigma}, \quad t>0
$$

where $\sigma=\left(m-1+\frac{2}{n}\right)^{-1}$.
Proof. One can observe that the proof of the Aronson-Bénilan inequality

$$
\Delta f \geq-\frac{\sigma}{t}, \quad \text { for } t>0
$$

in [2] can be slightly modified to show that

$$
\Delta f \geq-\frac{\sigma}{t+\tau}, \quad \text { for } \quad t>0
$$

provided that

$$
\Delta f \geq-\frac{\sigma}{\tau}, \quad \text { at } \quad t=0
$$

Setting $K=\sigma / \tau$, the result follows.
We prove next that if $f$ is root-concave, then the upper bound of the gradient is preserved under the flow.

Theorem 3.2. If $f$ is root-concave on $0 \leq t \leq T$ and satisfies

$$
|D f| \leq C, \quad \text { at } \quad t=0
$$

then

$$
|D f| \leq C, \quad t>0
$$

We show first the next simple lemma:

Lemma 3.3. Assume that $\sqrt{f}$ is concave on $0 \leq t \leq T$. Then for any point $P=\left(x_{0}, t_{0}\right)$ at the free-boundary of $f$ and any unit vector $V$ tangent to the boundary of the set

$$
\Omega\left(t_{0}\right)=\left\{x \in \mathbb{R}^{n}: f\left(x, t_{0}\right)>0\right\}
$$

at the point $P$, we have

$$
D_{i j}^{2} f V^{i} V^{j} \leq 0, \quad \text { at } \quad P
$$

Proof. The result follows by a simple approximation argument. Consider a sequance of points $P_{k}=\left(x_{k}, t_{0}\right)$ converging to the point $P$ and such that the sequence $c_{k}=f\left(P_{k}\right)$ decreases to zero. Let $V_{k}=\left(V_{k}^{i}\right)$ be a sequence of unit vectors tangent to the level set $f\left(x, t_{0}\right)=c_{k}$ of $f$ at $P_{k}$ and such that $V_{k} \rightarrow V$ as $k \rightarrow \infty$. Since

$$
D_{i j}^{2}(\sqrt{f}) \leq 0, \quad \text { at } \quad P_{k}
$$

we have

$$
D_{i j}^{2}(\sqrt{f}) V_{k}^{i} V_{k}^{j} \leq 0
$$

which implies that

$$
f\left(P_{k}\right) D_{i j}^{2} f\left(P_{k}\right) V_{k}^{i} V_{k}^{j} \leq \frac{1}{2} f_{i}\left(P_{k}\right) f_{j}\left(P_{k}\right) V_{k}^{i} V_{k}^{j}
$$

where $f_{i}=D_{i} f$. Since each vestor $V_{k}=\left(V_{k}^{i}\right)$ is tangent to the level set $f\left(x_{k}, t_{0}\right)=$ $c_{k}$ at $P_{k}$, we have

$$
f_{i}\left(P_{k}\right) f_{j}\left(P_{k}\right) V_{k}^{i} V_{k}^{j}=0
$$

inequality implying that

$$
f\left(P_{k}\right) D_{i j}^{2} f\left(P_{k}\right) V_{k}^{i} V_{k}^{j} \leq 0
$$

Because $f\left(P_{k}\right)>0$ we must have

$$
D_{i j}^{2} f\left(P_{k}\right) V_{k}^{i} V_{k}^{j} \leq 0
$$

which implies (3.7) by taking the limit $k \rightarrow \infty$.
We are now in position to prove Theorem 3.2.
Proof of Theorem 3.2. We will use the maximum principle on

$$
X=\frac{f_{i}^{2}}{2}-\epsilon t
$$

where $f_{i}=D_{i} f$, for $i=1,2, \ldots, n$ and the summation convention is used. Then we will send $\epsilon$ to zero to get the desired estimate. Since

$$
f_{t}=f f_{k k}+r f_{k}^{2}
$$

we compute

$$
\begin{equation*}
X_{t}=f_{i} f_{i t}=f_{i}\left(f f_{k k}+r f_{k}^{2}\right)_{i}=f_{i}^{2} f_{k k}+f f_{i} f_{i k k}+2 r f_{k} f_{i} f_{k i}-\epsilon \tag{3.1}
\end{equation*}
$$

On the other hand $X_{k}=f_{i} f_{i k}$ and $X_{k k}=f_{i} f_{i k k}+f_{i k}^{2}$, so that

$$
f X_{k k}=f f_{i k k}+f f_{i k}^{2} .
$$

Hence, using (3.1) we conclude that

$$
\begin{equation*}
X_{t}=f X_{k k}-f f_{i k}^{2}+f_{i}^{2} f_{k k}+2 r f_{k} X_{k}-\epsilon . \tag{3.2}
\end{equation*}
$$

Interior Estimate. At an interior maximum point $P$ of $X$ we must have

$$
f X_{k k} \leq 0
$$

while $X_{k}=0$ for eack $k$. Hence, from (3.2) we deduce that

$$
\frac{\partial X(P)}{\partial t} \leq f_{i}^{2} f_{k k}-\epsilon
$$

Therefore, it is enough to show that

$$
\Delta f=f_{k k} \leq 0
$$

at an interior maximum point $P$ of $X$. We can assume, by rotating the coordinates that

$$
\begin{equation*}
f_{n}>0 \quad \text { and } \quad f_{i}=0, \quad i=1, \ldots, n-1 \tag{3.3}
\end{equation*}
$$

at the point $P$. Moreover,

$$
X_{n}=f_{n} f_{n n}=0, \quad \text { at } \quad P
$$

which implies that

$$
f_{n n}=0 \quad \text { at } \quad P .
$$

It remains to show that

$$
f_{k k} \leq 0, \quad \forall k=1, \ldots, n-1
$$

But this follows directly from the root concavity inequality

$$
f f_{i j} V^{i} V^{j} \leq \frac{1}{2} f_{i} f_{j} V^{i} V^{j}
$$

by taking $V=\left(V^{i}\right)$ with $V^{i}=\delta^{i k}$ and using (3.3).

Boundary Estimate. Assume now that $X$ attains its maximum at a free-bounadry point $P=\left(x_{0}, t_{0}\right)$ and also assume that

$$
\begin{equation*}
f_{n}>0 \quad \text { and } \quad f_{i}=0, \quad i=1, \ldots, n-1 \tag{3.4}
\end{equation*}
$$

at the point $P$. Since, $f(P)=0$ we conclude by (3.2) and (3.4) that

$$
X_{t} \leq f_{n}^{2} f_{k k}+2 r f_{n} X_{n}-\epsilon
$$

at $P$. Also, since $f_{n}>0$ at the maximum point $P$ of $X$, we must have

$$
X_{n} \leq 0, \quad \text { at } \quad P
$$

concluding that

$$
\begin{equation*}
\frac{\partial X}{\partial t} \leq f_{n}^{2} f_{k k}-\epsilon, \quad \text { at } \quad P \tag{3.5}
\end{equation*}
$$

We will show that $f_{k k} \leq 0$. Indeed, , by (3.4), we have

$$
X_{n}=f_{i} f_{i n}=f_{n} f_{n n}
$$

and $P$. Since $X_{n} \leq 0$ at $P$ we conclude that $f_{n n} \leq 0$, at $P$. It remains to show that

$$
f_{i i} \leq 0, \quad i=1, \ldots, n-1
$$

But this follows immediately from Lemma 3.2, since each of the unit vectors $V_{k}=$ $\left(\delta_{i k}\right), k=1, \ldots, n-1$ are tangent of the boundary of the set

$$
\Omega\left(t_{0}\right)=\left\{f\left(\cdot, t_{0}\right)>0\right\}
$$

at $P$.
Lemma 3.4. Assume that $f$ satisfies the non-degeneracy condition

$$
\begin{equation*}
\alpha f_{t}+f \geq c>0, \quad \text { on } \quad f>0, t=0 \tag{3.6}
\end{equation*}
$$

and the inequality

$$
\Delta f \geq-K, \quad \text { on } t>0
$$

for some positive constants $c, \alpha$ and $K$. Then, at time $t>0$ we have

$$
(\alpha+t) f_{t}+f \geq c e^{-K t} \quad \text { on } \quad f>0
$$

Proof. Set

$$
F=(t+\alpha) f_{t}+f
$$

We will prove, using the maximum principle that

$$
F \geq c e^{-K t}, \quad \text { on } \quad\{f>0\}
$$

provided that $F \geq c$ on $\{f>0\}$ at time $t=0$. By direct a computation we find that $F$ evolves by

$$
F_{t}=f \Delta F+2 r D f \cdot D F+\Delta f F
$$

on $\{f>0\}$. Since $\Delta f \geq-K$, we obtain that for $F \geq 0$

$$
F_{t} \geq f \Delta F+2 r D f \cdot D F-K F
$$

and therefore $\tilde{F}=F e^{K t}+\epsilon t$ satisfies the differential inequality

$$
\begin{equation*}
\tilde{F}_{t} \geq f \Delta \tilde{F}+2 r D f \cdot D \tilde{F}+\epsilon \tag{3.7}
\end{equation*}
$$

with $r=1 /(m-1)>0$. It is clear, by (3.7), that the minimum of $\tilde{F}$ cannot be attained in the interior of the set $\{f>0\}$. Let $P$ be a free-boundary point where $\tilde{F}$ is minimum. By rotating the coordinates we can assume that at the point $P$, $D_{n} f>0$, while $D_{i} f=0$ for all $i=1, \ldots, n-1$. Then, since $\tilde{F}$ is minimum at $P$ we must have

$$
D_{n} F \geq 0, \quad \text { at } \quad P
$$

Therefore, $D f \cdot D \tilde{F} \geq 0$ at $P$ and hence by (3.7) we obtain

$$
\tilde{F}_{t} \geq \epsilon \quad \text { at } \quad P .
$$

We conclude that

$$
\tilde{F} \geq c, \quad \text { on } \quad f>0
$$

for $t>0$, provided that $\tilde{F} \geq c$ on $\{f>0\}$ at time $t=0$. This shows the desired result.

Corollary 3.5. Assume that at time $t=0$ the initial pressure $f$ is root-concave and satifies the non-degeneracy condition

$$
f+|D f|^{2} \geq c>0, \quad \text { on } \quad f>0
$$

and the lower bound on the Laplacian

$$
\Delta f \geq-K
$$

for some positive constants $c$ and $K$. Then, for $t>0, f$ satisfies the non-degeneracy estimate

$$
f+\left(t+\frac{1}{K+r}\right)\left(r+\frac{1}{2}\right)|D f|^{2} \geq c e^{-K t}, \quad \text { on } \quad f>0
$$

with $r=\frac{1}{m-1}$.

Proof. We will apply Lemma 3.4, to $F=(t+\alpha) f_{t}+f$, with $\alpha=\frac{1}{K+r}$. At $t=0$ we have

$$
F=\alpha f_{t}+f=\alpha f \Delta f+\alpha r|D f|^{2}+f
$$

and therefore, since $\Delta f \geq-K$ and $\alpha=\frac{1}{K+r}$, we have

$$
F \geq(1-\alpha K) f+\alpha r|D f|^{2}=\frac{r}{K+r}\left(f+|D f|^{2}\right)
$$

The non-degeneracy estimate on $f$ implies that at $t=0$

$$
F \geq \frac{c r}{K+r}, \quad \text { on } \quad\{f>0\}
$$

Hence, by Lemma 3.4, for $t>0$

$$
F \geq \frac{c r}{K+r} e^{-K t}, \quad \text { on } \quad\{f>0\}
$$

which implies that

$$
\begin{equation*}
(t+\alpha) f \Delta f+r(t+\alpha)|D f|^{2}+f \geq \frac{c r}{K+r} e^{-K t}, \quad \text { on } \quad\{f>0\} \tag{3.8}
\end{equation*}
$$

On the other hand, by Theorem 2.1, $f$ is root-concave for $t>0$, and therefore

$$
\begin{equation*}
f \Delta f \leq \frac{1}{2}|D f|^{2} \tag{3.9}
\end{equation*}
$$

Combining (3.8) and (3.9) we ontain

$$
f+(t+\alpha)\left(r+\frac{1}{2}\right)|D f|^{2} \geq \frac{c r}{K+r} e^{-K t}, \quad \text { on } \quad\{f>0\}
$$

as desired.
An immediate consequence Theorem 3.2 and Corrolary 3.5 is the following, important for our purposes, result:
Theorem 3.6. Assume that at time $t=0$ the function $f$ is root-concave and satisfies the upper gradient bound

$$
|D f| \leq C
$$

the non-degeneracy estimate

$$
f+|D f|^{2} \geq c>0, \quad \text { on } \quad\{f>0\}
$$

and the lower bound on the Laplacian

$$
\Delta f \geq-K
$$

for some positive constants $C, c$ and $K$. Then, given a free-boundary point $P=$ $P\left(x_{0}, t_{0}\right), 0 \leq t_{0} \leq T$, there exists positive constants $d_{0}$ and $c_{0}$, depending only on $C, c, K$ and $T$, such that for all $x$ in $\left\{f\left(\cdot, t_{0}\right)>0\right\}$ with $d\left(x, x_{0}\right)<d_{0}$, we have

$$
\left|D f\left(x, t_{0}\right)\right| \geq c_{0}>0
$$

Based on Theorem 3.6 we will prove the following result, which permits us to exchange coordinates in a uniform in size neighborhood of a free- boundary point $P$.

Theorem 3.7 Assume that at time $t=0$ the function $f$ is root-concave and satisfies the upper gradient bound

$$
|D f| \leq C
$$

the non-degeneracy estimate

$$
f+|D f|^{2} \geq c>0, \quad \text { on } \quad\{f>0\}
$$

and the lower bound on the Laplacian

$$
\Delta f \geq-K
$$

for some positive constants $C, c$ and $K$. Then, there exist positive constants $\rho$ and $c_{0}$, depending only on $C, c, K, T$ and the shape of the initial support, such that given a free-boundary point $P=P\left(x_{0}, t_{0}\right), \rho \leq t_{0} \leq T$, there exits a unit direction $\nu=\nu_{x_{0}}$, depending only on $x_{0}$, such that

$$
D f(x, t) \cdot \nu \geq c_{0}>0
$$

for all $x \in \Omega(t) \cap B_{\rho}(0), t_{0}-\rho \leq t \leq t_{0}$.
Since, by assumption, $f$ is smooth up to the interface at $t=0$, the initial support

$$
\Omega=\left\{x \in \mathbb{R}^{n}: f(x, 0)>0\right\}
$$

is a domain with smooth boundary. Moreover, since $f$ is root concave, the domain $\Omega$ is strictly convex. Let us also assume, without loss of generality, that $B_{1}(0) \subset$ $\Omega$, where $B_{1}(0)$ denotes the unit ball $B_{1}(0)=\left\{x \in \mathbb{R}^{n}:|x| \leq 1\right\}$. For each $x=(r, \theta) \in \mathbb{R}^{n}$ the half line $\{(\lambda r, \theta) ; \lambda>0\}$ intersects the boundary of the convex domain $\Omega$ at a unique point, which we will denote by $\bar{\theta}$. Let us denote by $\nu_{\bar{\theta}}$ the exterior unit normal to $\Omega$ at the point $\bar{\theta} \in \partial \Omega$. We define the vector field $\nu_{x}$ by

$$
\nu_{x}=\nu_{\theta}
$$

The proof of Theorem 3.7 is an immediate consequence of Theorem 3.6 and the next lemma:

Lemma 3.8. There exits numbers positive $\rho$ and $\eta_{0}$, depending only on the initial data $f(\cdot, 0)$, such that if $P=\left(x_{0}, t_{0}\right)$ is a free-boundary point of $f$ with $2 \rho \leq t_{0} \leq T$, then

$$
\cos \left\langle n_{x}(t), \nu_{x_{0}}\right\rangle \geq \eta
$$

for all $x \in B_{\rho}\left(x_{0}\right) \cap \Omega(t), t_{0}-\rho \leq t \leq t_{0}$, where $n_{x}(t)$ denotes the outer unit normal vector to the level set

$$
\Omega(t, x)=\left\{y \in \mathbb{R}^{n}: f(y, t) \geq f(x, t)\right\} .
$$

The proof of Lemma 3.8 will be based on the short time existence and the following geometric result, along the lines of Lemma 2.2 in [3]. For some $\rho>$ $0, \delta_{o}>0,\{x \mid f(x, \rho)>0\} \supset\left(1+\delta_{o}\right) \Omega=\left\{\left(1+\delta_{o}\right) x \mid x \in \Omega\right\}$ by the short time existence [8].

Proposition 3.9. Assume that $f$ is root-concave and let $x_{0}, x_{1}$ be two distinct points in $\mathbb{R}^{n} \backslash\left(1+\delta_{o}\right) \Omega$, where $\Omega$ denotes the initial support, such that $\left|x_{0}\right|,\left|x_{1}\right| \leq R$. Then, there exists a constant $c_{0}=c_{0}\left(R, \delta_{o}\right)$ with $0<c_{0}<1$, such that if

$$
\begin{equation*}
\cos \left\langle x_{1}-x_{0}, \nu_{x_{0}}\right\rangle \geq c_{0} \tag{3.10}
\end{equation*}
$$

then

$$
f\left(x_{1}, t\right) \leq f\left(x_{0}, t\right), \quad t \geq 0 .
$$

Proof. Since the domain $\Omega$ is convex, one can easily observe that there exists a constant $c_{0} \in(0,1)$, depending on $R$ and the shape of the domain $\Omega$, such that if

$$
\cos \left\langle x_{1}-x_{0}, \nu_{x_{0}}\right\rangle \geq c_{0}
$$

$\Omega$ lies one one side of the line bisecting the line segment $\overline{x_{0} x_{1}}$ vertically. Let us assume, for simplicity that the vertical bisector is the line $x_{n}=a$ and that

$$
\Omega \subset\left\{x: x_{n}<a\right\} .
$$

Define the function $\tilde{f}$ on $\left\{x: x_{n}<a\right\}, t>0$, by

$$
\tilde{f}(x, t)=f\left(x^{\prime}, 2 a-x_{n}, t\right)
$$

where we use the notation $x=\left(x^{\prime}, x_{n}\right)$. We will show by the comparison principle that at time $t>0$, we have $\tilde{f} \leq f$ on $\left\{x: x_{n}<a\right\}$. Indeed, both $\tilde{f}$ and $f$ are
solutions of equation $f_{t}=f \Delta f+r|D f|^{2}$. Moreover, they coincide at $x_{n}=a$, while at $t=0$, and and for $x \in\left\{x: x_{n}<a\right\}$, we have

$$
\tilde{f}(x, 0)=f\left(x^{\prime}, 2 a-x_{n}, 0\right)=0 \leq f(x, 0)
$$

since $\left(x^{\prime}, 2 a-x_{n}\right)$ lies outside the support $\Omega$ of $f$. Since both $f$ and $\tilde{f}$ are compactly supported, from the standard comparison principle we deduce that for $t>0$

$$
\tilde{f}(x, t) \leq f(x, t) \quad \text { on }\left\{x: x_{n}<a\right\}
$$

implying the desired inequality

$$
f\left(x_{1}, t\right)=\tilde{f}\left(x_{0}, t\right) \leq f\left(x_{0}, t\right)
$$

We will present now the proof of Lemma 3.7.
Proof of Lemma 3.8. Fix a point $P=\left(x_{0}, t_{0}\right)$ on the free-boundary of $f$ and denote by $\Omega(t)$ the set

$$
\Omega(t)=\left\{x \in \mathbb{R}^{n}: f(x, t)>0\right\}
$$

Let $x \in B_{\rho}\left(x_{0}\right) \cap \Omega(t), t_{0}-\rho \leq t \leq t_{0}$, where $\rho$ is a small number to be determined later. Define the cones

$$
C_{x}^{1}=\left\{y \in B_{\rho}\left(x_{0}\right):\left\langle y-x,-\nu_{x}\right\rangle \leq \delta_{0}\right\}
$$

and

$$
C_{x}^{2}=\left\{y \in B_{\rho}\left(x_{0}\right):\left\langle y-x, \nu_{x}\right\rangle \leq \delta_{0}\right\}
$$

where $\delta_{0} \in(0,1)$ is a contant sufficiently small, so that, by Proposition 3.8 we have:

$$
\begin{equation*}
C_{x}^{1} \subset\{y: f(y, t) \geq f(x, t)\} \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{x}^{2} \subset\{y: f(y, t) \leq f(x, t)\} \tag{3.12}
\end{equation*}
$$

for all $t_{0}-\rho \leq t \leq t_{0}$. Since the vector field $\nu_{x}$ is smooth, we can choose $\rho>0$, depending only on the initial data, such that

$$
\begin{equation*}
\left\langle\nu_{x}, \nu_{x_{0}}\right\rangle<\delta_{0} / 2 \tag{3.13}
\end{equation*}
$$

for all $x \in B_{\rho}\left(x_{0}\right)$. Thus, combining (3.11)-(3.13) we obtain that

$$
\tilde{C}_{x}^{1} \equiv\left\{y \in B_{\rho}\left(x_{0}\right):\left\langle y-x,-\nu_{x_{0}}\right\rangle \leq \delta_{0} / 2\right\} \subset\{y: f(y, t) \geq f(x, t)\}
$$

and

$$
\tilde{C}_{x}^{2} \equiv\left\{y \in B_{\rho}\left(x_{0}\right):\left\langle y-x, \nu_{x_{0}}\right\rangle \leq \delta_{0} / 2\right\} \subset\{y: f(y, t) \leq f(x, t)\}
$$

for all $t_{0}-\rho \leq t \leq t_{0}$. This in particular implies that

$$
\left\langle n_{x}(t), \nu_{x_{0}}\right\rangle \geq \frac{\pi}{2}-\frac{\delta_{0}}{2}, \quad \forall x \in B_{\rho}\left(x_{0}\right) \cap \Omega(t), \quad t_{0}-\rho \leq t \leq t_{0}
$$

where $n_{x}(t)$ denotes the outer unit normal vector to the level set

$$
\Omega(t, x)=\left\{y \in \mathbb{R}^{n}: f(y, t) \geq f(x, t)\right\} .
$$

We conclude that, there exists a poitive number $\eta$ such that

$$
\cos \left\langle n_{x}(t), \nu_{x_{0}}\right\rangle \geq \eta
$$

for all $x \in B_{\rho}\left(x_{0}\right) \cap \Omega(t), t_{0}-\rho \leq t \leq t_{0}$, showing the desired result.

## 4. Local Coordinate Change and Preliminary Results

Let us assume in this section that $f$ is a solution of the free-boundary problem

$$
\begin{cases}\frac{\partial f}{\partial t}=f \Delta f+r|D f|^{2} & (x, t) \in \mathbb{R}^{n} \times[0, T]  \tag{4.1}\\ f(x, 0)=f^{0} & x \in \mathbb{R}^{n}\end{cases}
$$

with $r=1 /(m-1)$, where $f^{0}$ is a non-negative and compactly supported function which satisfies the hypotheses of Theorem 1.1. As in Section 3, we denote by $\Omega(t)$, $0<t \leq T$, the set

$$
\Omega(t)=\left\{x \in \mathbb{R}^{n}: f(x, t)>0\right\} .
$$

Also, for $0<\tau<T$, let us denote by $\Omega_{\tau}$ the set

$$
\Omega_{\tau}=\left\{(x, t) \in \mathbb{R}^{n} \times(0, \tau): f(x, t)>0\right\}=\underset{0<t \leq \tau}{\cup} \Omega(t)
$$

and by $\Gamma_{\tau}$, the interface

$$
\Gamma_{\tau}=\bigcup_{0<t \leq \tau} \partial \Omega(t)
$$

We will introduce next a local coordinate change, used in [8], [9] which allows us to transform the free-boundary problem (4.1) near the interface to nonlinear degenerate problem with fixed boundary. Assume, for the moment, that $f$ is a $C^{1}$-function in its support and pick a point $P_{0}=\left(x_{0}, t_{0}\right)$ at the free-boundary $\Gamma_{T}$, with $0<t_{0}<T$. We can assume, by rotating the coordinates, that at the point $P_{0}$

$$
D_{n} f\left(P_{0}\right)=c_{0}>0
$$

Hence, there exists a number $\delta>0$ for which

$$
\begin{equation*}
D_{n} f(P) \geq c>0, \quad \forall P \in \mathcal{A}_{\delta}\left(P_{0}\right) \tag{4.2}
\end{equation*}
$$

where

$$
\mathcal{A}_{\delta}\left(P_{0}\right)=\left\{(x, t): x \in \Omega(t) \cap B_{\delta}\left(x_{0}\right) \quad t_{0}-\delta \leq t \leq t_{0}\right\}
$$

Hence, we can apply the Implicit Function Theorem, to solve the equation

$$
z=f\left(x^{\prime}, x_{n}, t\right), \quad\left(x^{\prime}, x_{n}, t\right) \in \mathcal{A}_{\delta}\left(P_{0}\right)
$$

with respect to $x_{n}$, yielding to a function

$$
x_{n}=h\left(x^{\prime}, z, t\right)
$$

To simplify the notation, lets us introduce the new coordinates

$$
\begin{equation*}
y_{i}=x_{i}, \quad i=1, \ldots n-1, \quad y_{n}=z, \quad t=t \tag{4.3}
\end{equation*}
$$

where time is still denoted by $t$. Denote by $R_{0}$ the point

$$
R_{0}=\left(y_{0}, t_{0}\right)=\left(y_{0}^{\prime}, 0, t\right)
$$

Then, we can choose $\rho>0$, sufficiently small, so that the function $z=h(y, t)$ is well defined in the parabolic cube

$$
\begin{equation*}
Q_{\rho}\left(R_{0}\right)=\left\{\left|y^{\prime}-y_{0}^{\prime}\right| \leq \rho, \quad 0 \leq y_{n} \leq \rho, \quad t_{0}-\rho^{2} \leq t \leq t_{0}\right\} \tag{4.4}
\end{equation*}
$$

One can show ([8], [9]) that the function $h(y, t)$ satisfies the equation
(4.5) $h_{t}=y_{n}\left(\Delta_{\mathbb{R}^{n-1}} h-\frac{2 \sum_{i=1}^{n-1} h_{i}}{h_{n}} h_{i n}+\frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}^{2}} h_{n n}\right)+r \frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}}$
where for $i=1, \ldots, n$ and $j=1, \ldots, n$, we use the notation $h_{i}=D_{y_{i}} h, h_{i j}=D_{y_{i} y_{j}}^{2} h$ and $\Delta_{\mathbb{R}^{n-1}}=\sum_{i=1}^{n-1} h_{i i}$. Equation (4.5) can also be expressed in divergence form ([9]) as

$$
\begin{equation*}
h_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} h-y_{n}^{\frac{m-2}{m-1}} D_{y_{n}}\left(y_{n}^{\frac{1}{m-1}} \frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}}\right) \tag{4.6}
\end{equation*}
$$

The linearization of equation (4.5) at a point $h$ is:

$$
\begin{equation*}
\tilde{h}_{t}=y_{n}\left(\Delta_{\mathbb{R}^{n-1}} h-\frac{2 \sum_{i=1}^{n-1} h_{i}}{h_{n}} \tilde{h}_{i n}+\frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}^{2}} \tilde{h}_{n n}\right)+\sum_{i=1}^{n} b_{i} \tilde{h}_{i} \tag{4.7}
\end{equation*}
$$

with

$$
\begin{equation*}
b_{i}=-\frac{2 r h_{i}}{h_{n}}+\frac{2 y_{n} h_{i}}{h_{n}^{2}} h_{n n}-\frac{2 y_{n}}{h_{n}} h_{i n}, \quad i=1, \ldots, n-1 \tag{4.8}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{n}=\sum_{i=1}^{n-1}\left(r \frac{1+h_{i}^{2}}{h_{n}^{2}}-\frac{2 y_{n}\left(1+h_{i}^{2}\right)}{h_{n}^{2}} h_{n n}+\frac{2 y_{n} h_{i}}{h_{n}^{2}} h_{i n}\right) \tag{4.9}
\end{equation*}
$$

The linearization of the divergence form equation (4.6) at a point $h$ is:

$$
\begin{equation*}
\tilde{h}_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} \tilde{h}+y_{n}^{\frac{m-2}{m-1}} D_{y_{n}}\left(y_{n}^{\frac{1}{m-1}} A_{i} \tilde{h}_{i}\right) \tag{4.10}
\end{equation*}
$$

with

$$
A_{i}=-\frac{2 \sum_{i=1}^{n-1} h_{i}}{h_{n}}, \quad i=1, \ldots, n-1
$$

and

$$
A_{n}=\frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}}
$$

It has been shown in [9] that the equation (4.10) has the form

$$
\begin{equation*}
u_{t}=y_{n}^{-\sigma} D_{i}\left(y_{n}^{1+\sigma} a^{i j} D_{j} u\right)+\sigma a^{n j} D_{j} u \tag{4.11}
\end{equation*}
$$

with $\sigma>-1$, where the matrix $\left(a^{i j}\right)=\left(a^{i j}(D h)\right)$ satisfies

$$
a^{i j} \xi_{i} \xi_{j} \geq \min \left(1, h_{n}^{-2}\right)|\xi|^{2}, \quad \text { and } \quad\left|a^{i j} \xi_{i} \eta_{j}\right| \leq 2 \frac{1+|D h|^{2}}{h_{n}^{2}}|\xi||\eta| .
$$

Hence

$$
\begin{equation*}
a^{i j} \xi_{i} \xi_{j} \geq \lambda|\xi|^{2} \quad \text { and } \quad\left|a^{i j} \xi_{i} \eta_{j}\right| \leq \lambda^{-1}|\xi||\eta| \tag{4.12}
\end{equation*}
$$

for some positive constant $c$, provided that

$$
h_{n}=D_{n} h \geq c>0 \quad \text { and } \quad|D h| \leq c^{-1}
$$

It is easy to observe that these bounds are satisfied by $h$ on the cube $Q_{\rho}\left(R_{0}\right)=$ $\left\{\left|y^{\prime}-y_{0}{ }^{\prime}\right| \leq \rho, 0 \leq y_{n} \leq \rho, t_{0}-\rho^{2} \leq t \leq t_{0}\right\}$, since $f \in C^{1}\left(\mathcal{A}_{\delta}\right)$ and satisfies (4.2).

We will use in the next section, the following result by Koch [9].
Theorem 4.1. (Hölder Regularity) Assume that $u$ is a solution of the equation (4.11) in the cube $Q_{\rho}=Q_{\rho}\left(R_{0}\right)$ with $\sigma>-1$ and coefficients which satisfy conditions (4.12). Then, there exists a number $\gamma>0$, depending only on $n, \sigma, \lambda$ such that $u \in C^{\gamma}\left(Q_{\delta}\right)$, with $\delta=\rho / 2$, and

$$
\|u\|_{C^{\gamma}\left(Q_{\delta}\right)} \leq C(n, \sigma, \lambda) \rho^{-\gamma}\left|Q_{\rho}\right|_{\sigma}^{-1} \int_{Q_{\rho}}|u| d \mu_{\sigma}
$$

where $d \mu_{\sigma}$ denotes the measure $d \mu_{\sigma}=y_{n}^{\sigma} d y d t$ and $\left.\mid Q_{\rho}\left(R_{0}\right)\right)\left.\right|_{\sigma}=\int_{Q_{\rho}\left(R_{0}\right)} d \mu_{\sigma}$.
We will also need the following generalization of Theorem 4.1, also proven in [9].

Theorem 4.2. (First Schauder Estimate) Let $u$ be a solution of the equation

$$
\begin{equation*}
u_{t}=y_{n}^{-\sigma} D_{i}\left(y_{n}^{1+\sigma} a^{i j} D_{j} u\right)+y_{n}^{-\sigma} D_{i}\left(y_{n}^{\sigma} f^{i}\right) \tag{4.13}
\end{equation*}
$$

in the cube $Q_{\rho}=Q_{\rho}\left(R_{0}\right)$ with $\sigma>-1$. Assume that the coefficients $a^{i j}$ satisfy conditions (4.12) and $f^{i} \in C^{\beta}\left(Q_{\rho}\left(R_{0}\right)\right)$, for some $\beta>0$. Then, there exists a number $0<\gamma<\beta$, depending only on $n, \sigma, \lambda, \beta$, for which $u \in C^{\gamma}\left(Q_{\delta}\right)$, with $\delta=\frac{\rho}{2}$ and

$$
\|u\|_{C^{\gamma}\left(Q_{\delta}\right)} \leq C(n, \sigma, \lambda) \rho^{-\gamma}\left|Q_{\rho}\right|_{\sigma}^{-1} \int_{Q_{\rho}}|u| d \mu_{\sigma}+\sum_{i=1}^{n}\left\|f^{i}\right\|_{C^{\gamma}\left(Q_{\rho}\right)}
$$

where $\left.d \mu_{\sigma}=y_{n}^{\sigma} d y d t, \mid Q_{\rho}\left(R_{0}\right)\right)\left.\right|_{\sigma}=\int_{Q_{\rho}\left(R_{0}\right)} d \mu_{\sigma}$.

The linearlized of the non-divergence form equation (4.7) has the form

$$
u_{t}=y_{n} a^{i j} u_{i j}+b^{i} u_{i}
$$

where the matix $\left(a^{i j}\right)$ satisfies conditions (4.12). In addition, if we assume that $y_{n} D_{i j}$ are continuous in $Q_{\rho}\left(R_{0}\right)$, then the coefficients $b^{i}$ given by (4.8) and (4.9) are bounded and in addition

$$
\begin{equation*}
b_{n} \geq \frac{r}{2 h_{n}^{2}} \geq \lambda>0 \tag{4.14}
\end{equation*}
$$

provided that $\rho$ and $\lambda$ are is sufficiently small.
In [8] Daskalopoulos and Hamilton showed a Schauder-type estimate for solutions of equation

$$
\begin{equation*}
u_{t}=y_{n} a^{i j} u_{i j}+b^{i} u_{i}+g \tag{4.15}
\end{equation*}
$$

where the coefficients ( $a^{i j}$ ) and $b^{i}$ satisfy conditions (4.12) and (4.14). Since the equation is degenerate the Hölder norms need to be scaled according to a singular metric. More precicely, let us consider the half space $\mathcal{H}=\left\{y_{n}>0\right\}$ and define on $\mathcal{H}$ the Riemannian metric

$$
d s^{2}=\frac{1}{y^{n}} \sum_{i=1}^{n} d y_{i}^{2}
$$

The distance between two points $x=\left(x^{\prime}, x_{n}\right)$ and $y=\left(y^{\prime}, y_{n}\right)$ in $\mathcal{H}$ is a function $s(x, y)$ which is equivalent to the function

$$
\bar{s}(x, y)=\frac{\left|x^{\prime}-y^{\prime}\right|+\left|x_{n}-y_{n}\right|}{\sqrt{x_{n}}+\sqrt{y_{n}}+\sqrt{\left|x^{\prime}-y^{\prime}\right|}} .
$$

For the parabolic problem we use the parabolic distance

$$
s((x, t),(y, s))=s(x, y)+\sqrt{|t-s|} .
$$

We denote, as in [8] by $C_{s}^{\alpha}$ the Banach space of all Hölder continuous functions with respect to the distance $s$, where the Hölder norm is also defined with respect to $s$. Suppose next that the set $\mathcal{A}$ is the closure of its interior, and the function $f$ on $\mathcal{A}$ has continuous derivatives $f_{t}, D_{i} f, D_{i j}^{2} f, i, j=1, \ldots, n$ in the interior of $\mathcal{A}$, and that

$$
f_{t}, D_{i} f \quad \text { and } \quad y_{n} D_{i j}^{2} f, \quad i, j=1, \ldots, n
$$

extend continuously to the boundary, and the extensions are Hölder continuous on $\mathcal{A}$ of class $\mathcal{C}_{s}^{\alpha}(\mathcal{A})$. We define $\mathcal{C}_{s}^{2+\alpha}(\mathcal{A})$ to be the Banach space of all such functions with norm

$$
\|f\|_{\mathcal{C}_{s}^{2+\alpha}(\mathcal{A})}=\|f\|_{C_{s}^{\alpha}(\mathcal{A})}+\sum_{i=1}^{n}\left\|D_{i} f\right\|_{C_{s}^{\alpha}(\mathcal{A})}+\sum_{i, j=1}^{n}\left\|y_{n} D_{i j}^{2} f\right\|_{C_{s}^{\alpha}(\mathcal{A})} .
$$

Define the box of side $\rho$ around a point $R_{0}=\left(y_{0}, t_{0}\right)$ to be

$$
\mathcal{B}_{r}\left(R_{0}\right)=\left\{\left|y_{i}-y_{0 i}\right| \leq \rho, \quad y_{n} \geq 0, \quad t_{0}-\rho \leq t \leq t_{0}\right\} .
$$

Theorem 4.3. ( Second Schauder Estimate ) For any $\alpha$ in $0<\alpha<$ and $\rho>0$, there exists a constant $C$ depending on $n, \lambda, \alpha$ and $\rho$ so that

$$
\|u\|_{C_{s}^{2+\alpha}\left(\mathcal{B}_{\rho / 2}\right)} \leq C\left(\|f\|_{C_{s}^{\circ}\left(\mathcal{B}_{\rho}\right)}+\|g\|_{C_{s}^{\alpha}\left(\mathcal{B}_{\rho}\right)}\right)
$$

for all solutions $u \in C_{s}^{2+\alpha}\left(\mathcal{B}_{\rho}\right)$ of equation (4.15)
The above theorem is proven in [8] in the case of dimension $n=2$. The proof of the Theorem in dimensions $n \geq 3$ is very similar, with the obvious changes.

Before we finish this section we will state, for the convenience of the reader, the short time $C^{\infty}$-Regularity result, proven in [8]. This will be used, together with theorems 4.1 and 4.2 in the proof of Therorem 1.1. Let $\Omega$ be a domain in $\mathbb{R}^{n}$. Imitating the case where $\Omega$ is the half-space $\mathcal{H}$, we define the distance function $s$ in $\Omega$ to coincide with the standard Euclidean distance at the interior of $\Omega$, and around any point $P \in \partial \Omega$, to be the pull back of the distance $s$ on the half space $\mathcal{H}$ via a map $\varphi: \mathcal{H} \rightarrow \Omega$ that straightens the boundary of $\Omega$ near $P$. The parabolic distance $\bar{s}$ is defined

$$
\bar{s}\left(\left(P_{1}, t_{1}\right),,\left(P_{2}, t_{2}\right)\right)=s\left(P_{1}, P_{2}\right)+\sqrt{\left|t_{1}-t_{2}\right|} .
$$

Similarly to the half-space case we can define the Banach space $C_{s}^{\alpha}(\Omega), C_{s}^{2+\alpha}(\Omega)$, as well as the spaces $C_{s}^{\alpha}(\mathcal{A}), C_{s}^{2+\alpha}(\mathcal{A})$, for a subeset $\mathcal{A}$ of $\Omega \times[0, \infty)$. The following two results are proven in [8]:

Theorem 4.4. Assume that the initial data $f^{0} \in C^{2+\alpha}(\Omega)$, and satisfies the nondegeneracy condition

$$
\begin{equation*}
f^{0}+\left|D f^{0}\right|^{2} \geq c>0 \tag{4.16}
\end{equation*}
$$

for some $\alpha>0$ and $c>0$. Then, there exists a number $T>0$ for which the solution $f$ of the initial value problem (4.1) belongs to the space $C_{s}^{2+\alpha}\left(\Omega_{\tau}\right)$, for all $\tau<T$.

Theorem 4.5. Assume that for some $T>0$ and some number $\alpha$ in $0<\alpha<1$, $f \in C_{s}^{2+\alpha}\left(\Omega_{T}\right)$ is a solution of the free-boundary problem (4.1) satisfying the nondegeneracy condition

$$
|D f(x, t)|+f(x, t) \geq c>0, \quad(x, t) \in \Omega_{T}
$$

Then, $f$ is smooth up to the interface on $0<t<T$ and in particular the freeboundary $\Gamma_{T}$ is smooth.

Combining the previous two Theorems we obtain:
Theorem 4.6. Assume that the initial data $f^{0} \in C_{s}^{2+\alpha}(\Omega)$, and satisfies the nondegeneracy condition (4.16), for some $\alpha>0$ and $c>0$. Then, there exists a number $T>0$ for which the solution $f$ of the initial value problem (4.1) is smooth up to the interface on $0<t<T$ and in particular the free-boundary $\Gamma_{T}$ is smooth.

## 5. All time $C^{\infty}$-Regularity

This section will be devoted to the proof of the Theorem 1.1. Using the notation of the previous section, we will first show the following result:

Theorem 5.1. Assume that $f^{0}$ is smooth in the closure its support $\Omega$ and that in addition $f^{0}$ is root-concave and satisfies the non-degeneracy condition

$$
\begin{equation*}
f^{0}+\left|D f^{0}\right|^{2} \geq c>0 \tag{5.1}
\end{equation*}
$$

for some $c>0$. Then, there exists a number $\beta>0$ such that the solution $f$ of the initial value problem (1.1) belongs to the class $C_{s}^{2+\beta}\left(\Omega_{T}\right)$, for all $0<T<\infty$.

Proof. By Theorem 4.5 there exists a maximal time $T>0$ for which $f$ is smooth up to the interface on $0 \leq t<T$. Assuming that $T<\infty$, we will show that at time $t=T$, the function $f$ satisfies the non-degeneracy condition

$$
\begin{equation*}
f(x, T)+|D f(x, T)|^{2} \geq c(T)>0 \tag{5.2}
\end{equation*}
$$

and also $f(\cdot, T) \in C_{s}^{2+\beta}(\Omega(T))$ with

$$
\begin{equation*}
\|f(\cdot, T)\|_{C_{s}^{2+\beta}(\Omega(T))} \leq C \tag{5.3}
\end{equation*}
$$

for some $\beta<\alpha$. Therefore, by Theorem 4.3, there exists a number $T^{\prime}>0$ for which $f$ is in $C_{s}^{2+\beta}\left(\Omega_{\tau}\right)$, for all $\tau<T+T^{\prime}$. Theorem 4.4 then implies that $f$ is smooth up to the interface on $0<t<T+T^{\prime}$, contradicting the fact that $T$ is maximal.

Condition (5.2) is implied by Corollary 3.5: Observe first that since $f$ is smooth up to the interface for $\tau<T$, we can choose $\tau_{0}$ small enough so that

$$
f\left(\cdot, \tau_{0}\right)+\left|D f\left(\cdot, \tau_{0}\right)\right|^{2} \geq \frac{c}{2}>0, \quad x \in \Omega\left(\tau_{0}\right)
$$

The Aronson-Bénilan inequality $\Delta f \geq-\sigma / t$, implies that at $t=\tau_{0}$ we have

$$
\Delta f \geq-K=-\frac{\sigma}{\tau_{0}}
$$

Hence, we can applly Corollary 3.5 to conclude that

$$
f+\left(t+\frac{1}{K+r}\right)\left(r+\frac{1}{2}\right)|D f|^{2} \geq \frac{c}{2} e^{-K\left(t-\tau_{0}\right)}
$$

on $\{f>0\}$ for $\tau_{0} \leq t \leq T$, proving (5.2).
We will next prove condition (5.3). Let $P_{0}=\left(x_{0}, t_{0}\right)$ be a free-boundary point with $t_{0}<T$. By Theorem 3.7, there exists a number $\delta>0$ depending only on $n, r, T$ and $f^{0}$ and unit direction $\nu=\nu_{x_{0}}$ such that

$$
\begin{equation*}
D f \cdot \nu \geq c_{0}>0, \quad \forall(x, t) \in \mathcal{A}_{\delta}\left(P_{0}\right) \tag{5.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{A}_{\delta}\left(P_{0}\right)=\left\{(x, t): x \in \Omega(t) \cap B_{\delta}\left(x_{0}\right), \quad t_{0}-\delta \leq t \leq t_{0}\right\} . \tag{5.5}
\end{equation*}
$$

We can assume, without loss of generality, that $\nu=e_{n}$, so that

$$
D_{n} f \geq c_{0}>0, \quad \forall(x, t) \in \mathcal{A}_{\delta}\left(P_{0}\right)
$$

Hence, we can perform the local coordinate change (4.3) on $\mathcal{A}_{\delta}\left(P_{0}\right)$ to obtain a function $x_{n}=h(y, t)$ defined on the parabolic cube

$$
Q_{\eta}\left(R_{0}\right)=\left\{\left|y^{\prime}-y_{0}^{\prime}\right| \leq \eta, \quad 0 \leq y_{n} \leq \eta, \quad t_{0}-\eta^{2} \leq t \leq t_{0}\right\}
$$

with $R_{0}=\left(y_{0}, t_{0}\right)=\left(x_{0}{ }^{\prime}, 0, t_{0}\right)$. To simplify the notation, we will denote, for any $\eta>0$ the cube $Q_{\eta}\left(R_{0}\right)$ by $Q_{\eta}$. Notice, that since $f$ is continuous on $\mathbb{R}^{n} \times\left[t_{0}-\delta, t_{0}\right]$, we can choose $\rho$ sufficiently small, depending on $\delta$ and the modulus of continuity of $f$, such that

$$
(x, t) \in \mathcal{A}_{\delta}\left(P_{0}\right) \quad \text { if } \quad(y, t) \in Q_{\eta} .
$$

We will show, using Theorem 4.1, that:

Lemma 5.2. There exists a numbers $\gamma>0$ and $C>0$, depending only on $n, r, T$ and the initial data $f^{0}$, such that the gradient Dh satisfies

$$
\begin{equation*}
\|D h\|_{C^{\gamma}\left(Q_{\rho}\right)} \leq C \tag{5.5}
\end{equation*}
$$

with $2 \rho=\eta$.
The significance of this Lemma is that the norm $\|D h\|_{C^{\gamma}\left(Q_{\rho}\right)}$ remains uniformly bounded, as $t_{0} \uparrow T$. Let us continue with the proof of Theorem 5.1 and leave the proof of the lemma for the end. Observe first that (5.5) implies that

$$
\|D h\|_{C_{s}^{\beta}\left(Q_{\rho}\right)} \leq C
$$

for some $\beta<\gamma$. Hence, the Schauder estimate of Theorem 5.3 applied to equation (4.5), implies that $h \in C_{s}^{2+\beta}\left(Q_{\rho / 2}\right)$ with

$$
\|h\|_{C_{s}^{2+\beta}\left(Q_{\rho / 2}\right)} \leq C
$$

Since $C$ remains uniformly bounded as $t \uparrow T$, we can go back to the original coordinates to finally concude that

$$
\|f\|_{C_{s}^{2+\beta}(\Omega(T))} \leq C
$$

finishing the proof of Theorem 5.1.
Before we prove Lemma 5.2 we will show the next simple Lemma:

Lemma 5.3. Under the hypotheses of Lemma 5.2, there exists a constant $c>0$, depending only on $n, r, T$ and the initial data $f^{0}$ such that

$$
|D h| \leq c^{-1}
$$

and

$$
D_{y_{n}} h \geq c>0
$$

in $Q_{\eta}=Q_{\eta}\left(R_{0}\right)$.
Proof. One can easily compute that

$$
D_{y_{i}} h=-\frac{D_{x_{i}} f}{D_{x_{n}} f}, \quad D_{y_{n}} h=\frac{1}{D_{x_{n}} f} .
$$

Since

$$
|D f| \leq C
$$

and

$$
D_{x_{n}} f \geq c>0
$$

in $\mathcal{A}_{\delta}\left(P_{0}\right)$ the lemma follows.
We will now prove Lemma 5.2:
Proof of Lemma 5.2. We will show that the derivatives $D_{y_{i}} h$ belong to the Hölder class $C^{\gamma}\left(Q_{\rho}\right)$, with $\rho=\delta / 2$, by differentiating equation (4.6) with respect to $y_{i}$ and applying Theorem 4.1.

Let us first show the conclusion of the lemma for the derivatives $u=D_{y_{i}} h=h_{i}$, $i=1, \ldots, n-1$. Differentiating (4.6) with respect to $y_{i}$, we find that $u=D_{y_{i}} h$ satisfies the equation

$$
u_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} u+y_{n}^{-\sigma} D_{y_{n}}\left(y_{n}^{1+\sigma} A_{i} u_{i}\right)
$$

with $\sigma=\frac{2-m}{m-1}$ and

$$
\begin{equation*}
A_{i}=-\frac{2 \sum_{i=1}^{n-1} h_{i}}{h_{n}}, \quad i=1, \ldots, n-1 \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{n}=\frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}} \tag{5.7}
\end{equation*}
$$

where to simplify the notation we denote by $h_{i}=D_{y_{i}} h$. It follows from (4.12) and Lemma 5.3 that the coefficients $A_{i}, i=1, \ldots, n$ satisfy the hypotheses of Theorem 4.1. Hence,

$$
\|u\|_{C^{\gamma}\left(Q_{\rho}\right)} \leq C(n, \sigma, \lambda) \rho^{-\gamma}\left|Q_{\delta}\right|_{\sigma}^{-1} \int_{Q_{\delta}}|u| y_{n}^{\sigma} d y d t
$$

Since $\sigma>-1$, the last estimate in combination with Lemma 5.3, implies that

$$
\begin{equation*}
\left\|D_{i} h\right\|_{C^{\gamma}\left(Q_{\rho}\right)} \leq C, \quad 1=1, \ldots, n-1 \tag{5.8}
\end{equation*}
$$

with $C$ depending only on $n, \sigma, T$ and $f^{0}$.
It remains to prove the same estimate for the derivative $u=D_{y_{n}} u$. Differentiating (4.6) with repsect to $y_{n}$ we find that $u=D_{y_{n}} h$ satisfies the equation

$$
\begin{equation*}
u_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} u+y_{n}^{-(1+\sigma)} D_{y_{n}}\left(y_{n}^{2+\sigma} A_{i} u_{i}\right)+\Delta_{\mathbb{R}^{n-1}} h, \tag{5.9}
\end{equation*}
$$

where the $A_{i}, i=1, \ldots, n$ are given by (5.6) and (5.7). To verify this, let us denote by

$$
Q=-\frac{1+\sum_{i=1}^{n-1} h_{i}^{2}}{h_{n}}
$$

and by

$$
F=D_{y_{n}} Q=A_{i} u_{i}
$$

where $u=D_{y_{n}} h=h_{n}$ and summation convention is used. Under this notation, (4.6) can be simply written as

$$
\begin{equation*}
h_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} h+y_{n}^{-\sigma} D_{y_{n}}\left(y_{n}^{1+\sigma} Q\right) \tag{5.9}
\end{equation*}
$$

Hence, differentiating (5.9) with respect to $y_{n}$ we obtain

$$
u_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} u+D_{y_{n}}\left[y_{n}^{-\sigma} D_{y_{n}}\left(y_{n}^{1+\sigma} Q\right)\right]+\Delta_{\mathbb{R}^{n-1}} h
$$

which results, after some calculations, to the equation

$$
u_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} u+(2+\sigma) F+y_{n} D_{y_{n}} F+\Delta_{\mathbb{R}^{n-1}} h .
$$

The last equation can be rewritten as

$$
u_{t}=y_{n} \Delta_{\mathbb{R}^{n-1}} u+y_{n}^{-(1+\sigma)} D_{y_{n}}\left(y_{n}^{2+\sigma} F\right)+\Delta_{\mathbb{R}^{n-1}} h
$$

where $F=D_{y_{n}} Q=A_{i} u_{i}$, yielding to (5.9). We observe next that equation (5.9) is of the form of equation (4.13), with coefficients ( $a^{i j}$ ) which satisfy condition (4.12), because of the bounds of Lemma 5.3, and

$$
f^{i}=D_{i} h, \quad i=1, \ldots, n-1 \quad f^{n}=0
$$

Hence, by Theorem 4.2 and (5.8) we obtain

$$
\left\|D_{n} h\right\|_{C^{\gamma}\left(Q_{\rho}\right)} \leq C
$$

with $C$ depending only on $n, \sigma, T$ and $f^{0}$, finishing the proof of the lemma.

## 6. Less Regular Initial data

In this section we will show Theorem 1.2. Let $f$ be a weak solution of the initial value problem (1.1) with continous initial data $f^{0}$. We would like to assume that $\phi=\sqrt{f^{0}}$ is weakly concave on its support $\Omega$, namely that it satisfies the inequality

$$
\begin{equation*}
\frac{\phi(x)+\phi(y)}{2}-\phi\left(\frac{x+y}{2}\right) \leq 0 \tag{6.1}
\end{equation*}
$$

Denoting by $d(x)$ the distance function from the boundary of $\Omega$ we will first show the following result:

Theorem 6.1. If $f^{0}$ is continous and and strictly positive on the compact domain $\Omega$ with $f^{0}=0$ at $\partial \Omega$ and in addition $\sqrt{f^{0}}$ is weakly concave in $\Omega$ and satisfies the non-degeneracy condition

$$
\begin{equation*}
f^{0}(x) \geq c d(x) \tag{6.2}
\end{equation*}
$$

for some conatnt $c>0$, then the weak solution $f$ of the initial value problem (1.1) has $f(\cdot, t)$ weakly concave for all $0<t<\infty$.

Proof. Let us approximate the initial data $f^{0}$ by a sequence of functions $f_{k}^{0}$, so that each $f_{k}^{0}$ is supported on a compact domains $\Omega_{k}, f_{k}^{0} \in C^{\infty}\left(\bar{\Omega}_{k}\right)$ and satisfies the estimates

$$
\begin{equation*}
D_{i j}^{2} \sqrt{f_{k}^{0}} \leq 0 \quad \text { on } \Omega_{k} \tag{6.2}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{k}^{0}(x)+\left|D f_{k}^{0}\right| \geq \tilde{c} \quad \text { on } \Omega_{k} \tag{6.3}
\end{equation*}
$$

for some constant $\tilde{c}>0$. It is easy to observe that such a approximation is possible, since $\sqrt{f^{0}}$ on $\Omega$ is weakly concave and satisfies (6.1). Moreover, $\left\{f_{0}^{k}\right\}$ can be chosen so that $f_{k}^{0} \rightarrow f^{0}$ uniformly on $\mathbb{R}^{n}$. Let $f^{k}$ be the unique weak solution of the initial value problem (1.1) with initial data $f_{k}^{0}$. Since $f_{k}^{0}$ satisfies conditions (6.2) and (6.3), it follows from Theorem 1.1 that the solution $f_{k}$ is smooth up to the interface for $0 \leq t<\infty$ and moreover $f_{k}(\cdot, t)$ is root-concave for all $t>0$. In particular, each $\sqrt{f_{k}}(\cdot, t)$ satisfies the inequality (6.1) namely

$$
\begin{equation*}
\frac{\sqrt{f_{k}}(x, t)+\sqrt{f_{k}}(y, t)}{2}-\sqrt{f_{k}}\left(\frac{x+y}{2}, t\right) \leq 0 \tag{6.4}
\end{equation*}
$$

for all $x, y$ in its support. Since, the sequence of solutions $f_{k}$ is uniformly bounded, it is equicontinuous on compact subsets of $\mathbb{R}^{n} \times(0, \infty)$. Therefore, using the equicontinuity result in [10] one can show, by standard arguments, that $f^{k}$ converges, uniformly on compact subsets of $\mathbb{R}^{n} \times(0, \infty)$, to the solution $f$. By taking the limit $k \rightarrow \infty$ in (6.4) we obtain that $\sqrt{f}(\cdot, t)$ is weakly concave, finishing the proof of the theorem.

We will assume next that $f^{0}$ belongs to the weighted Hölder space $C_{s}^{2+\alpha}(\Omega)$, as defined in Section 4. We will show that:

Theorem 6.2. Assume that the function $f^{0} \in C_{s}^{2+\alpha}(\Omega)$ is root-concave in $\Omega$ and satisfies the non-degeneracy condition

$$
f^{0}+\left|D f^{0}\right|^{2} \geq c>0
$$

for some $c>0$. Then, the solution $f$ of the initial value problem (1.1) is a smooth function smooth up to the interface $\Gamma$ and $f(\cdot, t)$ is root-concave, for all $0 \leq t<\infty$. In particular, the free boundary $\Gamma$ is a smooth surface.

Proof. The proof of this theorem follows quite immediately, by combining the short time regularity results in [8], Theorems 4.6 , and 4.7 with Theorems 1.1 and 6.1. By Theorems 4.6 and 4.7 , there exists a number $T>0$ for which $f$ is $C^{\infty_{-}}$ smooth up to the interface on $0<t<\tau$ and also $f \in C^{2+\alpha}\left(\Omega_{\tau}\right)$, for all $\tau<T$, where $\Omega_{\tau}=\left\{(x, t) \in \mathbb{R}^{n} \times[0, \tau]: f(x, t)>0\right\}$. Therefore, there exists a number $0<\tau<T$, such that $f(\cdot, \tau)$ is smooth on the closure of its support and satisfies the non-degeneracy condition

$$
f(\cdot, \tau)+|D f(\cdot, \tau)|^{2} \geq \tilde{c}>0
$$

with $\tilde{c}=\frac{c}{2}$. In addition, by Theorem 6.1, the function $f(\cdot, \tau)$ is root-concave on its support. Hence, we can apply Theorem 1.1 to conlcude that $f$ must remain smooth up to the interface, for all $0<t<T$, proving the desired result.

We will finish this paper with the proof of Theorem 1.2:
Proof of Theorem 1.2. Lets us approximate $f^{0}$ by a sequence of functions $f_{k}^{0}$ which are compactly supported, smooth on the closure of their support $\Omega_{k}$ and satisfy the gradient estimate

$$
\left|D f_{k}^{0}\right| \leq C \quad \text { on } \mathbb{R}^{n}
$$

the non-degeneracy estimate

$$
f_{k}^{0}+\left|D f_{k}^{0}\right|^{2} \geq c \quad \text { on } \Omega_{k}
$$

the lower bound on the Laplacian

$$
\Delta f_{k}^{0} \geq-K \quad \text { on } \mathbb{R}^{n}
$$

in the distributional sense and the root concavity estimate

$$
D_{i j}^{2} \sqrt{f_{k}^{0}} \leq 0 \quad \text { on } \Omega_{k}
$$

for some positive constants $c>0$ and $C>0$ which are independent of $k$. We can choose such a sequence $f_{k}^{0}$ so that $f_{0}^{k} \rightarrow f_{k}$ uniformly on $\mathbb{R}^{n}$. According Theorem 1.1, for each $k \in \mathbb{N}$, the solution $f_{k}$ of equation (1.1) with initial data $f_{k}^{0}$ is smooth up to the interface. In addition each $f_{k}(\cdot, t)$ is root-concave on its support for all $t>0$ and satisfies the following estimates, proven in Section 4:

$$
\left|D f_{k}\right| \leq C \quad \text { on } \mathbb{R}^{n}
$$

and

$$
\Delta f_{k} \geq-K \quad \text { on } \mathbb{R}^{n}
$$

in the distributional sense and

$$
f^{k}+\left(t+\frac{1}{K+r}\right)\left(r+\frac{1}{2}\right)\left|D f_{k}\right|^{2} \geq c e^{-K t} \quad \text { on } \Omega_{k} .
$$

Let $P=\left(x_{0}, t_{0}\right)$ be a point on the interface of $f$ with $t_{0}>0$. Denoting by $Q_{\delta}\left(x_{0}, t_{0}\right)$ the cylinder

$$
Q_{\delta}\left(x_{0}, t_{0}\right)=\left\{\left|x-x_{0}\right| \leq \delta, \quad t_{0}-\delta^{2} \leq t \leq t_{0}\right\}
$$

we will show the following claim:
Claim. There exist numbers $\delta>0, \tilde{c}>0$ and a unit direction $\nu_{x_{0}}$ such that

$$
D f^{k} \cdot \nu_{x_{0}} \geq \tilde{c}, \quad \text { on } Q_{\delta}\left(x_{0}, t_{0}\right) \cap\left\{f_{k}>0\right\}
$$

for all $k$ sufficiently large.
Assuming that the claim holds and that the unit direction $\nu_{x_{0}}$ is the unit vector $\nu_{x_{0}}=e_{n}$ parallel to the $x_{n}$-axis. Then, we can perform the local coodinate change (4.3) and apply Theorem 4.1, as in the proof of Lemma 5.2, to conclude that

$$
\left\|D f_{k}\right\|_{C^{\gamma}\left(Q_{\delta, k}\right)} \leq C
$$

for some $\gamma>0$, with $Q_{\delta, k}=Q_{\delta}\left(x_{0}, t_{0}\right) \cap\left\{f_{k}>0\right\}$ and $C$ is independent of $k$. We can now use the Schauder estimate, Theorem 4.3, to conclude that

$$
\begin{equation*}
\left\|f_{k}\right\|_{C_{s}^{2+\beta}\left(Q_{\delta, k}\right)} \leq C \tag{6.5}
\end{equation*}
$$

for some $\beta<\gamma$, where again $C$ is independent of $k$. On the other hand, since $f_{0}^{k} \rightarrow$ $f^{0}$ uniformly, one can show be standard arguments (see in the proof of Theorem 6.1) that $f_{k} \rightarrow f$ uniformly on compact sets of $\mathbb{R}^{n} \times(0$, infty $)$. In particular $f_{k} \rightarrow f$ on $\tilde{Q}_{\delta}=Q_{\delta}\left(x_{0}, t_{0}\right) \cap\{f>0\}$. By (6.5) we will have

$$
\|f\|_{C_{s}^{2+\beta}\left(\tilde{Q}_{\delta}\right)} \leq C
$$

which implies that $f \in C_{s}^{2+\beta}\left(\tilde{Q}_{\delta}\right)$. Hence, by Theorem 4.5, f is smooth up to the interface, showing the desired result.

## References

[1] S.B. Angenent and D.G. Aronson, The Focusing problem for the radially symmetric porous medium equation, Comm. Partial Differential Equations, Vol. 20, no. 7-8 (1995) pp 12171240.
[2] D.G. Aronson and P. Bénilan, Régularité des solutions de l'équation de milieux poreux dans $\mathbf{R}^{n}$, C.R. Acad. Sci. Paris Vol. 288 (1979) pp 103-105.
[3] D.G. Aronson and L.A. Caffarelli, The Initial Trace of a Solution of the Porous Medium equation, Trans. Amer. Math. Soc., Vol. 280 (1983) pp 351-366.
[4] L.A. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, Ind. Univ. Math. J., Vol. 29 (1980) pp 361-389.
[5] L.A. Caffarelli, J.L. Vázquez and N.I. Wolanski, Lipschitz continuity of solutions and interfaces of the n-dimensional porous medium equation, Ind. Univ. Math. J., Vol. 36 (1987) pp 373-401.
[6] L.A. Caffarelli and N.I. Wolanski, $C^{1, \alpha}$ regularity of the of the free boundary for the n- dimensional porous media equation, Comm. Pure and Appl. Math., Vol. 43 (1990) pp 885-902.
[7] P. Daskalopoulos and R. Hamilton, The Free Boundary for the N-dimensional Porous Medium Equation, Internat. Math. Res. Notices, Vol. 17 (1997) pp 817-831.
[8] P. Daskalopoulos and R. Hamilton, $C^{\infty}$-Regularity of the Free Boundary for the Porous Medium Equation, J. of Amer. Math. Soc., Vol. 11, No 4 (1998) pp 899-965.
[9] H. Koch, Non-Euclidean Singular Integrals and the Porous Medium Equation, Habilitation Thesis (1998).
[10] P. Sacks, Continuity of solutions of a singular parabolic equation, Nonlinear Anal. 7 (1983) pp 387-409.

