Free-Boundary Regularity on the Focusing Problem for the Gauss Curvature Flow with Flat Sides

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1. Introduction.

We consider, in this paper, a compact convex body M in $\mathcal{R}^3$ which is subject to wear under impact from any random angle. An example can be a stone on a beach impacted by the sea. Our objective is to study the deformation of the surface Σ of the body. The probability of impact at any point P on the surface Σ is proportional to the Gauss Curvature K. Therefore the surface evolves by the flow

$$\frac{\partial P}{\partial t} = K\,N$$

where N denotes the unit inward normal. The *Gauss Curvature Flow* was introduced by Firey [F], who showed that it shrinks smooth, compact, strictly convex and centrally symmetric hypersurfaces in $\mathcal{R}^3$ to round points. Tso [T] showed that if the initial surface Σ is smooth, compact and strictly convex, then the Gauss Curvature Flow admits a unique solution $\Sigma(t)$ which shrinks to a point at the exact time $T^* = V/4\pi$, where V is the volume enclosed by the initial surface Σ. Chow [C] proved that, under certain restrictions on the second fundamental form of the initial surface, the Gauss Curvature flow shrinks smooth compact strictly convex hypersurfaces to round points. Andrews [A] has recently shown that the Gauss Curvature flow shrinks compact convex hypersurfaces to round points.

In this work, we will consider the case where the initial surface has flat sides and as a consequence the parabolic equation describing the motion of the hypersurface becomes degenerate where the curvature becomes zero. Hence, according to Hamilton's results in [H], the junction Γ between each flat side and the strictly convex part of the surface, where the equation becomes degenerate, behaves like a free-boundary propagating with finite speed. Each flat side shrinks to a point in finite time and eventually the surface becomes strictly convex.

Daskalopoulos and Hamilton [DH] studied the solvability of the Gauss Curvature flow with flat sides and the regularity of the interface Γ, by viewing the flow as a free-boundary problem. Let us assume, for simplicity, that the initial surface Σ has only one flat side, namely that at time $t = 0$ we have

$$\Sigma = \Sigma_1 \cup \Sigma_2$$

where Σ_1 is the flat side and Σ_2 is the strictly convex part of the surface. The junction between the two sides is the curve

$$\Gamma = \Sigma_1 \cap \Sigma_2.$$

Since the equation is invariant under rotation, we can also assume, without loss of generality, that Σ_1 lies on the $z = 0$ plane and that Σ_2 lies above this plane. Then, the lower part of the surface Σ can be represented as the graph of a function

$$z = f(x, y)$$

over a compact domain $\Omega \subset \mathcal{R}^2$ containing the initial flat side Σ_1. The basic assumption in [DH] is that the function f *vanishes quadratically* at $z = 0$ and that the junction curve Γ is strictly convex. Namely, setting

$$g = \sqrt{2f}$$

it is assumed in [DH] that at time $t = 0$ the function g satisfies

$$|Dg(x, y)| \geq \lambda \qquad \text{and} \qquad D^2_{\tau\tau} g(x, y) \geq \lambda \qquad \forall (x, y) \in \Gamma \tag{1.1}$$

for some positive number $\lambda > 0$, where $D^2_{\tau\tau}$ denotes the second order tangential derivative at Γ. To explain the condition (1.1) we look at the rotationally symmetric case $z = f(r, t)$, where the equation which is satisfied by f becomes

$$f_t = \frac{f_r\, f_{rr}}{r(1 + f_r^2)^{3/2}}.$$

This equation can be modeled, near $r = 1$, on the simpler equation $f_t = f_r\, f_{rr}$. It is easy to compute that both functions $z = (r + 2t - 1)^2_+$ and $z = (r - 1)^3/6\,(1 - t)^2$ are solutions of the above equation. The first one vanishes quadratically at the interface $z = 0$ and the free-boundary starts moving immediately at time $t > 0$, while second one vanishes cubically at $z = 0$ and the free-boundary doesn't move. It has been recently shown by Chopp, Evans and Ishii [CEI] that if the surface Σ is of class $C^{3,1}$, which, in particular, implies that f vanishes cubically at the interface, then the flat region does not move at all for a positive interval time. Condition (1.1) guarantees that the interface Γ will start to move at any point at time $t = 0$.

One can choose the domain Ω so that when Σ evolves by the Gauss Curvature Flow, then at least for some short time $0 < t < T$, the lower part of the surface $\Sigma(t)$, can be represented as the graph of a function

$$z = f(x, y, t)$$

over Ω. This is because the results by Hamilton [H] guarantee that the lower part of the surface will not turn immediately vertical. Since $\Sigma(t)$ solves the Gauss Curvature Flow, it can be shown, by standard computation, that the function f satisfies the equation

$$f_t = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^{3/2}}. \tag{1.2}$$

The flat side will always lie on the $z = 0$ plane, while the strictly convex side will always have $z > 0$. It was shown in [DH] that if the initial surface satisfies (1.1) and g, Dg and $d\, D^2 g \in C^\alpha(\Omega)$, with d denoting the distance to the free-boundary, then the Gauss Curvature Flow admits a solution $\Sigma(t)$, in $0 < t \leq \tau$, for some time $\tau > 0$, the function $g = \sqrt{2f}$ is smooth up to the interface and the junction $\Gamma(t)$ between the flat and the strictly convex side is a *smooth curve*, in the time interval $0 < t \leq \tau$.

In this paper, we address the question of the regularity of the interface up to its *focusing time*, that is up to the time T when the flat side shrinks to a point. Andrews [A] showed that if the initial surface is

$C^{1,1}$, then the Gauss Curvature Flow admits a viscosity solution of class $C^{1,1}$. We conjecture that, under certain assumptions on the initial surface, the free-boundary Γ will be *smooth* for all time $0 < t < T$ and the function g will be smooth up to the interface. However, we will treat here only the case where the initial surface Σ is a *surface of revolution.*

Let us assume that the initial surface Σ is a surface of revolution around the $z-$ axis and that, as before, the flat side lies on the plane $z = 0$ while the strictly convex side has $z > 0$. Then, the lower part of the surface can be represented as the graph of a radial function $z = f(r)$, satisfying

$$f(r) \equiv 0, \qquad \text{for } 0 \le r \le r_0 \qquad \text{and} \qquad \lim r \to r_1 - 0 f_r(r) = +\infty \tag{1.3}$$

for some numbers $0 < r_0 < r_1$, since $f(r) \equiv 0$ at the flat side and $f_r(r) = +\infty$ at the point where the surface turns vertical.

Set $g = \sqrt{2f}$. Conditions (1.2) can now be expressed as

$$g_r(r_0) \ge \lambda > 0 \tag{1.4}$$

since $|Dg| = |g_r|$ and $g_{\tau\tau} = g_r/r$. Moreover, since f satisfies (1.3), g will satisfy

$$g(r) \equiv 0, \quad \text{for} \quad 0 \le r \le r_0 \qquad \text{and} \qquad \lim_{r \to r_1 - 0} g_r(r) = +\infty. \tag{1.5}$$

When Σ evolves by the Gauss Curvature Flow, then at time t the lower part of the surface $\Sigma(t)$ can be represented as the graph of the radial function $z = f(r,t)$ which evolves by the one-dimensional nonlinear equation

$$f_t = \frac{f_r\, f_{rr}}{r\,(1 + f_r^2)^{3/2}} \tag{1.6}$$

and has initial data $f(r,0) = f(r)$. The function $g = \sqrt{2f}$ will evolve by the equation

$$g_t = \frac{g\, g_r\, g_{rr} + g_r^3}{r\,(1 + g^2 g_r^2)^{3/2}} \tag{1.7}$$

with initial data $g(r,0) = g(r)$ satisfying conditions (1.4) and (1.5). Let us define T to be the *focusing time* of the flat side, manely the time

$$T = \max\{t > 0 : \; g(0,t) = 0\,\}. \tag{1.8}$$

Then, at time $t > T$ the surface $\Sigma(t)$ will become strictly convex and it will start moving away from the $z = 0$ plane. In other words, $g > 0$ for $t > T$, while $g(r,T) = 0$ only at the point $r = 0$. Our first result shows that the free-boundary $z = \gamma(t)$ is smooth, for all t in the time interval $0 < t < T$:

1.1. Theorem. *Assume that Σ is a surface of revolution around the $z-$axis, with a flat side, such that at time $t = 0$ the lower part of the surface is the graph of a function $z = f(r,0) \ge 0$, $0 \le r \le r_1$ with $z = 0$ at the flat side $0 \le r \le r_0$. Assume that at time $t = 0$, $g = \sqrt{2f}$ is smooth on $[r_0, r_1)$ and satisfies conditions (1.4) and (1.5). Then, the function $g = \sqrt{2f}$ will be smooth up to the interface $g = 0$, for all time $0 < t < T$, with T denoting the foccusing time of the flat side. In particular the interface $z = \gamma(t)$ will be smooth.*

Our second result describes the behavior of g at the foccusing time:

1.2. Theorem. *Under the same hypotheses as in Theorem 1.1, the solution g os (1.7) satisfies the derivative estimates*

$$C_1\, r^{\frac{2}{5}} \le g_r(r,t) \le C_2\, r^{\frac{1}{4}}, \qquad 0 \le t \le T \tag{1.9}$$

near the interface. This implies that at the foccusing time T of the flat side, the function g is of class $C^{1+\beta}$, for any $\beta < \frac{1}{4}$, and has no better regularity than of class $C^{1+\frac{2}{5}}$.

Remark. According the results in [DH], the smoothness assumption on g at the initial time $t = 0$ can be replaced by the regularity assumptions that $g, g_r, d\, g_{rr} \in C^{\alpha}([r_1, r_0))$, for some number $0 < \alpha < 1$, where d denotes the distance to the interface $z = \gamma(t)$.

The focusing problem for the porous medium equation

$$u_t = \Delta u^m, \qquad m > 0$$

has been studied by Angenent and Aronson, [AV1], [AV2]. In Section 7, we will show how the techniques in [AV1] can be used to obtain the exact behavior of radial solutions to the evolution Monge-Ampére equation

$$f_t = f_{xx} f_{yy} - f_{xy}^2 \tag{1.10}$$

at their focusing time. When f is radially symmetric, equation (1.10) takes the one dimensional form

$$f_t = \frac{f_r\, f_{rr}}{r}. \tag{1.11}$$

Also, the function $g = \sqrt{2f}$ satisfies the evolution equation

$$g_t = \frac{g\, g_r\, g_{rr} + g_r^3}{r}. \tag{1.12}$$

The result in Theorem 1.1 holds true in the case of the Monge-Ampére equation. We will show in Section 7, using the techniques in [AG] and [GV], that equation (1.11) possesses self-similar solutions of the form

$$\Phi_c(r,t) = \frac{r^2}{\sqrt{T-t}}\phi\Big(\frac{c(T-t)}{r^{\alpha^*}}\Big), \qquad c > 0 \tag{1.13}$$

with α^* a number in $\frac{3}{2} < \alpha^* < \frac{6}{5}$. As a consequence, one can use the techniques in [AA1], with minor modifications, to show that, as t approaches the focusing time T, any solution $g(r,t)$ of (1.11) converges, after rescaling, to one of the self-similar solutions Φ_c. However, because of the derivative term in the denominator of equation (1.7), one can show that equation (1.7) does not admit self-similar solutions of the form (1.13). The derivative estimates (1.9) provide the best information that we have on the behavior of the solution g of (1.7) at the focusing time $t = T$.

2. Basic Derivative Estimates.

In this section, we will prove certain basic estimates on the first and second derivatives of g. By a simple rescaling argument, we can assume without loss of generality, that

$$\max g(\cdot, t) \ge 2, \qquad \text{for } \ 0 \le t \le T, \tag{2.1}$$

where T is the focusing time of the flat side, defined by (1.8).

2.1 Lemma. *Under the hypotheses of Theorem 1.1 and condition (2.1), there exits a constant $C < \infty$ such that*

$$0 \le g_r(r,t) \le C, \qquad \text{on } \{\, g \le 1,\ 0 \le t \le T \,\}.$$

Proof. We first notice that $g_r \ge 0$, because the function $f = g^2/2$ is increasing, since f is convex and $f_r = 0$ at the interface. When $g = 1$, then $g_r = f_r/g = f_r$ and $f_r \le C$, since f is convex and bounded. Hence $g_r \le C$, on $g = 1$, $0 \le t \le T$. For the interior estimate, we first observe that we can approximate the solution g by a decreasing sequence of positive smooth increasing and strictly convex solutions g^ϵ. Indeed, let us approximate the initial data $g(\cdot, 0)$ by a decreasing sequence of positive smooth increasing and strictly convex functions $g^\epsilon(\cdot, 0)$. Since at $t = 0$ the function g is smooth up to the interface, and also $g_r \le C$, on $g = 1$, $0 \le t \le T$, we can take the sequence $g^\epsilon(\cdot, 0)$ such that

$$g^\epsilon_r(r,0) \le C, \qquad \text{on } \{\, g^\epsilon(\cdot,0) \le 1 \,\}$$

and

$$g^\epsilon_r \le C, \qquad \text{on } \{\, g^\epsilon = 1,\ 0 \le t \le T \,\}$$

with the constant C being independent of ϵ. We will show, using the maximum principle, that

$$g^\epsilon_r \le C, \qquad \text{on } \{\, g_\epsilon \le 1, 0 \le t \le T \,\}.$$

Set $G = g^\epsilon_r \ge 0$. To simplify the notation lets denote g^ϵ by g and set $I = 1 + g^2 g_r^2$. A direct computation shows that G satisfies the equation

$$G_t = \frac{gG}{r\, I^{\frac{3}{2}}}\, G_{rr} + \frac{gI - 3g^3G^2}{r\, I^{\frac{5}{2}}}\, G_r^2 + \frac{4G^2\, I - g\, G\, I - 6G^4 g^2}{r\, I^{\frac{5}{2}}}\, G_r - \frac{3G^6 g + G^3 I}{r\, I^{\frac{5}{2}}}.$$

Since,

$$\frac{3G^6 g + G^3 I}{r\, I^{\frac{5}{2}}} > 0$$

at the maximum of G, the maximum principle implies that G cannot attain an interior maximum. Hence $g^\epsilon_r \le C$, which implies the desired result.

The next Lemma provides a lower bound on the second derivative g_{rr} of g.

2.2 Lemma. *Under the hypotheses of Theorem 1.1, and condition (2.1), there exits a constant $C > 0$ such that*

$$g_{rr}(r,t) \ge -C\, r^{\frac{8}{5}}, \qquad \text{on } \{\, g \le 1,\ 0 \le t \le T \,\}.$$

Proof. Let $\{g^\epsilon\}$ be the approximation sequence constructed in the proof of Lemma 2.1. Throughout the proof of this Lemma, we will denote by C various constants which are independent of ϵ. We can choose the sequence $\{g^\epsilon\}$ such that at time $t = 0$,

$$\frac{g^\epsilon_{rr}(r,0)}{r^{\frac{8}{5}}} \ge -C, \qquad \text{on } \{\, g^\epsilon(\cdot,0) \le 1 \,\}.$$

This is because $|g_{rr}(r,0)| \le C$ and $g(r,0) = 0$ on $0 \le r \le r_0$. Set

$$G = \frac{g^\epsilon_{rr}}{r^{\frac{8}{5}}}.$$

We will show that there exists a constant C, independent of ϵ, such that

$$G \geq -C, \qquad \text{on } \{\, g^\epsilon \leq 1,\ 0 \leq t \leq T \,\}.$$

To simplify the notation, let us denote g^ϵ by g.

We begin by showing that the above inequality holds true when $g = 1$. Indeed, since $f = g^2/2$ is convex, we have

$$f_{rr} = g\, g_{rr} + {g_r}^2 \geq 0$$

and therefore, by Lemma 2.1, we obtain

$$g_{rr} \geq -\frac{g_r^2}{g} \geq -C.$$

On the other hand, there exits a constant $r_0 > 0$, independent of ϵ, such that when $g(r,t) = 1$, then $r \geq r_0$. This is because $g_r \leq C$, by Lemma 2.1. Hence,

$$G = \frac{g_{rr}}{r^{\frac{8}{5}}} \geq -C \qquad \text{on } \{\, g = 1,\ 0 \leq t \leq T \,\}.$$

We will show, using the maximum principle, that

$$G \geq -C, \qquad\qquad \text{on } \{\, g^\epsilon < 1,\ 0 < t \leq T \,\}.$$

Let us first show the computation in the simpler case where g satisfies the evolution Monge-Ampére equation (1.11). In this case, a direct calculation shows that G evolves as

$$G_t = \frac{g\, g_r\, G_{rr}}{r} + \frac{15 r^{\frac{13}{5}}\, g\, G + 25 r\, g_r^2 + 6 g\, g_r}{5 r^2}\, G_r + \frac{45 r\, g_r + 14\, g}{5 r^{\frac{2}{5}}}\, G^2 - \frac{6 g\, g_r}{25\, r^3}\, G + \frac{2 g_r^3}{r^{\frac{23}{5}}}. \tag{2.2}$$

It is enough to prove that G doesn't attain an interior minimum. Indeed, assuming, without loss of generality, that at an interior minimum point $G < 0$, then at this point we have $G_t \leq 0$, $G_{rr} \geq 0$, $G_r = 0$ while

$$\frac{45 r\, g_r + 14\, g}{5 r^{\frac{2}{5}}}\, G^2 > 0, \qquad -\frac{6 g\, g_r}{25\, r^3}\, G > 0, \qquad \frac{2 g_r^3}{r^{\frac{23}{5}}} > 0$$

which is impossible.

Let us now assume that g is a solution of the Gauss Curvature Flow (1.7). To simplify the notation, set $I = 1 + g^2 g_r^2$. Let $\delta > 0$ be a small number to be chosen later. When $g(r,t) \geq \delta$, then by Lemma 2.1, $r \geq \delta/C$ and hence, by the convexity of $f = g^2/2$ we obtain

$$G = \frac{g_{rr}}{r^{\frac{8}{5}}} \geq -\frac{g_r^2}{g\, r^{\frac{8}{5}}} \geq -C(\delta)$$

with $C(\delta)$ being independent ϵ. Hence, it is enough to restrict our attention to the interior region where $g < \delta$ and $0 < t \leq T$. After several direct computations, one finds that $G = g_{rr}/r^{\frac{8}{5}}$ satisfies an equation of the form

$$G_t = A(r,g,g_r)\, G_{rr} + B(r,g,g_r,G)\, G_r + C_1(r,g,g_r)\, G + C_2(r,g,g_r)\, G^2 + C_3(r,g,g_r)\, G^3 + D(r,g,g_r)$$

where

$$A(r,g,g_r) = \frac{g^5 g_r^5 + 2 g^3 g_r^3 + g g_r}{r I^{\frac{7}{2}}}$$

for all (r,t) and

$$
\begin{aligned}
C_1(r,g,g_r) &\le \frac{-6gg_r}{25r^3\, I^{\frac{7}{2}}} + O(g^2 g_r^3)\\
C_2(r,g,g_r) &= \frac{14g}{5r^{\frac{2}{5}}\, I^{\frac{7}{2}}} + O(r g_r^2 g^2)\\
C_3(r,g,g_r) &= \frac{-9r^{\frac{11}{5}} g^3 g_r}{I^{\frac{7}{2}}} + O(r^{\frac{11}{5}} g^5 g_r^3)\\
D(r,g,g_r) &= \frac{2g_r^3}{r^{\frac{23}{5}}\, I^{7/2}} + O(\frac{3g_r^7}{r^{\frac{13}{5}} I^{\frac{7}{2}}})
\end{aligned}
$$

in the region where $g < \delta$. We can choose $\delta > 0$, sufficiently small, such that

$$C_1 < 0, \qquad C_2 > 0, \qquad C_3 > 0$$

in the region where $\{\, g < \delta \,\}$. This is possible, since all the error terms in the approximations of C_1, C_2 and C_3 depend on g to a higher order than one. To control D, we first observe that we can find a number $r_0 > 0$, sufficiently small, such that if $r < r_0$, then $D > 0$. When $r \ge r_0$, we can estimate

$$D \ge -C_0$$

with the constant C_0 depending on r_0 and the upper bound of g_r. We can then conclude, that

$$\frac{\partial}{\partial t}\left(e^{(C_0+1)t}\, \frac{g_{rr}}{r^{\frac{8}{5}}} \right) > 0$$

at the minimum point of G. Hence the desired estimate follows from a direct application of the classical maximum principle.

3. Lipschitz Continuity of the Free-Boundary.

In this section, we will show that, under the assumptions of Theorem 1.1, the interface $\partial\{\, g(r,t) = 0 \,\}$ is a Lipschitz continuous curve $z = \gamma(t)$, on $0 \le t < T$. Our techniques are similar to the techniques in [CF].

3.1 Lemma. *Under the assumptions of Theorem 1.1 and condition (2.1), the limit*

$$g_r(\gamma(t)+0,t) = \lim_{r\to\gamma(t)+0} g_r(r,t)$$

exists, for $0 \le t < T$.

Proof. By Lemma 2.2, we have $g_{rr} \ge -C\, r^{8/5} \ge -\tilde{C}$ on $g \le 1$ and hence

$$(g + \tilde{C}\, r^2)_{rr} \ge 0$$

which means that the function $g + \tilde{C}\, r^2$ is convex, i.e., $(g + \tilde{C}\, r^2)_r$ is monotone increasing and hence the limit $g_r(\gamma(t)+0,t)$ exists.

3.2 Lemma. *Under the assumptions of Theorem 1.1 and condition (2.1), we have*

$$\gamma'(t+0) = -\frac{g_r^2(\gamma(t)+0)}{\gamma(t)}, \qquad \text{for } 0 \le t < T \tag{3.1}$$

Proof. Fix a number $t_0 \in [0, T)$.

Case 1. $g_r(\gamma(t_0)+0) = 0$: For any $\epsilon > 0$, there exists a linear function $h_0 = h_0(r)$ on $\gamma(t_0) \le r \le \gamma(t_0) + \epsilon$ with slope ϵ, such that $\{\, r : h_0(r) = 0 \,\} = \{\, r : g(r, t_0) = 0 \,\}$ and $h_0 \ge g$, for all r. Let h be a solution of (1.7) with initial data h_0 which is smooth up to $h = 0$, when $t \in [t_0, t_0 + \rho]$, for a small $\rho > 0$. The existence and regularity of h follow from the results in [DH]. Let us denote by $\eta(t)$ the free-boundary of h, namely the curve $\partial\{\, h = 0 \,\}$. Then

$$0 \ge \eta'(t_0) = -\frac{h_r^2(t_0)}{\eta(t_0)} = -\frac{h_r^2(t_0)}{\gamma(t_0)} \ge -\frac{\epsilon^2}{\gamma(t_0)}.$$

Since, $h \ge g$ we have $0 < \eta(t) \le \gamma(t)$ and hence

$$\eta'(t_0) \le \gamma'(t_0) \le 0.$$

Therefore

$$-\frac{\epsilon^2}{\gamma(t_0)} \le \gamma'(t_0) \le 0, \qquad \forall \epsilon > 0$$

which implies (3.1), by letting $\epsilon \to 0$.

Case 2. $g_r(\gamma(t_0)+0) > 0$: For any number $\epsilon > 0$, one can find, as in Case 1, solutions h_1 and h_2 of (1.7) which are smooth up to their interface on the time interval $[t_0, t_0 + \rho]$, for some $\rho > 0$, in addition they are linear at $t = t_0$, $\gamma(t_0) \le r \le \gamma(t_0) + \epsilon$ and they satisfy

$$(h_1)_r(\gamma(t_0)+0) = g_r(\gamma(t_0)+0) + \epsilon \quad \text{and} \quad (h_2)_r(\gamma(t_0)+0) = g_r(\gamma(t_0)+0) - \epsilon$$

and

$$h_2 \le g \le h_1.$$

Following a similar agrument as in Case 1, we can then show that

$$-\frac{g_r^2(\gamma(t_0)+0)}{\gamma(t_0)} - \epsilon \le \gamma'(t_0) \le -\frac{g_r^2(\gamma(t_0)+0)}{\gamma(t_0)} + \epsilon, \qquad \forall \epsilon > 0$$

which proves (3.1).

3.3. Lemma. *Under the hypotheses of Theorem 1.1 and condition (2.1), for every number $\delta > 0$, sufficiently small, there exists a constant $C_\delta > 0$, such that*

$$\gamma'' + C_\delta \gamma' = \mu \le 0, \qquad 0 \le t \le T - \delta$$

for some non-positive measure μ.

Proof. Fix a number t_0 in $[0, T - \delta]$.

Case 1 : $\gamma'(t_0 + 0, t_0) = 0$. Then, by Lemma 3.2, $g_r(\gamma(t_0) + 0, t_0) = 0$. Furthermore, since g_r is bounded, we have $|\gamma'| \le C_\delta$, on $[0, T - \delta]$. Hence,

$$|\int_0^{T-\delta} \gamma''(t)\, dt| \le C_\delta$$

which implies that γ'' is a measure. In addition $\gamma'' \le 0$, since $\gamma(t)$ is decreasing and $\gamma'(t_0 + 0) = 0$.

Case 2: $\gamma'(t_0 + 0) < 0$. Then, by Lemma 3.2, $g_r(\gamma(t_0) + 0, t_0) > 0$, and hence there exists a function h_0, smooth on $r > \gamma(t_0)$, such that $h_0 \le g$ at $t = t_0$, h_0^2 convex and satisfying

$$(h_0)_r = g_r \qquad \text{and} \qquad (h_0)_{rr} = (\lim_{r \to \gamma(t_0)+0} g_{rr})\,(1 - \epsilon)$$

for some $\epsilon > 0$ and small. By the results in [DH] , there exists a solution h of (1.7) which is smooth up to $h = 0$ when $t \in [t_0, t_0 + \rho]$, for a small $0 < \rho < \delta/2$. Let $\eta(t)$ be the free-boundary of h. Then, $\gamma(t) \le \eta(t)$, since $h \le g$ and $\gamma'(t_0) = \eta'(t_0)$. Hence

$$\gamma(t_0 + \rho) - \gamma(t_0) - \rho\,\gamma'(t_0 + 0) \le \eta(t_0 + \rho) - \eta(t_0) - \rho\,\eta'(t_0 + 0)$$

which implies that

$$\Phi_\rho = \frac{\gamma(t_0 + \rho) - \gamma(t_0) - \rho\,\gamma'(t_0 + 0)}{\rho^2/2} \le \eta''(t_0) + o(\rho).$$

To estimate $\eta''(t_0)$ we differentiate the equality $h(\eta(t), t) = 0$ twice with respect to t. Since, at $h = 0$

$$h_t = \frac{h\,h_r h_{rr} + h_r^3}{r\,(1 + h^2\,h_r^2)^{3/2}} = \frac{h_r^3}{r}$$

we find that

$$\eta'(t) = -\frac{h_t}{h_r} = -\frac{h_r^2}{r}.$$

Also, since $h_{rr} \ge -C$, by Lemma 2.2, we have

$$\eta''(t) = -\frac{h_{rr} h_r^3}{r^2} + \frac{h_r^4}{r^3} \le \frac{h_r^2}{r}\left[\frac{C}{r} + \frac{h_r^2}{r^2}\right].$$

Hence

$$\eta''(t) \le -C_\delta\,\eta'(t), \qquad t \in [t_0, t_0 + \rho]$$

where C_δ is a constant depending on δ. The last estimate holds because by the results in [DH] the function h is smooth up to the interface $z = \eta(t)$ on $[t_0, t_0 + \rho]$. Combining the above estimates we conclude that

$$\Phi_\rho \le \eta''(t_0 + 0) + o(\rho) \le -C_\delta\,\eta'(t_0 + 0) + o(\rho) = C_\delta - \gamma'(t_0 + 0) + o(\rho) \tag{3.2}$$

On the other hand, we have $\Phi_\rho \ge -C$, since $g_{rr} \ge -C$ by Lemma 2.2. Also, for every $\delta > 0$,

$$\int_0^{T-\delta} \Phi_\rho \le C$$

since g_r is bounded, by Lemma 2.1. It follows that Φ_ρ converges weakly to a measure μ_0, as $\rho \to 0$. Therefore $\Phi_\rho + C_\delta\,\gamma'(t_0 + 0)$ converges, as $\rho \to 0$, to a measure μ and $\mu \le 0$, by (3.2). This finshes the proof of the Lemma.

3.4. Corollary. *Under the same hypotheses as in Lemma 3.3, if $0 \le t_2 \le t_1 \le T - \delta$ then*

$$\gamma'(t_2 - 0)\,e^{C_\delta\,t_2} \le \gamma'(t_1 + 0)\,e^{C_\delta\,t_1}$$

where C_δ is the constant in Lemma 3.3

Proof. From Lemma 3.3 we have

$$\frac{d}{dt}(\gamma'(t)\,e^{C_\delta t}) = (\gamma''(t) + C_\delta\,\gamma'(t))\,e^{C_\delta t} \le 0$$

and therefore the result follows.

An immediate consequence of the above Corollary is:

3.5. Theorem. *Under the hypotheses of Theorem 1.1, the free-boundary $z = \gamma(t)$ is Lipschitz continuous on $0 \le t < T$.*

3.6. Corollary. *Under the same hypotheses as in Lemma 3.3, for every sufficiently small number $\delta > 0$, one can decompose the free-boundary $z = \gamma(t)$ as*

$$\gamma(t) = \xi(t) + \zeta(t), \qquad 0 \le t \le T - \delta$$

where ξ is concave and ζ is of class $C^{1,1}$.

Proof. For $\delta > 0$, let μ be the non-positive measure

$$\mu = \gamma'' + C_\delta \, \gamma'.$$

One can decompose the function $\gamma(t)$ on the time interval $0 \le t \le T - \delta$, as $\gamma(t) = \xi(t) + \zeta(t)$, where ξ and ζ are given by

$$\xi(t) = \int_0^t \int_0^\tau d\mu(s)\, d\tau$$

and

$$\zeta(t) = \gamma(0) + \gamma'(0+0)\, t - C_\delta \int_0^t (\gamma(s) - \gamma(0))\, ds.$$

In addition it follows from Lemma 3.3 and Theorem 3.5 that ξ is concave and ζ is $C^{1,1}$.

3.7. Corollary. *Under the hypotheses of Theorem 1.1, or every small number $\delta > 0$, there exists a constant c_δ, such that*

$$g_r(\gamma(t) + 0) \ge c_\delta > 0, \qquad 0 \le t \le T - \delta.$$

Proof. This is an immediate consequence of Lemma 3.2 since $g_r(\gamma(t)+0) > 0$ at $t = 0$, by assumption (1.4).

4. C^1-Continuity of the Free-Boundary

In this section, we will prove that the free-boundary $z = \gamma(t)$ is a C^1 curve, on the interval $0 < t < T$. We will also prove that g_r and $g\, g_{rr}$ are continuous up to the free-boundary $z = \gamma(t)$, for $0 < t < T$. Our techniques are similar to the techniques in [CF].

4.1. Lemma. *Under the hypotheses of Theorem 1.1 and condition (1.2), for every number $\delta > 0$, there exists a constant C_δ, such that*

$$|g_t(r,t)| \le C_\delta \qquad \text{and} \qquad |g_{tt}(r,t)| \le \frac{C_\delta}{d(r,\gamma(t))} \tag{4.1}$$

for all (r,t) in $\{\, g \le 1,\ 0 \le t \le T - \delta \,\}$.

Proof. It is enough to show these bounds for a point (r_0, t_0) in the set $\{\, 0 < g < 1,\ 0 < t \le T - \delta \,\}$ which is sufficiently close to the free-boundary $z = \gamma(t)$. Pick such a point (r_0, t_0) and for a number ρ in $0 < \rho < \delta/2$, let us denote by Q_ρ the cylinder $Q_\rho = \{(r,t) : |r - r_0| \le \rho,\ t_0 - \rho^2 \le t \le t_0 \,\}$. If we define

$$w(r,t) = d^{-1}\, g(r_0 + d\, r, t_0 + d\, t), \qquad (r,t) \in Q_\rho$$

with $d = r_0 - \gamma(t_0)$, then w satisfies the equation

$$w_t = \frac{w\, w_r w_{rr} + w_r^3}{(r_0 + dr)(1 + d^{-2} w^2 w_r^2)^{3/2}}. \tag{4.2}$$

Observe next that there exists a positive constant $c = c(\delta) > 0$ such that

$$c \le w(r,t) \le c^{-1} \qquad \text{and} \qquad c \le w_r(r,t) \le c^{-1}, \qquad \forall (r,t) \in Q_\rho.$$

This is implied by the bounds $0 < c_\delta \le g_r(r,t) \le C$ shown in Lemmas 2.1 and 3.7, since for $(r,t) \in Q_\rho$, $t \le T - \frac{\delta}{2}$. It follows from the above bounds that the equation (4.2) is non-degenerate on Q_ρ and therefore by the standard regularity theory of parabolic equations ([LSU]) the function w is smooth in $Q_{\frac{\rho}{2}}$. In particular the time derivatives w_t and w_{tt} are bounded on $Q_{\frac{\rho}{2}}$ by a constant C_δ, depending only on δ and as a result g_t and g_{tt} satisfy the bounds (4.1) at the point (r_0, t_0).

4.2. Theorem. *Under the hypotheses of Theorem 1.1, the free-boundary $z = \gamma(t)$ is a C^1 curve, for $0 < t < T$.*

Proof. We have already shown that the free-boundary $z = \gamma(t)$ is Lipschitz, strictly decreasing and that for all $0 < t < T$ both $\gamma'(t+0)$ and $\gamma'(t-0)$ exist, when $0 < t < T$. Assume that for some t_0 in $0 < t_0 < T$ we have $\gamma'(t_0+0) \ne \gamma'(t_0-0)$. It then follows from Corollary 3.6 that

$$a = \gamma'(t_0+0) < \gamma'(t_0-0) = b.$$

For $\alpha > 0$ and $\beta > 0$, set

$$A = (\gamma(t_0) + \alpha, t_0), \qquad B = (\gamma(t_0) + \alpha, t_0 + \beta), \qquad C = (\gamma(t_0) + \alpha, t_0 - \beta).$$

Since

$$\gamma(t_0 + \beta) = \gamma(t_0) + a\,\beta + o(\beta) \quad \text{and} \quad \gamma(t_0 - \beta) = \gamma(t_0) - b\,\beta + o(\beta)$$

and

$$g(B) \approx g_r(\gamma(t_0+\beta), t_0+\beta)\,(\gamma(t_0) + \alpha - \gamma(t_0+\beta)) + o(\gamma(t_0) + \alpha - \gamma(t_0+\beta))$$

we have

$$g(B) \approx \sqrt{-\gamma(t_0+\beta)\,\gamma'(t_0+\beta)}\,(\alpha - a\beta + o(\beta)) + o(\alpha - a\beta + o(\beta))$$

and hence

$$g(B) \approx \sqrt{-(\gamma(t_0) + a\,\beta + o(\beta))\,(-a + o(\beta))}\,(\alpha - a\beta + o(\beta)) + o(\alpha).$$

Similarly

$$g(C) \approx \sqrt{-(\gamma(t_0) - b\,\beta + o(\beta))\,(-b + o(\beta))}\,(\alpha + b\,\beta + o(\beta)) + o(\alpha)$$

and

$$g(A) \approx g_r(\gamma(t_0) + 0, t_0)\,\alpha + o(\alpha) \approx \sqrt{-\gamma(t_0)a}\,\alpha + o(\alpha).$$

We conclude that

$$|g(B) + g(C) - 2g(A)| \approx |(\sqrt{-\gamma(t_0)b} - \sqrt{-\gamma(t_0)a})\,\alpha + o(\alpha)| \approx \sqrt{\gamma(t_0)}\,(\sqrt{-b} - \sqrt{-a}) + o(\alpha).$$

On the other hand,

$$|g(B) + g(C) - 2g(A)| \le |g_{tt}(\gamma(t_0) + \alpha, t_0)|\,\beta^2 + o(\beta^2) \le \frac{C}{\alpha}\,\beta^2 + o(\beta^2)$$

and hence we must have

$$\sqrt{\gamma(t_0)}\,(\sqrt{-b}-\sqrt{-a})+o(\alpha)\le\frac{C}{\alpha}\,\beta^2+o(\beta^2)$$

which is impossible if $a>b$ and both α and β sufficiently small with β much smaller than α. Therefore $a=b$ which proves that γ is C^1.

4.3. Lemma. *Under the assumptions of Theorem 1.1, the function g_r is continuous up to the free-boundary $z=\gamma(t)$, for $0\le t<T$.*

Proof. Let $b=g_r(r_0+0,t_0)$ and $r_0=\gamma(t_0)$. We will show that there exists a function $\sigma(\epsilon)$, with $\sigma(\epsilon)\to 0$, as $\epsilon\to 0$, such that

$$|g_r(r,t)-b|<\sigma(\mathrm{dist}((r,t),(r_0,t_0))),\qquad \text{as } \mathrm{dist}((r,t),(r_0,t_0))\to 0.$$

Let us show first that $g_r(r,t)<b+\sigma(\mathrm{dist}((r,t),(r_0,t_0)))$. If not, then there exists $\delta>0$ and a sequence of points $(r_n,t_n)\to(r_0,t_0)$ for which $g_r(r_n,t_n)\ge b+\delta$. Since, $g_{rr}\ge -C$, by Lemma 2.2, we will then have

$$g(r,t_n)\ge g(r_n,t_n)+(b+\delta)(r-r_n)-\frac{C}{2}(r-r_n)^2.$$

Hence, by letting $n\to\infty$, we obtain

$$g(r,t_0)\ge(b+\delta)(r-r_0)-\frac{C}{2}(r-r_0)^2$$

impossible, since $g_r\to b$, as $r\to r_0+0$. We will show next that $g_r(r,t)>b-\sigma(\mathrm{dist}((r,t),(r_0,t_0)))$. Since, γ' is continuous, by Lemma 4.1, for any number $\epsilon>0$ there exists a number $\delta>0$ such that

$$|\gamma'(t)-\gamma'(t_0)|<\epsilon,\qquad \text{for } |t-t_0|<\delta.$$

Hence, from the relation $\gamma'(t_0)=-g_r^2(\gamma(t_0),t_0)/\gamma(t_0)$ we conclude that

$$|g_r(\gamma(t),t)-g_r(\gamma(t_0),t_0)|<\sigma_1(\epsilon)$$

with $\sigma_1(\epsilon)\to 0$, as $\epsilon\to 0$. Therefore

$$g_r(r,t)=g_r(\gamma(t),t)+\int_{\gamma(t)}^{r}g_{rr}(\eta,t)\,d\eta\ge g_r(\gamma(t),t)-C\,(r-\gamma(t))\ge g_r(r_0,t_0)-\sigma_1(\epsilon)-C\,\epsilon=b-\sigma(\epsilon)$$

which shows the desired estimate if $\sigma(\epsilon)=\sigma_1(\epsilon)+C\,\epsilon$.

4.4. Lemma. *Under the assumptions of Theorem 1.1, we have*

$$\limsup_{(r,t)\to(\gamma(t_0)+0,t_0)}g_t(r,t)\le\frac{g_r^3(\gamma(t_0),t_0)}{\gamma(t_0)},\qquad \text{for } 0<t_0<T.$$

Proof. Fix a time t_0 in $0<t_0<T$ and set $b=g_r(\gamma(t_0),t_0)$. If the conclusion of the Lemma fails, then there exists a number $\rho>0$ and a sequence of points $(r_n,t_n)\to((\gamma(t_0),t_0)$ such that

$$g_t(r_n,t_0)>\frac{b^3}{r_n}+\rho. \tag{4.3}$$

Set $\epsilon_n = r_n - \gamma(t_0)$, $\alpha_n = t_n - t_0$ and $\tilde{t}_n = t_0 + \alpha\,(\epsilon_n + \alpha_n)$, where $\alpha > 0$ is a small number to be determined later. We then have

$$g(r_n, \tilde{t}_n) = g(r_n, t_n) + g_t(r_n, t_n)\,\alpha(\epsilon_n + \alpha_n) + g_{tt}(r_n, t_0)\,\alpha^2(\epsilon_n + \alpha_n)^2 + o(\alpha^2(\epsilon_n + \alpha_n)^2). \tag{4.4}$$

To estimate $g(r_n, t_n)$ from below we write

$$g(r_n, t_n) = g(\gamma(t_n), t_n) + g_r(\gamma(t_n), t_n)(r_n - \gamma(t_n)) + \frac{g_{rr}(\gamma(t_n), t_n)}{2}(r_n - \gamma(t_n))^2 + o((r_n - \gamma(t_n))^2)$$

and use the estimate $g_{rr} \geq -C$ and the identity $g(\gamma(t_n), t_n) = 0$ to obtain

$$g(r_n, t_n) \geq g_r(\gamma(t_n), t_n)(r_n - \gamma(t_n)) - \frac{C}{2}(r_n - \gamma(t_n))^2 + o((r_n - \gamma(t_n))^2). \tag{4.5}$$

From Lemma 4.1 we have

$$g_{tt}(r_n, t_n) \geq \frac{-C}{r_n - \gamma(t_n)}. \tag{4.6}$$

Also from Lemma 3.2 we have

$$\begin{aligned} r_n - \gamma(t_n) &= r_n - \gamma(t_0) + \gamma(t_0) - \gamma(t_n) \\ &= \epsilon_n - \gamma'(t_0)(t_n - t_0) + o(t_n - t_0) = \epsilon_n + \frac{b^2}{\gamma(t_0)}\alpha_n + o(\alpha_n) \end{aligned} \tag{4.7}$$

since $\epsilon_n = r_n - \gamma(t_0)$, $\alpha_n = t_n - t_0$ and $\gamma'(t_0) = -b^2/\gamma(t_0)$. In addition, by Lemma 4.3 we have

$$g_r(\gamma(t_n), t_n) \geq g_r(\gamma(t_0), t_0)) + \sigma(\alpha_n + \epsilon_n) = b + \sigma(\alpha_n + \epsilon_n) \tag{4.8}$$

since $b = g_r(\gamma(t_0), t_0)$. Combining the estimates (4.3)-(4.8) we conclude

$$\begin{aligned} g(r_n, \tilde{t}_n) \geq (b + \sigma(\alpha_n + \epsilon_n))(r_n - \gamma(t_n)) + (\frac{b^3}{r_n} + \rho)\,\alpha(\epsilon_n + \alpha_n) - \frac{C}{2}(r_n - \gamma(t_n))^2 \\ - \frac{C}{r_n - \gamma(t_n)}\alpha^2\,(\epsilon_n + \alpha_n)^2 + o((r_n - \gamma(t_n))^2 + o(\alpha^2(\epsilon + \alpha_n)^2). \end{aligned} \tag{4.9}$$

On the other hand, using Lemma 4.2, we can estimate $g(r_n, \tilde{t}_n)$ from above as

$$g(r_n, \tilde{t}_n) = \int_{\gamma(\tilde{t}_n)}^{r_n} g_r(\xi, \tilde{t}_n)\,d\xi \leq (b + \sigma(\alpha(\epsilon_n + \alpha_n)))(r_n - \gamma(\tilde{t}_n))$$

where

$$\begin{aligned} r_n - \gamma(\tilde{t}_n) &= r_n - \gamma(t_0) + \gamma(t_0) - \gamma(\tilde{t}_n) \\ &= \epsilon_n - \gamma'(t_0)(\tilde{t}_n - t_0) + o(\tilde{t}_n - t_0) \\ &= \epsilon_n + \frac{b^2}{\gamma(t_0)}(\alpha_n + \alpha(\alpha_n + \epsilon_n)) + o(\alpha_n + \alpha(\alpha_n + \epsilon_n)). \end{aligned} \tag{4.10}$$

Hence

$$g(r_n, \tilde{t}_n) \leq (b + \sigma(\alpha(\alpha_n + \epsilon_n)))\left(\epsilon_n + \frac{b^2}{\gamma(t_0)}(\alpha_n + \alpha(\alpha_n + \epsilon_n)) + o(\alpha_n + \alpha(\alpha_n + \epsilon_n))\right). \tag{4.11}$$

To simplify the notation set $\lambda_n = \alpha_n + \epsilon_n$. Combining the estimates 4.7 and 4.9 - 4.11, we obtain, after several cancellations, that

$$(\epsilon_n + \frac{b^2}{\gamma(t_0)})\,\sigma(\lambda_n) - \tilde{C}\lambda_n^2 - \tilde{C}\alpha^2\lambda_n + o(\lambda_n^2) + o(\alpha^2\lambda_n^2) + \alpha\rho\lambda_n$$
$$\leq \sigma(\epsilon_n + \alpha\lambda_n)\left(\epsilon_n + \frac{b^2}{\gamma(t_0)}(\alpha_n + \alpha\lambda_n) + o(\alpha_n + \alpha\lambda_n\right) + b\lambda_n \approx \sigma(\lambda_n)\lambda_n + o(\lambda_n)$$

which implies, that

$$(\alpha\rho - \tilde{C}\alpha^2)\,\lambda_n + o(\lambda_n) \leq o(\lambda_n).$$

This is impossible if we set $\alpha = \min(1, \rho/2\tilde{C})$ and let $\lambda_n \to 0$. Therefore the assumption of the Lemma holds.

4.5. Lemma. *The function $g\,g_{rr}$ is continuous up to the free-boundary $z = \gamma(t)$ for $0 < t < T$.*

Proof. Pick a point (r_0, t_0) at the free-boundary $r_0 = \gamma(t_0)$ with $0 < t_0 < T$. Since $g_{rr} \geq -C$, and $g \to 0$, as $(r,t) \to (r_0, t_0)$, we must have

$$\liminf_{(r,t)\to(r_0,t_0)} g g_{rr} = 0.$$

Hence, from the equation (1.7) and the previous Lemma we deduce that

$$\limsup_{(r,t)\to(r_0,t_0)} g_t(r,t) = \frac{g_r^3(r_0,t_0)}{r_0}$$

which shows that g_t is continuous up to the free-boundary. Since

$$g_t = \frac{g g_r g_{rr} + g_r^3}{r(1 + g^2 g_r^2)^{3/2}}$$

and g_r is continous up to the free-boundary and bounded from below by a positive constant, we conclude that $g g_{rr}$ is continous up to the interface.

5. The C^∞-Regularity of the Free-Boundary.

This section is devoted to the proof of Theorem 1.1. To simplify the notation, we will assume, without loss of generality, that g satisfies condition (2.1). We have already shown, in the previous sections that $g_r \leq C$ and $g_{rr} \geq -C$, on $\{\, g \leq 1,\ 0 \leq t \leq T \,\}$ and also $g_r \geq c_\delta > 0$ on $\{\, g \leq 1,\ 0 \leq t \leq T - \delta \,\}$. The proof of Theorem 1.1 will be based on the above estimates and upon the next Lemma:

5.1. Lemma. *For any small number $\delta > 0$, there exists a constant C_δ such that*

$$g_{rr} \leq C_\delta, \qquad \text{on } \{\, g \leq 1,\ 0 \leq t \leq T - \delta \,\}. \tag{5.1}$$

Proof. We have shown in Lemmas 4.2 and 4.4 that the functions g_r and $g\,g_{rr}$ are continuous up to the interface on the set $\{\, g \leq 1,\ 0 \leq t \leq T - \delta \,\}$. Since this set is compact they will have a modulus of continuity. Hence, for a given $\epsilon > 0$ there exist functions $\delta(\epsilon)$ and $\eta(\epsilon)$ such that for any point (r_0, t_0) on the free-boundary with $0 \leq t_0 \leq T - \delta$ we have

$$|g_r - g_r(r_0,t_0)| < \epsilon \quad \text{and} \quad |g\,g_r| < \epsilon \tag{5.2}$$

on $R_{\delta,\eta} = [r_0, r_0+\delta) \times (t_0-\eta, t_0+\eta)$. Set $b = g_r(r_0,t_0)$, $t_1 = t_0-\eta$ and $t_2 = t_0+\eta$. Then, (5.2) implies that

$$(b-\epsilon)\,(r-\gamma(t)) < g(r,t) < (b+\epsilon)\,(r-\gamma(t)) \tag{5.3}$$

and

$$-\frac{b^2+2\epsilon}{\gamma(t)} \approx -\frac{(b+\epsilon)^2}{\gamma(t)} < \gamma'(t) < -\frac{(b-\epsilon)^2}{\gamma(t)} \approx -\frac{b^2-2\epsilon}{\gamma(t)} \tag{5.4}$$

since $\gamma'(t) = -g_r^2/\gamma(t)$. Set

$$\gamma^*(t) = \gamma(t_1) - a\,(t-t_1), \tag{5.5}$$

with $a = \frac{b^2+4\epsilon}{\gamma(t_0)}$, $t_1 = t_0-\eta$.

To estimate g_{rr} we will construct an appropriate barrier. Let us begin by computing the evolution of $G = g_{rr}$ on $R_{\delta,\eta}$. Since $g \approx b\delta$ on $R_{\delta,\eta}$, where $b = g_r(r_0,t_0)$, we find after direct calculations that G satisfies an equation of the form

$$L(G) \equiv G_t - \left\{A(r,g,g_r)\,G_{rr} + B(r,g,g_r,G)\,G_r + C_1(r,g,g_r)\,G^2 + C_2(r,g,g_r)\,G\right\} = D(r,g,g_r)$$

where, for $(r,t) \in R_{\eta,\delta}$, the coefficients A, B, C_1, C_2 and D satisfy

$$A \approx \frac{gg_r}{r} + \frac{o((r-\gamma(t))}{r}, \qquad B \approx \frac{5b^2}{r} + \frac{3b\delta}{r} - \frac{2b\delta}{r^2} + \frac{o(\delta)}{r}, \qquad C_1 \approx \frac{9b}{r} - \frac{2b\delta}{r} - \frac{o(\delta)}{r}$$

and

$$C_2 \approx \frac{-42b^6\delta}{r} - \frac{8b^2}{r^2} + \frac{2b^2\delta}{r^3} + \frac{o(\delta)}{r}, \qquad D \approx \frac{3b^7}{r} - \frac{6b^7\delta}{r^2} - \frac{2b^3}{r^3}$$

with $b = g_r(r_0,t_0)$. We can assume, by a simple scaling, that r is small compared to b and therefore deduce that $D \le 0$, which imples that $L(G) \le 0$.

Let $\gamma^*(t)$ be defined as in (5.5). To find a barrier ϕ of the form

$$\phi(r,t) = \frac{\alpha}{r-\gamma(t)} + \frac{\beta}{r-\gamma^*(t)}$$

we will choose suitable constants α and β, so that

$$L(\phi) \ge 0.$$

A direct calculation shows that

$$\begin{aligned} L(\phi) = {} & \frac{\alpha}{(r-\gamma(t))^2}\left[\gamma'(t) - \frac{A}{r-\gamma(t)} + (B-C_1) + C_2(r-\gamma(t))\right] \\ & + \frac{\beta}{(r-\gamma^*(t))^2}\left[\gamma^{*\prime}(t) - \frac{A}{r-\gamma^*(t)} + (B-C_1) + C_2(r-\gamma^*(t))\right] \end{aligned}$$

Using the the estimates (5.3),(5.4) and (5.5) we deduce that

$$L(\phi) \ge \frac{\alpha}{(r-\gamma(t))^2}\left[\frac{3b^2}{r} - \frac{9b\alpha}{r} + O(\delta,\epsilon)\right] + \frac{\beta}{(r-\gamma^*(t))^2}\left[\frac{2b^2}{r} - \frac{9b\,\beta}{r} + O(\delta,\epsilon)\right].$$

Therefore, we can choose $\beta > 0$ and $\alpha_0 > 0$ sufficiently small, depending only on $b = g_r(r_0, t_0)$ and r_0, such that $L(\phi) \geq 0$, for all $\alpha < \alpha_0$.

We next compare ϕ and $G = g_{rr}$ on the parabolic boundary of $R_{\eta,\delta}$. Since $g\, g_{rr} \leq \epsilon$ and $g \geq (b-\epsilon)\,(r - \gamma(t))$ in $R_{\eta,\delta}$, we have

$$g_{rr} \leq \frac{\epsilon}{(b-\epsilon)\,(r-\gamma(t))}.$$

In particular

$$g_{rr}(\gamma(t)+\delta, t) \leq \frac{\epsilon}{(b-\epsilon)\,\delta} \qquad \text{on } [t_1, t_2]$$

if ϵ is sufficiently small, depending on b. On the other hand we have

$$\phi(\gamma(t)+\delta, t) \geq \frac{\beta}{\gamma(t)+\delta-\gamma^*(t)}$$

and by (5.4)

$$\gamma(t) - \gamma^*(t) = \frac{b^2+4\epsilon}{r_0}(t-t_1) + \gamma(t) - \gamma(t_1) = [\frac{b^2+4\epsilon}{r_0} + \gamma'(\tilde{t})]\,(t-t_1) \leq 0(\epsilon)\,\eta$$

for some $\tilde{t} \in [t_1, t]$. Hence, we can make

$$\phi(\gamma(t)+\delta, t) \geq \frac{\beta}{2\delta}$$

by choosing ϵ sufficiently small and $\eta \leq \delta$, concluding that

$$g_{rr}(\gamma(t)+\delta, t) \leq \frac{\epsilon}{(b-\epsilon)\,\delta} \leq \phi(\gamma(t)+\delta, t) \qquad \text{for } t \in [t_1, t_2]$$

if ϵ is sufficiently small, depending on β. Finally, at $t = t_1$ we have

$$\phi(r, t_1) = \frac{\beta}{r-\gamma(t_1)} \geq \epsilon(b-\epsilon)(r-\gamma(t_1)) \geq g_{rr}(r, t_1)$$

if ϵ is sufficiently small, concluding that $G \leq \Phi$ at the parabolic boundary of $R_{\eta,\delta}$. The maximum principle then implies that

$$G = g_{rr} \leq \frac{\alpha}{r-\gamma(t)} + \frac{\beta}{r-\gamma^*(t)}, \qquad \text{on } R_{\eta,\delta}$$

for all $\alpha \leq \alpha_0$. Taking the limit $\alpha \to 0$, we finally conclude that

$$g_{rr} \leq \frac{\beta}{r-\gamma^*(t)}$$

in $R_{\eta,\delta}$, which immediately implies that

$$g_{rr}(r_0, t_0) \leq \frac{\beta}{r_0-\gamma^*(t_0)} = \frac{\beta}{r_0-\gamma(t_1)+b\,\eta} \leq \frac{\beta}{b\eta}$$

with the constants η and β depending only on $b = g_r(r_0, t_0)$. Since, we have already shown that

$$0 < c_\delta \leq g_r(\gamma(t), t) \leq c_\delta^{-1}, \qquad \text{on } 0 \leq t \leq T-\delta$$

the desired estimate $g_{rr} \leq C_\delta$ follows.

We are now in position to finish the proof Theorem 1.1.

Proof of Theorem. Fix a number $\delta > 0$. Since g_{rr} is bounded on $0 \le t \le T - \delta$, it follows that the functions g, g_r, and $g\, g_{rr}$ are of class C^α up to the free-boundary $z = \gamma(t)$, for some $0 < \alpha < 1$ and for $0 \le t \le T - \delta$. Since $g_r \ge c_\delta > 0$ on $0 \le t \le T - \delta$, the distance to the free-boundary $d = d(r,t)$ is proportional to the function $g(r,t)$, and hence $d\, g_{rr}$ is also of class C^α. Moreover, the time derivative g_t is also of class C^α, since

$$g_t = \frac{g\, g_r g_{rr} + g_r^3}{r\,(1 + g^2 g_r^2)^{3/2}}.$$

It follows by the regularity result in [DH], that g is of class C^∞ up to the free-boundary for all $0 < t \le T - \delta$. Since δ is arbitrary the result follows.

6. The behavior of Solutions near the Focusing Time.

This section is devoted to the proof of Theorem 1.2, which determines the behavior of the solution g of (1.7) at the focusing time T of the flat side. One can observe that all the results which are shown in the previous sections hold true, not only for solutions of equation (1.7) but also for solutions of the Monge Ampére equation (1.12). In this section, we will show that Theorem 1.2 is satisfies for solutions to both equations (1.7) and (1.12). To simplify the notation in the proofs, we will assume, without loss of generality, that condition (2.1) holds.

The first Lemma provides the upper bound on the derivative of g at the focusing time.

6.1. Lemma. *Under the assumptions of Theorem 1.2 and (2.1), the function g satisfies the derivative estimate*

$$g_r \le C\, r^{\frac{1}{4}} \qquad \text{on } \{\, g \le 1,\ 0 \le t \le T \,\}.$$

Proof. As in the proof Lemma 2.1, we first approximate g by a decreasing sequence g^ϵ of positive smooth increasing strictly convex solutions. Set

$$G = \frac{g_r^\epsilon}{r^a + \epsilon}$$

where $a > 0$ is to be determined. We will find an upper bound of G which is independent of ϵ. To simplify the notation, let us denote g^ϵ by g. We will show, using the maximum principle that G doesn't attain an interior maximum. Actually, using the result of Lemma 1.2, it is enough to show that G doesn't attain a maximum at a point (r,t) where $0 < t < T$ and $r \le \rho$, with ρ sufficiently close to $r = 0$.

We will demonstrate the proof in the case that g is a solution to the evolution Monge-Ampére equation (1.12). The computations for the Gauss Curvature Flow are more involved but very similar. When g evolves by (1.12) one can see, by direct calculations, that at a positive maximum point (r,t) of G with $r \le \rho$ one has

$$0 \le G_t(r,t) \le \frac{(4a-1)r^{3a} + (8a\epsilon - 3\epsilon)r^{2a} + (4a-3)\epsilon^2 r^a - \epsilon^3}{r^2(r^a+\epsilon)}\, G^3 + \frac{(2a^2-2a)r^{2a} + \epsilon(a^2-2a)r^a}{r^3(r^a+\epsilon)}\, g\, G^2.$$

Setting $a = \frac{1}{4}$ we obtain

$$0 \le G_t(r,t) \le \frac{-\epsilon\, r^{2a} - 2\epsilon^2 r^a - \epsilon^3}{r^2(r^a+\epsilon)}\, G^3 - \frac{3(r^{2a} + \epsilon r^a)}{8r^3(r^a+\epsilon)}\, g\, G^2 < 0$$

since $G > 0$ at the maximum point (r,t). This is impossible, proving that G cannot have a maximum point at (r,t).

We will prove next the lower bound on the derivative of g, at the focusing time.

6.2. Lemma. *Under the assumptions of Theorem 1.2 and (2.1), there exists a constant $c > 0$ such that*

$$g_r \geq c\, r^{\frac{2}{5}} \qquad \text{for} \quad r \geq \gamma(t)$$

on the set $\{g \leq 1,\ 0 \leq t \leq T\}$.

Proof. For numbers $a > 0$ and $A > 0$, to be determined later, set

$$G = \frac{g_r}{r^a} \qquad \text{and} \qquad \tilde{G} = G\, e^{At}.$$

It is then easy to see that

$$\tilde{G} \geq c > 0, \qquad \text{on } \{g = 1,\ 0 \leq t \leq T\}$$

since the bound in Lemma 2.1, implies that if $g(r,t) = 1$ then $r \geq r_0 > 0$. Therefore, to prove the lower bound on G, it is enough to show that $\tilde{G}$ doesn't attain its minimum at an interior point (r,t) where $g < 1$. To demonstrate the ideas, we will present first the proof in the simpler case where g satisfies the Monge-Ampére equation (1.12). Assume first that $\tilde{G}$ attains its minimum at an interior point (r,t), where $r > \gamma(t)$. Then, at this point we have

$$G_r = 0, \qquad G_{rr} \geq 0, \qquad \text{and} \qquad G_t + A\,G \leq 0.$$

Let us also assume, without loss of generality that at (r,t) we have $G < 1$. Computing the evolution of G we obtain, after several calculations, that at the point (r,t)

$$G_t \geq \frac{(4a-1)\, r^{2a}}{r^2}\, G^3 + \frac{2(a^2-a)\, r^a g}{r^3}\, G^2 = \frac{r^a G^2}{r^3}\left[(4a-1) r g_r + 2(a^2-a)\, g\right].$$

By the Mean Value Theorem, $g(r,t) = (r - \gamma(t))\, g_r(r^*,t) \leq r\, g_r(r^*,t)$, for some point $r^* \in [\gamma(t), r]$. Also, by Lemma 2.2,

$$g_r(r,t) - g_r(r^*,t) = g_{rr}(r^{**},t)(r - r^*) \geq -C\, r^{\frac{8}{5}}.$$

Therefore, at the point (r,t), we have

$$\begin{aligned} G_t &\geq \frac{r^a G^2}{r^3}\left[(2a^2+2a-1) r g_r + 2r(a^2-a)\,(g_r(r,t) - g_r(r^*,t)\right] \\ &\geq \frac{r^a G^2}{r^2}\left[(2a^2+2a-1) r^a G - C(a^2-a)\, r^{\frac{8}{5}}\right]. \end{aligned}$$

Choosing $a = \frac{2}{5}$, so that $2a^2 + 2a - 1 \geq 0$ and $\frac{8}{5} + a = 2$ we obtain

$$G_t \geq -C\, \frac{r^{\frac{8}{5}+a}}{r^2}(a - a^2)\, G^2 \geq -\tilde{C}\, G^2.$$

This contradicts the condition $G_t + A\,G \leq 0$ if $A = \tilde{C}$, since $G < 1$ at (r,t).

Assume next that the minimum of $\tilde{G} = e^{At} G$ is attained at a point $(\gamma(t), t)$ of the free-boundary. At the free-boundary where $g = 0$, $\tilde{G}$ evolves by

$$\frac{d\tilde{G}(\gamma(t),t)}{dt} = e^{At}\left[G_t + g_r\, \gamma'(t)\right] + A\, e^{At} G = \frac{4 r^{2a} G^2 (e^{At} G)_r}{r} + \frac{r^{2a}(4a-1) - r^{3a+1}}{r^2} e^{At} G^3 + A\, e^{At} G.$$

Since $\tilde{G}_r = (e^{At}G)_r \geq 0$ at a minimum point $(\gamma(t), t)$ of $\tilde{G}$, we conclude that at such a point

$$\frac{d\tilde{G}(\gamma(t),t)}{dt} \geq \frac{r^{2a}(4a-1) - r^{3a+1}}{r^2} e^{At} G^3 + A\, e^{At} G.$$

Therefore setting $a = 2/5$ we obtain

$$\frac{d\tilde{G}(\gamma(t),t)}{dt} \geq \frac{3}{5} r^{-\frac{6}{5}} G^3 e^{At} + [A - r^{\frac{1}{5}} G^2]\, G e^{At}.$$

We can assume, without loss of generality that $\tilde{G} < 1$ at the minimum point $(\gamma(t), t)$. Hence, from the above estimate we conclude that at the point $(\gamma(t), t)$, we must have

$$\frac{d\tilde{G}(\gamma(t),t)}{dt} > \frac{3}{5} r^{-\frac{6}{5}} G^3 e^{At} + [A - r^{\frac{1}{5}} e^{-2At}]\, G e^{At} > 0$$

if A is chosen sufficiently large. This leads to a contradiction, since at a minimum point of $\tilde{G}$ one has $\frac{d\tilde{G}}{dt} \leq 0$.

In the case of the Gauss Curvature Flow the computations are more involved but very similar. Set $I = 1 + g^2 g_r^2$. If

$$\tilde{G} = \frac{g_r}{r^{\frac{2}{5}}}\, e^{At}$$

assumes an interior minimum at (r, t) one can compute that at this point

$$\begin{aligned} G_t &\geq -CG^4 + \frac{3G^3}{5r^{\frac{6}{5}} I^{\frac{3}{2}}} - \frac{12gG^2}{25r^{\frac{13}{5}} I^{\frac{3}{2}}} = -CG^4 + \frac{G^2}{r^{\frac{13}{5}} I^{\frac{3}{2}}}[\frac{3rg_r}{5} - \frac{12g}{25}] \\ &\geq -CG^4 + \frac{G^2}{r^{\frac{13}{5}} I^{\frac{3}{2}}}[\frac{3rg_r}{5} - \frac{12g_r(r^*)}{25}] \geq -CG^4 + \frac{3G^2}{5r^{\frac{13}{5}} I^{\frac{3}{2}}}\, [g_r - g_r(r^*)] \end{aligned}$$

for some point r^* in $[\gamma(t), r]$ and use the inequality $g_{rr} \geq -Cr^{\frac{8}{5}}$, proven in Lemma 2.2, to deduce that

$$G_t \geq -CG^4 - C\frac{r^{\frac{13}{5}}}{r^{\frac{13}{5}} I^{\frac{3}{2}}}\, G^2 \geq -C\, G$$

if $\tilde{G} \leq 1$ at the minimum point (r, t). We can then conclude, as above, that $\tilde{G}$ cannot attain an interior minimum if A is chosen sufficiently large. To show that $\tilde{G}$ cannot attain its minimum at the free-boundary, one uses the same argument as in the case of the Monge-Ampére equation, shown above. This concludes the proof of the Lemma.

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Lemmas 6.1 and 6.2.

7. Self-Similar behavior of Solutions to the parabolic Monge-Ampére Equation.

In this last section we will sketch the proof of the self-similar behavior of solutions to the parabolic Monge-Ampére equation (1.12) at their focusing time. We will omit most of the details of the proofs, since they are very similar to the proofs of the analogous results in [AV] and [AA1].

We begin by showing that equation (1.12) admits self-similar solutions. To simplify the notation, let us translate the time variable so that the focusing time of the solution g of (1.12) is $T = 0$. Using the scaling of the equation we easily deduce that we should look for a self-similar solution Φ of the form

$$\Phi(r,t) = \frac{r^2}{\sqrt{-t}}\, \phi(\frac{t}{t^\alpha}) \qquad t \leq 0. \tag{7.1}$$

Since Φ must satisfy the equation (1.12), setting $s = \frac{t}{r^\alpha}$ and $-\xi = \ln(-s)$, we find that $\phi = \phi(\xi)$ must satisfy the equation

$$(\alpha^3\phi\phi' + 2\alpha^2\phi^2)\,\phi''(\xi) + 3\alpha^2\phi{\phi'}^2(\xi) + \alpha^3{\phi'}^3(\xi) + 20\alpha\phi^2\phi'(\xi) - 12\phi^3 - \phi' - \frac{1}{2}\phi = 0 \tag{7.2}$$

where

$$\alpha^3\phi\phi' + 2\alpha^2\phi^2 \geq 0$$

since $gg_r \geq 0$. Set $\psi(\xi) = \phi(\xi)$ and $\theta(\xi) = \phi'(\xi)$, so that the above equation becomes equivalent to the system

$$\begin{cases} \psi' = \theta \\ \alpha^2(\alpha\theta\psi + 2\psi^2)\,\theta' = \theta + \frac{1}{2}\psi - 12\psi^3 - 20\alpha\psi^2\theta - 2\alpha^2\psi\theta^2 - \alpha^3\theta^3. \end{cases} \tag{7.3}$$

Introducing the new variable τ so that

$$\frac{d\tau}{d\theta} = \frac{1}{\alpha\theta\psi + 2\psi^2} \geq 0$$

and defining $\rho(\tau) = \psi(\xi(\tau))$ and $\sigma(\tau) = \theta(\xi(\tau))$, we conclude that the system (7.3) is equivalent to

$$\begin{cases} \rho' = \sigma\,(\alpha\rho\sigma + 2\rho^2) \\ \alpha^2\,\sigma' = \sigma + \frac{1}{2}\rho - 12\rho^3 - 20\alpha\rho^2\sigma - 3\alpha^2\rho\sigma^2 - \alpha^3\sigma^3. \end{cases} \tag{7.4}$$

Finally, setting

$$\begin{cases} \tau = \alpha\,\omega \\ \sigma(\tau) = \frac{1}{\alpha}\,\zeta(\omega), \quad \rho(\tau) = \eta(\omega) \end{cases}$$

the system (7.4) becomes equivalent to

$$\begin{cases} \eta' = \zeta\,(\eta\zeta + 2\eta^2) = \eta\zeta\,(\zeta + 2\eta) \\ \xi' = \frac{1}{\alpha}\,\zeta + \frac{1}{2}\eta - 12\eta^3 - 20\eta^2\zeta - 9\eta\zeta^2 - \zeta^3. \end{cases} \tag{7.5}$$

The last system of equations is very similar to the one studied in [GV] with a little different constant. Hence, we can follow the proofs of the results in [GV] and [AA1] to establish the following result.

7.1. Theorem. *There exists a function ϕ, continuous on $(-\infty, 0]$ and constants $E > 0$ and $\alpha^* > 0$ such that*

$$\Phi(r,t) = \frac{r^2}{\sqrt{-t}}\,\phi(\frac{t}{r^{\alpha^*}}), \qquad t \leq 0$$

is a solution of (1.12) and

$$\begin{cases} \phi(s) > 0 & \text{for } -E < s < 0 \\ \phi(s) = 0 & \text{for } s \geq -E \end{cases}$$

It follows the free-boundary of Φ is given by

$$r = \left(\frac{-t}{E}\right)^{\frac{1}{\alpha^*}}.$$

For a constant $c > 0$ set

$$\Phi_c(r,t) = \frac{r^2}{\sqrt{-t}}\,\phi(\frac{ct}{r^{\alpha^*}}), \qquad t \leq 0.$$

Following the arguments of the proof of Theorem 1.1 in [AA1] one can prove the following result:

7.1. Theorem. *Let g be a solution of (1.12) which focuses at time $t = 0$. Assume that $g(r, t_0)$ has finite intersection points with $\Phi_c(r, t)$, for some $t_0 < 0$ and $c > 0$. Then, there exists a number $c^* > 0$ such that*

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{\frac{4-\alpha^*}{2\alpha^*}}} g(\frac{1}{\lambda^{\alpha^*}} r, \lambda t) = h_{c^*}(r, t).$$

In other words, for each $\eta < 0$, we have

$$\lim_{r \to 0} \frac{g(r, r\,\eta^{\alpha^*})}{r^{\frac{4-\alpha^*}{2}}} = \frac{\phi(c^* \eta)}{\sqrt{-\eta}}.$$

An immediate consequence of this Theorem is the next Corollary.

7.2. Corollary. *Under the hypotheses of Theorem 7.1, we have*

$$\lim_{r \to 0} \frac{g(r, 0)}{r^{\frac{4-\alpha^*}{2}}} = c^*.$$

Using Theorem 1.2 we can estimate the number α^*.

7.3. Corrolary. *The number α^* in Theorem 7.2 can be estimated as*

$$\frac{3}{2} \leq \alpha^* \leq \frac{6}{5}.$$

Proof. Since $C_1\, r^{\frac{2}{5}} \leq g_r \leq C_2\, r^{\frac{1}{4}}$, by Theorem 1.2, we must have

$$1 + \frac{2}{5} \leq \frac{4 - \alpha^*}{2} \leq 1 + \frac{1}{4}$$

which immediately implies the desired bound.

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