WORN STONES WITH FLAT SIDES THE REGULARITY OF THE INTERFACE

1. Introduction

In [?] Daskalopoulos and Hamilton considered the free-boundary problem associated to the Gauss Curvature Flow with Flat sides. This is the flow describing the deformation of the a compact convex body M in $\mathbb{R}^3$ which is subject to wear under impact from any random angle. An example can be a stone on a beach impacted by the sea. The probability of impact at any point P on the surface Σ is proportional to the Gauss Curvature K. Therefore the surface evolves by the flow

$$\frac{\partial P}{\partial t} = K\, N \tag{1.1}$$

where N denotes the unit inward normal.

The *Gauss Curvature Flow* was introduced by Firey [F], who showed that it shrinks smooth, compact, strictly convex and centrally symmetric hypersurfaces in $\mathbb{R}^3$ to round points.

Tso [T] showed that if the initial surface Σ is smooth, compact and strictly convex, then the Gauss Curvature Flow admits a unique solution $\Sigma(t)$ which shrinks to a point at the exact time $T^* = V/4\pi$, where V is the volume enclosed by the initial surface Σ.

Chow [C] proved that, under certain restrictions on the second fundamental form of the initial surface, the Gauss Curvature flow shrinks smooth compact strictly convex hypersurfaces to round points.

Andrews [A] has showed and that the Gauss Curvature flow shrinks compact convex hypersurfaces to round points.

In [?] Daskalopoulos and Hamilton considered the case where the initial surface has flat sides and as a consequence the parabolic equation describing the motion of the hypersurface becomes degenerate where the curvature becomes zero. Hence, according to Hamilton's results in [H], the junction Γ between each flat side and the strictly convex part of the surface, where the equation becomes degenerate, behaves like a free-boundary propagating with finite speed.

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Let us assume, for simplicity that the initial surface has only one flad side. The main objective of the work in [?] was to study the solvability and short time regularity of the interface Γ of the Gauss Curvature flow with flat sides, by viewing the flow as a *free-boundary* problem. Let us assume, for simplicity that the initial surface has only one flat side. It was shown in [?] that, under certain assumptions on the initial surface Σ which will guarantee that the junction curve Γ will start move at every point at time $t = 0$, the Gauss Curvature Flow admits a solution $\Sigma(t)$, in $0 < t \leq \tau$, for some time $\tau > 0$, and the junction $\Gamma(t)$ between the flat and strictly convex side is a *smooth curve*, for $0 < t \leq T$. Moreover, each strictly convex side is smooth up to the interface, for $0 < t \leq \tau$.

In this work we will show that under the same assumptions as in [?] the C^∞ regularity of the strictly convex side up to the interface and the free-boundary $\Gamma(t)$ is preserved up to the focusing time of the flat side.

In [?] ben Andrews has shown that the surface $\Sigma(t)$ is of class $C^{1,1}$ up to the time it shinks to a point. This beautiful result will be crucial in this paper.

Let us assume that $t = 0$ we have

$$\Sigma = \Sigma_0 \cup \Sigma_1$$

where Σ_0 is the flat side and Σ_1 is the strictly convex part of the surface. The junction between the two sides is the curve

$$\Gamma = \Sigma_0 \cap \Sigma_1.$$

Since the equation is invariant under rotation, we can also assume that Σ_0 lies on the $z = 0$ plane and that Σ_1 lies above this plane. Then, the lower part of the surface Σ can be written as the graph of a function

$$z = f(x)$$

over a compact domain $\Omega \subset \mathbb{R}^2$ containing the initial flat side Σ_0. We can choose the domain Ω to be the set

$$\Omega = \{x \in \mathbb{R}^2 : |Df|(x) < \infty\}$$

so that f turns vertical at the boundary $\partial\Omega$. Our basic assumption on the initial surface is that the function f *vanishes quadratically* at $z = 0$ and that the junction curve Γ is strictly convex. Namely, setting

$$g = \sqrt{2f}$$

we assume that at time $t = 0$ the function g satisfies

$$|Dg(x)| \geq \lambda \qquad \text{and} \qquad D^2_{\tau\tau} g(x) \geq \lambda \qquad \forall x \in \Gamma \tag{1.2}$$

for some positive number $\lambda > 0$, where $D^2_{\tau\tau}$ denotes the second order tangential derivative at Γ. As explained in [?] condition (1.2) guarantees that the interface Γ will start to move at any point at time $t = 0$ making the Gauss Curvature Flow to behave like a free-boundary problem.

Assume that at time $T_0 > 0$ the flat side $\Sigma_1(T_0)$ contains the disc

$$B_{\rho_0} = \{\, x \in \mathbb{R}^2 \,:\, |x| < \rho_0 \,\}$$

for some number $\rho_0 > 0$. For $0 < t \leq T_0$ the lower part of the surface $\Sigma(t)$ can be written as the graph of a function

$$z = f(x,t)$$

on the set

$$\Omega(t) = \{\, x \in \mathbb{R}^2 \,:\, |Df|(x,t) < \infty \,\}.$$

This is because the results in [?] guarantee the the lower part of the surface will not turn vertical before the flat side shrinks to a point. Since $\Sigma(t)$ solves the Gauss Curvature Flow, the function f will satisfy the equation

$$f_t = \frac{\det D^2 f}{(1 + |Df|^2)^{3/2}} \tag{1.3}$$

by a standard computation. On the flat side we will always have $z = 0$, while on the strictly convex side we will have that $z > 0$. The function $g = \sqrt{2f}$ satisfies the equation

$$g_t = \frac{g\,\det D^2 g + g_y^2 g_{xx} - 2 g_x g_y g_{xy} + g_x^2 g_{yy}}{(1 + g^2 |Dg|^2)^{3/2}}. \tag{1.4}$$

The following short time result was proven in [?]. We state it here for the convenience of the reader.

Theorem [DH](Short time Regularity) *Let Σ be a weakly convex, compact hypersurface in $\mathbb{R}^3$ so that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is flat and Σ_2 is strictly convex. Assume that at time $t = 0$ the lower part of Σ_2 can be written as the graph of a function f over a domain Ω containing the flat side and that the function*

$$g = \sqrt{2\,f}$$

is of class $C^{2+\alpha}$ up to the interface $z = 0$, for some $\alpha > 0$ and satisfies conditions (1.2)*. Then, there exists a time $\tau > 0$ for which the Gauss Curvature Flow* (1.1) *admits a solution $\Sigma(t)$ on $0 \le t \le \tau$. Moreover, the function*

$$z = g(x,t)$$

with $g = \sqrt{2f}$ is smooth up to the interface $z = 0$ for all $0 < t \le \tau$. In particular the junction $\Gamma(t)$ between the strictly convex and the flat side will be a smooth curve for all t in $0 < t \le \tau$.

The main objective of this work is to show that under the assumptions of Theorem [DH], the function g remains smooth up to the interface for all $0 < t \le T_0$.

Theorem 1.1. *Assume that at time T_0 the flat side of the surface Σ contains the disc $D_{\rho_0} = \{x \in \mathbb{R}^2 : |x| < \rho_0\}$. Then, under the assumptions of Theorem [DH], the function $g = \sqrt{2f}$ is smooth up to the interface $z = 0$ for all $0 < t \le T_0$. In particular the junction $\Gamma(t)$ between the strictly convex and the flat side will be a smooth curve for all t in $0 < t \le T_0$.*

The following Theorem due to Andrews [**?**] will be used in this paper:

Theorem [A]

2. Finite and Non-Degenerate Speed of the Free-Boundary

In this section we will show that the free-boundary $\Gamma(t)$ moves with finite and non-degenerate speed. This will follow from certain differential inequalities which will be proved using the scaling of the equation and the maximum principle.

Throughout this section we will assume that $z = f(x,t)$ is a solution of (1.3) such that $f(\cdot,t)$ is of class $C^{1,1}$ on the set

$$\Omega(t) = \{ x \in \mathbb{R}^2 : |Df(x)| < \infty \}$$

for all $0 \le t \le T$

and that $g = \sqrt{2f}$ is smooth up to the interface $\Gamma(t)$, on $0 \le t \le \tau$, for some $0 < \tau \le T$.

Lemma 2.1. *The function*

$$f_\epsilon(x,t) = \frac{1}{1+B\epsilon} f((1+\epsilon)x, (1-A\epsilon)t)$$

is a supersolution (subsolution) of (1.3) *if and only if*

$$-A + B - 4 > (<) \frac{3(B-1)\,|Df_\epsilon(x,t)|^2}{(1+|Df_\epsilon(x,t)|^2)}. \tag{2.1}$$

Proof. By direct compution we find that

$$\begin{aligned} f_{\epsilon t}(x,t) &= \frac{1-A\epsilon}{1+B\epsilon} f_t((1+\epsilon)x,(1-A\epsilon)t) \\ &= \frac{1-A\epsilon}{1+B\epsilon} \frac{\det(D^2 f((1+\epsilon)x,(1-A\epsilon)t))}{(1+|Df((1+\epsilon)x,(1-A\epsilon)t)|^2)^{3/2}} \\ &= \frac{1-A\epsilon}{1+B\epsilon} \frac{(1+B\epsilon)^2}{(1+\epsilon)^4} \frac{\det(D^2 f_\epsilon(x,t))}{(1+(\frac{1+B\epsilon}{1+\epsilon})^2\,|Df_\epsilon(x,t)|^2)^{3/2}}. \end{aligned}$$

Using the expansion

$$(\frac{1+B\epsilon}{1+\epsilon})^2 = 1+2(B-1)\epsilon+o(\epsilon^2)$$

we obtain, after several simple calculations, that

$$\begin{aligned} (f_\epsilon)_t(x,t) &= \frac{1-A\epsilon}{1+B\epsilon}\frac{(1+B\epsilon)^2}{(1+\epsilon)^4}\left[1-\epsilon\frac{3(B-1)\,|Df_\epsilon(x,t)|^2}{2(1+|Df_\epsilon(x,t)|^2)}+O(\epsilon^2)\right]\frac{\det(D^2 f_\epsilon(x,t))}{(1+|Df_\epsilon(x,t)|^2)^{3/2}} \\ &= \left[1+\epsilon\left(-A+B-4-\frac{3(B-1)\,|\nabla f_\epsilon(x,t)|^2}{2(1+|\nabla f_\epsilon(x,t)|^2)}\right)+O(\epsilon^2)\right]\frac{\det(D^2 f_\epsilon(x,t))}{(1+|Df_\epsilon(x,t)|^2)^{3/2}} \end{aligned}$$

and hence

$$f_{\epsilon t}(x,t) > (<) \frac{\det(D^2 f_\epsilon(x,t))}{(1+|Df_\epsilon(x,t)|^2)^{3/2}}$$

if and only if (2.1) holds.

The finite speed of the free-boundary will follow as a consequence of the following differential inequality:

Lemma 2.2. *If $B_{\rho_0} \subset \{x \in \Omega(t) : f(x,T)=0\}$, then there exists $\delta_0 > 0$ s.t.*

$$-B\,f(x,t) + x\cdot\nabla f(x,t) - At\,f_t(x,t) \geq 0 \tag{2.2}$$

on the set $\{(x,t):\ f(x,t)\leq\delta_0,\ 0\leq t\leq T\}$, for some constants $A,B>0$.

Proof. By the uniform continuity of $f(x,t)$ on $0 \leq t \leq T$, for a given $0<\eta<1$, there is a $0<\delta_0<<\eta$ such that

$$\{x:\ f(x,t)\leq\delta_0,\ 0\leq t\leq T\} \subset \{x:\ d(x,\Omega(t)) \leq \frac{\eta r_0}{2}\} \tag{2.3}$$

and

$$\{(1+\epsilon)x:\ d(x,\Omega(t)) \leq \frac{\eta r_0}{2}\} \subset \{x:\ d(x,\Omega(t)) \leq \eta r_0\} \tag{2.4}$$

for all $\epsilon << \delta_0$.

Let us consider the function

$$f_\epsilon(x,t) = \frac{1}{1+B\epsilon} f((1+\epsilon)x,(1-A\epsilon)t),$$

with $A = \delta_0^2$ and $B = 8$ and $\epsilon << \delta_0 << \eta$. We will show that f_ϵ is a supersolution of equation (1.3) on the set $\mathcal{A}_{\delta_0} = \{ x : f(x,t) \leq \delta_0, 0 \leq t \leq T \}$ such that $f_\epsilon \geq f$ at the parabolic boundary of $\mathcal{A}_{\delta_0}$. The comparison principle will then imply that $f_\epsilon \geq f$ on $\mathcal{A}_{\delta_0}$. Hence

$$\frac{d}{d\epsilon}_{|\epsilon=0} f_\epsilon \geq 0$$

on $\mathcal{A}_{\delta_0}$ proving the estimate (2.2).

By Lemma 2.1, in order to show that f_ϵ is a supersolution of (1.3) it is enough to prove that

$$-A + B - 4 > \frac{3(B-1)\,|Df_\epsilon(x,t)|^2}{(1 + |Df_\epsilon(x,t)|^2)}.$$

For $A = \delta_0^2 < 1$ and $B = 8$ the above inequality is satisfied if

$$|Df_\epsilon((1+\epsilon)x, (1-A\epsilon)t)|^2 \leq \frac{1}{7}, \qquad \text{on } \mathcal{A}_{\delta_0}.$$

Since

$$|Df_\epsilon((1+\epsilon)x, (1-A\epsilon)t)| = \frac{1+\epsilon}{1+B\epsilon}\,|Df((1+\epsilon)x, (1-A\epsilon)t)| \leq |Df((1+\epsilon)x, (1-A\epsilon)t)|$$

it is enough to show that

$$|Df((1+\epsilon)x, (1-A\epsilon)t)| \leq \frac{1}{\sqrt{7}}$$

on $\mathcal{A}_{\delta_0}$. Because the initial data is of class $C^{1,1}$, Andrew's result in [?] shows that $f(\cdot,t)$ is of class $C^{1,1}$ uniformly on $0 \leq t \leq T$. Hence, there exists a constant M independent of t such that

$$|Df(x,t)| \leq M\, d(x, \partial\Omega(t)). \tag{2.5}$$

Assume now that $f(x,t) \leq \delta_0$. Because f is convex and satisfies (1.3) the time derivative f_t is nonnegative and hence $f(x,(1-A\epsilon)t) \leq \delta_0$. By (2.3) and (2.4) we have $d((1+\epsilon)x, \partial\Omega((1-A\epsilon)t)) \leq \eta\, r_0$ which can be made arbitrarily small by choosing η small. Hence by (2.5) we can make $|Df(1+\epsilon)x, (1-A\epsilon)t)| \leq \frac{1}{\sqrt{7}}$ on $\mathcal{A}_{\delta_0}$.

It remains to show that $f_\epsilon \geq f$ on the parabolic boundary of $\mathcal{A}_{\delta_0}$. By simple differentiation we compute

$$\frac{d}{d\epsilon}_{|\epsilon=0} f_\epsilon(x,0) = -Bf(x,0) + x \cdot Df(x,0)$$

and therefore $\frac{d}{d\epsilon}_{|\epsilon=0} f_\epsilon(x,0) > 0$, on $\{x : f(x,0) \leq \delta_0\}$, for δ_0 sufficiently small, since $f(x,0) \approx C\, d(x, \Omega(0))^2$ by assumption. This implies that $f_\epsilon(x,0) \geq f(x,0)$ for small $\epsilon > 0$.

On the lateral boundary of $\mathcal{A}_{\delta_0}$ where $f(x,t)=\delta_0$ we have $d(x,\partial\Omega(t))\leq \eta r_0$, by (2.3). By the convexity of f the radial derivative f_r satisfies the estimate $f_r(x,t)\geq \frac{\delta_0}{\eta r_0}$ and therefore

$$x\cdot Df = r f_r(x,t) \geq r_0\,\frac{\delta_0}{\eta r_0} = \frac{\delta_0}{\eta}, \qquad \text{on } \partial_p \mathcal{A}_{\delta_0}.$$

We conclude that

$$-Bf(x,t) + x\cdot Df(x,t) - At\, f_t(x,t) \geq -B\delta_0 + \frac{\delta_0}{\eta} - \delta_0^2 T\, |f_t|_{L^\infty} > 0$$

if η is suffiecently small which proves that $f_\epsilon(x,t)\geq f(x,t)$ on the parabolic boundary of $\mathcal{A}_{\delta_0}$, finishing the proof of the lemma.

Let us express the interface $\Gamma(t)$ as a function $r=\gamma(\theta,t)$ where (r,θ) denote the polar coordinates.

Corollary 2.3. *If $B_{\rho_0}\subset\{x\in\Omega(t) : f(x,t)=0\}$, $0\leq t\leq T$, then there exists $A>0$ such that*

$$\gamma(\theta,t) \geq e^{-\frac{t-t_0}{At_0}}\gamma(\theta,t_0). \tag{2.6}$$

for all $0<t_0\leq t\leq T$. In particular, the free-boundary $r=\gamma(\theta,t)$ moves with finite speed, on $0\leq t\leq T$.

Proof. From inequality (2.2) we have

$$\begin{aligned} 0 &\geq \frac{Bf(x,t)}{At} - \frac{x}{At}\cdot Df(x,t) + f_t(x,t) \\ &\geq \frac{Bf(x,t)}{AT} - \frac{x}{At_0}\cdot Df(x,t) + f_t(x,t) \end{aligned}$$

and hence

$$\frac{d}{dt}\left(e^{\frac{B}{AT}(t-t_0)} f(e^{-\frac{t-t_0}{At_0}}x_0,t)\right) \leq 0 \tag{2.7}$$

which immediately implies the inequality (2.6).

Let us now express, in polar coordinates, by $r=\gamma_\epsilon(\theta,t)$ the ϵ-level set of the function f. Inequality (2.7) implies that $\gamma_\epsilon(\theta,t)$ has finite speed, as shown next:

Corollary 2.4. *If $B_{\rho_0}\subset\{x\in\Omega(t) : f(x,t)=0\}$, $0\leq t\leq T$, then there exists $A>0$ such that*

$$\gamma_\epsilon(\theta,t) \geq e^{-\frac{t-t_0}{At_0}}\gamma_\epsilon(\theta,t_0). \tag{2.8}$$

for all $0<t_0\leq t\leq T$. In particular, for each $\epsilon>0$, the ϵ-level set $r=\gamma_\epsilon(\theta,t)$ of f moves with finite speed, on $0\leq t\leq T$.

Proof. Indeed, assume that $r_0 = \gamma_\epsilon(\theta, t_0)$ and denote by x_0 the point $x_0 = (r_0, \theta)$. Then, by inequality (2.7) we have:

$$f(e^{-\frac{t-t_0}{At_0}} x_0, t) \leq f(x_0, t_0) = e^{-\frac{B}{AT}(t-t_0)} \epsilon \leq \epsilon$$

implying that

$$e^{-\frac{t-t_0}{At_0}} \gamma_\epsilon(\theta, t_0) = e^{-\frac{t-t_0}{At_0}} r_0 \leq \gamma_\epsilon(\theta, t)$$

as desired.

We will next show that the free-boundary moves with non-degenerate speed. This will follow from the next inequality:

Lemma 2.5. *There exist positive constants A, B and C for which*

$$-Bf(x,t) + x \cdot Df(x,t) - (C + At)\, f_t(x,t) \leq 0 \tag{2.9}$$

on $0 \leq t \leq T$.

Proof. By the short time regularity, $g = \sqrt{2f}$ is a smooth function up to the inteface $\Gamma(t)$ on $0 \leq t \leq \tau$, $\tau > 0$. Set $t^* = \tau/2$. We will show, using the comparison principle, that

$$f_\epsilon(x,t) = \frac{1}{1 + B\epsilon} f((1+\epsilon)x, (1 - A\epsilon)t - C\epsilon) \leq f(x,t) \tag{2.10}$$

on $\mathcal{A}_{t^*} = \{ f(x,t) \leq 1,\, t^* \leq t \leq T\}$, for an appropriate choice of constants A, B and C and ϵ sufficiently small. We will choose $A = B + 1$, for some constant B. Therefore $-A + B - 4 < 0$ which guarantees that f_ϵ is a subsolution of (1.3), by Lemma 2.1. Hence we only need to show that $f_\epsilon \leq f$ at the parabolic boundary of $\mathcal{A}_{t^*}$ which is equivalent to showing that the inequality (2.9) holds there. At $f = 1$ one can choose B sufficently large so that

$$-B\, f + x \cdot Df \leq 0$$

since $f(cdot, t) \in C^{1,1}$ uniformly in $0 \leq t \leq T$. Hence (**??**) holds at the lateral boundary of $\mathcal{A}_{t^*}$. To show that (**??**) holds on $\{ f(x,t) \leq 1, t = t^*\}$ we will actually show that

$$-Bg(x,t^*) + x \cdot Dg(x,t^*) - C\, g_t(x,t^*) \leq 0 \tag{2.11}$$

holds on $\{0 < g(x,t^*) \leq \sqrt{2}\}$. The function g satisfies equation (1.4) and it is smooth up to the interface on $0 \leq t \leq 2t^*$. Also, $|Dg|(x,t^*) \geq c > 0$ and $g_{\tau\tau} \geq c > 0$ at $\Gamma(t^*)$ by the initial assumptions on g. Hence, there exist numbers

$\rho > 0$ and $c_0 > 0$ such that $g_t \geq c_0 > 0$ on $\{g(x,t^*) > 0,\, d(x,\Gamma(t^*)) < \rho\}$ and therefore we can make

$$x \cdot Dg(x,t^*) - C\, g_t(x,t^*) \leq 0$$

on this set by choosing C sufficiently large. Having chosen C we can know choose B sufficiently large so that

$$-Bg(x,t^*) + x \cdot Dg(x,t^*) \leq 0$$

on $\{g(x,t^*) \leq \sqrt{2},\, d(x,\Gamma(t^*)) \geq \rho\}$. Combining both estimates we conlude that (??) holds $\{\, f(x,t) \leq 1, t = t^*\}$, which implies the validity of (2.10). Differentiating with respect to ϵ we conclude

$$\frac{d}{d\epsilon}_{|\epsilon=0} f_\epsilon(x,t) \leq 0$$

for $t^* \leq t \leq T$, which immediately implies inequality (2.9) on $t^* \leq t \leq T$. Since the estimate also holds on $0 \leq t \leq t^*$ the lemma follows.

Corollary 2.6. *There exist positive constants A and C such that*

$$\gamma(\theta,t) \leq e^{-\frac{t-t_0}{C+At_0}}\, \gamma(\theta,t_0) \tag{2.12}$$

for $t_0 \leq t \leq T$. In particular the free-boundary moves with non-degenerate speed.

Proof. From inequality (2.9) we have

$$\begin{aligned} 0 &\leq \frac{Bf(x,t)}{C+At} - \frac{x}{C+At} \cdot Df(x,t) + f_t(x,t) \\ &\leq \frac{Bf(x,t)}{C+At_0} - \frac{x}{C+AT} \cdot Df(x,t) + f_t(x,t) \end{aligned}$$

and hence

$$\frac{d}{dt}\left(e^{\frac{B(t-t_0)}{C+AT}} f(e^{-\frac{t-t_0}{C+At_0}} x_0, t)\right) \leq 0$$

which immediately implies the inequality (2.12).

3. Gradient Estimates

In this section we will establish estimates from above and bellow on the gradient $|Dg|$ of the solution g of (1.4). Lets assume that at $t = 0$ the function g satisfies the hypotheses of Theorem 1.1. and that g is a solution of (1.4) which is smooth up to the interface on $0 \leq t \leq T$. We can assume, without loss of generality, that

$$\max_{x\in\Omega(t)} g(x,t) \geq 2, \qquad \text{for } 0 \leq t \leq T. \tag{3.1}$$

The estimate from above follows as a straight forward consequence of the maximum principle.

Lemma 3.1. *Under the assumptions of Theorem 1.1 and* (3.1)*, there exists a constant C such that*

$$|Dg| \leq C, \qquad \text{on } 0 \leq g(\cdot,t) \leq 1, \ 0 \leq t \leq T.$$

Proof. Let us approximate f by a decreasing sequence of solutions f_ϵ of (1.3) which are positive, strictly convex and smooth on $\{ x \in \mathbb{R}^2 : |Df_e(x)| < \infty \}$, $0 \leq t \leq T$. Set $g_\epsilon = \sqrt{2 f_e}$. We can choose the $f_\epsilon' s$ such that $|Dg_e| \leq C$ at $t = 0$, on the set $\{ x : 0 \leq g_\epsilon \leq 1 \}$ and $Dg_\epsilon \leq C$ at $g_\epsilon = 1$, $0 \leq t \leq T$, for some uniform constant C. The last estimate holds because $|Dg| = |Df|/g \leq C$, at $g = 1$, $0 \leq t \leq T$, by the $C^{1,1}$ regularity of f proven in [?] and condition (3.1).

It is enough to show that

$$|Dg_\epsilon| \leq C, \qquad \text{on } 0 \leq g_\epsilon(\cdot,t) \leq 1, \ 0 \leq t \leq T.$$

To simplify the notation, we will denote g_ϵ by g, just assuming that g is a strictly positive and a smooth solution of (1.4) such that $f = g^2/2$ is convex. Set $X = |Dg|^2 = (g_x^2 + g_y^2)/2$. We will show, using the maximum principle that $X \leq \tilde{C}$, on $0 \leq g \leq 1, \ 0 \leq t \leq T$, provided that $X \leq \tilde{C}$ at $t = 0$ and $g = 1$, $0 \leq t \leq T$. Indeed, assume that X attains an interior maximum at the point $P_0 = (x_0, y_0, t_0)$. We can rotate the coordinates so that

$$g_x > 0 \qquad \text{and} \qquad g_y = 0, \qquad \text{at } P_0. \tag{3.2}$$

We will then have

$$X_x = g_x\, g_{xx} + g_y\, g_{xy} = g_x\, g_{xx} = 0$$

and

$$X_y = g_x\, g_{xy} + g_y\, g_{yy} = g_x\, g_{xx} = 0$$

at P_0, implying that

$$g_{xx} = g_{xy} = 0, \qquad \text{at } P_0. \tag{3.3}$$

The only non-zero second derivative of g is g_{yy} which is actually non-negative by the convexity of the level sets of g. Differentiating once more, we compute

$$X_{xx} = g_x\, g_{xxx} + g_y\, g_{xxy} + g_{xx}^2 + g_{xy}^2 = g_x\, g_{xxx} \leq 0$$

and

$$X_{yy} = g_x\, g_{xyy} + g_y\, g_{yyy} + g_{xy}^2 + g_{yy}^2 = g_x\, g_{xyy} + g_{yy}^2 \leq 0$$

at P_0 which implies that

$$g_x\, g_{xxx} \le 0 \qquad \text{and} \qquad g_x\, g_{xyy} \le 0, \qquad \text{at } P_0. \tag{3.4}$$

On the other hand, differentiating the equation

$$g_t = \frac{g \,\det D^2 g + g_y^2 g_{xx} - 2 g_x g_y g_{xy} + g_x^2 g_{yy}}{(1+g^2|Dg|^2)^{3/2}}$$

with respect to x and using (3.2) and (3.3), we find that at the point P_0

$$X_t = g_x\, g_{xt} = \frac{g g_x g_{yy} g_{xxx} + g_x^3 g_{xyy}}{(1+2g^2X)^{3/2}} - \frac{4 g g_x X + 2g^2 X_x}{(1+2g^2X)^{5/2}}.$$

Since $g_x > 0$, $X_x = 0$, $X \ge 0$ and $g_{yy} \ge 0$ at P_0, using (3.4) we finally conclude that $X_t \le 0$ at P_0 which implies the desired claim, therefore finishing the proof of the lemma.

We will next establish a lower bound on the gradient $|Dg|$ of g.

Lemma 3.2. *Under the assumptions of Theorem 1.1, if $B_{\rho_0} \subset \{(x,y) : g(x,y,T) = 0\}$ and g is smooth up to the interface on $0 \le t \le T$, then there exists a constant $c > 0$ depending only on ρ_0 and the initial data, such that*

$$|Dg| \ge c, \qquad \text{on} \quad \{\,(x,y,t) : g(x,y,t) > 0,\, 0 \le t \le T\,\}.$$

Proof. Consider the quantity $X = x\, g_x + y\, g_y$. Using the maximum principle, we will show that

$$X \ge c_T, \qquad \text{on} \quad \{\,(x,y,t) : g(x,t) > 0,\, 0 \le t \le T\,\} \tag{3.5}$$

provided that $X \ge c_0 > 0$ at $t = 0$. This will imply the lemma.

Let us assume first that X becomes minimum at and interior point P_0 i.e. $X(P_0) = \min_{t \le t_0} X(x,t)$. Since, both the equation (1.4) and the quantity X are rotationally invariant, we can assume without loss of generality that $y_0 = 0$ at P_0. Hence, at the point P_0 we will have:

$$X_x = x\, g_{xx} + g_x + y\, g_{yx} = x\, g_{xx} + g_x = 0$$

and

$$X_y = y\, g_{yy} + g_y + x\, g_{xy} = x\, g_{xy} + g_y = 0$$

implying that

$$g_{xx} = -\frac{g_x}{x} \qquad \text{and} \qquad g_{xy} = -\frac{g_y}{x}, \qquad \text{at } P_0. \tag{3.6}$$

Differentiating once more we obtain

$$X_{xx} = x g_{xxx} + 2 g_{xx} + y g_{xxy} = x g_{xxx} + 2 g_{xx}$$

and

$$X_{xy} = xg_{xxy} + 2g_{xy} + yg_{xxy} = xg_{xxy} + 2g_{xy}$$

and

$$X_{yy} = xg_{xyy} + 2g_{yy} + yg_{yyy} = xg_{xyy} + 2g_{yy}$$

implying, in particular, that

$$X_{xx} = xg_{xxx} + 2g_{xx} \geq 0 \quad \text{and} \quad X_{yy} = xg_{xyy} + 2g_{yy} \geq 0 \tag{3.7}$$

at the minumum point P_0.

On the other hand, differentiating in time t we compute

$$X_t = x\, g_{xt} + y\, g_{yt} = x\, g_{xt}$$

at P_0. To compute g_{xt} at P_0 we will differentiate the equation (1.4) with respect to x and use (3.6) and (3.7). To simplify the notation let us set $I = (1 + g^2 g_x^2 + g^2 g_y^2)$. Differentiating (1.4) with respect to x we obtain

$$\begin{aligned} X_t = x\, g_{xt} = I^{-\frac{3}{2}} \{ \, xg_{xxx}[gg_{yy} + g_y^2] - 2xg_{xxy}[gg_{xy} + g_x g_y] + xg_{xyy}[gg_{xx} + g_x^2] \, \} \\ - 3I^{-1} xg_t \, \{ \, g(g_x^2 + g_y^2) + g^2 (g_x g_{xx} + g_y g_{yx}) \, \}. \end{aligned}$$

Hence at the point P_0 where $X = x\, g_t$ and (3.7) hold, we compute after tidious calculations, that

$$X_t = I^{-\frac{3}{2}} (3xg_x - 2g) \det D^2 g - 2g_t - 3X\, I^{-1} \{ \, g|Dg|^2 + g^2 (g_x g_{xx} + g_y g_{yx}) \, \}.$$

We can substitute the second order derivatives g_{xx} and g_{xy} in the equation above by $g_{xx} = -\frac{g_x}{x}$ and $g_{xy} = -\frac{g_y}{x}$, hence obtaining that

$$X_t = I^{-\frac{3}{2}} (3xg_x - 2g) \, (-\frac{g_x g_{yy}}{x} - \frac{g_y^2}{x}) - 2g_t - 3X\, I^{-1} \{ \, g|Dg|^2 + g^2 (-\frac{g_x^2}{x} - \frac{g_y^2}{x}) \, \}.$$

or symplifying a little more

$$X_t = I^{-\frac{3}{2}} \, x^{-1} (2g - 3xg_x) \, (g_x g_{yy} + g_y^2) - 2g_t + X\, I^{-1} \, (\, g|Dg|^2 + g^2 \frac{|Dg|^2}{x} \,). \tag{3.8}$$

We wish to show that at the point P_0, where $x \geq \rho_0$, X satisfies

$$X_t \geq -C\, X \tag{3.9}$$

where C is a constant depending on ρ_0. To this end we will use Corollary 2.4 to show that

$$|g_t| \leq C\, g_x \qquad \text{and} \qquad |g_x\, g_{yy}| \leq C, \qquad \text{at } P_0 \tag{3.10}$$

To show the first estimate, let us denote by $\gamma_\epsilon(t)$ the ϵ-level set of g. Then, by Corollary 2.4

$$|\gamma_\epsilon'(t)| \leq C$$

for some constant C. Therefore, differentiating the equation $g(\gamma_\epsilon(t), t) = \epsilon$ with respect to t, we obtain

$$Dg \cdot \gamma_\epsilon'(t) + g_t = 0$$

which immediately implies that $|g_t| \leq C\,|Dg|$ at P_0. It remains to observe that at the point P_0 where $y = 0$ we have g_x is equal to the radial derivetive g_r and hence $|Dg| \leq C\, g_x$ for some constant C depending only on ρ_0. To prove the second estimate we write (1.4) at P_0 using (3.7) to obtain, after several calculations that

$$g_t = g_x \,\frac{(1-\frac{g}{x})\, g_x g_{yy} + C(\rho_0, |Dg|)}{1+g^2|Dg|^2}$$

where $C(\rho, |Dg|)$ is a constant depending only on ρ and the upper bound on $|Dg|$ poven in Lemma 3.1. Hence, we can solve the above equality with respect to $g_x\, g_{yy}$ and use the estimate $|Dg| \leq C\, g_x$ to conclude the bound $g_x g_{yy} \leq C$, therefore proving (3.9).

Assume next that the minimum of X occurs at a free-boundary point P_0 where $g = 0$. Since X is rotationally invariant we can assume this time that $g_y = 0$ at the point P_0. Then at P_0 we have:

$$X_x = x\, g_{xx} + g_x + y\, g_{yx} \geq 0$$

and

$$X_y = y\, g_{yy} + g_y + x\, g_{xy} = y\, g_{yy} + x\, g_{xy} = 0$$

implying that

$$x\, g_{xx} + g_x + y\, g_{yx} \geq 0 \qquad \text{and} \qquad g_{xy} = -\frac{y}{x}\, g_{yy} \qquad \text{at } P_0. \tag{3.11}$$

Also, the second derivative X_{yy} satisfies

$$X_{yy} = x g_{xyy} + 2 g_{yy} + y g_{yyy} \geq 0, \qquad \text{at } P_0. \tag{3.12}$$

On the other hand, $X_t = x\, g_{xt} + y\, g_{yt}$ and hence, differentiating eqrefeqn-g with respect to x and y, we find after several calculations that at the point P_0 where $g = 0$ we have

$$\begin{aligned} X_t = {} & x g_{xxx}[g g_{yy} + g_y^2] - 2x g_{xxy}[g g_{xy} + g_x g_y] + x g_{xyy}[g g_{xx} + g_x^2] + 3x g_x \det D^2 g \\ & + y g_{yyy}[g g_{xx} + g_x^2] - 2y g_{yyx}[g g_{xy} + g_x g_y] + y g_{yxx}[g g_{yy} + g_y^2] + 3y g_y \det D^2 g \end{aligned}$$

Hence, using that $g_y = 0$ at P_0 abd (3.12) we conclude, after several cancellations, that

$$X_t = g_x^2 \, [\, x g_{xyy} + y g_{yyy} \,] + 3x g_x \, [\, g_{xx} g_{yy} - g_{xy}^2 \,] \tag{3.13}$$

Let us denote by $\gamma(t)$ the free-boundary curve at time t. Then,

$$\frac{d}{dt} X(\gamma(t), t) = X_t + DX \cdot \gamma'(t).$$

Since $X_y = 0$ and $g_y = 0$ at P_0 and $g_t = -Dg \cdot \gamma'(t)$ at the free-boundary, we have

$$DX \cdot \gamma'(t) = X_x \, \frac{Dg \cdot \gamma'(t)}{g_x} = [\, x g_{xx} + g_x + y g_{xy} \,] \, (-\frac{g_t}{g_x})$$

and hence using that $g_t = g_x^2 g_{yy}$ at the free boudary point P_0 where $g = 0$ and also $g_y = 0$ we conclude that

$$DX \cdot \gamma'(t) = -[\, x g_{xx} + g_x + y g_{xy} \,] \, g_x g_{yy}. \tag{3.14}$$

Combining the inequalities (3.12) - (3.14) we finally obtain the estimate

$$\frac{d}{dt} X(\gamma(t), t) \geq -2 g_x^2 g_{yy} - 3x g_x \, [\, g_{xx} g_{yy} - g_{xy}^2 \,] - [\, x g_{xx} + g_x + y g_{xy} \,] \, g_x g_{yy}.$$

Substituting $g_{xy} = -\frac{y}{x} g_{yy}$ we find after some cancellations that

$$\frac{d}{dt} X(\gamma(t), t) \geq 2 g_x g_{yy} \, [\, x g_{xx} + y g_{xy} \,] + g_x^2 g_{yy}$$

and hence using once more the inequality $x \, g_{xx} + g_x + y \, g_{yx} \geq 0$ at P_0 we conclude that

$$\frac{d}{dt} X(\gamma(t), t) \geq -g_x^2 g_y y = -\frac{g_x g_{yy}}{x} \, X.$$

It is an immediate consequence of Corollary 2.3 that $|g_t / g_x| = |g_x g_{yy}| \leq C$ at the free-boundary point P_0. Therefore we finally obtain that

$$\frac{d}{dt} X(\gamma(t), t) \geq -C \, X, \qquad \text{at } P_0. \tag{3.15}$$

We have shown above that inequalities (3.9) or (3.15) hold respectively at a minimum interior or boundary point P_0 of X. This immediately implies that

$$\min_{\{g(\cdot, t) > 0\}} X(t) \geq \min_{\{g(\cdot, 0) > 0\}} X(0) \, e^{-Ct}$$

for all $0 \leq t \leq T$, from which the desired estimate follows.

As a consequence of Lemma 2.6 and 3.2 and Corollary 2.4, we obtain the following bound on the speed of the level sets of the function g. Denoting, in polar coordinates, by $r = \gamma_\epsilon(\theta, t)$ the ϵ-level set of the function g, we have the following:

Corollary 3.3. *Under the assumptions of Lemma 3.2 there exist positive constants C_1 and C_2, depending only on r_0 and the initial data, such that*

$$-C_2 \le (\gamma_\epsilon)_t(\theta,t) \le -C_1 < 0, \qquad 0 \le t \le T. \tag{3.16}$$

Proof. We have shown in Corollary 2.4 that

$$\gamma_\epsilon(\theta,t) \ge e^{-b_1(t-t_0)}\gamma_\epsilon(\theta,t_0), \qquad t_0 \le t \le T$$

for some positive constant $b_1 > 0$. This implies that

$$(\gamma_\epsilon)_t(\theta,t_0) = \lim_{t\to t_0} \frac{\gamma_\epsilon(\theta,t)-\gamma_\epsilon(\theta,t_0)}{t-t_0} \ge \lim_{t\to t_0} \frac{e^{-b_1(t-t_0)}-1}{t-t_0} = -b_1$$

which implies the left side of inequality (**??**) with $C-2=b_1$. To prove the other side, let us recall from the proof of Corollary 2.6 that

$$\frac{d}{dt}\left(e^{2a_2(t-t_0)} f(e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t),\theta,t)\right) \ge 0$$

for some positive constants a_2 and b_2. This implies that

$$f(e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0),\theta,t) \ge e^{2a_2(t-t_0)}\, g(\gamma_\epsilon(\theta,t_0),\theta,t_0)$$

showing that

$$g(e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0),\theta,t) \ge e^{a_2(t-t_0)}\, g(\gamma_\epsilon(\theta,t_0),\theta,t_0) = \epsilon\, e^{-a_2(t-t_0)}.$$

To simplify the notation, set $P(\theta,t) = (e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0),\theta,t)$. Then, for any small number $\delta > 0$ we have

$$g(e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0)+\delta,\theta,t) \ge g(P(\theta,t)) + g_r(P(\theta,t))\,\delta + o(\delta^2)$$

where, by Corollary 2.4 $g_r(P(\theta,t)) \ge c > 0$. Hence,

$$g(e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0)+\delta,\theta,t) \ge \epsilon + c\,\delta + o(\delta^2) \ge \epsilon = g(\gamma_\epsilon(\theta,t_0),\theta,t_0)$$

provided $\delta \ge \frac{\epsilon\, e^{-a_2(t-t_0)}}{c}$. Set

$$\delta = \frac{\epsilon\, e^{-a_2(t-t_0)}}{c} = \frac{\epsilon a_2(t-t_0)}{c} + o(t-t_0).$$

Then it follows from the above, that

$$\gamma_\epsilon(\theta,t) \le e^{-b_2(t-t_0)}\gamma_\epsilon(\theta,t_0) + \delta$$

and hence, as $t \to t_0$ we have

$$\frac{\gamma_\epsilon(\theta,t)-\gamma_\epsilon(\theta,t_0)}{t-t_0} \le \frac{e^{-b_2(t-t_0)}-1}{t-t_0}\,\gamma_\epsilon(\theta,t_0) + \frac{\epsilon a_2}{c} + o(1)$$

implying that

$$\dot\gamma_\epsilon(\theta,t) \le -b_2\,\gamma_\epsilon(\theta,t_0) + \frac{\epsilon a_2}{c}.$$

Since $\gamma_\epsilon(theta,t_0) \ge r_0$ we can choose ϵ suffciently small depending only on r_0 so that $b_2 r_0 - \frac{\epsilon a_2}{c} \ge b_2 r_0$, proving that $\dot\gamma_\epsilon(\theta,t) \le -b_2\, r_0$ on $0 \le t \le T$. Setting $C_1 = b_2\, r_0$ right side of inequality (3.16) follows.

4. Second Order Derivative Estimates

In this section we will establish certain bounds on the Gauss Curvature $K = \det(D^2 f)/(1+|Df|^2)$ and the second derivatives of the functions f and g. We will assume as in the previous section that $g = \sqrt{2f}$ satisfies the hypotheses of Theorem 1.1 and that g is smooth up to the interface on $0 \le t \le T$. By Theorem [A] the function f is of class $C^{1,1}$ and satisfies

$$||f||_{C^{1,1}} \le C, \qquad \text{on } 0 \le t \le T \tag{4.1}$$

where C depends only on the initial data. The first results provides a bound from above and bellow on $\frac{K}{g}$. Its proof is an immediate consequence of Lemma 3.2 and Corollary 3.3.

Lemma 4.1. *Under the assumptions of Theorem 1.1, if* $B_{\rho_0} \subset \{(x,y) : g(x,y,T) = 0\}$ *and* g *is smooth up to the interface on* $0 \le t \le T$*, then there exists a constant* $c > 0$ *depending only on* ρ_0 *and the initial data, such that the Gauss Curvature* $K = \det D^2 f/(1+|Df|^2)$ *satisfies the bound*

$$0 < c \le \frac{K}{g} \le \frac{1}{c}, \qquad \text{on } 0 \le t \le T \tag{4.2}$$

Proof. It is enough to establish the bound (4.2) near the interface. Since $f_t = K/(1+|Df|^2)^{1/2}$, $g_t = f_t/g =$ and $|Df|$ is bounded above near the interface, it is enough to estimate g_t from above and below.

Using the notation of Corollary 3.3, let $r = \gamma_\epsilon(\theta,t)$ denote, in polar coordinates, the ϵ-level set of g. Differentiating the equation $g(\gamma_\epsilon(\theta),\theta,t) = \epsilon$ with respect to θ we obtain

$$g_r \cdot \dot\gamma_\epsilon(\theta,t) + g_t = 0$$

which implies that

$$g_t = -g_r \cdot \dot\gamma_\epsilon(\theta,t).$$

By Lemmas 3.1 and 3.2 and the convexity of the level sets of g we have

$$0 < c < g_r < c^{-1}$$

while by Corollary 3.3

$$-C_2 \leq \dot{\gamma}_\epsilon(\theta,t) \leq -C_1 < 0, \qquad 0 \leq t \leq T.$$

Multiplying the above inequalities we obtain the estimate

$$C_1\, c < g_t = g_r \cdot \dot{\gamma}_\epsilon(\theta,t) < C_2\, c^{-1}$$

which immediately implies the desired result.

As an immediate Corollary of the bound (4.2) we obtain:

Corollary 4.2. *Under the assumptions of Lemma 4.1 the time derivative g_t of the solution g of* (1.4) *satisfies the bound*

$$0 < c \leq g_t \leq c^{-1} \tag{4.3}$$

Lemma 4.3. *Under the assumptions of Theorem 1.1, if $B_{\rho_0} \subset \{(x,y) : g(x,y,T) = 0\}$ and g is smooth up to the interface on $0 \leq t \leq T$, then there exists a constant $C > 0$ for which*

$$0 < g_{\tau\tau} \leq C$$

with τ denoting the tangential direction to the level sets derivative g.

Proof. Since the level sets of g are strictly convex, the derivetive $g_{\tau\tau}$ is strictly positive. To establish the bound from above, we will use the maximum principle on the quantity

$$X = g_y^2 g_{xx} - 2 g_x g_y g_{xy} + g_x^2 g_{yy} + \Delta f.$$

Denoting by ν and τ respectively, the normal and tangetial direction to the level sets of g, we can express the quantity X as

$$X = (g + g_\nu^2)\, g_{\tau\tau} + (g\, g_{\nu\nu} + g_\nu^2).$$

We have shown in the previous section that

$$0 < c \leq g_\nu \leq c^{-1}, \qquad \text{on } g > 0,\ 0 \leq t \leq T$$

for some $c > 0$. In addition, because $f \in C^{1,1}$ the Laplacian Δf is bounded. Therefore, an upper bound on X will imply the desired upper bound on $g_{\tau\tau}$. The purpose of adding the term Δf on X is to be able to control the sign of the error terms on the evolution equation of X.

Since $X = g_t + \Delta f$ at the free-boundary $g = 0$, Corollary 4.2 implies that

$$X \leq C, \qquad \text{at} \quad g = 0.$$

Hence, we can assume that X attains its maximum at an interior point $P_0 = (x_0, y_0, t_0)$. Also, since X is rotationally invariant, we can assume, without loss of generality, that $g_y = 0$ at the point P_0, i.e. $g_\nu = g_x$ at P_0. Hence $X = (g + g_x^2) g_{yy} + g g_{xx} + (g g_{xx} + g_x^2)$ at P_0. Since X has a maximum at P_0 we have

$$X_x = g g_{xxx} + (g + g_x^2) g_{xyy} + 2 g_x \det D^2 g + 3 g_x g_{xx} + g_x g_{yy} = 0$$

and

$$X_y = g g_{xxy} + (g + g_x^2) g_{yyy} + 2 g_x g_{xy} = 0,$$

implying that

$$g_{xyy} = -\frac{g g_{xxx} + 2 g_x \det D^2 g + 3 g_x g_{xx} + g_x g_{yy}}{g + g_x^2} \tag{4.4}$$

and

$$g_{yyy} = -\frac{g g_{xxy} + 2 g_x g_{xy}}{g + g_x^2} \tag{4.5}$$

We next compute the evolution equation of X from the evolution equation of g to show that $X_t \le K\,X$, for some constant K, at the point P_0. This will easily imply that $X \le C$, on $0 \le t \le T$, as desired.

For the convenience of the reader, let us first present the computations in the simpler case where f satisfies the evolution Monge-Ampére equation

$$f_t = \det D^2 f$$

and hence $g = \sqrt{2f}$ satisfies the equation

$$g_t = g \,\det D^2 g + g_y^2 g_{xx} - 2 g_x g_y g_{xy} + g_x^2 g_{yy} \tag{4.6}$$

To compute the evolution of X we differentiate twice the equation (1.4). Denoting by L the opetator

$$LX := X_t - \{\, (g g_{yy} + g_y^2)\, X_{xx} - 2(g g_{xy} + g_x g_y)\, X_{xy} + (g g_{xx} + g_x^2)\, X_{yy} \,\}$$

we find, after many tedious calculations, that at the maximum point P_0 where $g_y = 0$ and (4.4) and (4.5) hold, we have

$$\begin{aligned} LX = & -\frac{4g}{g_x^2 + g} f_{xxx}^2 - \frac{12 g g_x (\det D^2 g)}{g_x^2 + g} f_{xxx} - 4 f_{xxy}^2 + \frac{6 g_x^3 g_{xy}}{g_x^2 + g} f_{xxy} \\ & + \left(-\frac{4g^2}{g_x^2 + g} + 4g - 4g_x^2 \right) g_{yy}^2 + \left(-\frac{4g^2}{g_x^2 + g} + 3g_x^2 + 4g \right) (\det D^2 g)\, g_{yy} \\ & + \left(\frac{8g^2}{g_x^2 + 4} - 12g \right) (\det D^2 g)^2 \end{aligned}$$

After some more calculations and cancellations we obtain:

$$\begin{aligned}LX = & -\frac{4g}{g_x^2+g} f_{xxx}^2 - \frac{12 g g_x (\det D^2 g)}{g_x^2+g} f_{xxx} - 4 f_{xxy}^2 + \frac{6 g_x^3 g_{xy}}{g_x^2+g} f_{xxy} \\ & - 4 g_x^4 g_{yy}^2 + \left(-\frac{4g}{g_x^2+g}+1\right) (g \det D^2 g)\, g_{yy} + 3(g_x^2+g)\,(\det D^2 g)\, g_{yy} \\ & + \left(\frac{-4g^2 - 12 g g_x^2}{g_x^2+g}\right) (\det D^2 g)^2\end{aligned}$$

Rearranging the right hand side of the above inequality and deleting negative terms, we find that at the maximum point P_0, we have:

$$\begin{aligned}LX \le & -\frac{3g}{g_x^2+g} f_{xxx}^2 - \frac{12 g g_x (\det D^2 g)}{g_x^2+g} f_{xxx} - \frac{12 g g_x^2}{g_x^2+g} (\det D^2 g)^2 \\ & - 3 f_{xxy}^2 + \frac{6 g_x^3 g_{xy}}{g_x^2+g} f_{xxy} - \frac{3 g_x^6 g_{xy}^2}{(g_x^2+g)^2} \\ & + \left(-\frac{4g}{g_x^2+g}+1\right) (g \det D^2 g)\, g_{yy} \\ & + \frac{3 g_x^6 g_{xy}^2}{(g_x^2+g)^2} + 3(g_x^2+g)\,(\det D^2 g)\, g_{yy}\end{aligned}$$

The sum of the terms in the first two lines is equal to

$$-\frac{3g\,(f_{xxx} + 2 g_x \det D^2 g)^2}{g_x^2+g} - 3\left(f_{xyy} - \frac{g_x^3 g_{xy}}{g_x^2+g}\right)^2$$

which is negative. To estimate the rest of the terms, we use the estimate $c \le g_t \le c^{-1}$, proven in Corollary ..., implying that

$$c \le g\,(\det D^2 g) + g_x^2 g_{yy} \le c^{-1}$$

at the point P_0. Since we are trying to estimate the maximum of $X = g_x^2 g_{yy} + \Delta f$ from above, and Δf is bounded, we can assume, with no loss of generality, that at the point P_0

$$c^{-1} \le \frac{g_x^2 g_{yy}}{2} \le X \le 2\, g_x^2 g_{yy}$$

implying thst

$$g\,(\det D^2 g) \le -g_x^2 g_{yy} + c^{-1} \le -\frac{g_x^2 g_{yy}}{2}. \tag{4.7}$$

Therefore

$$\left(-\frac{4g}{g_x^2+g}+1\right) (g \det D^2 g)\, g_{yy} \le C\, X.$$

To estimate the last term, we will first bound the derivative g_{xy} using the inequality

$$0 \le g\,(\det D^2 g) + g_x^2 g_{yy} = (g_x^2 + g g_{xx})\, g_{yy} - g\, g_{xy}^2$$

which shows that

$$g\, g_{xy}^2 \le (g_x^2 + g g_{xx})\, g_{yy} = f_{xx}\, g_{yy} \le C\, g_{yy}. \tag{4.8}$$

¿From (4.7) and (4.8) we obtain the bounds

$$\det D^2 g \le -\frac{g_x^2 g_{yy}}{2g} \qquad \text{and} \qquad g_{xy}^2 \le \frac{C\, g_{yy}}{g}.$$

Hence,

$$\frac{3 g_x^6 g_{xy}^2}{(g_x^2+g)^2} + 3(g_x^2+g)\, (\det D^2 g)\, g_{yy} \le \frac{3}{g}\, (C_1\, X - C_2\, X^2)$$

where C_1 and C_2 are positive constants which depend only on c. Hence, we can make this term negative by assuming that $X \ge C_1/C_2$ at P_0. Combining the above bounds we conclude that, unless $X(P_0) \le K$ for some absolute constant K, we have $LX \le C\, X$ at the point P_0. Since,

$$(g g_{yy} + g_y^2)\, X_{xx} - 2(g g_{xy} + g_x g_y)\, X_{xy} + (g g_{xx} + g_y^2)\, X_{yy} \le 0$$

at P_0 this readily implies that $X_t \le C\, X$ at P_0 which is the desired bound.

We will know present the computations in the case of the Gauss Curvature Flow. To simplify the notation, let us set $I = 1 + g^2\, g_x^2$ and $J = g + g_x^2$. Also, until the end of this proof, C will denote various constants depending only on $||g||_{C^1}$ and $||f||_{C^{1,1}}$. Denoting by LX the operator

$$LX := X_t - I^{-3/2} \{\, (g g_{yy} + g_y^2)\, X_{xx} - 2(g g_{xy} + g_x g_y)\, X_{xy} + (g g_{xx} + g_x^2)\, X_{yy}\, \}$$

we find, after several calculations, that at the maximum point P_0, X satisfies the inequality

$$\begin{aligned} LX \le & -\frac{4g\, f_{xxx}^2}{J\, I^{3/2}} + \left(\frac{-12 g g_x (\det D^2 g)}{J\, I^{3/2}} + O(g) \right) f_{xxx} - \frac{4 f_{xxy}^2}{I^{3/2}} + \left(\frac{6 g_x}{I^{3/2}} + O(g) \right) g_{xy} f_{xxy} \\ & + C\, g_{yy} + \left(\frac{-4J}{I^{5/2}} + O(g) \right) g_{yy}^2 + \left(\left\{ \frac{3J}{I^{3/2}} + O(g) \right\} (\det D^2 g) + O(g) \right) g_{yy} \\ & + \left(\frac{-12 g g_x^2}{J I^{3/2}} + \frac{-4 g^2}{J I^{3/2}} + O(g^3) \right) (\det D^2 g)^2 + C\, g^2 |\det D^2 g| + C\, g^2. \end{aligned}$$

Here, and until the end of this proof, C will denote various constants depending only on $||g||_{C^1}$ and $||f||_{C^{1,1}}$. Also, we have denoted by $O(g)$ various terms satifsying $|O(g)| \le C\, g$. Completing the squares, as in the case of the evolution Monge-Ampére equation shown above, we find after many calculations that:

$$
\begin{aligned}
LX \leq & -\frac{3g}{J\,I^{3/2}}(f_{xxx}+2g_x)^2 - \frac{3}{I^{3/2}}\,(f_{xxy}+g_x g_{xy})^2 \\
& \left(-\frac{g f_{xxx}^2}{J\,I^{3/2}} + O(g)\, f_{xxx}\right) + \left(-\frac{f_{xxy}^2}{I^{3/2}} + O(g)\, g_{xy} f_{xxy}\right) \\
& + C\, g_{yy} + \left(\frac{-4J}{I^{5/2}} + O(g)\right) g_{yy}^2 + \left(\left\{\frac{3J}{I^{3/2}} + O(g)\right\} (\det D^2 g) + O(g)\right) g_{yy} \\
& + \left(\frac{-4g^2}{JI^{3/2}} + O(g^3)\right) (\det D^2 g)^2 + \frac{3 g_{xy}^2 g_x^2}{I^{3/2}} + C\, g^2 |\det D^2 g| + C\, g^2.
\end{aligned}
$$

The two terms on the first line are negative. Also,

$$
-\frac{g f_{xxx}^2}{J\,I^{3/2}} + O(g)\, f_{xxx} \leq -\frac{g f_{xxx}^2}{2J\,I^{3/2}} + Cg \leq C
$$

on $g \leq 1$ while

$$
-\frac{f_{xxy}^2}{I^{3/2}} + O(g)\, g_{xy} f_{xxy} \leq -\frac{f_{xxy}^2}{2I^{3/2}} + C g g_{xy} \leq C.
$$

Most of the remaining terms are either negative or can be estimated from $X = g_x^2\, g_{yy}$ and the bounds $0 < c \leq g_x^2$ and $|Dg| \leq C$, $|g\, D^2 g| \leq C$. We end up with the inequality

$$
LX \leq C\,(X+1) + 3I^{3/2}\, \left(J(\det D^2 g) + g_{xy}^2 g_x^2 g_{yy}\right).
$$

The last term can be shown to be nonnegative, exactly as in the case of the Monge-Ampére equation, using the estimate

$$
c \leq \frac{g(\det D^2 g) + g_x^2 g_{yy}}{I^{3/2}} \leq c^{-1}
$$

and provided that $X \geq C$ is sufficiently large. We conclude that at the maximum point P_0, either $X \leq C$, with C depeding only on $||g||_{C^1}$ and $||f||_{C^{1,1}}$, or

$$
LX \leq C\, X.
$$

This readily imlplies that $X_t \leq C\, X$ at P_0, from which the desired estimate follows.

Corollary 4.4. *Under the hypotheses of Lemma 4.3, there exist a constant $c > 0$ depending only on ρ and the initial data, for which the*

$$
g_{\tau\tau} \geq c > 0
$$

with τ denoting the tangential direction to the level sets of g.

Proof. We have shown in Lemma 4.1 that

$$\det D^2 f \geq c\, g.$$

We can express $\det D^2 f$ as $\det D^2 f = f_{\nu\nu} f_{\tau\tau} - f_{\nu\tau}^2$, where ν and τ denote the normal and tangential directions to the level sets of g respectively. Then

$$f_{\nu\nu} f_{\tau\tau} \geq c + f_{\nu\tau}^2 \geq c\, g$$

which imples the bound

$$f_{\tau\tau} \geq \frac{c\, g}{f_{\nu\nu}} \geq \tilde{c}\, g$$

since $f_{\nu\nu} \leq C$. Since $f_{\tau\tau} = g\, g_{\tau\tau} + g_\tau^2 = g\, g_{\tau\tau}$ we conclude that $g_{\tau\tau} \geq \tilde{c}$, for some positive constant $\tilde{c}$ depending only on the initial data and ρ.