ALL TIME EXISTENCE OF A SMOOTH SOLUTION FOR THE ONE-PHASE FREE BOUNDARY PROBLEM OF THE FLAME TYPE

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ABSTRACT. We consider the one-phase free boundary problem of the flame type for the heat equation: find (u,Ω) such that $\Omega = \{u>0\} \subset Q_T = \mathbb{R}^n \times (0,T)$ and

$$\begin{cases} u_t = \triangle u & \text{in } \Omega \\ u = 0 \quad \text{and} \quad |\nabla u| = 1 & \text{at } \partial\Omega \\ u(x,0) = u_o(x) & \text{on } \overline{\Omega_o}. \end{cases}$$

Under the condition that Ω_o is *convex* and $\log u_o$ is *concave*, we show that the convexity of $\Omega(t) = \{\, u(\cdot,t) > 0 \,\}$ and the concavity of $\log u(x,t)$ are preserved under the flow for $0 \le t \le T$, as long as $\partial\Omega(t)$ and u on $\overline{\Omega(t)}$ are smooth. As a consequence, we show the existence of a smooth up to the interface solution, on $0 < t < T_o$, with T_o denoting the extinction time of $\Omega(t)$.

1. INTRODUCTION

We consider in this paper the one-phase free boundary problem for the heat equation, describing the laminar flames as an asymptotic limit for the high activation energy [CV]. The classical formulation of the problem is the following: for a given initial data u_o whose positivity region is $\Omega_o = \{\, u_o > 0 \,\}$, find a domain $\Omega \subset Q_T = \mathbb{R}^n \times [0,T]$ and a function u which is smooth on Ω, up to $\partial\Omega$, such that

$$\begin{cases} u_t = \triangle u & \text{in } \Omega \\ u = 0 \quad \text{and} \quad |\nabla u| = 1 & \text{at } \partial\Omega \\ u(x,0) = u_o(x) & \text{on } \overline{\Omega_o} \end{cases} \qquad \textbf{(FB)}$$

We say the pair (u,Ω) is a solution of the problem **(FB)**. In general, the overdetermined boundary condition in **(FB)** does not give a solution

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on a fixed domain. Determining the domain Ω so that the solution u satisfies this overdetermined condition at $\partial\Omega$, is part of solving the problem. Hence, the problem **(FB)** is a *free boundary problem*. In addition, since u satisfies the heat equation only in the positive region $\Omega = \{ u > 0 \}$, without any condition on the possible negative region $\{u < 0\}$, **(FB)** is called *one-phase* free boundary problem.

Problem **(FB)**, in the mathematical frame work, was introduced by L. Caffarelli and J.L. Vasquez [CV]. It describes the propagation of the premixed equi-diffusional flames in the limit of high activation energy. The region $\Omega(t) = \{ x | \, u(x,t) > 0 \}$ represents the *unburnt (fresh) zone* at time t, the free boundary $\partial\Omega(t)$ corresponds to the *flame front* and $u = c(T_c - T)$ is the *normalized temperature*. We refer to the paper [V] of J.L. Vasquez, for further details.

The existence of a weak solution of the problem **(FB)** and the upper bound for the gradient of the weak solution, under suitable conditions on the initial data, has been established by L. Caffarelli and J.L. Vasquez in [CV]. Their techniques are based upon the singular pertubation method. The short time existence of a smooth solution of **(FB)**, before any singularities occur, has been shown by D. Andreucci and R. Gianni in [AG]. However, since advancing free boundaries may hit each other at certain time, one cannot expect the higher regularity of a weak solution for all times, without imposing any further geometric assumptions on the initial data. In other words, to avoid the development of singularities at all times, up to the extinction of the flame, certain geometric assumptions on the initial condition must be imposed.

We will prove in this paper, using the maximum principle, that the concavity of $\log u$ (i.e. log-concavity of u) in **(FB)** is preserved, under the assumption that the solution is smooth up to the inteface. As a consequence, we will show that for a given smooth and log-concave initial data, there exists a solution u in **(FB)** which is smooth up to the interface, up to the extinction time T_o of the flame.

A. Petrosyan [P] has recently shown, using very different techniques, that the log-concavity of a solution is preserved for a short time.

It is worth mentioning that a similar situation occurs in the porous medium equation

$$v_t = v\triangle v + r|\nabla v|^2, \qquad r > 0, \tag{1.1}$$

satisfied by the pressure v of a gas propagating through a porous medium: advancing free boundaries may hit each other creating singularities. However, it has been proved by Daskalopoulos, Hamilton and Lee in [DHL] that the concavity of $\sqrt{v}$ (i.e. root-concavity of v) is preserved under the flow and as a consequence solutions v with root-concave initial data remain smooth up to the interface at all times. In particular, the free-bounadry is smooth and $\partial v(\cdot, t)$ is convex, for all t, since it is the level set of $\sqrt{v}$.

Let u be the weak solution of **(FB)** given by L. Caffarelli and J.L. Vasquez [CV]. Set $\Omega = \{\,(x,t)|\,u(x,t) > 0\,\}$ and $\Omega(t) = \{\,x|\,u(x,t) > 0\,\}$. We impose the following conditions on the initial data:

Condition I: *The initial data $u_o(x)$ is compactly supported, smooth on the closure of its support $\Omega_o = \{\,x|\,u_o(x) > 0\}$, and log-concave (i.e. $\log u_o$ is concave).*

Since we are interested in the regularity up to just before the extinction of the fire, we also assume that:

Condition II: *There is $\rho_o > 0$ such that*

$$B_{\rho_o} \subset \Omega(t) \qquad \text{for } t \in [0, T_o]. \tag{1.2}$$

Our Main Theorem states as follows:

Theorem 1.1 (**Main Theorem**). *Under Conditions I,II, the weak solution u of* **(FB)** *is smooth and log-concave in $\overline{\Omega} \subset Q_{T_o}$. In particular, the free boundary $\partial\{\,u > 0\,\}$ is smooth in the space and time.*

For the proof of the Main Theorem, we will first establish uniform upper and lower bounds on the gradient $|\nabla u|$, when the solution u is log-concave and smooth up to the free boundary. This estimate is independent of the time interval. Second, we will show that the log-concavity of the solution $u(\cdot, t)$ is preserved as long as the solution u is smooth on its support and the free-noundary is smooth. Using the convexity of the free boundary we will next prove that the second derivatives of the solution are bounded. Let T be the first time when the smoothness breaks out. We will then show in the last section, that the uniform gradient and second derivative bounds imply that u is actually smooth on $\overline{\Omega(T)}$ at $t = T$, so that u will remain smooth for a while by the short time result. This will establish the C^∞-regularity of the solution, on $0 \le t < T_o$.

2. A-priori estimates for the gradient and u_t

Throughout this section we will assume that u is smooth up to the interface solution of **(FB)** with an initial data satisfying Conditions I and II imposed in the Introduction.

Lemma 2.1. *There exists a constant $C(u_o)$ for which*

$$|\nabla u| \le C(u_o), \qquad \text{on } \Omega.$$

Proof. Let $Z = |\nabla u|^2$. Then

$$Z_t = \triangle Z - 2\sum_{ij} |u_{ij}|^2 \le \triangle Z$$

by a simple computation. Hence, Z is a subsolution of the heat equation. By the maximum principle, the maximum of Z will be achieved at the free boundary or at $t = 0$. On the other hand, $Z = 1$ on the free boundary $\partial\Omega(t)$, by the given free-boundary $|\nabla u| = 1$. Hence Z is bounded by $\sup |\nabla u_o|^2 + 1$. □

Lemma 2.2. *If $\log u(\cdot, t)$ is concave for $t \in [0, T]$, then*

$$|\nabla u(x,t)| > c_o(u_o, \rho_o, \min_{0\le t\le T} \max_{x\in\Omega(t)} u(x,t))$$

for all $t \in [0, T]$.

Proof. We may assume, without loss of generality, that

$$\min_{0\le t\le T} \max_{x\in\Omega(t)} u(x,t) = 2.$$

Then, the log-concavity of u implies that

$$|\nabla u(x,t)| \ge 2c_o > 0, \qquad \text{on } \partial\{u \ge 1\}, \tag{2.1}$$

for some uniform $c_o > 0$. Moreover, each level set of $u(\cdot, t)$ convex and contains a uniform ball B_{ρ_o}, from Condition II. Hence, the normal direction to each level set of $u(\cdot, t)$ at every point is tilting from the radial direction with an angle uniformly away from the right angle. Therefore $|x \cdot \nabla u|$ is comparable to $|\nabla u|$.

Let $Z = (x \cdot \nabla u)^2 - c|\nabla u|^2$. By (2.1), $Z \geq c_o$ on $\partial\{u > 1\}$ and $\partial\Omega(t)$, when $Z = (x \cdot \nabla u)^2 - c|\nabla u|^2$, with $c > 0$ a uniformly small constant. Let us assume that the minimum of $Z(x,t)$ on $0 \leq t \leq T$, $x \in \Omega(t) \backslash \{u > 1\}$ is attained at an interior point (x_o, t_o). By rotating the coordinates, we may assume that e_n is the outward normal direction to the level set $\{x | \, u(x, t_o) = u(x_o, t_o)\}$ at x_0 and that $e_1, \cdots, e_{n-1}$ are its tangential directions. Then $u_i(x_o, t_o) = 0$ for $i = 1, \cdots, n-1$, $x_n \approx |x_o| \geq \rho_o$ and $Z \approx |\nabla u|^2 \approx u_n^2$. At the minimum point (x_o, t_o) of Z, we have

$$Z_j = 2(x \cdot \nabla u)\,(x \cdot \nabla u_j + u_j) - 2c\nabla u \cdot \nabla u_j = 0, \qquad j = 1, \cdots, n.$$

If $j \neq n$, then $x_n u_n \sum_{i=1}^n x_i u_{ij} - c u_n u_{nj} = 0$ implying

$$u_{nj} = \frac{-1}{(x_n^2 - c)} \sum_{i \neq n} x_i x_n u_{ij}, \qquad \text{for } j \neq n. \tag{2.2}$$

If $j = n$, then $x_n u_n (\sum_{i=1}^n x_i u_{in} + u_n) - c u_n u_{nn} = 0$ implying

$$u_{nn} = \frac{-1}{(x_n^2 - c)} \sum_{i \neq n} x_n x_i u_{in} - \frac{x_n u_n}{(x_n^2 - c)}.$$

We conclude that

$$u_{nn} = \frac{1}{(x_n^2 - c)^2} \sum_{i,j \neq n} x_n^2 x_i x_j u_{ij} - \frac{x_n u_n}{(x_n^2 - c)}. \tag{2.3}$$

Notice that by choosing the constant c sufficiently small, we can make $x_n - c > c/2$. On the other hand, a direct computation shows that

$$\begin{aligned} \triangle Z &= (2x \cdot \nabla u)\, [\, x \cdot \nabla(\triangle u) + 2\triangle u\,] \\ &\quad + \sum_j (x \cdot \nabla u_j + u_j)^2 - 2c\nabla u \cdot \nabla(\triangle u) - 2c\sum_{i,j} u_{ij}^2. \end{aligned}$$

Hence

$$\begin{aligned}
Z_t &= 2(x\cdot\nabla u)(x\cdot\nabla(\triangle u)) - 2c\nabla u\cdot\nabla(\triangle u)\\
&= \triangle Z - (4x\cdot\nabla u)\,(\triangle u) - \sum_j (x\cdot\nabla u_j + u_j)^2 + 2c\sum_{ij} u_{ij}^2\\
&= \triangle Z - 4x_n u_n \sum_{i\neq n} u_{ii} - \sum_{j\neq n}(\sum_{i\neq n} x_i u_{ij} + x_n u_{nj})^2 + 2c\sum_{i,j\neq n} u_{ij}^2\\
&+ 4c\sum_{j\neq n} u_{nj}^2 - 4x_n u_n u_{nn} - (\sum_{j\neq n} x_j u_{nj} + x_n u_{nn} + u_n)^2 + 2c u_{nn}^2\\
&\equiv \triangle Z + P(u_{ij}).
\end{aligned}$$

Expressing, $P(u_{ij})$ in the form

$$P(u_{ij}) = \sum_{i,j} A_{ij} u_{ij}^2 + \sum_{(i,j)\neq(k,l)} B_{ijkl} u_{ij} u_{kl} + \sum_{i,j} C_{ij} u_{ij} + D, \quad i,j = 1,...,n-1$$

we find by a direct differentiation and using (2.2) and (2.3) that

$$\begin{aligned}
\frac{\partial P}{\partial u_{ij}} =& -4x_n u_n \delta_{ij} - 2\left(\sum_{i\neq n} x_i u_{ij} + x_n u_{nj}\right)\left(x_i - \frac{x_n^2 x_i}{(x_n^2 - c)}\right) + 4c u_{ij}\\
&+ 8c u_{nj}\frac{-x_n x_i}{(x_n^2 - c)} + 4c u_{nn}\frac{x_n^2 x_i x_j}{(x_n^2 - c)^2} - 4x_n u_n \frac{x_n^2 x_i x_j}{(x_n^2 - c)^2}\\
&- 2\left(\sum_{j\neq n} x_j u_{nj} + x_n u_{nn} + u_n\right)\left(\frac{c x_n x_i x_j}{(x_n^2 - c)^2}\right).
\end{aligned}$$

Differentiating once more, we find that

$$\frac{\partial^2 P}{\partial u_{ij}^2} = -2\frac{x_i^2 c^2}{(x_n^2 - c)^2} + 4c + 8c\frac{x_n^2 x_i^2}{(x_n^2 - c)^2} + 4c\frac{x_n^4 x_i^2 x_j^2}{(x_n^2 - c)^4} - 2\left(\frac{c x_n x_i x_j}{(x_n^2 - c)^2}\right)^2.$$

Hence

$$A_{ij} = \frac{1}{2}\frac{\partial^2 P}{\partial u_{ij}^2} = 2c + 4c\frac{x_i^2}{(x_n^2 - c)} + 3c^2\frac{x_i^2}{(x_n^2 - c)^2} + 2c\frac{x_n^2 x_i^2 x_j^2}{(x_n^2 - c)^3} + c^2\frac{x_n^2 x_i^2 x_j^2}{(x_n^2 - c)^4}.$$

Also, for $(i,j)\neq(k,l)$, we have

$$\frac{\partial^2 P}{\partial u_{ij} u_{kl}} = -2c^2\frac{x_i x_k \delta_{lj}}{(x_n^2 - c)^2} + 8c\frac{x_n^2 x_i x_k \delta_{lj}}{(x_n^2 - c)^2} + 4c\frac{x_n^4 x_i x_j x_k x_l}{(x_n^2 - c)^4} - 2c^2\frac{x_n^2 x_i x_j x_k x_l}{(x_n^2 - c)^4}$$

implying that

$$B_{ijkl} = \frac{\partial^2 P}{\partial u_{ij} u_{kl}} = 8c\frac{x_i x_k \delta_{lj}}{(x_n^2 - c)} + 6c^2\frac{x_i x_k \delta_{lj}}{(x_n^2 - c)^2} + 4c\frac{x_n^2 x_i x_j x_k x_l}{(x_n^2 - c)^3} + 2c^2\frac{x_n^2 x_i x_j x_k x_l}{(x_n^2 - c)^4}.$$

Moreover

$$C_{ij} = \frac{\partial P}{\partial u_{ij}}|_{u_{ij}=0} = \tilde{C}_{ij}\, u_n$$

with

$$\tilde{C}_{ij} = -4x_n\delta_{ij} + 2c\frac{x_n^2 x_i}{(x_n^2-c)^2} - 4c\frac{x_n^3 x_i x_j}{(x_n^2-c)^3} - 4\frac{x_n^3 x_i x_j}{(x_n^2-c)^2} + 2c^2\frac{x_n x_i x_j}{(x_n^2-c)^3}$$

and

$$D = \left[\frac{c}{x_n^2-c} + \frac{cx_n^2}{(x_n^2-c)^2} + 8\frac{x_n^2}{(x_n^2-c)}\right] u_n^2 = \tilde{D}\, u_n^2.$$

Combining all the above we find that, at the point (x_o, t_o), where $\triangle Z \geq 0$, we have

$$\begin{aligned} Z_t =& \triangle Z + 2c\sum_{i,j\neq n} u_{ij}^2 + \left[\frac{4c}{(x_n^2-c)} + \frac{3c^2}{(x_n^2-c)^2}\right]\sum_{j\neq n}\left(\sum_{i\neq n} x_i u_{ij}\right)^2 \\ &+ \left[\frac{2cx_n^2}{(x_n^2-c)^3} + \frac{c^2x_n^2}{(x_n^2-c)^4}\right]\left(\sum_{i,j\neq n} x_i x_j u_{ij}\right)^2 + \sum_{i,j\neq n}\tilde{C}_{ij}u_n u_{ij} + \tilde{D}\, u_n^2 \\ \geq& 2c\sum_{i,j\neq n} u_{ij}^2 + \sum_{i,j\neq n}\tilde{C}_{ij}u_n u_{ij} + \tilde{D}\, u_n^2 \end{aligned}$$

By Cauchy-Schwarz, we finally conclude that

$$Z_t \geq -C\, u_n^2 = -C\, Z^2$$

at the point minimum point (x_o, t_o) of Z, where C is a constant depending only on the initial condition u_o and the number ρ. This implies that the interior minimum is uniformly bounded by a positive constant $c_o(u_o, \rho_o) > 0$. On the other hand, $Z \geq c_o > 0$ on $\partial(\Omega(t)\backslash\{u > 1\})$. Hence the conclusion follows from the maximum principle. $\square$

Lemma 2.3. *There exists a constant $C(u_o, \rho_o)$, for which*

$$|u_t(x,t)| = |\triangle u(x,t)| \leq C(u_o, \rho_o), \qquad \textit{on } \Omega.$$

Proof. Set

$$Z = e^{\alpha x\cdot\nabla u}\, \triangle u - \beta|\nabla u|^2$$

for some constants α and β to be determined later. If $|Z|$ is bounded above, then so is $|\triangle u|$, since $|\nabla u|$ is bounded. Let us assume that the minimum (maximum) of Z is achieved at the point (x_o, t_o) and that

$$Z(x_o, t_o) = M.$$

We first consider the case that (x_o, t_o) is an interior point i.e., it belongs in $\Omega(t_o)$. Then at (x_o, t_o), we have

$$Z_i = [(\triangle u)_i + \alpha(x \cdot \nabla u_i + u_i)]\, e^{\alpha x \cdot \nabla u} - 2\beta \nabla u \cdot \nabla u_i = 0$$

which implies that

$$\triangle u_i = -\alpha\,(x \cdot \nabla u_i + u_i) + 2\beta\, e^{-\alpha x \cdot \nabla u} \nabla u \cdot \nabla u_i. \tag{2.4}$$

The evolution equation for Z is:

$$Z_t = \triangle Z + 2\beta |D^2 u|^2 + e^{\alpha x \cdot \nabla u} \left[-\alpha (\triangle u)^2 - (\triangle u) \sum_i (\alpha x \cdot \nabla u_i + \alpha u_i)^2 \right]$$
$$+ e^{\alpha x \cdot \nabla u} \left[-2 \sum_i (\triangle u_i)\,(\alpha x \cdot \nabla u_i + \alpha u_i) \right]$$

and using (2.4) it becomes:

$$Z_t = \triangle Z + 2\beta |D^2 u|^2 - \alpha e^{\alpha x \cdot \nabla u} (\triangle u)^2 - \alpha^2 e^{\alpha x \cdot \nabla u} (\triangle u) \sum_i (x \cdot \nabla u_i + u_i)^2$$
$$+ 2\alpha^2 \sum_i (x \cdot \nabla u_i + u_i)^2 e^{\alpha x \cdot \nabla u} - 4\alpha\beta \sum_i \nabla u \cdot \nabla u_i\,(x \cdot \nabla u_i + u_i). \tag{2.5}$$

At the point (x_o, y_o) we have:

$$e^{\alpha x \cdot \nabla u}\, \triangle u = M + \beta\, |\nabla u|^2.$$

Also, we have

$$|x \cdot \nabla u_i + u_i\,| \le C\,(|D^2 u| + |\nabla u|)$$

and

$$|\Delta u| \le |\; D^2 u|$$

while, from Lemma 2.1, $|\nabla u|$ satisfies the bound

$$|\nabla u| \le C$$

for a constant C which depends only on the initial data. Hence, equation (2.5) implies that

$$Z_t = \triangle Z + 2\beta |D^2 u|^2 - \alpha^2 (M + \beta\, |\nabla u|^2) \sum_i (x \cdot \nabla u_i + u_i)^2 + R \tag{2.6}$$

with remainder

$$\begin{aligned} R = & -\alpha e^{\alpha x \cdot \nabla u} (\triangle u)^2 + 2\alpha^2 \sum_i (x \cdot \nabla u_i + u_i)^2 e^{\alpha x \cdot \nabla u} \\ & - 4\alpha\beta \sum_i \nabla u \cdot \nabla u_i \, (x \cdot \nabla u_i + u_i) \end{aligned} \tag{2.7}$$

satisfying the bound

$$|R| \le C\, \alpha\, (1 + \alpha + |\beta|)\, (|D^2 u|^2 + |\nabla u|^2) \tag{2.8}$$

for a constant C which depends only on the initial data and $\alpha > 0$ and β constants to be chosen in the sequel.

Let us first assume that (x_o, t_o) is an interior minimum of Z and that

$$Z(x_o, t_o) = \min_{x \in \Omega(t), 0 \le t \le T} Z(x, t) = M < 0.$$

Then $\triangle u \ge 0$ and $Z_t \le 0$ at (x_o, t_o). Let α and β be positive constants to be chosen to depend only on the initial data. We may assume, without loss of generality, that $M + \beta\, |\nabla u|^2 \le 0$ and that $|\nabla u| \le |D^2 u|$. If either of the two inequalities fail, then the desired lower bound on Z follows from the upper bound $|\nabla u| \le C$. Hence, from (2.6) and (2.7) we conclude that at the point (x_o, t_o):

$$0 \ge Z_t \ge 2\beta |D^2 u|^2 - C\, \alpha\, (1 + \alpha + |\beta|)\, |D^2 u|^2.$$

This leads us to a contradiction, by choosing α and β positive so that

$$2\beta - C\, \alpha\, (1 + \alpha + |\beta|) > 0.$$

Similar arguments show that Z cannot attain an interior maximum at (x_o, t_o), if $\alpha > 0$ and $\beta < 0$ are chosen appropriately, so that

$$2\beta + C\, \alpha\, (1 + \alpha + |\beta|) < 0.$$

Let us now consider the case where the minimum (maximum) point (x_o, t_o) of Z lies on the free boundary, i.e., $x_o \in \partial\Omega(t_o)$. We may assume, without loss of generality, that e_n is the outward normal direction to $\partial\Omega(t_o)$ at x_o and that $e_i, 1 \leq i \leq n-1$ are its tangential directions. The condition at free boundary may be expressed as $|\nabla u(x', \gamma(x',t)|^2 = 1$, where $x_n = \gamma(x',t)$ is the graph of the free boundary around (x_o, t_o). This in particular implies that $u_n = -1$ at (x_o, t_o). Notice also that, since $\Omega(t)$ is convex and contains the ball B_{ρ_o}, we have $x_n \approx |x| > 0$. Differentiating the $|\nabla u(x', \gamma(x',t)|^2 = 1$ with respect to time t, and using that $\gamma_t(x',t) = -u_t/u_n = \triangle u$ at (x_o, t_o), we obtain the idendity

$$\sum_{1\leq i\leq n-1} u_i \triangle u_i + u_n \triangle u_n + \triangle u \left(\sum_{1\leq i\leq n-1} u_i\, u_{in} + u_n u_{nn} \right) = 0.$$

Using the fact that $u_i = 0$ for $1 \leq i \leq n-1$, we conclude that

$$(\triangle u)_n + u_n u_{nn} \triangle u = 0 \qquad \text{or} \qquad (\triangle u)_n = u_{nn} \triangle u \tag{2.9}$$

at the point (x_o, t_o). Assume first that (x_o, t_o) is a minimum point. We may assume without loss of generality that at the point (x_o, t_o), $\Delta u = M < 0$. Hence, at this point, where also $u_n = -1$, $u_i = 0$, $u_{in} = 0$, for $1 \leq i \leq n-1$, we will have

$$\begin{aligned} 0 \geq Z_n &= [\triangle u_n + \alpha \triangle u\,(x \cdot \nabla u_n + u_n)]\; e^{\alpha x\cdot\nabla u} - 2\beta u_n u_{nn} \\ &= u_{nn} \left[(\alpha x_n + 1) \triangle u\, e^{-\alpha x_n} + 2\beta \right] - \alpha e^{-\alpha x_n} \triangle u \\ &= -u_{nn} \left[(\alpha x_n + 1)|M| e^{-\alpha x_n} - 2\beta \right] + \alpha e^{-\alpha x_n} |M|. \end{aligned}$$

Since, $x_n \equiv |x| \geq \rho_0$, in the case where for $u_{nn} \leq 0$, we can make

$$-u_{nn} \left[(\alpha x_n + 1)|M| e^{-\alpha x_n} - 2\beta \right] + \alpha e^{-\alpha x_n} |M| > 0$$

deriving a contradiction.

It remains to consider the case when $u_{nn} > 0$ at (x_0, t_0). Let us observe that since $u_i = 0, u_{ni} = 0$ for $1 \leq i \leq n-1$ at (x_o, t_o), we also have $(|\nabla u|^2)_i = 0$ for $1 \leq i \leq n-1$ at x_o. On the other hand, conditions $u_n = -1, u_{nn} > 0$ and $u_{ni} = 0$ at (x_o, t_o), imply that $(|\nabla u|^2)_n < 0$. Hence, by the convexity of the free boundary and the fact that $u_{ni} = 0$ for $1 \leq i \leq n-1$, $|\nabla u|^2$ achieves its local minimum at x_o. This implies that $\tilde{Z} = Z\,|\nabla u|^{2\mu}$, $\mu > 0$, still attains its local minimum at x_o. Therefore, assuming that at (x_o, t_o), $\triangle u = M < 0$, we conclude

$$\begin{aligned}
0 \geq \tilde{Z}_n = & \left[\left(\triangle u_n + \alpha \triangle u \, (x \cdot \nabla u_n + u_n) \right) e^{\alpha \cdot \nabla u} - 2\beta u_n u_{nn} \right] |\nabla u|^{2\mu} \\
& + 2\mu Z \, |\nabla u|^{2\mu - 2} \nabla u \cdot \nabla u_n \\
& = u_{nn} \left[(\alpha x_n + 1) \triangle u \, e^{-\alpha x_n} + 2\beta \right] - \alpha e^{-\alpha x_n} \triangle u - 2\mu Z \, u_{nn} \\
& \geq u_{nn} \left[-(\alpha x_n + 1)|M| e^{-\alpha x_n} + 2\beta + 2\mu |M| e^{-\alpha x_n} \right] \\
& \geq u_{nn} |M| e^{-\alpha x_n} \left[-(\alpha x_n + 1) + 2\mu \right] > 0
\end{aligned}$$

at x_o for $\mu > 0$ sufficiently large, a contradiction.

Finally, similar arguments show (x_o, t_o) is not a maximum point of $Z = e^{\alpha x \cdot \nabla u} \triangle u - \beta |\nabla u|^2$, if $\alpha > 0$ and $\beta < 0$ are chosen appropriately, depending only on the initial data. □

3. The log-concavity of the solution

Lemma 3.1. *Assume that $u(x,t)$ is smooth up to the interface for $0 \leq t \leq T$. If $\log u_o$ is strictly concave, then $\log u(\cdot, t)$ is concave for $0 \leq t \leq T$.*

Proof. We will proceed by contradiction. Let us assume that the maximum

$$v_{\alpha\alpha}(x_o, t_o) = \max_{x \in \overline{\Omega(t)}, 0 \leq t \leq T} \max_{\beta} v_{\beta\beta}(x,t) = \delta > 0 \tag{3.1}$$

is attained at the point (x_0, t_0) in the unit direction e_α and it is strictly positive. Then $-\delta$ is an eigenvalue for the matrix $D^2 v(x_o, t_o)$ and e_α is its corresponding eigenvector. So $v_{\alpha\beta}(x_o, t_o) = 0$ if $\alpha \neq \beta$. We can also assume, without loss of generality, that the number $\delta > 0$ in (3.1) is very small.

We begin by computing the evolution equations of $v = \log u$ and $v_{\alpha\alpha}$, from the evolution of u. Since $v_i = \frac{u_i}{u}$ and $v_{ii} = \frac{u_{ii}}{u} - \frac{u_i^2}{u^2}$, we find:

$$v_t = \triangle v + |\nabla v|^2$$

and

$$(v_{\alpha\alpha})_t = \triangle v_{\alpha\alpha} + 2\nabla v \cdot \nabla v_{\alpha\alpha} + 2\nabla v_\alpha \cdot \nabla v_\alpha.$$

Hence, at the maximum point (x_o, t_o) where $v_{\alpha\beta} = 0$, for $\alpha \neq \beta$, we have:

$$v_{\alpha\alpha t} = \triangle v_{\alpha\alpha} + 2\nabla v \cdot \nabla v_{\alpha\alpha} + 2v_{\alpha\alpha}^2.$$

In other words, setting $Z = v_{\alpha\alpha}$, we find that the point (x_o, t_o), Z satisfies the idendity

$$Z_t = \triangle Z + 2\nabla v \cdot \nabla Z + 2Z^2.$$

Let us first observe first that x_o cannot be an interior point of $\Omega(t_o)$, for $t_o > 0$. Indeed, at an interior maximum point one has

$$0 \leq Z_t \leq 2Z^2$$

which forces Z to stay non-positive. Assume next that the maximum happens at a point (x_o, t_o) on $\partial\{u > 0\}$ and that

$$\lim_{(x,t)\to(x_o,t_o)} Z = \delta > 0.$$

Since

$$Z = \frac{u_{\alpha\alpha}}{u} - \frac{u_\alpha^2}{u^2} = \frac{1}{u}\left[u_{\alpha\alpha} - \frac{u_\alpha^2}{u}\right] \to \delta \tag{3.2}$$

e_α should be a tangential direction to $\partial\Omega(t_o)$; otherwise $|\nabla u| > c_o > 0$ implies $u_\alpha(x_o, t_o) \neq 0$ and $Z \to -\infty$ which is a contradiction.

Let (y_o, s_o), be any other boundary point, i.e., $y_o \in \partial\Omega(s_o)$. Then

$$\lim_{(x,t)\to(y_o,s_o)} Z \leq \delta.$$

Let us assume, without loss of generality, that the normal direction at y_o to $\partial\Omega(s_o)$ is e_n. Then $u_i = 0$ for $i = 1 \cdots n-1$ and $-1/c_o < u_n < -c_o < 0$. To simplify the notation, let us denote by "lim" the limit along the normal direction e_n to y_o at $t = s_o$. By (3.1), we have:

$$v_{ii} = \frac{u_{ii}}{u} - \frac{u_i^2}{u^2} \leq Z \leq \delta, \qquad i = 1, \cdots, n-1.$$

Hence, using l'Hopital's rule we obtain:

$$0 = \lim \delta u \geq \lim\left[u_{ii} - \frac{u_i^2}{u}\right] = \lim\left[u_{ii} - \frac{2u_i u_{in}}{u_n}\right] = \lim u_{ii} = u_{ii}(y_o, s_o)$$

for $i = 1, \cdots, n-1$.

Let $x_n = \gamma(x', t)$ for $x = (x', x_n)$ be the equation for the free boundary around (y_o, s_o) i.e. $u(x', \gamma(x', t), t) = 0$ and $y_o = (y_o', \gamma(y_o', s_o))$.

Substituting $u_i(y_o, s_o) = \gamma_i(y'_o, s_o) = 0$ in $D_{ii}(u(x', \gamma(x', t), t)) = 0$, we obtain

$$\gamma_{ii}(y'_o, s_o) = \frac{u_{ii}(y_o, s_o)}{-u_n(y_o, s_o)} \leq 0 \tag{3.3}$$

for $i = 1, \cdots, n-1$. Therefore the free boundary $\partial\Omega(t)$ is convex for $0 \leq t \leq T$. The argument of Lemma 3.1 in [CS], then implies that

$$\limsup_{(x,t)\to(y_o,s_o)} \; v_{ii} \leq 0 \tag{3.4}$$

for all $y_o \in \partial\Omega(s_o), 0 \leq s_o \leq T$ and for $i = 1, \cdots, n-1$, which is a contradiction to our assumption $Z = v_{\alpha\alpha}(x_o, t_o) = \delta > 0$.

Let us outline the proof of (3.4) for the reader's convenience. First, we can approximate $\Omega(t)$ by strictly convex and (x,t)-smooth $\Omega_\varepsilon(t)$ and the initial data u_o by log-concave smooth $u_{\varepsilon,o}$ s.t. $\{u_{\varepsilon,o} > 0\} = \Omega_\varepsilon(0)$. As we can see in [CS], the strict convexity of $\partial\Omega(t)$ implies $u_{\varepsilon,ii}(x,t) < 0$ at the free boundary. Hence

$$\begin{aligned} \limsup_{x\to\partial\Omega(t)} v_{\varepsilon,ii} &= \limsup_{x\to\partial\Omega(t)} \left[\frac{u_{\varepsilon,ii}}{u_\varepsilon} - \frac{u^2_{\varepsilon,i}}{u^2_\varepsilon}\right] = \limsup_{x\to\partial\Omega(t)} \frac{1}{u_\varepsilon}\left[u_{\varepsilon,ii} - \frac{u_{\varepsilon,i}u_{\varepsilon,in}}{u_{\varepsilon,n}}\right] \\ &= \limsup_{x\to\partial\Omega(t)} \frac{u_{\varepsilon,ii}}{u_\varepsilon} = -\infty \end{aligned}$$

The maximum principle on $v_{\varepsilon,ii}$ implies that $v_{\varepsilon,ii} < 0$. By taking a limit in ε in the second differential quotient, we conclude (3.4). Q.E.D

4. $C^{1,1}$-ESTIMATE

We will show in this section that the second derivatives of the solution u of **(FB)** are uniformly bounded, due to the convexity of the free boundary:

Lemma 4.1. *There exists a constant $C(u_o, \rho_o)$ for which*

$$\max_{0\leq t\leq T} |u(x,t)|_{C^{1,1}(\Omega(t))} < C(u_o).$$

Proof. In order to show that every second derivative is bounded above, let us consider the quantity

$$Z = \max_\alpha u_{\alpha\alpha}.$$

We know that $Z \le C(u_o)$ at time $t=0$ and we wish to show that $Z \le C(u_o,\rho_o)$ for $0 \le t \le T$, for some, possibly larger constant $C(u_o,\rho_o)$, depending on u_o and ρ_o. As in Lemma 2.4, we only need to consider the case where the maximum of Z happens at a point (x_o,t_o) on the free boundary. Otherwise, as direct application of the maximum principle shows the desired estimate. By a coordinate change, we can assume that $e_1 \cdots, e_{n-1}$ are the tangential directions at x_o to $\partial\Omega(t_o)$ and e_n is the corresponding outward normal direction. We have shown in the proof of Lemma 2.3 that the free boundary condition $|\nabla u|^2 = 1$ implies that

$$(\triangle u)_n + u_n\, u_{nn}\,(\triangle u) = 0$$

at the interface. Since $u_n = -1$ at the point (x_o,t_o), we conclude that

$$(\triangle u)_n - \; u_{nn}\,(\triangle u) = 0 \tag{4.1}$$

at (x_o,t_o). Also, $u_{ii} \le 0$ from the convexity of the free boundary. By differentiating twice the free-boundary conditions $|\nabla u(\gamma(x',t),x',t)| = 1$ and $u(\gamma(x',t),x',t) = 0$ with respect to any tangential direction $1 \le i \le n-1$, we obtain

$$u_{nii} = -\gamma_{ii} u_{nn} + \nabla u_i \cdot \nabla u_i$$

and

$$\gamma_{ii} = -\frac{u_{ii}}{u_n} = u_{ii}.$$

Hence, from (4.1), at the point (x_o,t_o) we have:

$$\begin{aligned} 0 &= u_{nnn} + \sum_{i=1}^{n-1} u_{iin} - \; u_{nn}(\triangle u) \\ &= u_{nnn} - u_{nn} \sum_{i=1}^{n-1} \gamma_{ii} + \sum_{i=1}^{n-1} |\nabla u_i|^2 - 2\, u_{nn}(\triangle u) \\ &= u_{nnn} - u_{nn}\,[\triangle u - u_{nn}] + \sum_{i=1}^{n-1} |\nabla u_i|^2 - \; u_{nn}(\triangle u) \\ &= u_{nnn} + u_{nn}^2 - 2\, u_{nn}(\triangle u) + \sum_{i=1}^{n-1} |\nabla u_i|^2 \end{aligned} \tag{4.2}$$

If u_{nn} is non-positive, then every $u_{\alpha\alpha}$ is negative, i.e., $Z \le 0$ and hence bounded above. So, let us assume that u_{nn} is positive. Then $Z = u_{nn}$, since $u_{in} = 0, u_{ii} \leq 0$ for $1 \leq i \leq n-1$. Hence, u_{nn} attains its the maximum at x_o, implying that $u_{nnn} \geq 0$. From (4.2) we then conclude the inequality

$$u_{nn}^2 - 2\, u_{nn}(\triangle u) \leq 0$$

and dividing by $u_{nn} > 0$ we obtain

$$u_{nn} \leq 2\, \triangle u$$

at the point (x_o, t_o). The desired bound then follows by Lemma 2.3.

Finally, let us show that every second derivative of the solution u is bounded below. The interior bound follows easily by the maximum principle. On the other hand, on the free-boundary we have $|\triangle u| \leq C$, by Lemma 2.3, and $u_{ii} \leq 0, u_{nn} < C$. Hence $\min(u_{ii}, u_{nn}) > -C$ for a uniform constant C. $\square$

5. The coordinate change and the proof of Theorem 1.1

We begin this section by introducing the local change of coordinate, used in [DHL], which allows us to transform the free boundary problem near the interface into a nonlinear equation of divergence form on a domain with fixed boundary. In order to prove the Main Theorem 1.1, we will establish $C^{1,\alpha}$-estimates for the new nonlinear operator, with constants depending only on the uniform gradient estimates. Going back to the original coordinates, we will obtain a uniform $C^{1,\alpha}$-estimate for the solution u up to the free boundary and up to any time $T < T_o$. Hence, assuming that the the regularity of u or the free boundary breaks out at some time $T < T_o$, by the short time existence, the smoothness of u and the free boundary will persist even after $t = T$ up to $t = T_o$.

Let us assume, for the moment, that u is C^1 on the closure of its support and pick a point $P_o = (x_o, t_o)$ at the free boundary $\partial\Omega(t_o)$ with $0 \leq t_o \leq T$. We may assume that e_n is the outward normal direction at x_o to the free boundary $\partial\Omega(t_o)$ and $e_i, 1 \leq i \leq n-1$ are its tangential directions. From the convexity of each level set of $u(x,t)$ and the condition that the ball $B_{\rho_o}(0)$ is contained in the support $\overline{\Omega(t)}$ of $u(x,t)$, for $0 \leq t \leq T$, one can easily conclude that the vector e_n differs from the outward normal direction $\nu_x(t)$ to the level set $\{z | u(z,t) \geq u(x,t)\}$

of u at $P = (x,t)$, for P in a uniform neighborhood of P_0, by an angle uniformly smaller than $\frac{\pi}{2}$. In other words, there exists a uniform constant $\delta > 0$, for which

$$\cos\langle e_n, \nu_x(t)\rangle \geq c_o > 0,$$

for all $P = (x,t)$ in $\mathcal{A}_\delta(P_0) = \{(x,t) : x \in \Omega(t) \cap B_\delta(x_0) \quad t_0 - \delta \leq t \leq t_0\}$. Hence

$$-\frac{1}{c_o} < u_n = -|\nabla u| \, \cos\langle e_n, \nu_x(t)\rangle < -c_o < 0 \tag{5.1}$$

for $(x,t) \in \mathcal{A}_\delta(P_o)$.

Then, we can apply the Implicit Function Theorem to solve the equation

$$z = u(x', x_n, t), \qquad (x', x_n, t) \in \mathcal{A}_\delta(P_o)$$

with respect to x_n, yielding to the equation

$$x_n = v(x', z, t).$$

To simplify the notation, let us introduce the new coordinates

$$y_i = x_i, \quad i =, \cdots, n-1, \qquad y_n = z, \qquad t = t.$$

Since,

$$u_{x_n} = \frac{1}{v_{y_n}} \qquad u_{x_i} = -\frac{v_{y_i}}{v_{y_n}}, \qquad 1 \leq i \leq n-1, \qquad u_t = -\frac{v_t}{v_{y_n}}$$

the problem (**FB**) may be expressed, in the new coordinates, as

$$\begin{aligned} v_t = \sum_{i=1}^{n-1} v_{ii} + D_n \left(\frac{1 + \sum_{i=1}^{n-1} v_i^2}{-v_n} \right) \qquad & \text{on} \quad y_n > 0 \\ \frac{1 + \sum_{i=1}^{n-1} v_i^2}{v_n^2} = 1 \qquad & \text{at} \quad y_n = 0. \end{aligned} \tag{5.2}$$

Proof of Theorem 1.1. Let us assume that the regularity of u breaks out at some time $T < T_o$. The $C^{1,1}$-estimate on u, for $0 \leq t \leq T$, shown in Theorem 4.1, implies the $C^{1,1}$-estimate on v, for $0 \leq t \leq T$. Hence, we can apply Theorem 14.22 in [L] to the equation (5.2), to obtain a uniform $C^{2,\alpha}$-estimate on v and consequently on u, for $0 \leq t \leq T$. It follows that u satisfies at $t = T$ the two necessary conditions for the short time existence in [AG], namely the $C^{2,\alpha}$-bound and the nondegeneracy of u. Therefore $T = T_o$, otherwise u is smooth after $t = T$ which is a contradiction to the definition of T. $\square$

Remark 5.1. As we mentioned above, for the $C^{1,\alpha}$-estimate of v it is enough to use Theorem 14.22 in [L]. We will demonstrate next how one can obtain independently, in the case of spatial dimension $n = 2$, the $C^{1,\alpha}$-estimate of v, from the bound $|\nabla v| < C$, by the trancation method of De Giorgi.

Lemma 5.2. *Under the above notation, in dimension $n = 2$, and assuming that $|\nabla u| \le C$ on $\mathcal{A}_\delta(P_o)$, we have*

$$|u|_{C^{1,\alpha}(\mathcal{A}_\delta(P_o))} \le C$$

for some constant uniform constant $C(u_0)$.

Proof. It is enough to establish the $C^{1,\alpha}$ bound of v, at a point $y = y_o$, $t = t_o$. In the case of spatial dimension $n = 2$, equation (5.2) takes the simpler form:

$$\begin{aligned} v_t = v_{11} + D_2\left(\frac{1+v_1^2}{-v_2}\right) \qquad &\text{on} \quad y_2 > 0 \\ \frac{1+v_1^2}{v_2^2} = 1 \qquad &\text{at} \quad y_2 = 0. \end{aligned} \tag{5.3}$$

For the C^α-estimate of v_1, let $\xi = (\eta\,(v_1 - k)_+)_1$ be the our test function. Multiplying both sides of equation (5.3) by ξ and integrating by parts, we have:

$$\begin{aligned} \int_{B_\rho} v_t\,[\eta\,(v_1-k)_+]_1 &= \int_{B_\rho} v_{11}[\eta\,(v_1-k)_+]_1 + \int_{B_\rho} D_2\left(\frac{1+v_1^2}{-v_2}\right)[\eta\,(v_1-k)_+]_1 \\ &= \int_{B_\rho} v_{11}(\eta\,(v_1-k)_+)_1 - \int_{B_\rho} D_2\left(-\frac{2v_1v_{11}}{v_2} + \frac{1+v_1^2}{v_2^2}v_{21}\right)\eta\,(v_1-k)_+ \end{aligned}$$

Integrating by parts once more we obtain:

$$\begin{aligned} \int_{B_\rho} v_t\,[\eta\,(v_1-k)_+]_1 = \int_{B_\rho} v_{11}[\eta\,(v_1-k)_+]_1 + \int_{B_\rho}\left(-\frac{2v_1v_{11}}{v_2} + \frac{1+v_1^2}{v_2^2}v_{21}\right)[\eta\,(v_1-k)_+]_2 \\ + \int_{y_2=0}\left(-\frac{2v_1v_{11}}{v_2} + \frac{1+v_1^2}{v_2^2}v_{21}\right)\eta\,(v_1-k)_+ \end{aligned} \tag{5.4}$$

On the other hand, $1+v_1^2 = v_2^2$ on $y_2 = 0$. Differentiating this equation with respect to y_1, we find that $v_{21} = v_1 v_{11}/v_2$ on $y_2 = 0$. Hence:

$$\int_{y_2=0} \left(-\frac{2v_1v_{11}}{v_2} + \frac{1+v_1^2}{v_2^2} v_{21} \right) [\eta\,(v_1-k)_+] = -\int_{y_2=0} v_{21}\,[\eta\,(v_1-k)_+]$$
$$= -\int_{y_2=0} \left(\frac{v_1v_{11}}{\sqrt{1+v_1^2}} \right) [\eta\,(v_1-k)_+] = -\int_{B_\rho} D_2 \left[\left(\frac{v_1v_{11}}{\sqrt{1+v_1^2}} \right) [\eta(v_1-k)_+] \right]$$
$$= -\int_{B_\rho} \frac{v_1}{\sqrt{1+v_1^2}} [\eta\,(v_1-k)_+]\, v_{112} - \int_{B_\rho} \frac{v_1}{\sqrt{1+v_1^2}} v_{11}\,[\eta(v_1-k)_+]_2$$
$$-\int_{B_\rho} v_{11} \left[\frac{v_{12}}{\sqrt{1+v_1^2}} - \frac{v_1^2 v_{12}}{(1+v_1^2)3/2} \right] [\eta(v_1-k)_+]$$
$$= \int_{B_\rho} \left[\frac{v_1}{\sqrt{1+v_1^2}} \eta(v_1-k)_+ \right]_1 v_{12} - \int_{B_\rho} \frac{v_1}{\sqrt{1+v_1^2}} v_{11}\,[\eta(v_1-k)_+]_2$$
$$-\int_{B_\rho} \frac{v_{11}v_{12}}{\sqrt{1+v_1^2}} [\eta(v_1-k)_+] + \int_{B_\rho} \frac{v_1^2 v_{11} v_{12}}{(1+v_1^2)3/2} [\eta\,(v_1-k)_+]$$

After several cancellations, we conclude that

$$\int_{y_2=0} \left(-\frac{2v_1v_{11}}{v_2} + \frac{1+v_1^2}{v_2^2} v_{21} \right) [\eta\,(v_1-k)_+] = \int_{B_\rho} \frac{v_1\,[\eta_1 v_{12} - \eta_2 v_{11}]}{\sqrt{1+v_1^2}} (v_1-k)_+. \tag{5.5}$$

Let us set $\eta = \zeta^2$. The term $\int_{B_\rho} v_t\,[\eta\,(v_1-k)_+]_1$ can be estimated by Hölder's inequality, using the fact that v_t is bounded. Therefore, combining (5.4) and (5.5) and using again Hölder's inequality, we finally conclude that

$$\int_{A_{k,\rho}} \zeta^2 |\nabla v_1|^2 dy \le \gamma \left[\int_{A_{k,l}} \zeta^2 (v_1-k)_+^2 dy + \operatorname{mes} A_{k,\rho} \right] \tag{5.6}$$

for an uniform constant $\gamma > 0$, where $A_{k,\rho} = K_\rho \cap \{y | v_1(y) > k\}$. Analogously, by testing equation (5.3) against $\xi = \eta(-v_1-k)_+$, we derive inequality (5.6) for $-v_1$ also. ¿From the de Giorgi's argument , Theorem 7.2, Chapter 2 in [LU], we obtain a uniform bound for $|v_{y_1}|_\alpha$ up to the boundary $y_2 = 0$. In particular, equation (5.6) with $k = \min_{K_\rho} v_1$ implies that

$$\int_{K_\rho} \zeta^2 |\nabla v_1|^2 dy \le \gamma \rho^{n-2+2\alpha}. \tag{5.7}$$

On the other hand, solving equation (5.3) with respect to v_{22}, we find that

$$v_{22} = \frac{1+v_1^2}{v_2^2}(u_t - v_{11} + \frac{2v_1}{v_2}v_{12}). \tag{5.8}$$

Combining (5.7) and (5.8) we conclude that

$$\int_{K_\rho} \zeta^2 \, |\nabla v_2|^2 dy \le \gamma \rho^{n-2+2\alpha} \tag{5.9}$$

Hence, Morrey's estimate, Theorem 7.19 in[GT], implies the uniform $C^{1,\alpha}$-estimate of v up to the boundary $y_2 = 0$. □

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