GROMOV-WITTEN INVARIANTS OF VARIETIES WITH HOLOMORPHIC 2-FORMS

YOUNG-HOON KIEM AND JUN LI

Abstract. We show that a holomorphic two-form θ on a smooth algebraic variety X localizes the virtual fundamental class of the moduli of stable maps $\mathcal{M}_{g,n}(X,\beta)$ to the locus where θ degenerates; it then enables us to define the *localized GW-invariant*, an algebro-geometric analogue of the local invariant of Lee and Parker in symplectic geometry [15], which coincides with the ordinary GW-invariant when X is proper. It is deformation invariant. Using this, we prove formulas for low degree GW-invariants of minimal general type surfaces with $p_g > 0$ conjectured by Maulik and Pandharipande.

1. Introduction

In recent years, Gromov-Witten invariant (GW-invariant, for short) has played an important role in research in algebraic geometry and in Super-String theories. Various effective techniques have been contrived to compute these invariants, such as localization by torus action [8], degeneration method [10, 17, 19], quantum Riemann-Roch and Lefschetz [5], to name a few. For curves, these invariants have been completely determined in [23]. For higher dimensional case, much more remains to be done. In this paper, we provide a new technique, called the *localization by holomorphic two-form*, that is the algebro-geometric analogue of what was first discovered by Lee and Parker [15] in symplectic geometry.

The localization by holomorphic two-form is a localization theorem on virtual cycles. As is known, central to the construction of GW-invariants is to replace the fundamental class of the moduli of stable maps by its *virtual fundamental cycle*. In this paper, we show that a holomorphic two-form $\theta \in H^0(\Omega^2_X)$ on a smooth quasi-projective variety X localizes (or forces) the virtual fundamental cycle to (support in) the locus of those stable maps $f : C \to X$ whose images lie in the degeneracy locus of θ.

Based on this localization by holomorphic two-form, for a pair of a smooth quasi-projective variety and a holomorphic two-form (X, θ) with certain properness requirement, we shall define the so-called *localized GW-invariant*. This invariant is deformation invariant; when the degeneracy locus of θ is smooth, it is equivalent to the localized invariant of its normal bundle; when X is proper, it coincides with the ordinary GW-invariant.

Applying this to a smooth minimal general type surface S with positive $p_g = h^2(\mathcal{O}_S)$, we conjecture that its GW-invariants are obtained from the localized GW-invariants of the total space of a theta characteristic of a smooth curve D of genus $K_S^2 + 1$. In the ideal case, the localized GW-invariants are explicitly related to the twisted GW-invariants of D, which includes all GW-invariants of S *without* descendant insertions. This conjecture was partially proved by Lee-Parker for the case when S has smooth canonical divisors [15]. For others, we prove a degeneration formula that allows us to reduce the complete set of GW-invariants to the twisted GW-invariants of curves and of some low degree relative localized invariants. For the case of degrees 1 and 2, we work out their details and verify the formulas conjectured by Maulik and Pandharipande [22, (8) and (9)].

This localized invariant has recently been employed by W-P. Li and the second author to study the GW-invariants of the Hilbert schemes of points on surfaces [18].

We now provide a more detailed outline of this paper. In section 2, we work out the details of the localization by a holomorphic two-form and the localized GW-invariants. Let X be a smooth quasi-projective variety equipped with a (nontrivial) holomorphic two-form $\theta \in H^0(\Omega_X^2)$. Then θ gives rise to a homomorphism, called a *cosection*

$$\sigma : \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}}$$

from the obstruction sheaf of the moduli space $\mathcal{M} = \mathcal{M}_{g,n}(X,\beta)$ of stable maps to its structure sheaf. Let $Z(\sigma)$ be the locus over which σ is not surjective. We then show that we can canonically construct a *localized* virtual fundamental class

$$[\mathcal{M}]^{\rm vir}_{\rm loc} \in H_*^{BM}(Z(\sigma))$$

satisfying the following properties: it is deformation invariant under a mild condition on the cosections; *its pushforward to $H_*(\mathcal{M})$ coincides with the ordinary virtual class $[\mathcal{M}]^{\rm vir}$ in case the degeneracy locus $Z(\sigma)$ is proper.*

This immediately leads to

Theorem 1.1. *For a pair (X,θ) of a smooth projective variety and a holomorphic two-form, the virtual fundamental class of $\mathcal{M}_{g,n}(X,\beta)$ vanishes unless β is represented by a θ-null stable map.*

Here we say a stable map $f: C \to X$ with fundamental class $\beta = f_*[C]$ is θ-null if the image of *df* lies in the degeneracy locus of θ. (See section 3 for details.)

This theorem recovers the vanishing results of J. Lee and T. Parker [15]. For instance, if X is $2m$-dimensional and $D = zero(\theta^m)$, then all the GW-invariants vanish unless β is in the image of $H_2(D,\mathbb{Z})$ in $H_2(X,\mathbb{Z})$ by the inclusion map. In particular, if X is a holomorphic symplectic manifold (i.e. $D = \emptyset$), all GW-invariants vanish.

In section 3, we define the localized GW-invariants as the integral over the localized virtual fundamental class of tautological classes. By construction, when X is proper, the localized GW-invariants coincide with the ordinary GW-invariants. The localized invariants are invariant under a class of deformation relevant to our study.

For a smooth minimal general type surface S with $p_g > 0$, we shall take X to be the total space of any theta characteristic L on a smooth projective curve D of genus $h = K_S^2 + 1$ satisfying $h^0(L) \equiv \chi(\mathcal{O}_S) \mod (2)$. It is easy to see that X has a holomorphic two-form non-degenerate away from the zero section of $X \to D$.

Conjecture 1.2. *Let S be a smooth minimal general type surface with positive $p_g > 0$. Its GW-invariants $\langle \cdots \rangle^S_{\beta,g}$ vanish unless β is a non-negative integral multiple of $c_1(K_S)$. In case $\beta = dc_1(K_S)$ for an integer $d > 0$, we let (D,L) be a pair of a smooth projective curve of genus $K_S^2 + 1$ and its theta characteristic with parity $\chi(\mathcal{O}_S)$, and let X be the total space of L. Then there is a canonical homomorphism $\rho: H^*(S,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ so that for any classes $\gamma_i \in H^*(S,\mathbb{Z})$ and* *integers $\alpha_i \geq 0$, $i = 1, \cdots, n$,*

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{S}_{\beta,g} = \langle \tau_{\alpha_1}(\rho(\gamma_1)) \cdots \tau_{\alpha_n}(\rho(\gamma_n)) \rangle^{X,\mathrm{loc}}_{d[D],g}.$$

This conjecture is proved by Lee-Parker [15] when S has a smooth canonical divisor. In §3.3, we provide a different proof by showing invariance of the localized virtual fundamental class under deformation of S to the normal bundle X of a smooth canonical curve. In general, by studying the deformation of the (analytic) germ of a canonical divisor, we have confirmed the conjecture for a wider class of surfaces, including the case when S has a reduced canonical divisor. We shall address this in [12].

In light of this Conjecture, the GW-invariants of minimal general type surfaces are reduced to the localized GW-invariants of the total space X of a theta-characteristic L over a smooth curve D. Let (π, f) represent the universal stable map to D. In case $R^1\pi_* f^* L$ is locally free, we prove that the localized GW-invariants of X are the twisted GW-invariants of D with appropriate sign modifications (Proposition 3.15). This in particular enables us to recover the following result of Lee-Parker [15].

Theorem 1.3. *Let X be the total space of a theta-characteristic L on a smooth curve D. Then the localized genus g GW-invariant of X with homology class $d[D]$ is*

$$\langle 1 \rangle^{X,\mathrm{loc}}_{d,g} = \sum_{u:d\text{-fold étale cover of } D} \frac{(-1)^{h^0(u^*L)}}{|\mathrm{Aut}(u)|}$$

for $g = d(h-1)+1$ where h is the genus of D. All the other localized invariants are zero.

For the GW-invariants *with* descendant insertions, we reduce the problem to a simpler one by using a degeneration formula for the (relative) localized GW-invariants by degenerating X into a union of $Y_1 \cong X$ and $Y_2 = \mathbb{P}^1 \times \mathbb{C}$. By working out the relative localized invariants for low degree, in section 4 we prove

Theorem 1.4. *Let $X \to D$ be a theta characteristic over a smooth curve of genus h. Let $\gamma \in H^2(D, \mathbb{Z})$ be the Poincaré dual of a point in D. Then the degree one and two GW-invariants*[1] *with descendants are*

$$\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \rangle^{X,\bullet}_{[D],\mathrm{loc}} = (-1)^{h^0(L)} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i+1)!} (-2)^{-\alpha_i};$$

$$\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \rangle^{X,\bullet}_{2[D],\mathrm{loc}} = (-1)^{h^0(L)}\, 2^{h+n-1} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i+1)!} (-2)^{\alpha_i}.$$

The above two formulas are conjectured by Maulik and Pandharipande [22].

In section 5, we discuss a few special cases where our localization argument may enable us to compute the GW-invariants of three-folds. For instance, if X is a $\mathbb{P}^1$-bundle over a surface with $|K_S| \neq \emptyset$, we can combine the virtual localization by torus action ([8]) and our localization by holomorphic two-form, to reduce the computation of GW-invariants of X to the curve case where the degeneration method [17, 23] applies effectively.

[1] See §4.1 for the definition of GW-invariants with not necessarily connected domains.

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2. Localizing virtual cycles by cosections of obstruction sheaves

In this section, we shall show that for a Deligne-Mumford stack with a perfect-obstruction theory, a meromorphic cosection of its obstruction sheaf will localize its virtual cycle to the degeneracy locus of the cosection.

More precisely, we let M be a Deligne-Mumford stack endowed with a perfect obstruction theory and let $\mathcal{O}b$ be its obstruction sheaf [20, 3]. We call σ a meromorphic *cosection* of $\mathcal{O}b$ if there is a dense open subset $U \subset M$ so that σ is a sheaf homomorphism

$$\sigma : \mathcal{O}b|_U \longrightarrow \mathcal{O}_U. \tag{2.1}$$

We define its degeneracy locus as the union

$$Z(\sigma) = (M - U) \cup \{s \in U \mid \sigma(s) = 0 : \mathcal{O}b \otimes k(s) \longrightarrow k(s)\}.$$

The main result of this section is

Lemma 2.1 (Localization Lemma)**.** *Let M be a Deligne-Mumford stack endowed with a perfect obstruction theory and suppose the obstruction sheaf admits a meromorphic cosection σ. Then there is a canonical cycle*

$$[M]^{\mathrm{vir}}_{\mathrm{loc}} \in H^{BM}_*\big(Z(\sigma)\big)$$

whose image under the obvious $i_*\colon H^{BM}_*\big(Z(\sigma)\big) \to H^{BM}_*(M)$ is the virtual cycle

$$[M]^{\mathrm{vir}} = i_*[M]^{\mathrm{vir}}_{\mathrm{loc}} \in H^{BM}_*(M).$$

Here by a perfect obstruction theory we mean either the one defined by Tian and the second author in [20] using relative obstruction theory or by Behrend and Fantechi in [3] using cotangent complex. As pointed out by Kresch [13], the two constructions are equivalent and produce identical virtual cycles.

Further, the localized cycle $[M]^{\mathrm{vir}}_{\mathrm{loc}}$ is deformation invariant under a technical condition.

2.1. Virtual normal cones and cosections.

To prove this lemma, we shall first prove that the virtual normal cone associated to the obstruction theory of M lies in the kernel cone of the cosection. We begin with proving a fact about the normal cone, which was essentially proved in [24].

Lemma 2.2. *Let $W \subset V$ be a closed subscheme of a smooth scheme V defined by the vanishing $s = 0$ of a section s of a vector bundle E on V; let $C_W V$ be the normal cone to W in V, embedded in E via the section s. Suppose the cokernel*

$$\mathcal{A} = \operatorname{coker}\{ds : \mathcal{O}_W(T_V) \longrightarrow \mathcal{O}_W(E)\}$$

admits a surjective sheaf homomorphism $\sigma : \mathcal{A} \to \mathcal{O}_W$. Then the cone $C_W V$ lies entirely in the subbundle $F \subset E$ that is the kernel of the composite

$$\mathcal{O}_W(E) \xrightarrow{\ \phi\ } \mathcal{A} \xrightarrow{\ \sigma\ } \mathcal{O}_W. \tag{2.2}$$

Proof. To prove the lemma, we shall view $C_W V \subset E$ as the specialization of the section $t^{-1}s \subset E$ as $t \to 0$. More precisely, we consider the subscheme

$$\Gamma = \{(t^{-1}s(w), t) \in E \times (\mathbb{A}^1 - 0) \mid w \in V,\ t \in \mathbb{A}^1 - 0\}.$$

For $t \in \mathbb{A}^1 - 0$, the fiber Γ_t of Γ over $t \in \mathbb{A}^1$ is merely the section $t^{-1}s$ of E. We let $\bar{\Gamma}$ be the closure of Γ in $E \times \mathbb{A}^1$. By definition, the central fiber $\bar{\Gamma} \times_{\mathbb{A}^1} 0 \subset E$ is the normal cone $C_W V$ alluded before. Clearly, $C_W V$ is of pure dimension $\dim V$.

Now let $N \subset C_W V$ be any irreducible component and let $\alpha \in N$ be a general closed point of N. Suppose α does not lie in the zero section of E. Then we can find a regular irreducible curve C and a morphism $\iota : C \to \bar{\Gamma}$ that passes through α, say $\iota(0) = \alpha$. We let

$$\eta : C \longrightarrow V \quad \text{and} \quad \zeta : C \longrightarrow \mathbb{A}^1$$

be the projections induced by those from $E \times \mathbb{A}^1$ to V and to $\mathbb{A}^1$; we require that ζ dominates $\mathbb{A}^1$. We then choose a uniformizing parameter ξ of C at 0 so that $\zeta^*(t) = \xi^n$ for some n. Because $\iota(0) = \alpha$, $\xi^{-n} s \circ \eta(\xi)$ specializes to α; hence $s \circ \eta$ has the expression

$$s \circ \eta = \alpha \xi^n + O(\xi^{n+1}),$$

and thus $\eta^{-1}(W) = \operatorname{Spec} \mathbb{C}[\xi]/(\xi^n)$. In particular, pulling back the exact sequence

$$\mathcal{O}_W(T_V) \xrightarrow{ds} \mathcal{O}_W(E) \longrightarrow \mathcal{A} \longrightarrow 0$$

via the induced morphism

$$\bar{\eta} = \eta|_{\eta^{-1}(W)} : \bar{\eta}^{-1}(W) = \operatorname{Spec} \mathbb{C}[\xi]/(\xi^n) \longrightarrow W,$$

we obtain

$$\bar{\eta}^*(T_V) \xrightarrow{\bar{\eta}^*(ds)} \bar{\eta}^*(E) \xrightarrow{\bar{\eta}^*(\phi)} \bar{\eta}^*(\mathcal{A}) \longrightarrow 0 \tag{2.3}$$

in which $\bar{\eta}^*(\phi)$ is the pullback of ϕ in (2.2) and

$$\bar{\eta}^*(ds) = d(\alpha\xi^n + O(\xi^{n+1})) \equiv n\alpha\xi^{n-1} d\xi \mod \xi^n.$$

On the other hand, because (2.3) is exact, the composition

$$\bar{\eta}^*(\sigma) \circ \bar{\eta}^*(\phi) \circ \bar{\eta}^*(ds) = 0. \tag{2.4}$$

Now suppose $(\sigma \circ \phi)(\alpha) \neq 0$. Then the vanishing (2.4) implies that $\xi^{n-1} = 0 \in \mathbb{C}[\xi]/(\xi^n)$, a contradiction. Therefore, α lies in the kernel of (2.2), and so does the cone $C_W V$. This proves the lemma. □

In case σ is a meromorphic cosection of E and U is the largest open subset over which σ is defined and surjective, we define the (cone) kernel $E(\sigma)$ of σ to be the union of the restriction to $M - U$ of E with the kernel of $\sigma|_U$:

$$E(\sigma) = E|_{M-U} \cup \operatorname{Ker}(\sigma : E|_U \longrightarrow \mathcal{O}_U).$$

Corollary 2.3. *Let $W \subset V$ be as in Lemma 2.1 except that σ is only assumed to be a meromorphic cosection. Then the cone $C_W V$ lies entirely in the kernel $E(\sigma)$.*

In the following, we shall construct a localized Gysin map by intersecting with smooth sections that "almost" split the cosection σ.

2.2. **Localized Gysin maps.** Let $\sigma : E \to \mathcal{O}_M$ be a meromorphic cosection of the vector bundle E over a quasi-projective complex scheme M with $Z(\sigma) \subset M$ and $E(\sigma)$ its degeneracy locus and its kernel cone. For the bundle $E \to M$ we recall that the topological Gysin map

$$s_E^! : A_*E \longrightarrow A_*M$$

is defined by intersecting any cycle $W \in Z_*E$ with the zero section s_E of E. To define the localized Gysin homomorphism, we shall use smooth section of E that almost lifts $1 \in \Gamma(\mathcal{O}_M)$.

For this, we first pick a splitting of σ away from the degeneracy locus $Z(\sigma)$. Because σ is surjective away from $Z(\sigma)$, possibly by picking a hermitian metric on E we can find a smooth section $\breve{\sigma} \in C^\infty\bigl(E|_{M-D(\sigma)}\bigr)$ so that $\sigma \circ \breve{\sigma} = 1$.

Next, we pick a sufficiently small (analytic) neighborhood $\mathcal{U}$ of $Z(\sigma) \subset M$ which is properly homotopy equivalent to $Z(\sigma)$. Because M is quasi-projective, such a neighborhood $\mathcal{U}$ always exists. We then extend $\breve{\sigma}|_{M-\mathcal{U}}$ to a smooth section $\breve{\sigma}_{ex} \in C^\infty\bigl(E\bigr)$ and pick a smooth function $\rho\colon M \to \mathbb{R}^{>0}$ so that

$$\xi = \rho \cdot \breve{\sigma}_{ex} \in C^\infty\bigl(E\bigr)$$

is a small perturbation of the zero section of E.

Now let $W \subset E(\sigma)$ be any closed subvariety. By fixing a stratification of W and of M by complex subvarieties, we can choose the extension $\breve{\sigma}_{ex}$ and the function ρ so that the section ξ intersects W transversely. As a consequence, the intersection $W \cap \xi$, which is of pure dimension, has no real codimension 1 strata. Henceforth, it defines a closed oriented Borel-Moore chain cycle in M. But on the other hand, since $\sigma\circ\xi|_{M-\mathcal{U}} = \rho \in C^\infty\bigl(M-\mathcal{U}\bigr)$, ξ is disjoint from W over $M-\mathcal{U}$. Thus $W\cap\xi \subset E|_{\mathcal{U}}$ is a closed Borel-Moore cycle in $E|_{\mathcal{U}}$. In this way, under the projection $\pi_{\mathcal{U}}\colon E|_{\mathcal{U}} \to \mathcal{U}$, we obtain a class

$$[\pi_{\mathcal{U}}(W \cap \xi)] \in H_*^{BM}(\mathcal{U}).$$

Applying the standard transversality argument, one easily shows that this class is independent of the choice of $\mathcal{U}$ and the section ξ; thus it only depends on the cycle W we begin with. Furthermore, because $\mathcal{U}$ is properly homotopy equivalent to $Z(\sigma)$, $H_*^{BM}(Z(\sigma)) \cong H_*^{BM}(\mathcal{U})$. Therefore, the newly constructed class can be viewed as a class in $H_*^{BM}(Z(\sigma))$.

Definition-Proposition 2.4. *We define the localized Gysin map*

$$s_{E,\mathrm{loc}}^! : Z_*E(\sigma) \longrightarrow H_*^{BM}(Z(\sigma))$$

to be the linear map that sends any subvariety $W \subset Z_*E(\sigma)$ to the cycle $[\pi_{\mathcal{U}}(W \cap \xi)] \in H_*^{BM}(Z(\sigma))$. It sends any two rationally equivalent cycles in $Z_*E(\sigma)$ to the same homology class in $H_*^{BM}(Z(\sigma))$; therefore it factors through a homomorphism from the group of cycle classes:

$$s_{E,\mathrm{loc}}^! : A_*E(\sigma) \longrightarrow H_*^{BM}(Z(\sigma)).$$

Proof. The proof is standard and shall be omitted. □

Next, we shall investigate the case for a DM-stack with perfect-obstruction theory and a cosection of its obstruction sheaf.

2.3. **DM-stack with perfect obstruction theory.** Let M be a DM-stack with perfect obstruction theory and with a cosection $\sigma : \mathcal{O}b_M \to \mathcal{O}_M$ of its obstruction sheaf; let E be a vector bundle on M whose sheaf of sections $\mathcal{E}$ surjects onto $\mathcal{O}_M$; by the construction of virtual cycle, the obstruction theory of M provides a unique cone cycle $W \in Z_*E$, the virtual normal cone of M.

Lemma 2.5. *Let $\tilde\sigma : \mathcal{E} \to \mathcal{O}_M$ be the composite of σ with the quotient homomorphism $\mathcal{E} \to \mathcal{O}b_M$. Then the (virtual) normal cone $W \in Z_*E$ lies in the (cone) kernel $E(\tilde\sigma)$ of $\tilde\sigma$.*

Proof. Let $U \subset M$ be the largest open subset over which $\sigma : \mathcal{O}b_M \to \mathcal{O}_M$, and hence the composite $\tilde\sigma : \mathcal{E} \to \mathcal{O}_M$, is surjective. We let $F \subset E|_U$ be the kernel subbundle of $\tilde\sigma$. To prove the lemma we only need to show that the restriction of the cone W over U is entirely contained in F. Since this is a local property, we only need to prove this over every closed point $p \in U$. And by replacing a neighborhood of $p \in U$ by its étale covering, we can assume without loss of generality that M is a scheme.

To proceed, we recall the construction of the cone W at p. We let $\hat M$ be the formal completion of M at p with $\rho : \hat M \to M$ the tautological morphism; we let E_p be the fiber of E at p; let n and (z) be

$$n = \operatorname{rank} E + \operatorname{vir.dim} M, \quad (z) = (z_1, \cdots, z_n).$$

Then the obstruction theory of M at p provides a Kuranishi map $f \in \mathbf{k}[\![z]\!] \otimes E_p$ so that $\hat M$ is isomorphic to the subscheme $(f = 0) \subset \hat V = \operatorname{Spec} \mathbf{k}[\![z]\!]$. We claim that we can choose f and two isomorphisms ψ_1 and ψ_2 as shown so that the cokernel $\operatorname{coker}(df)$ fits into the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_{\hat M}(T_{\hat V}) & \xrightarrow{df} & \mathcal{O}_{\hat M} \otimes E_p & \xrightarrow{\mathrm{pr}} & \operatorname{coker}(df) & \longrightarrow & \mathcal{O}_{\hat M} \\
 & & \cong\big\downarrow \psi_1 & & \cong\big\downarrow \psi_2 & & \big\| \\
 & & \rho^*\mathcal{E} & \longrightarrow & \rho^*\mathcal{O}b_M & \xrightarrow{\rho^*(\sigma)} & \mathcal{O}_{\hat M}.
\end{array} \tag{2.5}$$

Before we prove the claim, we shall see how this leads to the proof of Lemma. Once we have this diagram, then since $\rho(p) \in U$, the pullback sheaf homomorphism $\rho^*(\sigma) \circ \psi_2$ is surjective. Hence by the proof of Lemma 2.2, the fiber of the normal cone $C_{\hat M}\hat V$ over p is entirely contained in the kernel vector space:

$$C_{\hat M}\hat V \times_{\hat M} p \subset F_p = \ker\{\rho^*(\sigma) \circ \psi_2 \circ \mathrm{pr}|_p : E_p \longrightarrow \mathbb{C}\}.$$

On the other hand, under the isomorphism ψ_1 the vector space F_p is isomorphic to the restriction to p of the kernel vector bundle $F = \ker\{\tilde\sigma|_U : \mathcal{E}|_U \longrightarrow \mathcal{O}_U\}$, and the fiber of the cone $C_{\hat M}\hat V$ over p is identical to the fiber over p of the cone W [20, Lemma 3.3]. Therefore, $W|_p \subset F|_p$. Since p is arbitrary, this proves that the support of W over U is entirely contained in the subbundle $F \subset E|_U$.

We now prove the claim. Indeed, the existence of ψ_1 and ψ_2 follows from the definition of the perfect obstruction theory based on the cotangent complex of M phrased in [3]. In case we use the perfect obstruction theory phrased in [20], their existence follows directly from the proof of Lemma 2.5 in [20], once the following technical requirement is met[2].

[2] This requirement is the consistency of the properties (1) and (3) for $k = 1$ in [20, Lemma 2.5]. It was used but not checked. Here we provide the details of it.

To state and verify this requirement, we let $p \in M$ be as before (and M is a scheme as assumed) and let $S \subset M$ be an affine neighborhood of p. Roughly speaking, the requirement is that we can find a complex of locally free sheaves $[\mathcal{E}_1 \xrightarrow{\sigma} \mathcal{E}_2]$ so that in addition to that the cokernel of σ and its dual $\sigma^\vee$ are $\mathcal{O}b_S$ and Ω_S, respectively, the homomorphism σ defines the obstruction to first order extensions.

We now set up more notation. We embed S as a closed subscheme of a smooth affine scheme V of $\dim V = \dim T_pS$. We let $A = \Gamma(\mathcal{O}_S)$, let $B = \Gamma(\mathcal{O}_V)$, let $\iota^* : B \to A$ be the quotient homomorphism with $I \subset B$ the ideal $I = \iota^{*-1}(0)$; thus $B/I = A$. We then consider the trivial (ring) extension of A by $\Omega_B \otimes_B A$: $A_2 = A \oplus \Omega_B \otimes_B A$; we let $d : B \to \Omega_B$ be the differential and consider the homomorphism $\xi : B \to A_2$ defined via $b \mapsto (\iota^* b, db)$. Then by taking $J \subset A_2$ the ideal generated by $\xi(I)$ and letting $A_1 = A_2/J$, the homomorphism ξ descends to $\xi_1 : A \to A_1$. Since $J^2 = 0$, the triple (A_1, A_2, ξ_1) associates to an obstruction class

$$o \in Ob_A \otimes_A J, \quad Ob_A = \Gamma(S, \mathcal{O}b_M),$$

to lifting ξ_1 to $A \to A_2$. Now let E_2 be a free A-module making Ob_A its quotient module. We lift o to an $\hat{o} \in E_2 \otimes_A J$, which via $E_2 \otimes_A J \subset E_2 \otimes_B \Omega_B$ defines a homomorphism

$$\psi : E_2^\vee \longrightarrow \Omega_B \otimes_B A.$$

We claim that the cokernel of ψ and $\psi^\vee$ are Ω_A and Ob_A, respectively. After this, the complex $[\Omega_B^\vee \otimes_B A \to E_2]$ will satisfy the requirement for Lemma 2.5 in [20] for $k = 1$; its inductive proof provides us with the Kuranishi map we seek for.

We now prove the claim. We let R be the cokernel of ψ and shall prove that as quotient sheaves of $\Omega_B \otimes_B A$, $R = \Omega_A$. First, by viewing J as an A-module, it is a submodule of $\Omega_B \otimes_B A$ satisfying $\Omega_B \otimes_B A/J = \Omega_A$. Because ψ is defined by an element in $E_2 \otimes_A J$, $\psi(E_2^\vee) \subset J$. Hence Ω_A is canonically a quotient sheaf of R, say via $\tau : R \to \Omega_A$. To show that τ is an isomorphism, we let $T = A \oplus R$, let $T_0 = A \oplus \Omega_A$, let $K = \ker(\tau)$, and let $f : A \to A \oplus \Omega_A$ be the tautological homomorphism defined via $a \mapsto (a, da)$. Then the triple (T, T_0, f) associates to an obstruction class $\bar{o} \in Ob_A \otimes_A K$ to lifting f to $A \to A \oplus R$. However, by the base change property, $\bar{o}$ is the image of o under $Ob_A \otimes_A J \to Ob_A \otimes_A K$. Since $\hat{o}$ is a lift of $o \in Ob_A \otimes_A J$, by the commutativity

$$\begin{array}{ccc} E_2 \otimes_A J & \longrightarrow & E_2 \otimes_A K \\ \big\downarrow & & \big\downarrow \\ Ob_A \otimes_A J & \longrightarrow & Ob_A \otimes_A K \end{array}$$

and by the fact that the element $\hat{o}$ has vanishing image under $E_2 \otimes_B \Omega_B \to E_2 \otimes_A R$, we see immediately that $\bar{o} = 0$. Thus f lifts to $A \to A \oplus R$. By the property of the cotangent module, this lifting is given by a homomorphism of A-modules $\varphi : \Omega_A \to R$ so that the composite $\tau \circ \varphi : \Omega_A \to \Omega_A$ is the homomorphism associated to $f : A \to A \oplus \Omega_A$, thus $\tau \circ \varphi = id_A$. It follows that Ω_A is a direct summand of R. But for $\mathfrak{m} \subset A$ the maximal ideal of $p \in S$, we have $R \otimes_A A/\mathfrak{m} = \Omega_A \otimes A/\mathfrak{m}$; thus possibly after shrinking S, $\Omega_A = R$. This proves that $R = \Omega_A$.

It remains to show that $\psi^\vee : \Omega_B^\vee \otimes_B A \to E_2$ has cokernel Ob_A. The proof is similar. We first show that the composite $\Omega_B^\vee \otimes_B A \to E_2 \to Ob_A$ is trivial. Let $T_0' = A$ and let $T_1' = A \oplus \Omega_B \otimes_B A$. The identity $id : A \to T_0'$ obviously lifts to $A \to T_1'$. Thus the obstruction $\tilde{o}$ to lifting id to $A \to T_1'$ is trivial. But by the base change property, $\tilde{o}$ is the image of $o \in Ob_A \otimes_A J$ under $Ob_A \otimes_A J \to Ob_A \otimes_B \Omega_B$, which also is the composite $\Omega_B^\vee \otimes_B A \to E_2 \to Ob_A$. Thus this composite vanishes, which proves that as quotient modules of E_2, $\operatorname{coker}(\psi^\vee)$ surjects onto Ob_A.

To prove $\operatorname{coker}(\psi^\vee) = Ob_A$, we shall use the property that S has a perfect obstruction theory. Namely, there is a free A-module E_1 and a homomorphism $\eta\colon E_1 \to E_2$ such that $\operatorname{coker}(\eta) = \mathcal{O}b_A$ and $\operatorname{coker}(\eta^\vee) = \Omega_A$. We now let $Q_1 = \operatorname{Im}(\psi^\vee)$ and $Q_2 = \operatorname{Im}(\eta)$, both as submodules of E_2. Because $\operatorname{coker}(\psi^\vee) \cong \operatorname{coker}(\eta^\vee)$, for every integer m, $Q_1 \otimes_A A/\mathfrak{m}^m$ and $Q_2 \otimes_A A/\mathfrak{m}^m$ have the same dimension as vector spaces; the same holds true for $\operatorname{Tor}^1(Q_1, A/\mathfrak{m}^m)$ and $\operatorname{Tor}^1(Q_2, A/\mathfrak{m}^m)$. Thus

$$\dim_{\mathbb{C}} Ob_A \otimes A/\mathfrak{m}^n = \dim_{\mathbb{C}} \operatorname{coker}(\psi^\vee) \otimes_A A/\mathfrak{m}^n;$$

since one is the quotient of the other, this implies that $Ob_A \otimes_A A/\mathfrak{m}^n = \operatorname{coker}(\psi^\vee) \otimes_A A/\mathfrak{m}^n$ for all m. Hence after shrinking S if necessary, $\operatorname{coker}(\psi^\vee) = Ob_A$. This completes the proof of the Lemma. □

With this lemma, we are ready to construct the localized virtual cycle

$$[M]^{\mathrm{vir}}_{\mathrm{loc}} \in H^{BM}_*(Z(\sigma)).$$

We first write $W = \sum m_i W_i$ as the weighted sum of (reduced) irreducible closed substacks $W_i \subset E$. To each such W_i, we let M_i be the image stack of $W_i \to M$ and pick a quasi-projective Z_i together with a proper and generically finite $\rho_i : Z_i \longrightarrow M_i$. We then pull back the pair $W_i \subset E|_{M_i}$ and the cosection:

$$\tilde{W}_i = W_i \times_M Z_i \subset \tilde{E}_i = E \times_M Z_i, \quad \tilde{\sigma}_i = \rho_i^*(\tilde{\sigma}) : \tilde{E}_i \longrightarrow \mathcal{O}_{Z_i}.$$

By the previous lemma, the cycle $\tilde{W}_i$ lies in the cone kernel $\tilde{E}_i(\tilde{\sigma}_i)$. Hence we can apply the localized Gysin map to the class $[\tilde{W}_i]$ to obtain

$$s^!_{\tilde{E}_i,\mathrm{loc}}[\tilde{W}_i] \in H^{BM}_*(Z(\tilde{\sigma}_i)).$$

We now let $\eta_i\colon Z(\tilde{\sigma}_i) \to Z(\sigma)$ be the induced map. Because ρ_i is proper, η_i is also proper. Thus it induces a homomorphism of Borel-Moore homology

$$\eta_{i*} : H^{BM}_*(Z(\tilde{\sigma}_i)) \longrightarrow H^{BM}_*(Z(\sigma)).$$

Finally, we let $\deg(\rho_i)$ be the degree of ρ_i, and define the localized virtual cycle

$$[M]^{\mathrm{vir}}_{\mathrm{loc}} = \sum \frac{m_i}{\deg(\rho_i)} \eta_{i*}\big(s^!_{\tilde{E}_i,\mathrm{loc}}[\tilde{W}_i]\big) \in H^{BM}_*(Z(\sigma)).$$

2.4. **Deformation invariance of the localized virtual cycles.** Like the ordinary virtual cycle, the localized virtual cycle is expected to remain constant under deformation of complex structures. In the following, we shall prove this under a technical assumption.

We let $t \in T$ be a pointed smooth affine curve; let $\pi\colon M \to T$ be a DM-stack over T with a perfect obstruction theory and obstruction sheaf $\mathcal{O}b_M$; we suppose M has a perfect relative obstruction theory with relative obstruction sheaf $\mathcal{O}b_{M/T}$, as defined in [20]. By definition, $\mathcal{O}b_M$ and $\mathcal{O}b_{M/T}$ fits into the exact sequence

$$\longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}b_{M/T} \longrightarrow \mathcal{O}b_M \longrightarrow 0. \tag{2.6}$$

Note that the restriction of $\mathcal{O}b_{M/T}$ to each fiber $M_t = M \times_T t$ is the obstruction sheaf $\mathcal{O}b_{M_t}$ of M_t.

We now suppose there is a cosection

$$\sigma : \mathcal{O}b_{M/T} \longrightarrow \mathcal{O}_M.$$

We let $Z(\sigma)$ be the union of $Z(\sigma_t) \subset M_t$ for all $t \in T$.

Proposition 2.6. *Suppose $Z(\sigma)$ is proper over T; suppose the cosection σ lifts to a cosection $\sigma' : \mathcal{O}b_M \to \mathcal{O}_M$. Then the localized virtual cycles $[M_t]^{\mathrm{vir}}_{\mathrm{loc}}$ are constant in t as classes in $H_*(Z(\sigma))$.*

Proof. We shall prove the proposition by showing that applying the localized Gysin map to the rational equivalence used in deriving the deformation invariance of the ordinary virtual cycles will provide us with the homologous relation necessary for the constancy of the classes $[M_t]^{\mathrm{vir}}_{\mathrm{loc}} \in H_*(Z(\sigma))$.

Without loss of generality, we can assume that M is a quasi-projective scheme and E is a vector bundle on M whose sheaf of sections $\mathcal{E} = \mathcal{O}_M(E)$ makes $\mathcal{O}b_M$ its quotient sheaf[3]. Then according to the construction of virtual cycles, the obstruction theory of M provides us with a unique cone cycle $W \in Z_* E$ whose intersection with the zero section of E is the virtual cycle of M.

For us, we shall use the localized Gysin map to derive a localized virtual cycle of M. We let $\tilde{\sigma}' : \mathcal{E} \to \mathcal{O}_M$ be the composite of $\mathcal{E} \to \mathcal{O}b_M$ with σ', and let $E(\tilde{\sigma}')$ be the kernel cone of $\tilde{\sigma}'$. Then Lemma 2.5 tells us that W is a cycle in $Z_* E(\tilde{\sigma}')$. Because $\mathcal{O}b_{M/T} \to \mathcal{O}b_M$ is surjective, $Z(\sigma') = Z(\sigma)$, which by assumption is proper over T. Thus by applying the localized Gysin map we obtain a homology class

$$s^!_{E,\mathrm{loc}}([W]) \in H^{BM}_*(Z(\sigma)).$$

Further, because T is smooth, for each closed point $t \in T$ the inclusion $t \hookrightarrow T$ defines a Gysin homomorphism

$$t^! : H^{BM}_*(Z(\sigma)) \longrightarrow H_*(Z(\sigma_t)).$$

Here the image lies in the ordinary homology group because $Z(\sigma)$ is proper over T.

By the elementary property of Gysin homomorphism, if we let $\iota_t : Z(\sigma_t) \to Z(\sigma)$ be the inclusion, then the classes

$$\iota_{t*}\left(t^!\left(s^!_{E,\mathrm{loc}}([W])\right)\right) \in H_*(Z(\sigma))$$

are constant in t. Therefore, to prove the proposition we only need to prove that

$$t^!\left(s^!_{E,\mathrm{loc}}([W])\right) = [M_t]^{\mathrm{vir}}_{\mathrm{loc}} \in H_*(Z(\sigma_t)). \tag{2.7}$$

Ordinarily, we prove this by applying the Gysin map to a rational equivalence between the cycle W and the cycle used to define the virtual cycle $[M_t]^{\mathrm{vir}}$. In our case, to get an identity in ordinary homology classes, we need to make sure that the rational equivalence lies inside a cone so that Proposition 2.4 can be applied.

To this end, a quick review of the construction of the rational equivalence is in order. First, using the homomorphism $\mathcal{O}_{M_t} \to \mathcal{O}b_{M_t}$ induced by the first arrow in (2.6), and possibly after replacing $\mathcal{E} \to \mathcal{O}b_M$ by another quoient sheaf, we can write $\mathcal{O}b_{M_t}$ as a quotient sheaf of $\mathcal{O}_{M_t} \oplus \mathcal{E}_t$ for $\mathcal{E}_t = \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{O}_{M_t}$. Then the composition

$$\mathcal{O}_{M_t} \oplus \mathcal{E}_t \xrightarrow{\phi} \mathcal{O}b_{M_t} \xrightarrow{\sigma_t} \mathcal{O}_{M_t}$$

defines a cosection ζ_t of $\mathcal{O}_{M_t} \oplus \mathcal{E}_t$. Because σ lifts to σ', $\mathcal{O}_{M_t}$ lies in its kernel. Hence, over $M_t - Z(\sigma_t)$ the kernel of ζ_t is a direct sum of the trivial line bundle $\mathbf{1}$ on M_t with $F_t = F|_{M_t}$, the restriction of the kernel F of $\sigma : E \to \mathcal{O}_M$. As before, we write $(\mathbf{1} \oplus E_t)(\zeta_t)$ for the kernel cone of ζ_t.

With the quotient sheaf $\mathcal{O}_{M_t} \oplus \mathcal{E}_t \to \mathcal{O}b_{M_t}$, the obstruction theory of M_t provides us a unique cone cycle $W'_t \subset \mathbf{1} \oplus E_t$ whose support, according to Lemma 2.5, lies

[3]The general case can be treated using the technique developed in [17].

in $(\mathbf{1}\oplus E_t)(\zeta_t)$. Because $Z(\sigma_t)$ is proper, the localized Gysin map then defines the localized virtual cycle

$$[M_t]^{\text{vir}}_{\text{loc}} = s^!_{\mathbf{1}\oplus E_t,\text{loc}}[W'_t] \in H_*(Z(\sigma_t)).$$

The cycle W'_t, up to a factor $\mathbb{A}^1$, is rationally equivalent to the normal cone $C_{W\times_T t}W \subset E_t \oplus \mathbf{1}$. As argued in [20, page 146], the rational equivalence lies in the total space of the vector bundle $L := \mathbf{1}\oplus\mathbf{1}\oplus E_t$. We let

$$\pi_{23} : L \to \mathbf{1}\oplus E_t \quad \text{and} \quad \pi_{13} : L \longrightarrow \mathbf{1}\oplus E_t, \tag{2.8}$$

be the projections to the indicated factors of $L = \mathbf{1}\oplus\mathbf{1}\oplus E_t$. We let $B_1 = \pi_{23}^* W'_t$; let $B_2 = \pi_{13}^* C_{W\times_T t}W$; let

$$\tilde\zeta : L \xrightarrow{pr_3} E_t \xrightarrow{\tilde\sigma'} \mathcal{O}_{M_t}$$

and let $L(\tilde\zeta)$ be the kernel cone of the homomorphism $\tilde\zeta$. Clearly, both B_1 and $B_2 \in Z_* L(\tilde\zeta)$, and

$$s^!_{L,\text{loc}}[B_1] = [M_t]^{\text{vir}}_{\text{loc}} \quad \text{and} \quad s^!_{L,\text{loc}}[B_2] = t^![M]^{\text{vir}}_{\text{loc}}$$

On the other hand, as shown in [20, page 146] (see also [3, 13]) there is a canonical rational equivalence $Q \in R_* L$ (here $R_* L$ is the set of rational equivalence in L) so that $\partial Q = B_1 - B_2$. We claim that

$$\operatorname{Supp}(Q) \subset L(\tilde\zeta). \tag{2.9}$$

Note that once this is proved, then B_1 and B_2 are rationally equivalent cycles in $Z_* L(\tilde\zeta)$. By Proposition 2.4, we will have

$$s^!_{L,\text{loc}}[B_1] = s^!_{L,\text{loc}}[B_2] \in H_*(Z(\sigma)).$$

We now prove (2.9). We let $p \in M_t - Z(\sigma_t)$ be any closed point. Following the notation introduced in the proof of Lemma 2.5 we let $f \in \mathbf{k}[\![z]\!]\otimes E_p$ be a Kuranishi map of M at p. Then $\hat M = (f=0) \subset \hat V = \operatorname{Spec}\mathbf{k}[\![z]\!]$ and the fiber over p of the cone W is identical to the closed fiber of the normal cone $C_{\hat M}\hat V$ to $\hat M$ in $\hat V$.

The fiber $W'_t|_p$ has a similar description. We pick an $\tilde h \in \mathcal{O}_T$ so that $\tilde h^{-1}(0) = \{t\}$ and pick a lifting $h \in \mathcal{O}_{\hat V}$ of $\tilde h$. Then as shown in [20, page 144], the pair $(h,f) \in \mathbf{k}[\![z]\!]\otimes(\mathbb{C}\oplus E_p)$ is a Kuranishi map of M_t at p. Hence the formal completion $\hat M_t$ of M_t at p is isomorphic to $(h=f=0) \subset \hat V_t = \operatorname{Spec}\mathbf{k}[\![z]\!]/(h)$. Likewise, the fiber at p of the cone W'_t is identical to the closed fiber of the normal cone $C_{\hat M_t}\hat V_t$.

Because f is a Kuranishi map of M at p, as argued in (2.5), we have an isomorphism $\operatorname{coker}(df) \cong \rho^*\mathcal{O}b_M$ and the surjective composite homomorphism

$$\mathcal{O}_{\hat M}\otimes E_p \longrightarrow \operatorname{coker}(df) \;\cong\; \rho^*\mathcal{O}b_M \xrightarrow{\rho^*(\sigma')} \mathcal{O}_{\hat M}. \tag{2.10}$$

We let $\mathcal{K}_0 \subset \mathcal{O}_{\hat M}\otimes E_p$ be the kernel (subsheaf) of (2.10) and let $\mathcal{K}_1 \subset \mathcal{O}_{\hat M}\otimes E_p$ be its complementary subsheaf:

$$\mathcal{O}_{\hat M}\otimes E_p = \mathcal{K}_0 \oplus \mathcal{K}_1.$$

Because $\mathcal{O}_{\hat M}\otimes E_p$ is locally free, we can extend this decomposition to that over $\hat V$,

$$\mathcal{O}_{\hat V}\otimes E_p = \mathcal{K}_0 \oplus \mathcal{K}_1 \tag{2.11}$$

still labeled as $\mathcal{K}_0$ and $\mathcal{K}_1$. Because $\mathbf{k}[\![z]\!]\otimes E_p = \Gamma(\mathcal{O}_{\hat V}\otimes E_p)$, f can be viewed as a section of $\mathcal{O}_{\hat V}\otimes E_p$, thus decomposed as (f_0, f_1) under the decomposition (2.11).

Sublemma 2.7. *The subscheme $\hat{M}_{red} \subset \hat{V}$, which is $\hat{M}$ endowed with the reduced scheme structure, is identical to the subscheme $\hat{M}'_{red} = (f_0 = 0)_{red} \subset \hat{V}$; the support of the normal cone $C_{\hat{M}}\hat{V}$ to $\hat{M}$ in $\hat{V}$ coincides with the support of the normal cone $C_{\hat{M}'}\hat{V}$, both as subcones in $\hat{M} \times E_p$. The same conclusion holds with $\hat{M} \subset \hat{V}$ replaced by $\hat{M}_t \subset \hat{V}_t$ and $\hat{M}'$ replaced by $\hat{M}'_t = (h = f_0 = 0) \subset \hat{V}_t$.*

We shall prove the Sublemma momentarily.

Granting the Sublemma, we now prove the Proposition. To begin with, we let $\hat{K}_0$ and $\hat{K}_1$ be the associated vector bundles on $\hat{M}_t$ of the sheaf $\mathcal{K}_0$ and $\mathcal{K}_1$, let $\hat{\mathbf{1}}$ be the trivial line bundle on $\hat{M}_t$, and let $\hat{L}$ be the direct sum bundle $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus (\hat{K}_0 \oplus \hat{K}_1)$. As before, we let

$$\hat{\pi}_{13} : \hat{L} \longrightarrow \mathbf{1} \oplus (\hat{K}_0 \oplus \hat{K}_1); \quad \hat{\pi}_{23} : \hat{L} \longrightarrow \hat{\mathbf{1}} \oplus (\hat{K}_0 \oplus \hat{K}_1)$$

be the projections similar to those in (2.8). Then by working with subschemes M_t and $\hat{M} \subset \hat{V}$, we obtain a rational equivalence $\hat{Q}$ in $\hat{L}$ so that

$$\partial \hat{Q} = \hat{\pi}_{23}^* C_{\hat{M}_t} \hat{V}_t - \hat{\pi}_{13}^* C_{C_{\hat{M}}\hat{V} \times_T t} C_{\hat{M}} \hat{V}.$$

Working with the subschemes $\hat{M}'_t$ and $\hat{M}' \subset \hat{V}$ instead, we obtain another cone $\hat{Q}'$ in $\hat{L}$ so that

$$\partial \hat{Q}' = \hat{\pi}_{23}^* C_{\hat{M}'_t} \hat{V}_t - \hat{\pi}_{13}^* C_{C_{\hat{M}'}\hat{V} \times_T t} C_{\hat{M}'} \hat{V}.$$

However, since $\hat{M}'$ is defined by the vanishing of f_0, which is a section of $\mathcal{K}_0$ over $\hat{V}$, the support of the rational equivalence $\hat{Q}'$ naturally lies in $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{K}_0$. On the other hand, following the explicit construction of the rational equivalence by Kresch [13], the support of the rational equivalence $\hat{Q}$ only depends on the supports of the subschemes $\hat{M}$ and $\hat{M}_t$, and on the supports of the cones $C_{\hat{M}_t}\hat{V}_t$ and $C_{C_{\hat{M}}\hat{V} \times_T t} C_{\hat{M}} \hat{V}$. Therefore, though $\hat{Q}$ may be different from $\hat{Q}'$, their supports are identical. Consequently, the support of $\hat{Q}$ lies entirely in $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{K}_0$.

Finally, because the fiber over p of the rational equivalence Q is identical to that of $\hat{Q}$, we conclude that the support of Q over $M - Z(\sigma_t)$ lies inside the subbundle $\mathbf{1} \oplus \mathbf{1} \oplus F$. This proves the proposition. □

Proof of Sublemma. We first show that $\hat{M}_{red} = \hat{M}'_{red}$. Suppose not. Then we can find a map $\rho : \mathbf{k}[\![t]\!] \to \hat{M}$ that does not factor through $\hat{M}' \subset \hat{M}$. Because $\hat{M}$ and $\hat{M}'$ are defined by $f_0 = f_1 = 0$ and by $f_0 = 0$ respectively, we must have $f_0 \circ \rho \equiv 0$ while $f_1 \circ \rho \not\equiv 0$. Hence for some n and $a \neq 0 \in \mathcal{K}_1 \otimes k(\bar{p})$, $f_1 \circ \rho(t) \equiv at^n \mod t^{n+1}$. Thus by the proof of Lemma 2.2, $a \in C_{\hat{M}}\hat{V}$, violating the conclusion of Lemma 2.2 that $C_{\hat{M}}\hat{V} \subset \hat{K}_0$. For the same reason, the proof of Lemma 2.2 shows that $C_{\hat{M}}\hat{V} \subset \hat{K}_0$ forces that as sets, $C_{\hat{M}}\hat{V} \times_{\hat{M}} \bar{p} = C_{\hat{M}'}\hat{V} \times_{\hat{M}} \bar{p}$.

The same argument works for the case $\hat{M}_t$ and $\hat{M}'_t \subset \hat{V}_t$. This proves the Sublemma. □

3. Localized GW-invariants

We continue to let X be a smooth quasi-projective variety endowed with a holomorphic two-form θ. This form induces a cosection of the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ of the moduli space $\mathcal{M}_{g,n}(X, \beta)$. We assume that the degeneracy locus $Z(\sigma)$ of σ is proper. In this case, we can pair the localized virtual cycle with the tautological classes on $\mathcal{M}$ to define the localized GW-invariants of (X, θ).

In this section, we shall also prove the deformation invariance of such invariants under some technical condition; we shall derive a formula of the localized invariants in a special case, sufficient to recover all GW-invariants of surfaces without descendants, first proved by Lee-Parker [15].

3.1. **Cosection of the obstruction sheaf.** Let X be a smooth complex quasi-projective variety endowed with a nontrivial holomorphic two-form $\theta\in\Gamma(\Omega^2_X)$. The two-form θ can be viewed as an anti-symmetric homomorphism

$$\hat\theta: T_X\longrightarrow \Omega_X,\quad (\hat\theta(v),v)=0 \tag{3.1}$$

from the tangent bundle to the cotangent bundle. Such a two-form will induce a cosection of the obstruction sheaf of the moduli space $\mathcal{M}_{g,n}(X,\beta)$. For convenience, in case the data g, n, X and β are understood implicitly, we shall abbreviate $\mathcal{M}_{g,n}(X,\beta)$ to $\mathcal{M}$.

Let S be an open subset of $\mathcal{M}$; let $f:\mathcal{C}\to X$ and $\pi:\mathcal{C}\to S$ be the universal family over S; let

$$R^1\pi_*f^*T_X\longrightarrow R^1\pi_*f^*\Omega_X\longrightarrow R^1\pi_*\Omega_{\mathcal{C}/S}\longrightarrow R^1\pi_*\omega_{\mathcal{C}/S}, \tag{3.2}$$

be the sequence of homomorphisms in which the first arrow is induced by (3.1) and the second is induced by $f^*\Omega_X\to\Omega_{\mathcal{C}/S}$. Because $R^1\pi_*\omega_{\mathcal{C}/S}\cong\mathcal{O}_S$, this sequence provides us with a canonical homomorphism

$$\sigma_f: R^1\pi_*f^*T_X\longrightarrow\mathcal{O}_S. \tag{3.3}$$

Lemma 3.1. *The composition*

$$\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/S},\mathcal{O}_\mathcal{C})\longrightarrow R^1\pi_*f^*T_X\longrightarrow\mathcal{O}_S, \tag{3.4}$$

in which the first arrow is induced by $f^*\Omega_X\to\Omega_{\mathcal{C}/S}$ *and the second arrow by* σ_f, *is a trivial homomorphism.*

Proof. The natural homomorphism $f^*\Omega_X\to\Omega_{\mathcal{C}/S}$ coupled with $\hat\theta$ induces a sequence of homomorphisms and their composite

$$\Theta:\Omega^\vee_{\mathcal{C}/S}\longrightarrow f^*T_X\xrightarrow{\hat\theta} f^*\Omega_X\longrightarrow\Omega_{\mathcal{C}/S}.$$

Because $\hat\theta$ is anti-symmetric, this composite is also anti-symmetric

$$(\Theta(v),v)=0,\quad \forall v\in\Omega^\vee_{\mathcal{C}/S}.$$

Because $\Omega_{\mathcal{C}/S}$ has rank one, Θ is trivial over the locus where $\Omega_{\mathcal{C}/S}$ is locally free.

We now prove that the composite

$$\bar\Theta:\omega^\vee_{\mathcal{C}/S}\longrightarrow\Omega_{\mathcal{C}/S} \tag{3.5}$$

of the tautological $\omega^\vee_{\mathcal{C}/S}\to\Omega^\vee_{\mathcal{C}/S}$ with Θ is trivial. Because Θ vanishes at general points, we only need to show that $\bar\Theta$ is trivial at a node, say q, of the a fiber of $\mathcal{C}/S$. Since the latter is a local problem, we can assume that $\mathcal{C}$ is a family of affine curves. Then by shrinking S if necessary we can realize $\mathcal{C}/S$ as a subfamily of $\tilde{\mathcal{C}}/\tilde S$ via $S\subset\tilde S$ and $\mathcal{C}=\tilde{\mathcal{C}}\times_{\tilde S}S$ so that the node q is smoothed within the family $\tilde{\mathcal{C}}$ and the morphism $f:\mathcal{C}\to X$ is extended to $\tilde f:\tilde{\mathcal{C}}\to X$.

For the family $\tilde f$, we form the similar homomorphism

$$\tilde\Theta:\omega^\vee_{\tilde{\mathcal{C}}/\tilde S}\longrightarrow\Omega_{\tilde{\mathcal{C}}/\tilde S}. \tag{3.6}$$

Following the definition, its restriction to $\mathcal{C}/S \subset \tilde{\mathcal{C}}/\tilde{S}$ is the homomorphism $\bar{\Theta}$ in (3.5). But for $\tilde{\Theta}$, it must be trivial since it is trivial at general points and since $\Omega_{\tilde{\mathcal{C}}/\tilde{S}}$ is torsion free. Therefore, (3.5) must be trivial as well.

If we apply $\mathcal{E}xt^1_\pi(-,\mathcal{O}_\mathcal{C})$ to $\bar{\Theta}$, we obtain the vanishing of the composition

$$\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/S},\mathcal{O}_\mathcal{C}) \longrightarrow R^1\pi_* f^*T_X \to R^1\pi_*\omega_{\mathcal{C}/S} \cong \mathcal{O}_S$$

which is exactly (3.4) by definition. □

Because the cokernel of $\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/S},\mathcal{O}_\mathcal{C}) \to R^1\pi_* f^*T_X$ is the restriction to S of the obstruction sheaf $\mathcal{O}b_\mathcal{M}$ of $\mathcal{M}_{g,n}(X,\beta)$, the preceding lemma gives us a canonical cosection $\sigma_S : \mathcal{O}b_M|_S \to \mathcal{O}_S$. Because this construction is canonical, these σ_S for $S \subset \mathcal{M}$ descend to form a sheaf homomorphism

$$\sigma : \mathcal{O}b_\mathcal{M} \longrightarrow \mathcal{O}_\mathcal{M}. \tag{3.7}$$

This cosection is surjective away from those stable morphisms that are θ-null.

Definition 3.2. *A stable map $u : C \to X$ is called* θ-null *if the composite*

$$u^*(\hat{\theta}) \circ du : T_{C_{reg}} \longrightarrow u^*T_X|_{C_{reg}} \longrightarrow u^*\Omega_X|_{C_{reg}}$$

is trivial over the regular locus C_{reg} of C.

Lemma 3.3. *The degeneracy locus of the cosection σ of $\mathcal{O}b_\mathcal{M}$ is the collection of θ-null stable morphisms in $\mathcal{M}$.*

Proof. By definition, σ at $[u : C \to X] \in \mathcal{M}$ is the composition

$$H^1(C,u^*T_X) \xrightarrow{\hat{\theta}} H^1(C,u^*\Omega_X) \longrightarrow H^1(C,\Omega_C) \longrightarrow H^1(C,\omega_C) = \mathbb{C}$$

whose Serre dual is

$$\mathbb{C} = H^0(C,\mathcal{O}_C) \longrightarrow H^0(f^*T_X \otimes \omega_C) \longrightarrow H^0(f^*\Omega_X \otimes \omega_C).$$

Because $\mathcal{O}_C$ is generated by global sections, the composite of the above sequence is trivial if and only if the composite

$$T_C \otimes \omega_C|_{C_{\rm reg}} \longrightarrow f^*T_X \otimes \omega_C|_{C_{\rm reg}} \longrightarrow f^*\Omega_X \otimes \omega_C|_{C_{\rm reg}}$$

is trivial. But this is equivalent to u being θ-null. This proves the lemma. □

We summarize the above as follows.

Proposition 3.4. *Any holomorphic two-form $\theta \in H^0(\Omega^2_X)$ on a smooth quasi-projective variety X induces a cosection $\sigma : \mathcal{O}b_\mathcal{M} \longrightarrow \mathcal{O}_\mathcal{M}$ of the obstruction sheaf $\mathcal{O}b_\mathcal{M}$ of the moduli space $\mathcal{M}$. Its degeneracy locus $Z(\sigma)$ is the set of all θ-null stable maps in $\mathcal{M}$.*

3.2. **Localized GW-invariants.** Using the cosection σ constructed, we shall construct the localized GW-invariants of a pair (X,θ) of a smooth quasi-projective variety and a two-form $\theta \in H^0(\Omega^2_X)$.

Definition 3.5. *We say β is θ-proper if for any g, the subset of θ-null stable morphisms in $\mathcal{M}_{g,n}(X,\beta)$ is proper.*

Let $\beta \in H_2(X,\mathbb{Z})$ be a θ-proper. For any pair g and n, we continue to denote by $\mathcal{M} = \mathcal{M}_{g,n}(X,\beta)$; we let $\sigma : \mathcal{O}b_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ be the cosection constructed in the previous subsection. Because the degeneracy locus $Z(\sigma)$ is the subset of θ-null stable morphisms, it is proper because β is θ-proper. Therefore, by the result of the previous section, we have the localized virtual cycle

$$[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in H_*(Z(\sigma), \mathbb{Q}).$$

We now define the localized GW-invariants of the θ-proper class β as follows.

We let

$$\mathrm{ev} : \mathcal{M} \longrightarrow X^n$$

be the evaluation morphism, let $\gamma_1, \cdots, \gamma_n \in H^*(X)$, let $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}^{\geq 0}$, and let ψ_i be the first Chern class of the relative cotangent line bundle of the domain curves at the i-th marked point. Because the localized virtual cycle $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ is an ordinary homology class, via the tautological $H_*(Z(\sigma), \mathbb{Q}) \to H_*(\mathcal{M}, \mathbb{Q})$, we can view it as a homology class in $H_*(\mathcal{M}, \mathbb{Q})$ [4].

We define the localized GW-invariant of X with descendants to be

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{X,\mathrm{loc}}_{g,\beta} = \int_{[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}} \mathrm{ev}^*(\gamma_1 \times \cdots \times \gamma_n) \cdot \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$$

In case X is proper, every class is θ-proper.

Proposition 3.6. *If X is proper, the localized GW-invariant coincides with the ordinary GW-invariant of X.*

Proof. This follows directly from that when $\mathcal{M}$ is proper, the localized virtual cycle $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ coincides with the virtual cycle $[\mathcal{M}]^{\mathrm{vir}}$ as homology classes in $H_*(\mathcal{M})$. □

This immediately gives the following vanishing results of J. Lee and T. Parker [15, 14] using deformation of almost complex structures.

Theorem 3.7. *Let X be a smooth projective variety endowed with a holomorphic two-form θ. The virtual fundamental class of the moduli space $\mathcal{M}_{g,n}(X, \beta)$ vanishes unless the class β can be represented by a θ-null stable morphism.*

Let D be the closed subset of $p \in X$ such that the anti-symmetric homomorphism $\hat{\theta}(p) : T_{X,p} \to T^\vee_{X,p}$ induced by θ is not an isomorphism. Then for any $\beta \neq 0$ and $\gamma_i \in H^*(X)$, the GW-invariant $\langle \prod \tau_{\alpha_i}(\gamma_i) \rangle^X_{g,\beta}$ vanishes if one of the classes γ_i is Poincaré dual to a cycle disjoint from D.

Proof. Since $\mathcal{M} = \mathcal{M}_{g,n}(X, \beta)$ is proper, the class $[\mathcal{M}]^{\mathrm{vir}}$ coincides with the localized virtual class $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$, which as shown before is a class supported in the set of θ-null stable maps in $\mathcal{M}$. If β can not be represented by a θ-null stable morphism, then the set of θ-null stable morphisms in $\mathcal{M}$ is empty. Therefore, $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} = 0$, proving the first statement.

For the second statement, just observe that the image of a θ-null stable map should be contained in D when $\beta \neq 0$. □

Corollary 3.8. *Let (X, θ) be a pair of a smooth projective variety of dimension $2m$ and a holomorphic two-form θ; let θ^m be its top wedge. Then the degeneracy locus D of θ is $D = zero(\theta^m)$. Suppose $\beta \in H_2(X, \mathbb{Z})$ is not in the image of $H_2(D, \mathbb{Z})$ by the inclusion $D \hookrightarrow X$, then the virtual fundamental class $[\mathcal{M}_{g,n}(X, \beta)]^{\mathrm{vir}}$ is zero. In particular, when θ is nondegenerate, $[\mathcal{M}_{g,n}(X, \beta)]^{\mathrm{vir}} = 0$ for $\beta \neq 0$.*

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[4] It may happen that $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \neq 0$ while its image in $H_*(\mathcal{M})$ is trivial. In this case we notice that the localized GW-invariant just defined vanishes automatically. Thus viewing $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}$ as a class is $H_*(\mathcal{M})$ won't cause any trouble as long as the numerical GW-invariants are concerned.

3.3. **Deformation invariance of the localized GW-invariants.** Like the ordinary GW-invariants, the localized GW-invariants are expected to remain constant under deformation of complex structures. In the following, we shall prove this for the circumstances relevant to our study.

We consider a smooth family $\mathcal{X}/T$ of quasi-projective varieties over a connected smooth affine curve T; we assume that this family admits a regular homomorphic two-form $\theta \in \Gamma(\mathcal{X}, \Omega^2_{\mathcal{X}/T})$. We let $\beta \in H_2(\mathcal{X}, \mathbb{Z})$ be a (fiber) curve class and let

$$\mathcal{M} = \mathcal{M}_{g,n}(\mathcal{X}/T, \beta)$$

be the moduli space of stable morphisms to fibers of $\mathcal{X}/T$ of fundamental class β.

Let $f : \mathcal{C} \to \mathcal{X}$ and $\pi : \mathcal{C} \to \mathcal{M}$ be the universal family of this moduli stack; let $\kappa \in H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}/T})$ be the Kodaira-Spencer class of the first order deformation of $\mathcal{X}/T$— it is the extension class of the exact sequence of sheaves of tangent bundles

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/T} \longrightarrow \mathcal{T}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow 0.$$

As shown in [20], the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ of $\mathcal{M} = \mathcal{M}_{g,n}(\mathcal{X}/T, \beta)$ and the relative obstruction sheaf $\mathcal{O}b_{\mathcal{M}/T}$, which is the sheaf whose restriction to each fiber

$$\mathcal{M}_t = \mathcal{M} \times_T t = \mathcal{M}_{g,n}(\mathcal{X}_t, \beta)$$

is the obstruction sheaf $\mathcal{O}b_{\mathcal{M}_t}$ of $\mathcal{M}_t$, fit into the exact diagram:

$$\begin{array}{ccccc} \pi_* f^* \mathcal{O}_{\mathcal{X}} & \xrightarrow{f^*\kappa} & R^1\pi_* f^* \mathcal{T}_{\mathcal{X}/T} & \xrightarrow{\text{surj}} & \ker\big(R^1\pi_* f^*\mathcal{T}_{\mathcal{X}} \to R^1\pi_* f^*\mathcal{O}_{\mathcal{X}}\big) \\ \| & & \text{surj}\downarrow & & \text{surj}\downarrow \\ \mathcal{O}_{\mathcal{M}} & \longrightarrow & \mathcal{O}b_{\mathcal{M}/T} & \xrightarrow{\text{surj}} & \mathcal{O}b_{\mathcal{M}} \end{array} \tag{3.8}$$

Applying the previous construction, we check that the form θ induces a cosection of $R^1\pi_* f^* \mathcal{T}_{\mathcal{X}/T}$ that descends to a cosection

$$\sigma : \mathcal{O}b_{\mathcal{M}/T} \longrightarrow \mathcal{O}_{\mathcal{M}}.$$

The restriction of σ to each fiber $\mathcal{M}_t$ is the previously constructed cosection σ_t of $\mathcal{O}b_{\mathcal{M}_t}$. We let $Z(\sigma)$ be the union of $Z(\sigma_t) \subset \mathcal{M}_t$ for all $t \in T$.

Suppose $Z(\sigma)$ is proper over T, for each $t \in T$ we can define the localized GW-invariants of $\mathcal{X}_t$:

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{\mathcal{X}_t, \mathrm{loc}}_{g,\beta}, \quad \gamma_i \in H^*(\mathcal{X}, \mathbb{Z}).$$

The deformation invariance principle states that the above is independent of t.

In this section, we shall prove this principle for the localized GW-invariants for the circumstances relevant to our study.

According to Proposition 2.6, the constancy of the localized GW-invariants follows from the lifting of the homomorphism σ to a homomorphism

$$\sigma' : \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}},$$

which by the lower exact sequence in (3.8) amounts to the vanishing of the composite

$$\mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}b_{\mathcal{M}/T} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{M}}. \tag{3.9}$$

We will prove the vanishing of this composite under certain additional technical conditions.

The first case we will investigate concerns a smooth family of projective varieties $\mathcal{X}/T$ over a connected smooth affine curve T.

Lemma 3.9. *Under the assumption, the composite*

$$\mathcal{O}_{\mathcal{M}} = \pi_* f^* \mathcal{O}_{\mathcal{X}} \xrightarrow{f^*\kappa} R^1\pi_* f^* \mathcal{T}_{\mathcal{X}/T} \longrightarrow \mathcal{O}b_{\mathcal{M}/T} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{M}}$$

is a lift of a section of $\mathcal{O}_T$ *via the pullback* $p^*\mathcal{O}_T \to \mathcal{O}_{\mathcal{M}}$ *for* $p\colon \mathcal{M} \to T$.

Proof. Let $\alpha \in \Gamma(\mathcal{O}_{\mathcal{M}})$ be this section. Since T is reduced, we only need to check that to each closed $t \in T$ the restriction $\alpha_t = \alpha|_{\mathcal{M}_t}$ is a constant section of $\mathcal{O}_{\mathcal{M}_t}$.

For this, we contract the Kodaira-Spencer class $\kappa_t \in H^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t})$ of the family $\mathcal{X}/T$ with the holomorphic two-form $\theta_t \in \Gamma(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ to obtain a class $\theta_t(\kappa_t) \in H^1(\mathcal{X}_t, \Omega_{\mathcal{X}_t})$. Clearly, if we represent κ_t by a $\mathcal{T}_{\mathcal{X}_t}$-valued anti-holomorphic 1-form, $\theta_t(\kappa_t)$ becomes a $\overline{\partial}$-closed $(1,1)$-form, thus representing a Dolbeault cohomology class. Then since $\mathcal{X}_t$ is Kähler, by Hodge theory there is a d-closed $(1,1)$-form u and a $(1,0)$-form v so that $\theta_t(\kappa_t) = u + \overline{\partial} v$. But then, since the section $\alpha_t \in \Gamma(\mathcal{O}_{\mathcal{M}_t})$ is defined via the push-forward $\pi_* f^* \theta_t(\kappa_t)$, it is equal to $\pi_* f^* u + \pi_* f^* \overline{\partial} v = \pi_* f^* u$, because $\pi_* f^* \overline{\partial} v = 0$; since u is a cohomology class in $H^2(\mathcal{X}_t, \mathbb{C})$, α_t is identical to the constant $\int_\beta u \in \mathbb{C}$. This proves the lemma. □

Corollary 3.10. *Suppose further that the class* β *can be represented by a* θ_t*-null stable map. Then the pairing* $\int_\beta u$ *vanishes; so does the section* α_t*; and so does the composite (3.9).*

The second case is the deformation to the normal cone of a smooth canonical curve in a surface X. Let (X, θ) be a pair of a smooth surface and a holomorphic two-form with smooth degeneracy locus $D = \theta^{-1}(0)$. By blowing up $X \times \mathbb{A}^1$ along $D \times 0$ we obtain a family of proper surfaces $\pi\colon Z \to \mathbb{A}^1$. We then let $\tilde{Z} \subset Z$ be the complement of the proper transform of $X \times 0$ in Z. The restricted family $\tilde{Z} \to \mathbb{A}^1$ is the union of $X \times (\mathbb{A}^1 - 0)$ with the total space of the normal bundle $N_{D/X}$. In this case, since the sheaf of the tangent bundle $\mathcal{T}_Z$ restricted to the exceptional divisor $E \subset Z$ is isomorphic to $\mathcal{T}_E \oplus \mathcal{O}_E(-1)$, the Kodaira-Spencer class κ_0 of the family $\tilde{Z}/\mathbb{A}^1$ along the central fiber is trivial. Thus the conclusion of Corollary 3.10 holds for this family. We remark that by adjunction formula, $N_{D/X}^{\otimes 2} \cong K_D$. Thus $\tilde{Z}_0$ is the total space of a theta characteristic of D.

The last case we shall consider concerns the total space of theta characteristics of smooth curves. Recall that a theta characteristic of a smooth curve D is a line bundle L so that $L^{\otimes 2} \cong K_D$. Give such a pair (D, L), the total space X of the line bundle L is a surface whose canonical line bundle K_X is the pullback of L under the tautological projection $\pi\colon X \to D$. Thus the assignment that sends any $x \in X$ to the same $x \in L_{\pi(x)}$ defines a canonical section $\theta \in \Gamma(X, K_X)$; the vanishing locus of θ is exactly the zero section of π. We call this two-form the *standard holomorphic two-form* on X.

In the following, we let D_t be a smooth family of curves and L_t be a family of theta characteristics of D_t. The total spaces $\mathcal{X}_t$ of L_t form a smooth family $\mathcal{X}/T$ with $\theta \in \Gamma(\mathcal{X}, \Omega^2_{\mathcal{X}/T})$ the standard relative holomorphic two-form. We let $\beta \in H_2(\mathcal{X}, \mathbb{Z})$ be the class generated by the zero section of one of $\mathcal{X}_t$. As before, we denote $\mathcal{M}_{g,n}(\mathcal{X}/T, d\beta)$ by $\mathcal{M}$.

Lemma 3.11. *The conclusion of Lemma 3.9 holds for the family* $\mathcal{X}$ *and the form* θ.

Proof. To prove the lemma, we first check that for the Kodaira-Spencer class $\kappa_0 \in H^1(\mathcal{X}_0, T_{\mathcal{X}_0})$ of the family $\mathcal{X}/T$ and $(f, \mathcal{C})$ the universal family of $\mathcal{M}_0 =$

$\mathcal{M}_{g,n}(\mathcal{X}_0, d\beta)$, we have that the composite

$$\mathcal{O}_{\mathcal{M}_0} = \pi_* f^* \mathcal{O}_{\mathcal{X}_0} \longrightarrow R^1\pi_* f^* \mathcal{T}_{\mathcal{X}_0} \longrightarrow \mathcal{O}b_{\mathcal{M}_0} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{M}_0} \tag{3.10}$$

is locally constant.

We first describe the Kodaira-Spencer class κ_0, which depends on the family D_t. For simplicity, we shall work with analytic charts of D_0. We pick an analytic open $U \subset D_0$ so that U is isomorphic to the unit disk $\Delta \subset \mathbb{C}$. We then let $V = D_0 - A$ with $A \subset U$ a compact subset so that $U - A$ is isomorphic to an annulus. The two open sets U and V form an open covering of D_0. Since $H^1(T_U) = H^1(T_V) = 0$, $H^0(T_{U\cap V}) \to H^1(T_{D_0})$ is surjective. Hence, for small t the family D_t can be realized by an analytic deformation of the gluing map

$$U \supset U \cap V \xrightarrow{\cong} U \cap V \subset V.$$

In concrete terms, if we let z and w be the analytic coordinates of U and V near $U \cap V$, and let $z = f(w, 0)$ be the identity map of $U \cap V$ in coordinate variables z and w, then D_t can be realized by gluing U and V via $z = f(w, t)$ with $f(w, t)$ an analytic deformation of $f(w, 0)$.

To proceed, we need the transition function of $\mathcal{X}_t$. Because $D_t = U \cup V$, the surface $\mathcal{X}_t$ is the union of the total space of $K_U^{\frac{1}{2}}$ and $K_V^{\frac{1}{2}}$. As to the transition function of $\mathcal{X}_t$, we let $\xi = (dz)^{\frac{1}{2}}$ and $\eta = (dw)^{\frac{1}{2}}$ be bases of $K_U^{\frac{1}{2}}$ and $K_V^{\frac{1}{2}}$ over $U \cap V$. Then by adopting the convention that $f_w = \frac{\partial f}{\partial w}$ and $\dot{f} = \frac{df}{dt}$, the two pairs of local charts (z, ξ) and (w, η) are related by

$$z = f(w, t) \quad \text{and} \quad \xi = (f_w)^{\frac{1}{2}}\eta.$$

Accordingly, the Kodaira-Spencer class of the first order deformation of $\mathcal{X}_t$ at $t = 0$ can be represented by Čech 1-cocycle

$$\kappa_0(U \cap V) = \big(\frac{d}{dt}(f) \cdot \frac{\partial}{\partial z} + \frac{d}{dt}((f_w)^{\frac{1}{2}}\eta) \cdot \frac{\partial}{\partial \xi}\big)|_{t=0} = \big(\dot{f} \cdot \frac{\partial}{\partial z} + \frac{\xi \dot{f}_w}{2f_w} \cdot \frac{\partial}{\partial \xi}\big)|_{t=0}.$$

Because over $K_U^{\frac{1}{2}} \cap K_V^{\frac{1}{2}}$, the standard holomorphic two-form is $\theta_0 = \xi \, d\xi \wedge dz$, the contraction is

$$\theta_0(\kappa_0) = -\xi \dot{f} \, d\xi + \frac{1}{2}\xi^2 \dot{f}_w f_w^{-1} dz.$$

Therefore, using $\dot{f}_z = \dot{f}_w \frac{\partial w}{\partial z} = -\dot{f}_w f_w^{-1}$,

$$\partial(\theta(\kappa_0)) = -\xi \dot{f}_z dz \wedge d\xi + \xi \dot{f}_w f_w^{-1} d\xi \wedge dz = 0.$$

Combined with the fact that $\bar{\partial}\theta(\kappa_0) = 0$, we see that the form $\theta_0(\kappa_0)$ is d-closed.

The lemma now follows easily. We let $p : \mathcal{X}_0 \to D_0$ be the projection and let $\mathcal{N} \subset \mathcal{M}_0$ be the (analytic) open subset consisting of those $h : C \to \mathcal{X}_0$ so that $p \circ h : C \to D_0$ are unramified over $U \cap V$. We then pick an oriented embedded circle $S^1 \subset U \cap V$ that separates the two boundary components of $U \cap V$. An easy argument shows that the homomorphism (3.10) is the function, up to sign,

$$(f, C) \in \mathcal{M}_0 \longmapsto \int_{f^{-1}(S^1)} \theta_0(\kappa_0) \in \mathbb{C}.$$

Because $\theta_0(\kappa_0)$ is d-closed, this integral only depends on the topological class of $f^{-1}(S^1)$, hence must be locally constant over $\mathcal{N}$. But then this constant must be zero since it vanishes on those h so that $h(C) \subset D_0 \subset \mathcal{X}_0$, and since by dilation along fibers of L_0 each $h : C \to \mathcal{X}_0$ can be deformed to a stable map from C to $D_0 \subset \mathcal{X}_0$ within $\mathcal{N}$. This shows that (3.10) is zero over $\mathcal{N}$.

To complete the proof, we observe that for each stable map in $\mathcal{M}_0$, we can choose $U \subset D_0$ so that this stable map lies in the $\mathcal{N}$ associated to U. Therefore (3.10) must be zero on all $\mathcal{M}_0$, completing the proof of the lemma. □

3.4. **Reduction for surfaces.** Let S be a smooth general type minimal surface equipped with a holomorphic two-form θ whose vanishing locus is a canonical curve $B \in |K_S|$. In case B is smooth, B is connected and by the adjunction formula, $K_B = K_S^{\otimes 2}|_B$, the genus of B is $g(B) = K_S^2{+}1$ and the normal bundle $N_{B/S} \cong K_S|_B$ is a theta characteristic of B, namely $N_{B/S}^{\otimes 2} \cong K_B$, of parity ([15, 22])

$$h^0(B, N_{B/S}) \equiv \chi(\mathcal{O}_S) \mod 2.$$

Conversely, for a smooth projective curve D of genus $K_S^2 + 1$ and a theta characteristic L of D of parity $\chi(\mathcal{O}_S) \mod 2$, we take $p\colon X \to D$ to be the total space of the line bundle L and let $\theta \in H^0(X, \Omega_X^2)$ be the standard two-form on X. Since $\theta^{-1}(0) = D$ is proper, the localized GW-invariants of X are well-defined.

The GW-invariants of S are expected to be determined by the numerical date K_S^2 and $\chi(\mathcal{O}_S)$.

Conjecture 3.12. *Let S be a smooth minimal general type surface with positive $p_g > 0$. Its GW-invariants $\langle \cdots \rangle_{\beta,g}^S$ vanish unless β is a non-negative integral multiple of $c_1(K_S)$. In case $\beta = dc_1(K_S)$ for an integer $d > 0$, we let (D, L) be a pair of a smooth projective curve of genus $K_S^2 + 1$ and its theta characteristic with parity $\chi(\mathcal{O}_S)$, and let X be the total space of L. Then there is a canonical homomorphism $\rho\colon H^*(S, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ so that for any classes $\gamma_i \in H^*(S, \mathbb{Z})$ and integers $\alpha_i \geq 0$, $i = 1, \cdots, n$,*

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle_{\beta,g}^S = \langle \tau_{\alpha_1}(\rho(\gamma_1)) \cdots \tau_{\alpha_n}(\rho(\gamma_n)) \rangle_{d[D],g}^{X,\mathrm{loc}}.$$

In case S has a smooth canonical divisor $D \in |K_S|$, this is proved by Lee-Parker using symplectic geometry in [15]. In §3.3 above, we proved this algebraically by showing invariance of the localized virtual fundamental class under deformation of S to the normal bundle X of D. More generally, by constructing a deformation of complex structures of an analytic neighborhood U of $D \in |K_S|$ in S, we can confirm this conjecture for a wider class of surfaces. We shall address this in our forthcoming paper [12].

In light of this conjecture, in the remainder of this paper, we shall concentrate on studying the localized GW-invariants of a surface $p : X \to D$ that is the total space of a theta characteristic L of a smooth projective curve D together with its standard holomorphic two-form θ on X.

3.5. **Relation with the twisted invariants.** The localized GW-invariant $\langle \cdot \rangle_{d[D],g}^{X,\mathrm{loc}}$ is expected to relate to the twisted GW-invariants of the curve D.

To begin with, the projection $p\colon X \to D$ induces a morphism from the moduli of stable morphisms to X to the moduli of stable morphisms to D

$$\bar{p} : \mathcal{M}_{g,n}(X, d) \longrightarrow \mathcal{M}_{g,n}(D, d), \quad d > 0.$$

Here since $H_2(X, \mathbb{Z})$ is generated by the zero section $D \subset X$, we will use integer d to stand for the class $d[D]$. At individual map level, it is clear that in case $h\colon C \to X$

is a stable map, then $p\circ h\colon C\to D$ is a stable map and the original h is determined by a global section $H^0(C,(p\circ h)^*L)$. Thus fibers of $\bar p$ are vector spaces.

We next put this into family. For convenience, in the following, we shall fix g,n and d and abbreviate $\mathcal{M}_{g,n}(X,d)$ and $\mathcal{M}_{g,n}(D,d)$ to $\mathcal{M}$ and $\mathcal{N}$, respectively. We let

$$\tilde f:\tilde{\mathcal{C}}\longrightarrow X \quad\text{and}\quad \tilde\pi:\tilde{\mathcal{C}}\longrightarrow \mathcal{M}$$

be the universal family of $\mathcal{M}$ and let

$$f:\mathcal{C}\longrightarrow D \quad\text{and}\quad \pi:\mathcal{C}\longrightarrow \mathcal{N} \tag{3.11}$$

be the universal family of $\mathcal{N}$. Because $X\to D$ has affine fibers, the composite $p\circ\tilde f:\tilde{\mathcal{C}}\to D$ is a family of stable morphisms as well. Therefore, $\tilde{\mathcal{C}}=\mathcal{C}\times_{\mathcal{N}}\mathcal{M}$ and $p\circ\tilde f$ is the pullback of f.

We next pick two vector bundles E_1 and E_2 on $\mathcal{N}$ and a sheaf homomorphism

$$\alpha:\mathcal{O}_{\mathcal{N}}(E_1)\longrightarrow \mathcal{O}_{\mathcal{N}}(E_2)$$

whose cohomology gives us the complex $\pi_! f^*L$. By viewing $q\colon E_1\to\mathcal{N}$ as the total space of E_1 with q the projection, we can form the pullback bundle q^*E_2 on E_1 and the associated section $\bar\alpha\in\Gamma(E_1,q^*E_2)$.

Lemma 3.13. *The vanishing locus $\bar\alpha^{-1}(0)$ is the moduli stack $\mathcal{M}$.*

Proof. The proof is straightforward and will be omitted. □

It was hoped that the localized GW-invariants of X can be recovered by the twisted GW-invariants of D. As will be shown later, this unfortunately fails in general. However, in case the sheaf $R^1\pi_*f^*L$ is locally free, this is true up to sign.

In the remainder of this section, we assume that $R^1\pi_*f^*L$ is locally free. Accordingly, we can take $E_1=\pi_*f^*L$, $E_2=R^1\pi_*f^*L$ and $\alpha=0$. Thus $\mathcal{M}=\bar\alpha^{-1}(0)$ is the total space of E_1 and $q=\bar p$.

In the following, we shall investigate the cosection σ more closely.

Lemma 3.14. *Let the notation be as above and suppose $d>0$. Then there are two canonical homomorphisms ν and $\bar\theta^\vee$ as shown*

$$\mathcal{O}b_{\mathcal{M}}\xrightarrow{\ \nu\ } R^1\tilde\pi_*(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}})\xrightarrow{\ \bar\theta^\vee\ }\mathcal{O}_{\mathcal{M}} \tag{3.12}$$

so that their composite is the associated cosection σ of $\mathcal{O}b_{\mathcal{M}}$. Further, ν is surjective and the middle term above is isomorphic to $\bar p^*E_1^\vee$.

Proof. Because the two-form θ on X is a section of $p^*L=\Omega_X^2$, it provides a section and its dual:

$$\bar\theta\in H^0\big(\mathcal{M},\tilde\pi_*\tilde f^*p^*L\big)\quad\text{and}\quad \bar\theta^\vee\in \operatorname{Hom}\big(R^1\tilde\pi_*(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}),\mathcal{O}_{\mathcal{M}}\big). \tag{3.13}$$

Note that although $\tilde\pi_*\tilde f^*p^*L=\bar p^*\pi_*f^*L$, $\bar\theta$ is not a pullback from a section over $\mathcal{N}$ since it is injective on $\mathcal{M}-\mathcal{N}$ while vanishing on $\mathcal{N}$. Since $L\cong L^\vee\otimes K_D$, we further have a morphism

$$\tilde f^*p^*L\cong \tilde f^*p^*L^\vee\otimes\tilde f^*p^*K_D\longrightarrow \tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}$$

that induces a homomorphism of vector bundles

$$R^1\tilde\pi_*\tilde f^*p^*L\longrightarrow R^1\tilde\pi_*(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}); \tag{3.14}$$

this homomorphism is surjective because $H^0_C(\tilde f^*p^*L)\to H^0_C(\tilde f^*p^*L^\vee\otimes\omega_C)$, which is the Serre dual of $H^1_C(\tilde f^*p^*L)\to H^1_C(\tilde f^*p^*L^\vee\otimes\omega_C)$, is injective for any stable map $C\to X$ of degree $d>0$.

Obviously, this homomorphism is the pullback of the corresponding *surjective* homomorphism

$$E_2=R^1\pi_*f^*L\longrightarrow R^1\pi_*(f^*L^\vee\otimes\omega_{\mathcal{C}/\mathcal{N}})=E_1^\vee \tag{3.15}$$

over $\mathcal{N}$ via $\bar p$. We let V be the kernel vector bundle of this homomorphism.

We claim that the cosection $\sigma: Ob_{\mathcal{M}}\longrightarrow\mathcal{O}_{\mathcal{M}}$ defined in (3.7) factors through the homomorphism $\bar\theta^\vee$ in (3.13). Indeed, from the natural morphisms

$$\tilde f^*(T_X\otimes p^*L)=\tilde f^*(T_X\otimes\Omega^2_X)\longrightarrow\tilde f^*\Omega_X\longrightarrow\Omega_{\tilde{\mathcal{C}}/\mathcal{M}}\longrightarrow\omega_{\tilde{\mathcal{C}}/\mathcal{M}}$$

we obtain a homomorphism $\tilde f^*T_X\to\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}$ and thus a homomorphism

$$R^1\tilde\pi_*\tilde f^*T_X\longrightarrow R^1\tilde\pi_*(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}). \tag{3.16}$$

Similar to the proof of Lemma 3.1, its composition with the natural homomorphism

$$\mathcal{E}xt^1_{\tilde\pi}(\Omega_{\tilde{\mathcal{C}}/\mathcal{M}},\mathcal{O}_{\tilde{\mathcal{C}}})\longrightarrow R^1\tilde\pi_*\tilde f^*T_X$$

is zero. Thus (3.16) lifts to a homomorphism

$$\nu: Ob_{\mathcal{M}}\longrightarrow R^1\tilde\pi_*(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}), \tag{3.17}$$

which by construction satisfies $\bar\theta^\vee\circ\nu=\sigma$, as desired.

Finally, because $R^1\tilde\pi_*\tilde f^*p^*L\to\mathcal{O}b_{\mathcal{M}}$ composed with (3.17) is (3.14) that is surjective, the arrow ν is surjective as well. The last isomorphism in the statement of the lemma follows from the Serre duality. □

As to the arrow $\bar\theta^\vee$, since $R^1\tilde\pi_*\big(\tilde f^*p^*L^\vee\otimes\omega_{\tilde{\mathcal{C}}/\mathcal{M}}\big)$ is dual to $\tilde\pi_*\tilde f^*p^*L=\bar p^*\pi_*f^*L=\bar p^*E_1$ and since $\mathcal{M}$ is the total space of E_1, an obvious choice of a cosection like $\bar\theta^\vee$ is via the standard pairing $E_1^\vee\times E_1\to\mathbb{C}$. A direct check shows that this is the case.

We now state and prove the main result of this section that relates the localized GW-invariants to the twisted GW-invariants of D.

Proposition 3.15. *Let the notation be as before and suppose $R^1\pi_*f^*L$ is a locally free sheaf on $\mathcal{N}$. Suppose $d>0$. Then the localized virtual cycle satisfies*

$$[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}}=\sum_{\mathcal{N}_i\subset\mathcal{N}}(-1)^{h^0(f^*L)}[\mathcal{N}_i]^{\mathrm{vir}}\cap e(V),$$

where V is the kernel vector bundle of (3.15)*; the summation is over all connected components $\mathcal{N}_i$ of $\mathcal{N}$. Here $h^0(f^*L)$ denotes the rank of π_*f^*L over each component and $e(V)$ is the Euler class of V.*

Proof. Let $\mathcal{O}_{\mathcal{N}}(\tilde F)\to\mathcal{O}b_{\mathcal{M}}$ be an epimorphism of a locally free sheaf on $\mathcal{M}$ to the obstruction sheaf of $\mathcal{M}$. The obstruction theory of $\mathcal{M}$ provides us with a virtual normal cone $C_{\mathcal{M}}\subset\tilde F$. Since the restriction $\mathcal{O}b_{\mathcal{M}}|_{\mathcal{N}}$ splits as the direct sum $\mathcal{O}b_{\mathcal{N}}\oplus E_2$, $\tilde F|_{\mathcal{N}}$ is an extension of E_2 by $F=\mathrm{Ker}(\tilde F|_{\mathcal{N}}\twoheadrightarrow E_2)$, and F surjects onto $\mathcal{O}b_{\mathcal{N}}$.

Because ν is surjective, by composing ν with the homomorphism $\mathcal{O}_{\mathcal{M}}(\tilde F)\to\mathcal{O}b_{\mathcal{M}}$ we obtain a surjective homomorphism of vector bundles $\tilde F\to\bar p^*E_1^\vee$. We let

$W \subset \tilde{F}$ be its kernel. By the proof of Lemma 2.2, the support of the cone $C_{\mathcal{M}}$ lies in the subvector bundle $W \subset \tilde{F}$.

We now look at the localized virtual cycle $[\mathcal{M}]^{\rm vir}_{\rm loc}$. First, by our construction it is the image of $[C_{\mathcal{M}}]$ under the localized Gysin map

$$[\mathcal{M}]^{\rm vir}_{\rm loc} = s^!_{\tilde{F},{\rm loc}}[C_{\mathcal{M}}] \in H_*(\mathcal{N}, \mathbb{Q}).$$

(See notation in section 2.) Since $C_{\mathcal{M}}$ lies in the kernel of $\tilde{F} \to \bar{p}^* E_1^\vee$ while the cosection $\tilde{F} \to \mathcal{O}b_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ factors through $\tilde{F} \to \bar{p}^* E_1^\vee$, we can deform the bundle $\tilde{F}$ to $W \oplus \bar{p}^* E_1^\vee$ so that $C_{\mathcal{M}} \subset W$ and $\bar{p}^* E_1^\vee \to \mathcal{O}_{\mathcal{M}}$ remain unchanged. Then we must have

$$s^!_{\tilde{F},{\rm loc}}[C_{\mathcal{M}}] = s^!_{W \oplus \bar{p}^* E_1^\vee,{\rm loc}}[C_{\mathcal{M}}] = s^!_W[C_{\mathcal{M}} \times_{\mathcal{M}} s^!_{\bar{p}^* E_1^\vee,{\rm loc}}[0_{\bar{p}^* E_1^\vee}]].$$

Now suppose $\mathcal{N}$ is connected and $r = \operatorname{rank} E_1$. An easy computation shows that

$$s^!_{\bar{p}^* E_1^\vee,{\rm loc}}[0_{\bar{p}^* E_1^\vee}] = (-1)^r[\mathcal{N}],$$

where $[\mathcal{N}]$ is the fundamental class of $\mathcal{N}$. Then since over $\mathcal{N} \subset \mathcal{M}$, $W = V \oplus F$ and $C_{\mathcal{M}} \times_{\mathcal{M}} \mathcal{N}$ coincides with the virtual normal cone $C_{\mathcal{N}} \subset F$ of $\mathcal{N}$, we have

$$s^!_W[C_{\mathcal{M}} \times_{\mathcal{M}} s^!_{\bar{p}^* E_1^\vee,{\rm loc}}[0_{\bar{p}^* E_1^\vee}]] = (-1)^r e(V) \cap s^!_F[C_{\mathcal{N}}] = (-1)^r e(V) \cap [\mathcal{N}]^{\rm vir}.$$

This proves the proposition in case $\mathcal{N}$ is connected.

The general case can be treated in exactly the same way by working over each individual connected component of $\mathcal{N}$. □

Corollary 3.16. *Suppose furthermore that $\pi_* f^* L$ is a trivial vector bundle on $\mathcal{N}$, then*

$$[\mathcal{M}]^{\rm vir}_{\rm loc} = (-1)^{h^0(f^*L)}[\mathcal{N}]^{\rm vir} \cap c_l(-\pi_! f^* L)$$

for l the rank of $-\pi_! f^* L = R^1\pi_* f^* L - \pi_* f^* L$.

Proof. Observe that $e(V) = c_l(-\pi_! f^* L)$ under the assumptions. □

3.6. **The case of étale coverings.** The above formula enables us to recover the GW-invariants of a surface S without descendant insertions, first proved by Lee-Parker [15].

As before, let S be a smooth minimal general type surface with $p_g \neq 0$ and let $\theta \in H^0(\Omega^2_S)$ be a general member; let $(\beta, g) \in H_2(S, \mathbb{Z}) \times \mathbb{Z}^{\geq 0}$. By Theorem 3.12, the GW-invariants vanish unless $\beta = dK_S$. In case $\beta = dK_S$, for $\gamma_i \in H^*(S, \mathbb{Q})$,

$$\langle \gamma_1, \cdots, \gamma_n \rangle^S_{\beta,g} = 0 \tag{3.18}$$

if one of $\gamma_i \in H^{\geq 3}(S, \mathbb{Q})$, by Theorem 3.7 (2). On the other hand, if all $\gamma_i \in H^{\leq 2}(S, \mathbb{Q})$ for all i and $\gamma_i \in H^{\leq 1}(S, \mathbb{Q})$ for at least one i, then (3.18) holds by dimension count. Hence we are reduced to consider the case $\gamma_i \in H^2(S, \mathbb{Q})$. But then according to Conjecture 3.12, we expect

$$\langle \gamma_1, \cdots, \gamma_n \rangle^S_{dK_S,g} = d^n \prod_{i=1}^n (\gamma_i \cdot K_S) \cdot \langle 1 \rangle^S_{dK_S,g} = d^n \prod_{i=1}^n (\gamma_i \cdot K_S) \cdot \langle 1 \rangle^{X,{\rm loc}}_{d,g} \tag{3.19}$$

where X is the total space of a theta characteristic of parity $\chi(\mathcal{O}_S)$ on a smooth curve D of genus $h = K_S^2 + 1$. As mentioned above, this equality holds true for a wide class of minimal surfaces of general type with $p_g > 0$, including those which admit smooth canonical curves. Obviously, the right hand side of (3.19) is trivial unless the virtual dimension of $\mathcal{M}_{g,0}(X,d)$ is zero, which happens exactly when $g-1 = dK_S^2 = d(h-1)$.

We now consider the only non-trivial case $g = d(h-1)+1$. This choice of g forces any $u \in \mathcal{M}_{g,0}(D,d)$ to be a d-fold étale cover of D. Since an étale cover $u : C \to D$ is a smooth and isolated point in $\mathcal{M}_{g,0}(D,d)$, a direct application of Theorem 3.12 and Corollary 3.16 shows

Proposition 3.17 ([15]).

$$\langle 1\rangle_{d,g}^{X,\mathrm{loc}} = \sum_{u:d\text{-fold étale cover of } D} \frac{(-1)^{h^0(u^*L)}}{|\mathrm{Aut}(u)|}.$$

4. Low-degree GW-invariants of surfaces

In the remainder of this paper, we shall use degeneration to determine low degree GW-invariants of surfaces with positive p_g. To keep this paper in a manageable size, we will only state the degeneration formula for localized GW-invariants and prove the closed formulas for degree one and degree two localized GW-invariants. These formulas were conjectured by Maulik and Pandharipande [22].

4.1. **A degeneration formula.** As degeneration formulas are more easily expressible for the GW-invariants of stable morphisms with not necessary connected domains, we shall work with such moduli spaces in the remainder of this paper. As such, for integers n, χ and homology class $\beta \in H_2(Y,\mathbb{Z})$ of a smooth projective variety Y, we shall denote by

$$\mathcal{M}_{\chi,n}(Y,\beta)^\bullet$$

the moduli of stable morphisms $f : C \to Y$ to Y from not necessarily connected n-pointed nodal curves C of Euler characteristic $\chi(\mathcal{O}_C) = \chi$ and of fundamental class $f_*([C]) = \beta$, such that the restriction of f to each connected component of C is non-constant. As usual, we call f stable when the automorphism group of f is finite.

Now suppose Y has a holomorphic two-form θ. The preceding discussion of localized GW-invariants can be adopted line by line to this moduli space. Consequently, we shall denote the resulting GW-invariants, of stable maps with not necessarily connected domain but with non-constant restriction to any connected components, by

$$\langle \prod_{j=1}^{n} \tau_{\alpha_j}(\gamma_j)\rangle_{\beta,\chi,\mathrm{loc}}^{Y,\bullet}.$$

For relative invariants, we fix a pair (Y,E) with $E \subset Y$ a smooth divisor; we fix the data (χ,β,n) plus a partition η of $\int_E \beta$; namely, $\eta = (0 \le \eta_1 \le \cdots \le \eta_\ell)$ with $\sum \eta_i = \int_E \beta$. Following the convention, we shall write $\ell = \ell(\eta)$ and $|\eta| = \sum \eta_i$. We then form the moduli of relative stable morphisms that consists of morphisms

$$f : (C, p_1, \cdots, p_n, q_1, \cdots, q_\ell) \to Y$$

from $(n+\ell)$-pointed curves to Y as before with one additional requirement: as divisors

$$f^{-1}(E) = \sum \eta_i q_i.$$

For such f, we define its automorphisms be $\varphi : C \xrightarrow{\cong} C$ so that φ fixes p_i and q_j, and $f \circ \varphi = f$. We call f stable if the automorphism group of f is finite.

This moduli space is a Deligne-Mumford stack; it is not proper in general. In [16], the second author constructed its compactification by considering all relative stable maps to the semi-stable models of (Y, E). The resulting moduli space $\mathcal{M}^{(Y,E)}_{\chi,n}(\beta,\eta)^\bullet$ is proper, has perfect obstruction theory, and has two evaluation morphisms: one ordinary and one special:

$$ev(f) = \big(f(p_1), \cdots, f(p_n)\big) \in Y^n, \quad \tilde{ev}(f) = (f(q_1), \cdots, f(q_\ell)) \in E^{\ell(\eta)}.$$

For classes $\gamma_i \in H^*(Y)$ and non-negative integers α_i, the relative GW-invariants

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n)\rangle^{(Y,E),\bullet}_{\chi,\beta,\eta}$$

is the direct image

$$\tilde{ev}_*\left(\int_{[\mathcal{M}^{(Y,E)}_{\chi,n}(\beta,\eta)^\bullet]^{\mathrm{vir}}} ev^*(\gamma_i \times \cdots \times \gamma_n)\cdot \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\right) \in H_*(E^{\ell(\eta)}).$$

The relative invariants allow us to use degeneration to study the GW-invariants of smooth varieties. Suppose a smooth variety W specialize to a union of two smooth varieties $Y_1 \cup Y_2$ intersecting transversally at $E = Y_1 \cap Y_2$, the GW-invariants of W can be recovered by the relative GW-invariants of the pairs (Y_1, E) and (Y_2, E).

In this section, we shall state a parallel degeneration theory for the localized GW-invariants of X that is the total space of a theta characteristic over a smooth curve D.

To this end, we continue denoting by X the total space of a theta characteristic L of a smooth curve D; we pick a point $q \in D$ and denote by E the fiber of X over q. We then blow up $X \times \mathbb{A}^1$ along $E \times 0$, resulting a family $\mathcal{X}$ over $\mathbb{A}^1$ whose fiber over $t \neq 0$ is the original X; its central fiber $\mathcal{X}_0$ is the union of X with $E \times \mathbb{P}^1$, intersecting transversally along $E \subset X$ and $E \times 0 \subset E \times \mathbb{P}^1$. To distinguish the $X \subset \mathcal{X}_0$ from $\mathcal{X}_t$, we shall denote by $Y_1 \subset \mathcal{X}_0$ the component X and denote $E \times \mathbb{P}^1 \subset \mathcal{X}_0$ by Y_2. We continuous to denote $E = Y_1 \cap Y_2$. Since X is the total space of the line bundle L over D, $\mathcal{X}$ is the total space of a line bundle $\mathcal{L}$ over $\mathcal{D}$ of which $\mathcal{D}$ is the blowing up of $D \times \mathbb{A}^1$ along $(q, 0)$ and $\mathcal{L}$ is the pull back of L via the composite of the projections $\mathcal{X} \to \mathcal{D} \to D$. We denote the projection by $p\colon \mathcal{X} \to \mathcal{D}$ and denote $D_i = p(Y_i) \subset \mathcal{D}_0$.

Our next step is to extend the standard holomorphic two-form θ of X to the family $\mathcal{X}$. First, since $\mathcal{X}$ is the total space of $\mathcal{L}$ over $\mathcal{D}$, $\mathcal{L}$ admits a tautological section $id \in \Gamma(\mathcal{X}, p^*\mathcal{L})$ that takes value $v \in \mathcal{L}$ at the point $v \in \mathcal{X}$. On the other hand, the isomorphism $L^{\otimes 2} \cong K_D$ induces an isomorphism $\mathcal{L}^{\otimes 2} \to \omega_{\mathcal{D}/T}(-D_2)$. In this way, the tautological section id composed with $p^*\mathcal{L} \to p^*\mathcal{L}^\vee \otimes \omega_{\mathcal{D}/T}$ and the isomorphism

$$\wedge^2\Omega_{\mathcal{X}/T}(\log \mathcal{X}_0) \cong p^*\mathcal{L}^\vee \otimes \omega_{\mathcal{D}/T},$$

provide us with a two-form

$$\Theta \in \Gamma(\mathcal{X}, \wedge^2\Omega_{\mathcal{X}/T}(\log \mathcal{X}_0)).$$

Its restriction to $\mathcal{X}_t$ for $t \neq 0$ is the standard two-form θ; its restriction to Y_1 is the one induced by θ via

$$\Omega^2_{Y_1} \longrightarrow \Omega^2_{Y_1}(E) \cong \wedge^2\Omega_{\mathcal{X}/T}(\log \mathcal{X}_0) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{Y_1};$$

it vanishes along Y_2.

Because the restriction $\theta_1 = \Theta|_{Y_1}$ vanishes along E, it defines a sheaf homomorphism

$$\theta_1 : T_{Y_1}(\log E) \longrightarrow \Omega^1_{Y_1}.$$

Thus for any stable map $f:C \to Y_1$ with $f^{-1}(E)$ a divisor in C, θ_1 induces a sheaf homomorphism

$$f^*T_{Y_1}(\log E) \longrightarrow f^*\Omega^1_{Y_1} \longrightarrow \omega_C; \tag{4.1}$$

this homomorphism will define us the localized relative GW-invariants of (Y_1, E).

Like the case of ordinary GW-invariants, the localized relative GW-invariants relies on a cosection of the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ of the moduli of the relative stable morphisms[5] $\mathcal{M}^{(Y_1,E)}_{\chi,n}(d,\eta)^\bullet$. For us, it is instructive to see how the two-form θ_1 induces such a cosection. Without getting into the details of the notion of pre-deformable morphisms that is essential to the construction of the moduli $\mathcal{M}^{(Y_1,E)}_{\chi,n}(d,\eta)^\bullet$, we shall describe the obstruction space and the cosection at a closed point $\xi \in \mathcal{M}^{(Y_1,E)}_{\chi,n}(d,\eta)^\bullet$ that is represented by a morphism $f : C \to Y_1$. In this case, as shown in [17], the obstruction space $Ob_\xi = \mathcal{O}b_{\mathcal{M}} \otimes \mathbf{k}(\xi)$ is the cokernel

$$\mathrm{Ext}^1(\Omega_C(R), \mathcal{O}_C) \longrightarrow H^1(C, f^*T_{Y_1}(\log E)) \longrightarrow Ob_\xi \longrightarrow 0,$$

where R is the divisor $f^{-1}(E) \subset C$; like before, the homomorphism (4.1) induces a homomorphism

$$H^1(C, f^*T_{Y_1}(\log E)) \longrightarrow H^1(C, \omega_C) = \mathbb{C}$$

that lifts to a homomorphism $Ob_\xi \to \mathbb{C}$; again, this homomorphism is trivial if and only if $f(C) \subset D_1 \subset Y_1$. Likewise, this construction carries to the family case to give us a cosection of the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$; the cosection is surjective away from those relative stable maps whose image lie in the zero section of Y_1.

We state the property of localized relative GW-invariants of (Y_1, E) as a proposition; its proof will appear in a separate paper [11].

Proposition 4.1. *Let (Y_1, E), let θ_1 and let $\mathcal{M} = \mathcal{M}^{(Y_1,E)}_{\chi,n}(d,\eta)$ be as before. Let $\tilde{ev} : \mathcal{M} \to E^\ell$, $\ell = \ell(\eta)$, be the special evaluation map and let $E_0 = E \cap D_1$. Then the holomorphic two-form θ_1 induces a canonical cosection $\sigma_1 : \mathcal{O}b_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ that is surjective away from $\tilde{ev}^{-1}(E_0^\ell)$. Consequently, its localized virtual cycle $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in H_*(E_0^\ell) \subset H_*(E^\ell)$ and the localized relative GW-invariants take values*

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{(Y_1,E),\bullet}_{\chi,\eta,\mathrm{loc}} \in H_*(E^\ell).$$

Here we omit the explicit reference to d in the notation for the localized GW-invariants because η is a partition of d.

For the pair (Y_2, E), since the form Θ restricts to zero along Y_2, we shall take its ordinary relative GW-invariants. Though Y_2 is not proper, because $Y_2 = \mathbb{A}^1 \times \mathbb{P}^1$ and E is one of the $\mathbb{A}^1$ in the product, the special evaluation morphism

$$\tilde{ev} : \mathcal{M}^{(Y_2,E)}_{\chi,n}(d,\eta)^\bullet \longrightarrow E^\ell$$

is proper. Consequently, the relative GW-invariants of (Y_2, E) is well-defined and take values

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{(Y_2,E),\bullet}_{\chi,\eta} \in H^{BM}_*(E^\ell).$$

[5]Here as before $d \in H_2(Y_1, \mathbb{Z})$ is the class generated by d-fold of the zero section of $Y_1 \to D$; η is a partition of d.

In [11], we shall use the intersection pairing

$$\star : H_*(E^l,\mathbb{Q}) \times H_*^{BM}(E^l,\mathbb{Q}) \longrightarrow \mathbb{Q}$$

to relate the localized GW-invariants of X with the pairings of the localized relative GW-invariants of (Y_1,E) and the relative GW-invariants of (Y_2,E). For any partition η we define $\mathbf{m}(\eta)=\prod \eta_i$.

Theorem 4.2 ([11]). *For any integers $\alpha_1,\cdots,\alpha_n\in\mathbb{Z}^{\geq 0}$, any splitting $n_1+n_2=n$ and classes $\gamma_{i(\leq n_1)}\in H^{\geq 1}(Y_1)$ and $\gamma_{j(>n_1)}\in H^{\geq 1}(Y_2)$, we have*

$$\langle \prod_{j=1}^n \tau_{\alpha_j}(\gamma_j)\rangle^{X,\bullet}_{\chi,d,\mathrm{loc}} = \sum \frac{\mathbf{m}(\eta)}{|\operatorname{Aut}(\eta)|}\cdot \langle \prod_{j=1}^{n_1} \tau_{\alpha_j}(\gamma_j)\rangle^{(Y_1,E),\bullet}_{\chi_1,\eta,\mathrm{loc}} \star \langle \prod_{j=n_1+1}^{n} \tau_{\alpha_j}(\gamma_j)\rangle^{(Y_2,E),\bullet}_{\chi_2,\eta}.$$

Here the summation is taken over all integers χ_1 and χ_2, partitions η of d subject to the constraint

$$\chi_1+\chi_2-l(\eta)=\chi. \tag{4.2}$$

The $\operatorname{Aut}(\eta)$ is the subgroup of permutations in $S_{\ell(\eta)}$ that fixes η.

4.2. **Low degree GW-invariants with descendants.** For a smooth projective surface S with $p_g>0$, in the previous section we have computed its GW-invariants *without* descendant insertions; in this section we shall compute its degrees one and two GW-invariants with *descendants*.

According to Conjecture 3.12, we only need consider the localized GW-invariants

$$\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{\chi,d[D],\mathrm{loc}} = \langle \tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_n}(\gamma_n)\rangle^{X,\bullet}_{\chi,d[D],\mathrm{loc}},\quad \text{for } \gamma_i\in H^*(X), \tag{4.3}$$

of the total space X of a theta characteristic L over a smooth projective curve D of genus $h:=K_S^2+1$ and $h^0(L)\equiv\chi(\mathcal{O}_S) \bmod 2$. Following the convention, since (4.3) is possibly non-trivial only when

$$-\chi = d\,K_S^2+\sum_{i=1}^n \alpha_i,\quad \alpha_i\in\mathbb{Z}_{\geq 0}, \tag{4.4}$$

we shall omit the reference to χ in the notation of (4.3) with the understanding that it is given by (4.4).

Let $\gamma\in H^2(D,\mathbb{Z})$ be the Poincaré dual of a point in D. The main result of this section is the following theorem, conjectured by Maulik and Pandharipande [22, (8)-(9)]

Theorem 4.3. *Let $X\to D$ and $h=g(D)$ be as before. Then the degree one and two GW-invariants with descendants are*

$$\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{[D],\mathrm{loc}} = (-1)^{h^0(L)}\prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{-\alpha_i}; \tag{4.5}$$

$$\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}} = (-1)^{h^0(L)}\,2^{h+n-1}\prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}. \tag{4.6}$$

The first identity will follow directly from Corollary 3.16 and the second will be proved using degeneration.

We begin with the degree one case. Let $-\chi = K_S^2 + \sum \alpha_i$,

$$\mathcal{M} = \mathcal{M}_{\chi,n}(X,[D])^\bullet \quad \text{and} \quad \mathcal{N} = \mathcal{M}_{\chi,n}(D,[D])^\bullet;$$

let $f:\mathcal{C} \to D$ with $\pi:\mathcal{C} \to \mathcal{N}$ be the universal family. Because maps in $\mathcal{N}$ has degree one over D, a direct application of the base change property shows that $\pi_* f^* L \cong H^0(L) \otimes \mathcal{O}_{\mathcal{N}}$ and that $R^1\pi_* f^* L$ is locally free. By Corollary 3.16,

$$\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{[D],\mathrm{loc}} = (-1)^{h^0(L)} \int_{[\mathcal{N}]^{\mathrm{vir}}} \prod \psi_i^{\alpha_i} ev_i^*(\gamma) \cap c_{top}(-\pi_! f^* L),$$

is a twisted GW-invariant of D with sign $(-1)^{h^0(L)}$. From [6], we can readily evaluate them and obtain (4.5). We omit the straightforward computation here.

We next prove (4.6). We begin with a few special cases.

Lemma 4.4 ($h = 0$ case)**.** *Let Y_0 be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Then*

$$\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{Y_0,\bullet}_{2[\mathbb{P}^1],\mathrm{loc}} = 2^{n-1} \prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}. \tag{4.7}$$

Proof. Since $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-1)$ is rigid, the moduli space of stable maps to $\mathcal{O}_{\mathbb{P}^1}(-1)$ is proper and the localized GW-invariant coincides with the twisted GW-invariant of $\mathbb{P}^1$ by Corollary 3.16. Hence, (4.7) follows from the differential equations for the twisted invariants in [6]. □

Lemma 4.5. *We have*

$$\langle \tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}} = (-1)^{h^0(L)} \left(\frac{2^h}{-3}\right) \tag{4.8}$$

We postpone the proof of this lemma until later.

Lemma 4.6. *Let (Y_1, E) and (Y_2, E) be the relative pairs resulting from the degeneration constructed in the previous subsection. Then*

$$\langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}} = (-1)^{h^0(L)}\, 2^h\, [pt^2] \quad \textit{and} \quad \langle \tau_1(\gamma)\rangle^{(Y_2,E),\bullet}_{(1,1)} \star [pt^2] = -1/6.$$

Proof. We first look at the first identity. It is easy to see, from the construction of localized relative invariants (Proposition 4.1), that $\langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}}$ is a scalar multiple of $[pt^2]$. Thus, an easy virtual dimension counting shows that the only stable maps that contribute to this invariant must have $-\chi = 2(h-1)$. By (4.4), the composites of these stable maps with $p:X \to D$ are étale covers of D. Hence the (relevant) moduli space of relative stable maps to (Y_1, E) is a disjoint union of 2^{2h} vector spaces, each consists of all liftings of an étale cover of D. By our proof of Proposition 3.15, we obtain

$$\langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}} = \sum_{u:\text{2-fold étale covers of } D} (-1)^{h^0(u^*L)}\, [pt^2].$$

It is known that étale double covers of D are parameterized by the set of order 2 line bundles on D, and exactly $2^{h-1}(2^h+1)$ of them satisfy $h^0(u^*L) \equiv h^0(L) \bmod 2$ [9]. Therefore we have

$$\langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}} = (-1)^{h^0(L)} \big(2^{h-1}(2^h+1) - 2^{h-1}(2^h-1)\big)\, [pt^2] = (-1)^{h^0(L)} 2^h [pt^2].$$

This proves the first equation.

Since Y_2 is the total space of the trivial line bundle over $\mathbb{P}^1$, any stable map in $\mathcal{M}^{Y_2,E}_{\chi,n}(2[D],(1,1))^\bullet$ with two distinct intersection points with E has two irreducible components, one with the marked point and the other without. Therefore, because

$$\langle\tau_1(\gamma)\rangle^{(Y_2,E)}_{(1)} \star [pt] = \langle\tau_1(\gamma)\rangle^{Y_0,\bullet}_{[\mathbb{P}^1],\mathrm{loc}} = -\frac{1}{12} \quad \text{and} \quad \langle 1\rangle^{(Y_2,E)}_{(1)} \star [pt] = 1,$$

we have

$$\langle\tau_1(\gamma)\rangle^{(Y_2,E),\bullet}_{(1,1)} \star [pt^2] = \Big(\langle\tau_1(\gamma)\rangle^{(Y_2,E)}_{(1)} \star [pt]\Big)\Big(\langle 1\rangle^{(Y_2,E)}_{(1)} \star [pt]\Big) + \\ + \Big(\langle 1\rangle^{(Y_2,E)}_{(1)} \star [pt]\Big)\Big(\langle\tau_1(\gamma)\rangle^{(Y_2,E)}_{(1)} \star [pt]\Big) = -\frac{1}{6}$$

□

We next prove (4.6), assuming (4.8). By the degeneration formula, we have

$$\langle\prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}} = \frac{1}{2}\ \langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}} \star \langle\prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{(Y_2,E),\bullet}_{(1,1)} + \\ +2\ \langle 1\rangle^{(Y_1,E),\bullet}_{(2),\mathrm{loc}} \star \langle\prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{(Y_2,E),\bullet}_{(2)}. \tag{4.9}$$

In particular, from (4.8) and Lemma 4.6, we have

$$(-1)^{h^0(L)}\binom{2^h}{-3} = \langle\tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}} = \frac{1}{2}\,(-1)^{h^0(L)}\binom{2^h}{-6} + 2\langle 1\rangle^{(Y_1,E),\bullet}_{(2),\mathrm{loc}} \star \langle\tau_1(\gamma)\rangle^{(Y_2,E),\bullet}_{(2)}.$$

Comparing this with the case where $D = \mathbb{P}^1$ ($h = 0$), we see that the relative invariants $\langle 1\rangle^{(Y_1,E),\bullet}_{(1,1),\mathrm{loc}}$ and $\langle 1\rangle^{(Y_1,E),\bullet}_{(2),\mathrm{loc}}$ are exactly those for $D = \mathbb{P}^1$, multiplied by $(-1)^{h^0(L)}\ 2^h$. Therefore by (4.9) and (4.7), we have

$$\begin{aligned}\langle\prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}} &= (-1)^{h^0(L)}\ 2^h\ \langle\prod_{i=1}^n \tau_{\alpha_i}(\gamma)\rangle^{Y_0,\bullet}_{2[\mathbb{P}^1],\mathrm{loc}} \\ &= (-1)^{h^0(L)}\ 2^{h+n-1}\ \prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}.\end{aligned}$$

This proves Theorem 4.3.

4.3. **Proof of Lemma 4.5.** It remains to prove (4.8). We first claim that for $-\chi = 2(h-1)+1$, the moduli space $\mathcal{N} = \mathcal{M}_{\chi,1}(D, 2[D])^\bullet$ has exactly $2^{2h}+1$ connected components. Obviously, one of them consists of all double covers of D branched at two points; we denote this component by $\mathcal{N}_0$. Each of the other components is distinguished by an étale double cover $u\colon C \to D$: elements in this component are formed by adding genus one ghost components to u. We denote this component by $\mathcal{N}_u$ and denote by $\mathcal{M}_u$ the corresponding component in $\mathcal{M} = \mathcal{M}_{\chi,1}(X, 2[D])$. Because there are 2^{2h} étale double covers of D, there are 2^{2h} of such components.

It is easy to see that the contribution to the localized GW-invariant $\langle\tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}}$ from the component $\mathcal{M}_u$ is

$$\langle\tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}}[\mathcal{M}_u] = (-1)^{h^0(u^*L)}\ \left(-\frac{1}{12}\right).$$

Here the sign $(-1)^{h^0(u^*L)}$ is due to Proposition 3.5 and the factor $-1/12$ is from the formula (4.5) for the degree one case. Because exactly $2^{h-1}(2^h+1)$ of the 2^{2h} étale double covers $u: C \to D$ satisfy $h^0(u^*L) \equiv h^0(L) \bmod 2$, the total contribution to $\langle \tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}}$ from these irreducible components is

$$(-1)^{h^0(L)}\left(-\frac{1}{12}\right)\cdot\left(2^{h-1}(2^h+1)-2^{h-1}(2^h-1)\right) = (-1)^{h^0(L)}\left(-\frac{2^h}{12}\right).$$

Therefore to prove (4.8), it suffices to show

Lemma 4.7. *The contribution to* $\langle \tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}}$ *from the connected component* $\mathcal{M}_0$ *is*

$$\langle \tau_1(\gamma)\rangle^{X,\bullet}_{2[D],\mathrm{loc}}[\mathcal{M}_0] = (-1)^{h^0(L)}\left(\frac{2^h}{-3}\right) - (-1)^{h^0(L)}\left(-\frac{2^h}{12}\right) = (-1)^{h^0(L)}(-2^{h-2}).$$

Unlike the previous case, we cannot apply Proposition 3.15 directly to evaluate the above quantity because $\pi_* f^*L$ is not locally free over $\mathcal{N}_0$. Nevertheless, by a detailed investigation of its failure to being locally free we can identify the extra contribution, thus proving the lemma.

It is easy to get the contribution to the twisted GW-invariant from the component $\mathcal{N}_0$ of branched double covers. From [6] and [23], it is straightforward to deduce that the degree two GW-invariants of D twisted by the top Chern class of $-\pi_! f^*L$ is

$$\langle \tau_1(\gamma); c_{top}(-\pi_! f^*L)\rangle^{D,\bullet}_{2,\chi} = \left(h-\frac{8}{3}\right)2^{2h-3};$$

the contribution to the twisted GW-invariant from any of the irreducible components $\mathcal{N}_u$ is $-\frac{1}{12}$. Therefore, the contribution to the twisted GW-invariant from $\mathcal{N}_0$ is

(4.10)
$$\langle \tau_1(\gamma); c_{top}(-\pi_! f^*L)\rangle^{D,\bullet}_{2,\chi}[\mathcal{N}_0] = \left(h-\frac{8}{3}\right)2^{2h-3} - 2^{2h}\left(-\frac{1}{12}\right) = (h-2)\,2^{2h-3}.$$

Our next step is to take D to be a *hyperelliptic* curve with $\delta: D \to \mathbb{P}^1$ double cover; we then cut down the space $\mathcal{N}_0$ by the insertion $\tau_1(\gamma) = \psi_1 ev_1^*(\gamma)$. As is known, a double cover $u: C \to D$ branched at two points $q_1, q_2 \in D$ is characterized by a line bundle ξ on D satisfying $\xi^2 \cong \mathcal{O}_D(q_1+q_2)$: the curve C is a subscheme of the total space of ξ defined by $\{t \in \xi \mid t^2 = v\}$ for v a section in $H^0(\mathcal{O}_D(q_1+q_2))$ vanishing at q_1+q_2; the map is that induced by the projection $\xi \to D$.

We now let

$$f: \mathcal{C} \to D, \quad \pi: \mathcal{C} \to \mathcal{N}_0, \quad s: \mathcal{N}_0 \to \mathcal{C} \;\; \text{the section of marked points}$$

be the universal family of $\mathcal{N}_0$; we let $\mathrm{ev}_1: \mathcal{N}_0 \to D$ as before be the evaluation morphism. We can cut down $\mathcal{N}_0$ by $\psi_1 ev_1^*(\gamma)$ as follows: we pick a general point $q \in D$ and form the subscheme $ev_1^{-1}(q)$; it represents the class $ev_1^*(\gamma)$. For ψ_1, we observe that the natural homomorphism $s^*f^*\Omega_D \to s^*\omega_{\mathcal{C}/\mathcal{N}_0}$ induces a section

$$\phi \in \Gamma\big(\mathcal{N}_0, s^*(\omega_{\mathcal{C}/\mathcal{N}_0} \otimes f^*\Omega_D^\vee)\big). \tag{4.11}$$

This section vanishes at a map $u = (\xi, q_1+q_2) \in \mathcal{N}_0$ if and only if the marked point is one of the two branched points q_1 and q_2. For convenience, we denote

$\mathcal{Z} = \{\phi = 0\}$; for $q \in D$, we denote $\mathcal{Z}_q = \mathcal{Z} \times_D q$ with $\mathcal{Z} \to D$ induced by evaluation map ev_1. This way, the class $\psi_1 ev_1^*(\gamma)$ is represented by the substack

$$\mathcal{M}_{0,q} = \mathcal{M} \times_{\mathcal{N}_0} \mathcal{Z}_q.$$

Lemma 4.8. *The stack $\mathcal{M}_{0,q}$ has an induced perfect obstruction theory; its obstruction sheaf has a cosection consistent with that of $\mathcal{M}_0$; the degree of its localized virtual cycle equals the contribution $\langle \tau_1(\gamma)\rangle^{X,\bullet}_{2[D],loc}[\mathcal{M}_0]$.*

Proof. Our first step is to argue that there are line bundles L_1 and L_2 on $\mathcal{M}_0$ and their respective sections s_1 and s_2 so that $(s_1 = s_2 = 0) = \mathcal{M}_{0,q}$. Indeed, because of our construction that $\mathcal{Z} \subset \mathcal{N}_0$ is a Cartier divisor and $\mathcal{Z}_q = \mathcal{Z} \times_D q$, $\mathcal{Z}_q \subset \mathcal{N}_0$ is cut out by two sections of two line bundles on $\mathcal{N}_0$. Then since $\mathcal{M}_{0,q} = \mathcal{M}_0 \times_{\mathcal{N}_0} \mathcal{Z}_q$, $\mathcal{M}_{0,q}$ is cut out by the vanishing of the pullback of these sections. This proves the claim.

We let the two pullback line bundles be L_1 and L_2 and let the two pullback sections be s_1 and s_2. According to [17], (s_1, s_2) induces an obstruction theory of $(s_1 = s_2 = 0) = \mathcal{M}_{0,q}$; its obstruction sheaf $\mathcal{O}b_{\mathcal{M}_{0,q}}$ and the obstruction sheaf $\mathcal{O}b_{\mathcal{M}_0}$ of $\mathcal{M}_0$ fits into the exact sequence

$$\longrightarrow \mathcal{O}_{\mathcal{M}_{0,q}}(L_1) \oplus \mathcal{O}_{\mathcal{M}_{0,q}}(L_2) \longrightarrow \mathcal{O}b_{\mathcal{M}_{0,q}} \longrightarrow \mathcal{O}b_{\mathcal{M}_0} \otimes_{\mathcal{O}_{\mathcal{M}_0}} \mathcal{O}_{\mathcal{M}_{0,q}} \longrightarrow 0; \tag{4.12}$$

the obstruction theory of $\mathcal{M}_{0,q}$ is perfect as well.

In [17, Lemma 4.6], it is proved that

$$[\mathcal{M}_{0,q}]^{\mathrm{vir}} = c_1(L_1) \cup c_1(L_2)[\mathcal{M}_0]^{\mathrm{vir}}.$$

To apply to the localized GW-invariants, we need to define the localized virtual cycle $[\mathcal{M}_{0,q}]^{\mathrm{vir}}_{\mathrm{loc}}$; show that applying the localized first Chern class $c_1(L_i, s_i)$ we will have

$$[\mathcal{M}_{0,q}]^{\mathrm{vir}}_{\mathrm{loc}} = c_1(L_1, s_1) \cup c_1(L_2, s_2)[\mathcal{M}_0]^{\mathrm{vir}}_{\mathrm{loc}}. \tag{4.13}$$

For this, let $\sigma : \mathcal{O}b_{\mathcal{M}_0} \to \mathcal{O}_{\mathcal{M}_0}$ be the cosection. Because of (4.12), $\mathcal{O}b_{\mathcal{M}_{0,q}}$ has an induced cosection, say σ_q. Since the degeneracy locus $Z(\sigma) \subset \mathcal{M}_0$ is proper, $Z(\sigma_q) = Z(\sigma) \cap \mathcal{M}_{0,q}$ is also proper. Further, by combining the argument of [17, Lemma 4.6] and the proof of Proposition 2.6, one proves the identity (4.13). Since the argument is routine, we shall omit the details here. This proves the lemma. □

In light of this lemma, we need to investigate the stack structure of $\mathcal{M}_{0,q}$. According to Lemma 3.13, it suffices to investigate the locus $\Lambda \subset \mathcal{Z}$ near which the sheaf $R^1\pi_* f^* L$ on $\mathcal{N}_0$ is non-locally free. For a double cover $u : C \to D$ in $\mathcal{Z}$ given by $(\xi, q_1 + q_2)$, since

$$H^0(C, u^*L) = H^0(D, L) \oplus H^0(D, L \otimes \xi^{-1}), \tag{4.14}$$

it is easy to see that $R^1\pi_* f^* L$ fail to be locally free at u exactly when $h^0(L \otimes \xi^{-1}) \neq 0$. Let

$$\eta \neq 0 \in H^0(D, L \otimes \xi^{-1})$$

and let $H = \delta^* \mathcal{O}_{\mathbb{P}^1}(1)$. Then as $L^2 \cong K_D = (h-1)H$ and $\xi^{\otimes 2} = \mathcal{O}(q_1 + q_2)$, we have

$$(h-1)H = q_1 + q_2 + 2\eta^{-1}(0), \tag{4.15}$$

which, because D is hyperelliptic, is possible only if either $q_1 = q_2$ or they are conjugate to each other. In particular, when $u \in \Lambda \cap \mathcal{Z}_q$, one of q_i must be q; thus $\Lambda \cap \mathcal{Z}_q$ is finite.

Were this intersection empty, by Corollary 3.16, the localized GW-invariant (4.7) would have been equal to the twisted GW-invariant multiplied by $(-1)^{h^0(L)}$. To find the correction caused by these exceptional points, we need a detailed analysis of the virtual normal cone of the moduli space $\mathcal{M}_0$ near the fibers over Λ. Since the cases $q_2 = q_1$ and $= \bar{q}_1$ ($\bar{q}_1$ is the conjugate point of q_1) require independent analysis, we shall investigate them separately. We denote by $\Lambda'' \subset \Lambda$ the subset of elements associated to $q_1 = q_2$; we denote $\Lambda' = \Lambda - \Lambda''$.

For general $q \in D$, it is easy to enumerate the set $\Lambda' \cap \mathcal{Z}_q$. Let $u = (\xi, q + \bar{q})$ be any point in this set. Since $q + \bar{q} = H$, (4.15) reduces to $2\eta^{-1}(0) = (h-2)H$. Let

$$r = h^0(L \otimes \xi^{-1}) - 1 = h^1(L \otimes \xi^{-1}) - 2, \tag{4.16}$$

which ranges between 0 and $(h-2)/2$. Since D is hyperelliptic, $\eta^{-1}(0)$ must be $p_1 + \cdots + p_{h-2}$ for points p_i on C so that $p_{i+r} = \bar{p}_i$ for $i \le r$ and p_j, $j \ge 2r+1$, are distinct ramification points of $\delta : D \to \mathbb{P}^1$. The r in (4.16) decomposes $\Lambda' \cap \mathcal{Z}_q$ into union $\cup \Lambda'_r$; elements in Λ'_r are uniquely determined by the distinct ramified points $p_{j>2r}$ since

$$\xi = L(-p_1 - \cdots - p_{h-2}) = L - rH - p_{2r+1} - \cdots - p_{h-2}. \tag{4.17}$$

Therefore, Λ'_r consists of $\binom{2h+2}{h-2-2r}$ elements, where $2h+2$ is the number of ramification points of δ.

We next claim that the scheme structure of $\mathcal{M}_{0,q} = \mathcal{M}_0 \times_{\mathcal{N}_0} \mathcal{Z}_q$ near the fiber over $u = (\xi, q + \bar{q}) \in \Lambda'_r$ is (analytically) isomorphic to

$$\mathbf{A}^l \times \left\{ z w_1 = z^3 w_2 = \cdots = z^{2r+1} w_{r+1} = 0 \right\} \subset \mathbf{A}^{l+r+2}, \quad l = h^0(L). \tag{4.18}$$

Since $\mathcal{Z}_q$ is smooth near u, the local defining equation of $\mathcal{M}_{0,q}$ is determined by a locally free resolution of $\bar{\pi}_! \bar{f}^* L$ for $\bar{f} : \bar{\mathcal{C}} \to D$ and $\bar{\pi} : \bar{\mathcal{C}} \to \mathcal{Z}_q$ the restrictions of f and π to $\mathcal{Z}_q$. On the other hand, from (4.14), we see immediately that $\bar{\pi}_* \bar{f}^* L = H^0(L) \otimes \mathcal{O}$. By Riemann-Roch, away from $\mathcal{M}_{0,q} \times_{\mathcal{Z}} \Lambda$, $R^1 \bar{\pi}_* \bar{f}^* L$ is locally free and has rank $l + 1$. Therefore, $R^1 \bar{\pi}_* \bar{f}^* L$ is a direct sum of its torsion free part $\mathcal{R}$ and its torsion part. Since $h^1(L \otimes \xi^{-1}) = r + 2$ by Riemann-Roch [1], in a formal neighborhood of $u \in \mathcal{Z}_q$ with z the local coordinate of $\mathcal{Z}_q$ at u, we can find positive integers $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{r+1}$ and express the torsion part of $R^1 \bar{\pi}_* \bar{f}^* L$ as

$$\mathbb{C}[[z]]/(z^{\alpha_1}) \oplus \cdots \oplus \mathbb{C}[[z]]/(z^{\alpha_{r+1}}). \tag{4.19}$$

Thus a (locally free) resolution of $\bar{\pi}_! \bar{f}^* L$ can be chosen as

$$\operatorname{diag}(z^{\alpha_1}, \cdots, z^{\alpha_{r+1}}) : \bigoplus_{i=1}^{r+1} \mathcal{O} \longrightarrow \left(\bigoplus_{i=1}^{r+1} \mathcal{O} \right) \oplus \mathcal{R}.$$

By Lemma 3.13, if we let $(z_1, \cdots, z_l, w_1, \cdots, w_{r+1})$ be the coordinate for the vector space $H^0(C, f^* L) = H^0(L) \oplus H^0(L \otimes \xi^{-1})$, the local defining equation of $\mathcal{M}_{0,q}$ near fibers over u can be chosen as

$$z^{\alpha_1} w_1 = z^{\alpha_2} w_2 = \cdots = z^{\alpha_{r+1}} w_{r+1} = 0.$$

It remains to determine the integers α_i. For this we need to investigate whether a line $\mathbb{C}\eta \subset H^0(L \otimes \xi^{-1})$ can be extended to a submodule

$$\mathbb{C}[z]/(z^n) \subset \pi_* f^* L / z^n \pi_* f^* L. \tag{4.20}$$

Let

$$H^0(L \otimes \xi^{-1}) = F_{r+1} \supseteq F_r \supseteq \cdots \supseteq F_1 \supseteq F_0 = \{0\}$$

be a flag with $F_{r+1-k} = H^0(L \otimes \xi^{-1}(-kq))$. It is a complete flag since $h^0(L \otimes \xi^{-1}(-kq)) = \max(r+1-k, 0)$, using (4.17). For $0 \le k \le r$, let $\eta \in F_{r+1-k} - F_{r-k}$ be a general element and (p_i) be the zeros of η. Then after reshuffling if necessary, we have

$$p_1 = \cdots = p_k = q,\ p_{r+1} = \cdots = p_{r+k} = \bar q,\ p_{k+1}, \cdots, p_r, p_{r+k+1}, \cdots, p_{2r} \notin \{q, \bar q\}.$$

Suppose η fits into a $\mathbb{C}[z]/(z^n)$-modules as in (4.20), then there are morphisms

$$\mathbf{q},\ \mathbf{q}' \quad \text{and} \quad \mathbf{p}_i : \operatorname{Spec} \mathbb{C}[z]/(z^n) \longrightarrow D,$$

which extend the points $q, \bar q$ and p_i respectively, such that $\mathbf{q}$ is constant, $\mathbf{q}'$ has non-vanishing first order variation, and that as families of divisors on D parameterized by $\operatorname{Spec} \mathbb{C}[z]/(z^n)$, the relation (4.15) holds:

$$(h-1)H = \mathbf{q} + \mathbf{q}' + 2\mathbf{p}_1 + \cdots + 2\mathbf{p}_{h-2}.$$

Because $\bar q$ is not a branched point of $\delta : D \to \mathbb{P}^1$, our local coordinate z for $\mathcal{Z}_q$ at u can be thought of as a local coordinate for D near $\bar q$ and also a local coordinate for $\mathbb{P}^1$ via δ. Without loss of generality, we can choose $\mathbf{q}'$ so that $\delta \circ \mathbf{q}'(z) = z$. Suppose $\delta \circ \mathbf{p}_i = p_i(z)$. Then the above identity on divisors over $\operatorname{Spec} \mathbb{C}[z]/(z^n)$ is equivalent to

$$w \prod_{i=1}^{k} (w - p_i(z))^2 \equiv (w - z) \prod_{i=1}^{k} (w - p_{r+i}(z))^2 \mod z^n, \tag{4.21}$$

for a formal variable w.

Shortly, we shall show that (4.21) is solvable if and only if $n \le 2k+1$. Once this is done, we see that because $\dim F_k = k$, we have $\alpha_1 = 1$, $\alpha_2 = 3$, $\ldots$, $\alpha_r = 2r+1$. This will provide us the structure result of $\mathcal{M}_{0,q}$ over an element in Λ'_r.

We now prove (4.21) is solvable if and only if $n \le 2k+1$. We first compare the constant coefficients in w of (4.21); we obtain

$$z \prod_{i=1}^{k} p_{r+i}^2(z) \equiv 0 \mod z^n. \tag{4.22}$$

To proceed, we shall show that for $n = 2k+1$, (4.21) holds true if and only if $p_{r+i}(z) \equiv_{(z^2)} c_i z$ for uniquely determined complex numbers $c_i \neq 0$, $1 \le i \le k$. (Here we use $\equiv_{(z^n)}$ to mean equivalence modulo z^n.) Together with (4.22), this immediately implies the claim.

Let $n = 2k+1$; let f, g and h be defined by

$$f = \prod_{i=1}^{k} (1 - p_i(z)/w) = \sum_{j=0}^{k} A_j w^{-j}, \quad g = \prod_{i=1}^{k} (1 - p_{r+i}(z)/w) = \sum_{j=0}^{k} B_j w^{-j}$$

and $h = \sqrt{1 - z/w} \cdot g = \sum_{j=0}^{\infty} C_j w^{-j}$, with A_i, B_i and C_i analytic functions of z. Dividing (4.21) by w^{2k+1}, we see immediately that that $C_j \equiv_{(z^n)} A_j$ for $0 \le j \le k$ and that $h^2 - f^2 = 2f(h-f) + (h-f)^2$ has no terms w^{-j} with $j \le 2k$, modulo z^n as usual. Since f is monic and $h - f$ has no terms w^{-j} with $j \le k$, $h - f$ has no terms w^{-j} with $j \le 2k$. This implies that $C_j \equiv_{(z^n)} 0$ for $k < j \le 2k$. If we let $\sqrt{1 - z/w} = \sum D_j w^{-j}$, we obtain

$$0 \equiv_{(z^n)} C_j = \sum_{i=0}^{k} D_{j-i} B_i = D_j + \sum_{j=1}^{k} D_{j-i} B_i$$

for $k < j \le 2k$, and thus we have a matrix equation

$$(G_1\ G_2\ \cdots\ G_k)B \equiv_{(z^n)} -G_{k+1}, \tag{4.23}$$

where G_j is the column vector $(D_j, D_{j+1}, \cdots, D_{j+k-1})^t$ and B is the column vector $(B_k, B_{k-1}, \cdots, B_1)^t$. Since $D_j = -2^{1-2j}\frac{1}{j}\binom{2j-2}{j-1}z^j$, the determinant of the Hankel matrix $(G_1\ G_2\ \cdots\ G_k)$ is $[(-1)^k/2^{2k^2-k}]z^{k^2}$ and the determinant of $(G_2\ G_3\ \cdots\ G_{k+1})$ is $[(-1)^k/2^{2k^2+k}]z^{k^2+k}$. (See [25] for the computation of these Hankel determinants.) To prove the solvability of (4.21) for $n = 2k+1$, we can replace $\equiv_{(z^n)}$ by the honest equality. By Cramer's rule, the matrix equation $(G_1\ G_2\ \cdots\ G_k)B = -G_{k+1}$ has a unique solution $B_k = (-4)^{-k}z^k$ and $B_j = \beta_j z^j$ for some $\beta_j \in \mathbb{C}$, $j \le k$. The equation $A_j = C_j$ for $j \le k$ implies $A_j = \gamma_j z^j$ for some $\gamma_j \in \mathbb{C}$. Therefore $p_{r+i}(z) = c_i z$ for some $c_i \ne 0$ solves (4.21) and (4.22).

To see the insolvability of (4.21) for $n > 2k+1$, we observe from (4.23) that the least order terms of B_j are uniquely determined as above and thus $p_{r+i}(z) \equiv_{(z^2)} c_i z$ for $c_i \ne 0$. Then (4.22) cannot be satisfied. This completes our description of the formal neighborhoods of exceptional points in Λ'_r.

The structure of Λ'' is similar. First, $\Lambda'' \cap \mathcal{Z}_q$ is the disjoint union of Λ''_r for $0 \le r \le \frac{h-3}{2}$; each Λ''_r has $\binom{2h+2}{h-3-2r}$ elements; the local defining equation for the stack $\mathcal{M}_{0,q}$ near a fiber over a point in Λ''_r is isomorphic to

$$\mathbf{A}^l \times \left\{z^2 w_1 = z^4 w_2 = \cdots = z^{2(r+1)} w_{r+1} = 0\right\} \subset \mathbf{A}^{l+r+2}. \tag{4.24}$$

Combined, we see that $\mathcal{M}_{0,q}$ has one irreducible component V_0 that dominates $\mathcal{Z}_q$: it is the vector bundle $\bar\pi_* \bar f^* L$ over $\mathcal{Z}_q$; other irreducible components lie over points in $\Lambda \cap \mathcal{Z}_q$: one for each $u \in \Lambda'_r \cap \mathcal{Z}_q$—we denote this component by $V_u \subset \bar p^{-1}(u)$, where $\bar p\colon \mathcal{M}_{0,q} \to \mathcal{Z}_q$ is the projection as before. Finally, we comment that points in V_u have automorphism groups $\mathbb{Z}_2$ since the only marked points are branch points for these stable maps.

We now prove Lemma 4.5. We let E be a vector bundle on $\mathcal{M}_{0,q}$ so that its sheaf of sections surjects onto $\mathcal{O}b_{\mathcal{M}_{0,q}}$; we let $W \subset E$ be the virtual normal cone of the obstruction theory of $\mathcal{M}_{0,q}$. Of all the irreducible components of W, one dominates V_0. It is a sub-bundle of $E|_{V_0}$; its normal bundle in $E|_{V_0}$ is the pullback of the torsion free part $\mathcal{R}$ of $R^1\bar\pi^*\bar f^*L$. Therefore, the contribution to the localized GW-invariant of this component is $(-1)^l$ times the first Chern class of $\mathcal{R}$. Here $l = h^0(L)$ is the dimension of the fiber of $V_0 \to \mathcal{Z}_q$; the sign $(-1)^l$ is due to the reason similar to the proof of Proposition 3.15; the degree of $\mathcal{R}$ is the difference of $c_1(-\bar\pi_! \bar f^* L)$ and the total degree of the torsion part of $R^1\bar\pi_*\bar f^*L$. By (4.18), the latter at a point in $\Lambda'_r \subset \Lambda'$ is (4.19) with $\alpha_i = 2i-1$; thus the total degree of the torsion part lying over Λ' is

$$\sum_{r=0}^{[\frac{h-2}{2}]} \frac{1}{2}\cdot a_{2r}\cdot \binom{2h+2}{h-2-2r}, \quad a_{2r} = 1+3+5+\cdots+(2r+1).$$

Here the factor $\frac{1}{2}$ is from the trivial $\mathbb{Z}_2$ action. Similarly, the degree of the torsion part lying over Λ'' is

$$\sum_{r=0}^{[\frac{h-3}{2}]} \frac{1}{2}\cdot a_{2r+1}\cdot \binom{2h+2}{h-3-2r}, \quad a_{2r+1} = 2+4+6+\cdots+(2r+2).$$

Hence, by Proposition 3.15 and (4.10), the contribution to the localized GW-invariant of this irreducible component is

$$(-1)^l\Big[(h-2)2^{2h-3}-\sum_{j=0}^{h-2}\binom{2h+2}{h-2-j}a_j\Big].$$

The other irreducible components are supported over V_u for $u\in\mathcal{Z}_q\cap\Lambda$. Let $m=\operatorname{rank}E$. Over a $u\in\mathcal{Z}_q\cap\Lambda'_r$, the virtual normal cone W has $(r+1)$ irreducible components lying over $V_u\subset\bar p^{-1}(u)$: they are indexed by $0\le i\le r$; the i-th is supported on a rank $m+i-(l+r+1)$ subbundle of E, of multiplicity $2i+1$, over

$$V_{u,i}=\mathbf{A}^l\times\{w_1=\cdots=w_i=0,\, z^{2i+1}=0\}\subset V_u\subset\bar p^{-1}(u).$$

Thus following the proof of Proposition 3.15, the total contribution to the localized GW-invariant of these components is

$$b_{2r}\cdot\frac{(-1)^l}{2}=\sum_{i=0}^{r}(-1)^{l+r+1-i}\,(2i+1)\,\frac{1}{2}.$$

As mentioned, the sign is from Proposition 3.15; $(2i+1)$ is the multiplicity; $1/2$ is from the trivial $\mathbb{Z}_2$ action.

By the same reason, there are $(r+1)$ irreducible components of W over a point $u\in\Theta''_r$; they are indexed by $0\le i\le r$; the i-th is a rank $m+i-(l+r+1)$ sub-bundle of E over $\mathbf{A}^{l+r+1-i}$ with multiplicities $2i+2$. The contribution to the localized GW-invariant of these components is

$$b_{2r+1}\cdot\frac{(-1)^l}{2}=\sum_{i=0}^{r}(-1)^{l+r+1-i}\,(2i+2)\,\frac{1}{2}.$$

Combining the above, the contribution to the localized GW-invariant of the component $\mathcal{N}_0$ of branched double covers is

$$(-1)^{h^0(L)}\Big[(h-2)2^{2h-3}-\sum_{j=0}^{h-2}\binom{2h+2}{h-2-j}\cdot\frac{a_j-b_j}{2}\Big].$$

Now it is an elementary combinatorial exercise to check that this coincides with the desired $(-1)^{h^0(L)}(-2^{h-2})$. This completes our proof of Lemma 4.7.

5. Remarks on three-fold case

We shall conclude our paper by commenting on how our method of localization by holomorphic two-form can be applied to study GW-invariants of three-folds.

Let X be a smooth projective three-fold over a smooth projective surface S

$$p:X\longrightarrow S;$$

we assume S has a holomorphic two-form θ whose vanishing locus is D; θ pulls back to a holomorphic two-form $\tilde\theta$ on X. Let $\beta\in H_2(X,\mathbb{Z})$ and $u:C\to X$ be a stable map such that $u_*[C]=\beta$. Then u is $\tilde\theta$-null only if for any irreducible component C' of C its image $p(C')$ is either a point or is contained in D. Now suppose $p_*(\beta)\neq 0\in H_2(S,\mathbb{Z})$; since C is connected, u is $\tilde\theta$-null stable map if and only if $u(C) \subset Y = p^{-1}(D)$. Here $Y = p^{-1}(D)$ is defined by the Cartesian square

$$\begin{array}{ccc} X & \xleftarrow{\ i\ } & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xleftarrow{\ j\ } & D \end{array}$$

Hence, the virtual fundamental class of the moduli space $\mathcal{M}_{g,n}(X, \beta)$ is supported in the locus of stable maps to Y. In particular, if β is not in the image of $H_2(Y, \mathbb{Z})$, the GW-invariants vanish.

Now suppose $\beta' = j_*(d[D])$ for some $d \in \mathbb{Z} - \{0\}$. Under favorable circumstances as in Proposition 3.15, we can express the virtual fundamental class $[\mathcal{M}_{g,n}(X, \beta)]^{\mathrm{vir}}$ in terms of $[\mathcal{M}_{g,n}(Y, \beta_Y)]^{\mathrm{vir}}$ for suitable $\beta_Y \in H_2(Y, \mathbb{Z})$. In this case, the GW-invariants of the three-fold X can be computed in terms of the GW-invariants of the surface Y. In case Y also admits a holomorphic two-form, for instance, $Y = D \times E$ for an elliptic curve E, then we can apply our localization by holomorphic two-form again and reduce the computation further to the curve case. In case Y is a ruled surface over D, then by deforming Y to $\mathbb{P}(\mathcal{O} \oplus L)$ for some line bundle L on D, we may assume Y admits a torus action. The virtual localization formula [8] then can be applied to this case to evaluate the GW-invariants of Y in terms of those of D.

Another important case is when X is a ruled three-fold over a surface S with a canonical divisor D. In this case, we can proceed as follows: using the circle action on the fibers of $X \to S$ we can apply the virtual localization first and reduce the computation of the invariants of X to S. After that, we can apply our localization principle by holomorphic two-form to further reduce the computation to the curve case. As is indicated by our degree two calculation of the GW-invariants of surfaces, further works must be done to carry this through. Nevertheless, it is worth pursuing due to the importance of the ruled three-folds in the future investigation of GW-invariants of general three-folds.

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Department of Mathematics and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail address: kiem@math.snu.ac.kr

Department of Mathematics, Stanford University, Stanford, USA
E-mail address: jli@math.stanford.edu