THE EQUIVARIANT COHOMOLOGY RING OF THE MODULI SPACE OF VECTOR BUNDLES OVER A RIEMANN SURFACE

Young-Hoon Kiem

Abstract. We prove (obvious analogues of) the Mumford conjecture and the structure theorem for the equivariant cohomology ring

$$H^*_{\mathcal{G}}(\mathcal{C}^{ss}) = H^*_{SU(2)}(Hom(\pi_1(X), U(2)))$$

of the moduli space of rank 2 semistable vector bundles of even degree over a Riemann surface X. Our theorem completely determines the ring structure of the equivariant cohomology.

1. Introduction

Let M_d be the moduli space of rank 2 semistable holomorphic vector bundles of degree d over a Riemann surface X of genus g. Thanks to many authors including those named below, we now have a clear understanding of the cohomology ring of the moduli space when the degree d is odd. In this case, it is a smooth projective varitiety and there is a universal bundle $\mathcal{U} \to M_d \times X$.

Fix a holomorphic line bundle L over X of odd degree d and let M_L be the moduli space of rank 2 semistable bundles with determinant L over X. Then we get a cohomologically trivial fibration

$$\begin{array}{ccc} M_L & \longrightarrow & M_d \\ & & \downarrow \\ & & Jac \end{array}$$

over the Jacobian of degree d by taking determinant [T2]. Hence,

$$H^*(M_d) = H^*(M_L) \otimes H^*(Jac).$$

The Künneth components of the first Chern class of the universal bundle generate the cohomology ring $H^*(Jac)$ and those of the second Chern class generate $H^*(M_L)$ [AB][N]. If we put $d = 4g-3$, the push-forward $f_!\mathcal{U}$ is a vector bundle of rank $2g-1$ by Riemann-Roch where $f : M_d \times X \to M_d$ is the projection onto the first component. Therefore $c_r(f_!\mathcal{U}) = 0$ for $r \geq 2g$. Mumford conjectured that the relations from the vanishing of the Chern classes form a complete set of relations and it was proved by Kirwan [K2]. Moreover, Baranovsky, King & Newstead, Siebert & Tian, and Zagier found a (minimal) finite set of relations that generate the whole

1991 *Mathematics Subject Classification.* 14D20, 14F43, 70S15, 81T13.

relation ideal [Ba][KN][ST][Z]. Their results actually enable us to compute the cup product and can be summarized in the "structure theorem".

As the moduli space is a smooth projective variety, the ring structure is uniquely determined by the intersection pairing by Poincare duality. Formulas for the intersection pairing were deduced by Donaldson and Thaddeus [D][T1]. Witten generalized the formulas to arbitrary rank case when the degree is coprime to the rank. Recently, Jeffrey & Kirwan, and Liu proved his formulas [JK][Liu].

Contrary to the spectacular achievements for the odd degree case, little has been known for the even degree case. In this case, the moduli space is a singular projective variety that can be viewed as a quotient space. Therefore, we can think of at least three cohomology groups: equivariant, intersection, and ordinary. The Betti numbers have been known thanks to the works of Atiyah & Bott, Kirwan, and Cappell & Lee & Miller, respectively [AB][K1][CLM]. In this paper, we compute the ring structure of the equivariant cohomology. Namely, we prove analogues of the Mumford conjecture and the structure theorem.

Our method is parallel to the odd degree case. In section 2, we recall various facts about the moduli space of rank 2 vector bundles of even degree over a fixed Riemann surface X of genus g. From the perspective of [AB], it is the symplectic quotient $\mathcal{C}^{ss}//\mathcal{G} = \mathcal{A}_{flat}/\mathcal{G}$ of the space $\mathcal{C}^{ss}$ of semistable holomorphic structures on a fixed rank 2 complex Hermitian vector bundle E of even degree, by the $U(2)$- gauge group $\mathcal{G}$. As constants act trivially and the quotient is singular, we think of the homotopy quotient of $\mathcal{C}^{ss}$ by the group $\overline{\mathcal{G}} = \mathcal{G}/U(1)$. The equivariant cohomology $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ is the cohomology of this homotopy quotient

$$\mathcal{C}^{ss}_{\overline{\mathcal{G}}} = \mathcal{C}^{ss} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}.$$

There is a universal bundle $\mathcal{U}$ for this homotopy quotient and the Künneth components of the first and second Chern classes of the universal bundle generate the equivariant cohomology by the arguments of Atiyah and Bott [AB].

Fix a line bundle L of even degree and consider the moduli space $\mathcal{C}^{ss}_L//\mathcal{G}$ of rank 2 semistable bundles with determinant L. It turns out that

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}) = H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L) \otimes H^*(Jac)$$

where Jac is the Jacobian.

Let the degree be $4g-2$ and $f : \mathcal{C}^{ss}_{\overline{\mathcal{G}}} \times X \to \mathcal{C}^{ss}_{\overline{\mathcal{G}}}$ be the projection onto the first component. As the universal bundle is holomorphic in the X direction, we can consider the push-forward $f_!\mathcal{U}$ of the universal bundle, which is a bundle of rank $2g$ by Riemann-Roch. Therefore, the Chern classes $c_r(f_!\mathcal{U})$ vanish for $r > 2g$.

In section 3, we first deduce an expression for the Chern polynomial of $f_!\mathcal{U}$ and then read off the "Mumford relations" to (1)show that the relations coming in this fashion generate all the others (the Mumford conjecture), and (2)find a finite set of relations among them, which generate all the relations (the structure theorem).

The moduli space of rank 2 vector bundles with determinant L is homeomorphic[1] to the quotient

$$R := Hom(\pi_1(X), SU(2))/SU(2)$$

[1]It is in fact diffeomorphic as a stratified symplectic space. (See [SL].)

where $SU(2)$ acts by conjugation. Moreover, we have the following isomorphism:

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L) \cong H^*_{SU(2)}(R^{\#}_{SU(2)})$$

where $R^{\#}_{SU(2)} = Hom(\pi_1(X), SU(2))$. As a corollary of the structure theorem, we show that the kernel of the restriction map $H^*_{SU(2)}(R^{\#}_{SU(2)}) \to H^*_{SU(2)}(R^{\#}_{red})$ is generated by a single class ξ where $R^{\#}_{red}$ is the subset of reducible homomorphisms. The class is, up to constant multiple, the V class in [CLM].

Our method can be applied equally well to the odd degree case and provides a simple proof of the Mumford conjecture and the structure theorem [Ki1]. In fact, a stronger version of the Mumford conjecture was proved in [Ki1]. Namely, the Mumford relations from the first vanishing Chern class only, generate the whole relation ideal. However, they are NOT independent over $\mathbb{Q}[\alpha, \beta]$.

As a further application of the structure theorem for the equivariant cohomology, in [Ki3], we deduce various structures of the intersection cohomology of the singular moduli space, including the intersection pairing, from the equivariant cohomology by using a splitting, constructed in [Ki2], of the Kirwan map from the equivariant cohomology to the intersection cohomology.

Every cohomology group in this paper has rational coefficient.

Acknowledgement. I am very grateful to my advisor, Professor Ronnie Lee, for encouragement and valuable advices. I would like to thank the referee for careful reading and comments.

2. The equivariant cohomology of the moduli space

In this section, we recall various facts about the equivariant cohomology of the moduli space of rank 2 semistable vector bundles over a Riemann surface X of genus g.

Let $\mathcal{C}$ be the space of holomorphic structures on a fixed rank 2 complex Hermitian vector bundle E of even degree, $4g-2$. It is an affine space of Cauchy-Riemann operators based on $\Omega^{0,1}(End\, E)$ and therefore contractible. If we denote by $\mathcal{C}^{ss}$ the subspace of semistable holomorphic structures, then the moduli space of semistable bundles can be identified with the symplectic quotient $\mathcal{C}^{ss}//\mathcal{G}$ where $\mathcal{G}$ is the gauge group of the principal $U(2)$-bundle associated to E. (See [AB].) Obviously, constants commute with Cauchy-Riemann operators and thus act trivially on $\mathcal{C}^{ss}$. We put $\overline{\mathcal{G}}_c = \mathcal{G}_c/\mathbb{C}^*$. Then $\overline{\mathcal{G}}_c$ acts freely on the open dense subset $\mathcal{C}^s$ of stable holomorphic structures, but not on $\mathcal{C}^{ss}$. So, we consider the homotopy quotient

$$\mathcal{C}^{ss}_{\overline{\mathcal{G}}} = \mathcal{C}^{ss} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}.$$

In this paper, the equivariant cohomology means the cohomology of the homotopy quotient.

Even though there does not exist any (holomorphic) universal bundle for the moduli space $\mathcal{C}^{ss}//\mathcal{G}_c$ (See [Ra], Thm 2), we do have a (topological) universal bundle for the homotopy quotient as follows (See [AB], p579): Let W be the obvious universal vector bundle over $\mathcal{C}^{ss} \times X$. By taking quotient of the pullback of $\mathbb{P}W$, over $\mathcal{C}^{ss} \times E\overline{\mathcal{G}} \times X$, by $\overline{\mathcal{G}}$, we get a projective bundle $\mathbb{P}\mathcal{U}$ over $\mathcal{C}^{ss}_{\overline{\mathcal{G}}} \times X$. This bundle lifts to a vector bundle $\mathcal{U}$ because the obstruction for the lifting vanishes by the existence of a vector bundle W' over

$$\mathcal{C}^{ss}_{\mathcal{G}} \times X \subset \mathcal{C}_{\mathcal{G}} \times X = B\mathcal{G} \times X = Map_{4g-2}(X, BU(2)) \times X$$

which is the pullback of the universal bundle over $BU(2)$ via the evaluation map

$$ev : Map_{4g-2}(X, BU(2)) \times X \to BU(2).$$

The universal bundle $\mathcal{U}$ is continuous in the $\mathcal{C}^{ss}_{\mathcal{G}}$ direction and holomorphic in the X direction.

The moduli space of vector bundles can be also thought of as the moduli space of flat connections as follows [AB]: Let $\mathcal{A}$ be the space of connections on the principal $U(2)$-bundle associated to E. It is an affine space based on the space of $u(2)$-valued 1-forms. By taking the $(0,1)$-part of a connection, we get a map $\mathcal{A} \to \mathcal{C}$. The Morse stratification on $\mathcal{A}$ with respect to the norm square of curvature is equivalent to the Shatz stratification of $\mathcal{C}$ and the map induces an identification

$$\mathcal{A}_{flat}/\mathcal{G} = \mathcal{C}^{ss}//\mathcal{G}_c$$

where $\mathcal{A}_{flat}$ is the subspace of flat connections. Moreover, if we let $\mathcal{G}_0 = \{g : X \to U(2) | g(p) = id\}$ with $p \in X$, then $\mathcal{G}_0$ acts freely on $\mathcal{A}_{flat}$ and

$$\mathcal{A}_{flat}/\mathcal{G}_0 = Hom(\pi_1(X), U(2)) =: R^{\#}_{U(2)}.$$

Therefore, $\mathcal{C}^{ss}_{\overline{\mathcal{G}}} = \mathcal{A}_{flat} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}} = R^{\#}_{U(2)} \times_{PU(2)} EPU(2)$ and thus

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}) = H^*_{\overline{\mathcal{G}}}(\mathcal{A}_{flat}) \cong H^*_{PU(2)}(R^{\#}_{U(2)}) = H^*_{SU(2)}(R^{\#}_{U(2)})$$

With the above identification, we can construct

$$End\mathcal{U} \to (R^{\#}_{U(2)} \times_{PU(2)} EPU(2)) \times X$$

as follows: Let $\tilde{X}$ be the universal cover of X. Consider

$$(R^{\#}_{U(2)} \times End(\mathbb{C}^2)) \times_{\pi_1(X)} \tilde{X} \to R^{\#}_{U(2)} \times X$$

where $g \in \pi_1(X)$ maps $(\phi, v) \in R^{\#}_{U(2)} \times End(\mathbb{C}^2)$ to $(\phi, Ad\,\phi(g)\,v)$. It is a vector bundle of rank 4 which induces $End\mathcal{U}$ over $(R^{\#}_{U(2)} \times_{PU(2)} EPU(2)) \times X$ by pulling back and taking quotient.

Now, we consider the special structure: Fix a line bundle L of degree $4g-2$ over X. Let $\mathcal{C}^{ss}_L$ denote the subspace of semistable holomorphic structures with determinant L. Then we have the following fibration by taking determinant

$$\begin{array}{ccc} \mathcal{C}^{ss}_L & \longrightarrow & \mathcal{C}^{ss} \\ & & \downarrow \\ & & Jac_{4g-2} \end{array}$$

By the same arguments as for $\mathcal{C}^{ss}$, the homotopy quotient $(\mathcal{C}_L^{ss})_{\overline{\mathcal{G}}} = \mathcal{C}_L^{ss} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}$ is homotopically equivalent to $R^{\#}_{SU(2)} \times_{PU(2)} EPU(2)$ where $R^{\#}_{SU(2)} = Hom(\pi_1(X), SU(2))$. Therefore,

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}_L^{ss}) = H^*_{PU(2)}(R^{\#}_{SU(2)}) = H^*_{SU(2)}(R^{\#}_{SU(2)}).$$

As $U(2) = SU(2) \times_{\mathbb{Z}/2} U(1)$,

$$R^{\#}_{U(2)} = R^{\#}_{SU(2)} \times_{(\mathbb{Z}/2)^{2g}} Jac$$

and

$$R^{\#}_{U(2)} \times_{PU(2)} EPU(2) = (R^{\#}_{SU(2)} \times_{PU(2)} EPU(2)) \times_{(\mathbb{Z}/2)^{2g}} Jac.$$

According to [AB], $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ is generated by the classes α, β, ψ_i, d_i defined by the Künneth decomposition of the Chern classes as follows:

$$c_1(\mathcal{U}) = (4g-2) \otimes [X] + \sum_{i=1}^{2g} d_i \otimes e_i + x \otimes 1$$

$$c_2(End(\mathcal{U})) = 2\alpha \otimes [X] + 4 \sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1$$

where $\{e_i\}$ is a symplectic basis of $H^1(X)$ so that $e_i e_{i+g} = [X]$. Similarly, $H^*_{\overline{\mathcal{G}}}(\mathcal{C}_L^{ss})$ is generated by the (restricted) classes α, β, ψ_i.

Because $\mathbb{Z}/2$ is the center of $SU(2)$, $(\mathbb{Z}/2)^{2g}$ action preserves

$$End\mathcal{U} \to (R^{\#}_{SU(2)} \times_{PU(2)} EPU(2)) \times X.$$

Hence, it acts trivially on α, β, ψ_i and it also acts trivially on the cohomology of the Jacobian. Therefore, we get

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}) = H^*_{\overline{\mathcal{G}}}(\mathcal{C}_L^{ss}) \otimes H^*(Jac).$$

To understand the ring structure of the cohomology, we have only to understand the relations for $H^*_{\overline{\mathcal{G}}}(\mathcal{C}_L^{ss})$. Let

$$f : \mathcal{C}^{ss}_{\overline{\mathcal{G}}} \times X \to \mathcal{C}^{ss}_{\overline{\mathcal{G}}}$$

be the projection onto the first component. Then $f_!\mathcal{U}$ is a vector bundle of rank $2g$ by Riemann-Roch and semistability. Therefore, $c_r(f_!\mathcal{U})$ are relations for $r \geq 2g+1$. By choosing a basis of $H^*(Jac)$, we get 2^{2g} sequences of relations in $H^*_{\overline{\mathcal{G}}}(\mathcal{C}_L^{ss})$ via the above decomposition. Those relations are called the *Mumford relations*.

In the next section, we will prove that the Mumford relations generate the whole space of relations in $\mathbb{Q}[\alpha, \beta] \otimes \Lambda^*(\psi_i)$. (Mumford's conjecture.) Moreover, we will find a finite number of classes that generate all the other relations and prove the "structure theorem".

3. The structure theorem

We first compute the Chern polynomial for $f_!\mathcal{U}$ and then read off the Mumford relations in order to prove the structure theorem. The Mumford conjecture is a consequence of our theorem.

Recall that

$$c_1(\mathcal{U}) = (4g-2)\otimes[X] + \sum_{i=1}^{2g} d_i\otimes e_i + x\otimes 1$$

$$c_2(End(\mathcal{U})) = 2\alpha\otimes[X] + 4\sum_{i=1}^{2g}\psi_i\otimes e_i - \beta\otimes 1.$$

As observed by Zagier ([Z], p557), for our purpose, we may assume that $x=0$. Let $\gamma = -2\sum_{i=1}^{g}\psi_i\psi_{i+g}$ and $\xi = \alpha\beta + 2\gamma$. Then α, β, γ generate the invariant part $[H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L)]^{inv}$ of $H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L)$ with respect to the $Sp(2g)$ action on ψ_i's and so do α, β, ξ.

Let

$$\lambda_j = (2g-1-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}})\otimes[X] + \sum_{i=1}^{2g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})\otimes e_i + (-1)^j\frac{\sqrt{\beta}}{2}\otimes 1.$$

Then one can check that $\lambda_1+\lambda_2 = c_1(\mathcal{U})$ and $\lambda_1\,\lambda_2 = c_2(\mathcal{U})$. Hence,

$$\begin{aligned}
&ch(\mathcal{U}) = e^{\lambda_1}+e^{\lambda_2}\\
&= \sum_{j=1,2}(1+(2g-1-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}})\otimes[X])\\
&(1+\sum_{i=1}^{2g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})\otimes e_i + \frac{1}{2}(\sum_{i=1}^{2g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})\otimes e_i)^2)exp((-1)^j\frac{\sqrt{\beta}}{2})\\
&= \sum_{j=1,2}(1+\sum_{i=1}^{2g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})\otimes e_i\\
&+(2g-1-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2}-(-1)^j\frac{\psi_{i+g}}{\sqrt{\beta}}))\otimes[X])exp((-1)^j\frac{\sqrt{\beta}}{2})
\end{aligned}$$

and by Grothendieck-Riemann-Roch

$$\begin{aligned}
&ch(f_!(\mathcal{U})) = f_*(ch(\mathcal{U})(1-(g-1)[X])\\
&= \sum_{j=1,2}(g-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2}-(-1)^j\frac{\psi_{i+g}}{\sqrt{\beta}}))exp((-1)^j\frac{\sqrt{\beta}}{2})\\
&= \sum_{j=1,2}\sum_{n\geq 0}\frac{(g-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}})((-1)^j\sqrt{\beta}/2)^n}{n!}\\
&-(n+1)\frac{(\sum_{i=1}^{g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2}-(-1)^j\frac{\psi_{i+g}}{\sqrt{\beta}}))((-1)^j\sqrt{\beta}/2)^n}{(n+1)!}.
\end{aligned}$$

We now use a trick of Zagier in [Z]. Noticing

$$log\prod(1+u_k)=\sum_{n\geq 1}\frac{(-1)^{n-1}\sum u_k^n}{n}$$

$$\sum e^{u_k}=\sum_{n\geq 0}\frac{\sum u_k^n}{n!},$$

we deduce that

$$log\, c(f_!\mathcal{U})=\sum_{j=1,2}(g-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}})log(1+(-1)^j\frac{\sqrt{\beta}}{2})- \\ \frac{\sum_{i=1}^{g}(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2}-(-1)^j\frac{\psi_{i+g}}{\sqrt{\beta}})}{1+(-1)^j\frac{\sqrt{\beta}}{2}}.$$

Therefore,

$$\begin{aligned} c(f_!\mathcal{U})_{-2t} &= \sum_{r\geq 0} c_r(f_!\mathcal{U})(-2t)^r \\ &= (1-\beta t^2)^g(\frac{1+t\sqrt{\beta}}{1-t\sqrt{\beta}})^{\frac{\xi}{2\beta\sqrt{\beta}}}exp(\frac{At+2Bt^2-2\gamma t/\beta}{1-\beta t^2}) \\ &= \Phi(t)\,G(t) \end{aligned}$$

where

$$G(t)=(1-\beta t^2)^g exp(\frac{At+2Bt^2-2\gamma t^3}{1-\beta t^2}),$$

$$\Phi(t)=\sum_{n=0}^{\infty}c_n t^n=exp(\alpha t+\xi\sum_{k\geq 1}\frac{\beta^{k-1}t^{2k+1}}{2k+1})=e^{-\frac{2\gamma t}{\beta}}(\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t})^{\frac{\xi}{2\beta\sqrt{\beta}}},$$

$$A=\sum_{i=1}^{g}d_i d_{i+g},\quad B=\sum_{i=1}^{g}-d_i\psi_{i+g}+d_{i+g}\psi_i.$$

From Riemann-Roch, $f_!\mathcal{U}$ is a vector bundle of rank $2g$ and therefore $\Phi(t)\,G(t)$ is a polynomial of degree$\leq 2g$.

To read off relations in the generators, we need to generalize a lemma of Zagier ([Z], p559). Let $\wedge^*H^3=\oplus_{l=0}^{g}\oplus_{k=0}^{g-l}\gamma^k Prim_l$ be the Lefshetz decomposition of the exterior algebra of $H^3_{\mathcal{G}}(\mathcal{C}_L^{ss})=\mathbb{Q}\{\psi_1,\cdots,\psi_{2g}\}$. Now, let $\sigma_l=\sum_{|I|=l}a_I\psi_I\in Prim_l(\psi_i)$ be a primitive element and put $\tilde{\sigma}_l=\sum_{|I|=l}a_I d_I\in Prim_l(d_i)$. Then we have the following

Lemma 1.

$$\frac{A^{g-l-p}B^{l+2p}}{(g-l-p)!(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g}d_i d_{i+g}]=\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l$$

$$\frac{A^iB^j}{i!j!}\tilde{\sigma}_l/[\prod_{i=1}^{g}d_i d_{i+g}]=0\ \textit{otherwise},$$

where $\cdot/[\prod_{i=1}^{g} d_i d_{i+g}]$ *means the coefficient of* $[\prod_{i=1}^{g} d_i d_{i+g}]$.

Proof. The second statement is quite obvious.

Considering the Sp_{2g} action, we may assume that $\sigma_l = \psi_{g-l+1}\cdots\psi_g$ since $Prim_l$ is an irreducible module. Now, Zagier's original lemma says

$$\frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!} = \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}/[\prod_{i=1}^{g-l} d_i d_{i+g}].$$

Thus,

$$\begin{aligned}
\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l &= \frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!}\sigma_l \\
&= \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}\sigma_l/[\prod_{i=1}^{g-l} d_i d_{i+g}] \\
&= \frac{(\sum_{i=1}^{g} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p+l}}{(g-l-p)!(2p+l)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]
\end{aligned}$$

as one can check directly. So we are done.

For $\tilde{\sigma}_l \in Prim_l(d_i)$,

$$\begin{aligned}
&G(t)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= \sum_{r,s}(1-\beta t^2)^g \frac{A^r t^r}{r!(1-\beta t^2)^r}\frac{2^s B^s t^{2s}}{s!(1-\beta t^2)^s} exp(\frac{-2\gamma t^3}{1-\beta t^2})\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{-p} 2^{l+2p} t^{g+l+3p}\frac{A^{g-l-p}}{(g-l-p)!}\frac{B^{l+2p}}{(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{-p} 2^{l+2p} t^{g+l+3p}\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l \\
&= 2^l t^{g+l} exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p \frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \\
&= 2^l t^{g+l}\sigma_l.
\end{aligned}$$

As a consequence,

$$2^l t^{g+l}\Phi(t)\sigma_l = c(f!(\mathcal{U}))_{-2t}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]$$

and thus $\Phi(t)\sigma_l$ is a polynomial of degree $\le g-l$.

Proposition 1. $\oplus_{l=0}^{g} Prim_l \otimes I_{g-l}$ *is a subspace of the relation ideal, where* I_n *is the ideal of* $\mathbb{Q}[\alpha,\beta,\xi]$ *generated by* $\{c_i | i \ge n+1\}$.

Recall that c_n was defined to be the n-th coefficient of $\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = exp(\alpha t + \xi\sum_{k\ge 1}\frac{\beta^{k-1}t^{2k+1}}{2k+1})$. One can readily check that the sequence $\{c_n\}$ is determined by the following recursion formula;

$$nc_n = \alpha c_{n-1} + (n-2)\beta c_{n-2} + 2\gamma c_{n-3}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \frac{\alpha^2}{2}$, $c_3 = \frac{\alpha^3+2\xi}{3!}$, etc. Therefore, I_n is in fact generated by just three elements c_{n+1}, c_{n+2}, c_{n+3}.

Now, put $c_{n,k} = \sum_{i=0}^{k} \frac{1}{i!}\binom{n-1-i}{k-i}(2\gamma)^i\beta^{k-i}c_{n-k-i}$, for $0 \leq k < n$. Then by modifying Theorem 4 in [Z], we get the following

Lemma 2.

$$(-1)^k c_{n,k} = \sum_{i=0}^{\infty}(-1)^i(\binom{n-k+i}{i} + \binom{n-k+i-1}{i-1})c_{k-i}c_{n+i}.$$

In particular, $c_{n,k}$ belongs to the ideal generated by c_n, c_{n+1}, c_{n+2}.

Proof. One can check, as in [Z], that both sides satisfy $kc_{n,k} = (n-1)\beta c_{n-1,k-1} + 2\gamma c_{n-2,k-1}$.

Lemma 3. *$c_n \equiv \alpha^n/n!$ modulo the ideal generated by ξ.*

Proof. It follows immediately from an induction with the recursion formula.

It is also easy to check the following variation of a lemma in [KN].

Lemma 4. *The leading term of $c_{n,k}$ is, up to constant, $\xi^k\alpha^{n-2k}$ for $0 \leq k \leq [\frac{n}{2}]$, and $\xi^{n-k}\beta^{2k-n}$ for $[\frac{n}{2}] < k < n$, with respect to the reverse lexicographical order where $\alpha > \xi > \beta$. If we put $c_{n,\frac{n}{2}} = 12\gamma c_{n,\frac{n-3}{2}} - \frac{n-1}{2}\alpha^2 c_{n,\frac{n-1}{2}}$, for odd n, then its leading term is up to constant $\xi^{\frac{n+1}{2}}$.*

Proof. Obvious from the definition of $c_{n,k}$ and Lemma 3.

From the above lemmas, we deduce that $\mathbb{Q}[\alpha, \beta, \xi]/I_g$ is a quotient of the vector space spanned by

$$\{\alpha^i\beta^j\xi^k \mid (1)\, i + 2k \leq g, \quad (2) \text{ if } k \geq 1 \text{ then } j + 2k \leq g\}.$$

Recall that $deg\,\alpha = 2$, $deg\,\beta = 4$, $deg\,\xi = 6$. We can compute the Poincare series for this graded vector space.

Lemma 5.

$$P_t(\mathbb{Q}\{\alpha^i\beta^j\xi^k \mid \textit{(1) } i+2k \leq g, \quad \textit{(2) if } k \geq 1 \textit{ then } j+2k \leq g\}) = \frac{\frac{1-t^{6g+6}}{1-t^6} - t^{2g+2}\frac{1-t^{2g+2}}{1-t^2}}{(1-t^2)(1-t^4)}.$$

Proof. Combinatorial exercise.

As a consequence,

$$P_t(\mathbb{Q}[\alpha, \beta, \xi]/I_g) \leq \frac{\frac{1-t^{6g+6}}{1-t^6} - t^{2g+2}\frac{1-t^{2g+2}}{1-t^2}}{(1-t^2)(1-t^4)}.$$

Therefore, we have

Lemma 6.

$$P_t(\oplus_{l=0}^g Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l}) \le \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Proof. Combinatorial exercise.

As $\oplus_{l=0}^g Prim_l \otimes I_{g-l}$ is a subspace of the relation ideal, there is a surjection $\oplus_{l=0}^g Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l} \to H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L)$. From Atiyah-Bott's equivariant Morse theoretic argument, it is well-known that

$$P_t(H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L)) = \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Therefore, together with the previous lemma, we deduce that $\oplus_{l=0}^g Prim_l \otimes I_{g-l}$ is, in fact, equal to the whole relation ideal. Since all the relations were derived from the Chern class of the pushforward bundle, we conclude that the obvious analogue of the Mumford conjecture is true. In summary, we have the following "structure theorm".

Theorem 1.

$$H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L) = H^*_{SU(2)}(R^\#_{SU(2)}) \cong \oplus_{l=0}^g Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l}.$$

Let $R^\#_{red}$ be the set of homomorphisms of $\pi_1(X)$ into $SU(2)$ whose images are abelian. Then the inclusion $R^\#_{red} \hookrightarrow R^\# := R^\#_{SU(2)}$ induces a homomorphism $H^*_{SU(2)}(R^\#) \to H^*_{SU(2)}(R^\#_{red})$. Here, according to [CLM], $H^*_{SU(2)}(R^\#_{red})$ is the $\mathbb{Z}/2$-invariant part of the (skew-)commutative algebra freely generated by q_i $(1 \le i \le 2g)$ and r, of degree 1 and 2 respectively. Moreover, α restricts to $-2w$, β to $4r^2$, ψ_i to $-2rq_i$ and γ to $4r^2w$ respectively, where $w = -2\sum_{i=1}^g q_iq_{i+g}$. We can now compute the kernel of the homomorphism.

Corollary 1. *The kernel of the homomorphism* $H^*_{SU(2)}(R^\#) \to H^*_{SU(2)}(R^\#_{red})$ *is generated by the single element* $\xi = \alpha\beta + 2\gamma$.

Proof. Trivially, ξ is in the kernel. We have

$$H^*_{SU(2)}(R^\#_{red}) = [\oplus_{l=0}^g Prim_l\{q_1,\cdots,q_{2g}\} \otimes \mathbb{Q}[r,w]/w^{g-l+1}]^{\mathbb{Z}/2}.$$

Because the homomorphism respects the symplectic action, we have only to consider the invariant part. From Lemma 3, one can easily deduce that

$$(\mathbb{Q}[\alpha,\beta,\xi]/I_{g-l})/(\xi) = \mathbb{Q}[\alpha,\beta]/\alpha^{g-l+1},$$

the right hand side of which obviously injects into $[\mathbb{Q}[r,w]/w^{g-l+1}]^{\mathbb{Z}/2}$. This completes the proof.

Note that ξ is actually the V class in [CLM] up to constant multiple.

Let $R = R^\#/SU(2)$, which is diffeomorphic to the moduli space M_L. Then the natural map $R^\# \times_{SU(2)} ESU(2) \to R$ induces a homomorphism $H^*(R) \to H^*_{SU(2)}(R^\#)$.

Corollary 2. *The image in $H^{6g-6}_{SU(2)}(R^{\#})$ of a top degree class in $H^{6g-6}(R)$ is a constant multiple of $\alpha^{g-2}\beta^{g-2}\xi$ where $R = R^{\#}/SU(2)$ and $R_{red} = R^{\#}_{red}/SU(2)$.*

Proof. As the image of a top degree class by the composition map $H^{6g-6}(R) \to H^{6g-6}(R_{red}) = 0 \to H^{6g-6}_{SU(2)}(R^{\#}_{red})$ is zero, the image of the class by the composition map $H^{6g-6}(R) \to H^{6g-6}_{SU(2)}(R^{\#}) \to H^{6g-6}_{SU(2)}(R^{\#}_{red})$ is also zero. Thus, the image in $H^{6g-6}_{SU(2)}(R^{\#})$ is in the kernel of the restriction map considered in Corollary 1. And from Theorem 1, Corollary 1, and our choice of the basis described between Lemma 4 and 5, we deduce that the kernel has only one generator in dimension $6g-6$, namely $\alpha^{g-2}\beta^{g-2}\xi$.

Remark 1. Our method applies equally well to the odd degree case as worked out in [Ki1]. It has been conjectured in [KN][T2] from the shape of the Harder-Narasimhan formula that the Mumford relations from the first vanishing Chern class only, freely generate the space of relations as a $\mathbb{Q}[\alpha,\beta]$-module. This was sometimes called the strong Mumford conjecture. Surprisingly, our computation disproves the conjecture and proves a weaker version. Namely, they do generate the whole space of relations as an ideal in $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$. Moreover, in the course of the proof, we (re-)proved the structure theorem and the Mumford conjecture in one stroke. See [Ki1] for details.

Remark 2. $\Phi(t)$ can be realized as the generating function for the Chern classes of a vector bundle Q over $(\mathcal{C}^{ss}_L)_{\overline{\mathcal{G}}}$ as in [ST]. Since the complex structure of X does not matter for our purpose, we may assume that X is a hyperelliptic smooth projective curve, i.e. there is an involution $i : X \to X$ such that the quotient $X/i = \mathbf{P}^1$. In particular, X has $2g+2$ Weierstrass points, $W = \{w_1, w_2, \cdots, w_{2g+2}\}$, or the branch points of the quotient map $X \to X/i$.

We recall a fact about semistable vector bundles. Let E be a rank 2 complex vector bundle over $T \times X$ and $det(E) = p_1^*M \otimes p_2^*L$ for a line bundle M over any variety T, where p_i is the projection onto the i-th factor, $i = 1, 2$. Suppose $E_t = E|_{t\times X}$ is semistable for all $t \in T$. In this case we have the following exact sequence, [DR]

$$0 \to (p_1)_*(E\otimes\hat{i}^*E) \to \sum_{1\le i\le 2g+1} E\otimes E|_{T\times w_i} \to R^1(p_1)_*(E\otimes\hat{i}^*E\otimes p_2^*L_W^{-1}) \to 0$$

where L_W is the line bundle whose divisor is W.

Let $S(E) = (p_1)_*(E\otimes\hat{i}^*E)^\natural$ be the i-antiinvariant subbundle of $(p_1)_*(E\otimes\hat{i}^*E)$. Then $S(E)$ is a bundle of rank $g+1$ by Lemma 2.2 of [DR]. Therefore,

$$S(E) \hookrightarrow \sum_{1\le i\le 2g+1} \Lambda^2 E|_{T\times w_i} = p_1^*M \otimes (\sum_{1\le i\le 2g+1} L_{w_i})_T$$

and thus we have, over T,

$$S(E)\otimes p_1^*M^{-1} \hookrightarrow (\sum_{1\le i\le 2g+1} L_{w_i})_T.$$

Therefore, we get a map $\phi_E : T \to Gr(g+1, 2g+1)$.

Now, we apply the above discussion to the universal bundle $\mathcal{U} \to (\mathcal{C}_L^{ss})_{\overline{\mathcal{G}}} \times X$. From the previous paragraph, we get a map $\phi = \phi_{\mathcal{U}} : (\mathcal{C}_L^{ss})_{\overline{\mathcal{G}}} \to Gr(g+1, 2g+1)$. The natural quotient bundle Q over the Grassmannian has rank g.

Now, one can modify the computation of Siebert and Tian to obtain

$$ch((p_1)_*(\mathcal{U} \otimes \hat{i}^*\mathcal{U})^\natural) = g + 1 - \alpha - \xi \sum_{k=1}^{\infty} \frac{\beta^{k-1}}{(2k+1)!}.$$

Therefore,

$$ch(\phi^* Q) = 2g + 1 - ch(\phi^* S) = g + \alpha + \xi \sum_{k=1}^{\infty} \frac{\beta^{k-1}}{(2k+1)!}$$

and thus $ch_0 = g$, $ch_1 = \alpha$, $ch_{2k} = 0$, and $ch_{2k+1} = \frac{\xi\beta^{k-1}}{(2k+1)!}$, for $k \geq 1$, where $ch_i = ch_i(\phi^* Q)$.

From a well-known identity involving Chern classes and Chern characters, we conclude that

$$\begin{aligned} nc_n &= ch_1 c_{n-1} + ch_3 c_{n-3} + ch_5 c_{n-5} + \cdots \\ &= \alpha c_{n-1} + \frac{\xi}{3!} c_{n-3} + \frac{\xi\beta}{5!} c_{n-5} + \frac{\xi\beta^2}{7!} c_{n-7} + \cdots . \end{aligned}$$

From this, one can easily deduce the following recursion formula

$$nc_n = \alpha c_{n-1} + (n-2)\beta c_{n-2} + 2\gamma c_{n-3}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \frac{\alpha^2}{2}$, $c_3 = \frac{\alpha^3 + 2\xi}{3!}$, etc. The generating function for this sequence is, as one can check easily,

$$\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = exp(\alpha t + \xi \sum_{k \geq 0} \frac{\beta^{k-1} t^{2k+1}}{2k+1}) = e^{-\frac{2\gamma t}{\beta}} (\frac{1 + \sqrt{\beta} t}{1 - \sqrt{\beta} t})^{\frac{\xi}{2\beta\sqrt{\beta}}}.$$

REFERENCES

[AB] M. Aityah, R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London, Series A, 308 (1982), 523-615.

[Ba] V. Baranovsky, *Cohomology ring of the moduli space stable vector bundles with odd determinant*, Izv. Russ. Acad. Nauk. 58/4 (1994), 204-210.

[CLM] S. Cappell, R. Lee, E. Miller, *The action of the Torelli group on the homology of representation spaces is nontrivial*, to appear in Topology.

[D] S. Donaldson, *Gluing techniques in the cohomology of the moduli spaces*, Topological methods in modern mathematics, Publish or perish, 1991.

[DR] U. Desale, S. Ramanan, *Classification of vector bundles of rank 2 over hyperelliptic curves*, Inv. Math. 38 (1976), 161-185.

[JK] L. Jeffrey, F. Kirwan, *Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface*, Ann. Math. 148 (1998), 109-196.

[Ki1] Y.-H. Kiem, *On the redundancy of the Mumford relations*, to appear in Journal of Algebraic Geometry.

[Ki2] Y.-H. Kiem, *Intersection cohomology of quotients of nonsingular varieties*, preprint.

[Ki3] Y.-H. Kiem, *Equivariant and intersection cohomology of moduli spaces of vector bundles*, preprint.

[K1] F. Kirwan, *On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface*, Proc. Lon. Math. Soc. (3) 53 (1986), 237-266.

[K2] F. Kirwan, *The cohomology rings of moduli spaces of bundles over Riemann surfaces*, Jour. A.M.S. 5 (1992), 853-906.

[KN] A. King, P. Newstead, *On the cohomology ring of the moduli space of rank 2 vector bundles on a curve*, preprint.

[Liu] K. Liu, *Heat Kernel and moduli spaces*, Math. Res. Letters 3 (1996), 743-762.

[N] P. Newstead, *Characteristic classes of stable bundles over an algebraic curve*, Trans. Amer. Math. Soc. 169 (1972), 337-345.

[Ra] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. 200 (1973), 69-84.

[SL] R. Sjamaar, E. Lerman, *Stratified symplectic spaces and reduction*, Ann. Math. 134 (1991), 375-422.

[ST] W. Siebert, G. Tian, *Recursive relations for the cohomology ring of moduli spaces of stable bundles*, Turkish J. Math. 19, No 2 (1995), 131-144.

[T1] M. Thaddeus, *Conformal field theory and the cohomology ring of the moduli space of stable bundles*, J. Diff. Geom. 35 (1992), 131-149.

[T2] M. Thaddeus, *An introduction to the topology of the moduli space of stable bundles on a Riemann surface*, Geometry and Physics- Proceedings of the conference at Aarhus university, Marcel Dekker, 1997, 71-99.

[Z] D. Zagier, *On the cohomology of moduli spaces of rank two vector bundles over curves*, Progress in Mathematics 129, Birkhäuser, 1995, 533-563.

Department of Mathematics, Yale University, New Haven, CT 06520-8283
E-mail address: young.kiem@yale.edu