ON THE REDUNDANCY OF THE MUMFORD RELATIONS

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Abstract. We provide an answer to a question raised by Thaddeus [T], about the Mumford relations for the cohomology of the moduli space of stable vector bundles over a Riemann surface. Namely, we prove that the Mumford relations from the first vanishing Chern class only, generate the whole relation ideal in $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$. However, they are not independent over $\mathbb{Q}[\alpha,\beta]$ and the $\mathbb{Q}[\alpha,\beta,\gamma]$-module generated by them is a proper subspace of the relation ideal.

1. Introduction and Statement of the main Theorem

Let M be the moduli space of rank 2 stable holomorphic vector bundles of degree $4g-3$ over a Riemann surface X of genus $g\geq 2$. Fix a line bundle L of degree $4g-3$ over X and let M_L be the moduli space of stable vector bundles with determinant L. Clearly, M_L is a subspace of M and is the fiber of the determinant map $det : M\to Jac_{4g-3}$ over L. As is well known, both spaces are smooth and M is in fact isomorphic to $M_L\times Jac_0$ modulo a finite group action. As the finite group acts trivially on the cohomology, $H^*(M)\cong H^*(M_L)\otimes H^*(Jac)$ as rings. (See [T].)

There is a universal bundle U over $M\times X$ and it restricts to a universal bundle for M_L. The Künneth components α, β, ψ_i, of the second Chern class of $End(U)$ form a complete set of generators for $H^*(M_L)$ via $c_2(End(U)) = 2\alpha\otimes[X] + 4\sum_{i=1}^{2g}\psi_i\otimes e_i - \beta\otimes 1$ as proved by Newstead [N]. Moreover, if we put $c_1(U) = (4g-3)\otimes[X]+\sum_{i=1}^{2g} d_i\otimes e_i + x\otimes 1$, then d_i, α, β, ψ_i form a complete set of generators for $H^*(M)$.

Mumford conjectured that the Künneth components of $c_r(f_!(U))\in H^*(M_L)\otimes H^*(Jac)$ for $r>2g-1$ form a complete system of relations for $H^*(M_L)$, where $f : M\times X\to M$ is the obvious projection and it was proved by Kirwan in [K]. But it turns out to be too big and a lot of relations are redundant.

At the end of his excellent survey article [T], Thaddeus raised the following question (see also [KN]): "..... But there is considerable redundancy among the Mumford relations. In fact, the shape of Harder-Narasimhan formula suggests that the cohomology ring is isomorphic to the quotient of $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$,....., by the ideal freely generated over $\mathbb{Q}[\alpha,\beta]$ by the Mumford relations for $r=2g$ only,..... This question remains open." We provide an answer to this question.

It is not literally true in the sense that the Mumford relations are not independent over $\mathbb{Q}[\alpha,\beta]$. For example, the coefficient of $d_{g+1}d_{g+2}\cdots d_{2g-1}$ in $c_{2g}(f_!U)$ is up to constant $(\alpha^2-\beta)\psi_1\psi_2\cdots\psi_{g-1}$ and the coefficient of $d_gd_{g+1}d_{g+2}\cdots d_{2g-1}d_{2g}$ is up to constant $\alpha\psi_1\psi_2\cdots\psi_{g-1}$ as one can deduce from the computation in section 3.[1]

However, we can prove the following

Theorem. *The Mumford relations from only $c_{2g}(f_!U)$ generate the whole relation ideal in $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$.*

Therefore, the relations from $c_r(f_!U)$, for $r>2g$, are all redundant.

2. The structure theorm

In this section, we recall and generalize Zagier's techniques [Z], to prove the "structure theorem" for $H^*(M_L)$.

Let α, β, ψ_i, d_i are as in the previous section and let $\gamma=-2\sum_{i=1}^g\psi_i\psi_{i+g}$ and $\xi=\alpha\beta+2\gamma$. From the description of the Chern classes of the universal bundle in the previous section, one can find an expression for the Chern characters and then apply Grothendieck -Riemann-Roch theorem to get an expression for $ch(f_!(U))_t$ and finally one for $c(f_!(U))_t=\sum_{i\geq 0}c_i(f_!U)t^i$ (cf. [Z], p557.);

$$\begin{aligned} c(f_!(U))_{-2t} &= (1-\beta t^2)^{g-\frac{1}{2}}\left(\frac{1+t\sqrt{\beta}}{1-t\sqrt{\beta}}\right)^{\frac{\xi}{2\beta\sqrt{\beta}}} exp\left(\frac{At+2Bt^2-2\gamma t/\beta}{1-\beta t^2}\right) \\ &= \Phi(t)\,G(t) \end{aligned}$$

where $f: M\times X\to M$ is the projection onto the first component and

$$G(t)=(1-\beta t^2)^g exp\left(\frac{At+2Bt^2-2\gamma t^3}{1-\beta t^2}\right),$$

$$\Phi(t)=\sum_{n=0}^{\infty}c_nt^n=(1-\beta t^2)^{-\frac{1}{2}}exp\left(\alpha t+\xi\sum_{k\geq 1}\frac{\beta^{k-1}t^{2k+1}}{2k+1}\right)=(1-\beta t^2)^{-\frac{1}{2}}e^{-\frac{2\gamma t}{\beta}}\left(\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t}\right)^{\frac{\xi}{2\beta\sqrt{\beta}}},$$

[1] The referee suggested another argument for $\mathbb{Q}[\alpha,\beta]$-dependence of the Mumford relations from $c_{2g}(f_!U)$ as follows. Because of the Harder-Narasimhan formula, they are dependent over $\mathbb{Q}[\alpha,\beta]$ if and only if the $\mathbb{Q}[\alpha,\beta]$-module generated by them is a proper subspace of the relation ideal. Since $dimM_L=6g-6$ and $deg\,\gamma^g=6g$, γ^g is a relation but clearly it is not contained in the $\mathbb{Q}[\alpha,\beta]$-module generated by the Mumford relations from $c_{2g}(f_!U)$.

$$A = \sum_{i=1}^{g} d_i d_{i+g}, \quad B = \sum_{i=1}^{g} -d_i\psi_{i+g} + d_{i+g}\psi_i.$$

Again from Riemann-Roch, $f_!U$ is a vector bundle of rank $2g-1$ and therefore $c(f_!U)_{-2t} = \Phi(t)\,G(t)$ is a polynomial of degree$\leq 2g-1$.

Now, the question is how to read the Mumford relations off the expression. Our strategy is the following: Let $\{u_n\}$ be a basis of $H^*(Jac)$ and let $\{u_n^*\}$ be the dual basis with respect to the top degree class $[\prod_{i=1}^g d_i d_{i+g}]$, i.e. $u_i^* u_j/[\prod_{i=1}^g d_i d_{i+g}] = \delta_{ij}$. (Poincare duality! Here, $\cdot/[\prod_{i=1}^g d_i d_{i+g}]$ means the coefficient of the top degree class.) Then for any $z \in H^*(M) = H^*(M_L) \otimes H^*(Jac)$, the coefficient of u_n^* in z is $zu_n/[\prod_{i=1}^g d_i d_{i+g}]$. Therefore, $\{c(f_!U)u_n/[\prod_{i=1}^g d_i d_{i+g}]\}$ give us all the Mumford relations.

For that purpose, we need to generalize a lemma of Zagier ([Z], p559). Let $\wedge^* H^3 = \oplus_{l=0}^g \oplus_{k=0}^{g-l} \gamma^k Prim_l$ be the Lefshetz decomposition of the exterior algebra of $H^3(M_L) = \mathbb{Q}\{\psi_1, \cdots, \psi_{2g}\}$. Now, let $\sigma_l = \sum_{|I|=l} a_I \psi_I \in Prim_l\{\psi_1, \cdots, \psi_{2g}\}$ be a primitive element and put $\tilde{\sigma}_l = \sum_{|I|=l} a_I d_I \in Prim_l\{d_1, \cdots, d_{2g}\}$. Then we have the following

Lemma.

$$\frac{A^{g-l-p}B^{l+2p}}{(g-l-p)!(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = \frac{(\frac{\gamma}{2})^p}{p!}\sigma_l.$$

$$\frac{A^i B^j}{i!j!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = 0 \ \textit{otherwise.}$$

Proof. The second statement is obvious.

Considering Sp_{2g} action, we may assume that $\sigma_l = \psi_{g-l+1}\cdots\psi_g$ since $Prim_l$ is an irreducible module. Now, Zagier's original lemma claims

$$\frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!} = \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}/[\prod_{i=1}^{g-l} d_i d_{i+g}].$$

Thus,

$$\begin{aligned}
\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l &= \frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!}\sigma_l \\
&= \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}\sigma_l/[\prod_{i=1}^{g-l} d_i d_{i+g}] \\
&= \frac{(\sum_{i=1}^{g} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p+l}}{(g-l-p)!(2p+l)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]
\end{aligned}$$

as one can check directly. So we are done. □

Therefore, for $\tilde{\sigma}_l \in Prim_l(d_i)$,

$$
\begin{aligned}
&G(t)A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= \sum_{r,s}(1-\beta t^2)^g \frac{A^r t^r}{r!(1-\beta t^2)^r}\frac{2^s B^s t^{2s}}{s!(1-\beta t^2)^s} exp(\frac{-2\gamma t^3}{1-\beta t^2})A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{k-p}2^{l+2p}t^{g+l+3p-k}\frac{A^{g-l-p}}{(g-l-p-k)!}\frac{B^{l+2p}}{(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{k-p}2^{l+2p}t^{g+l+3p-k}\frac{(g-l-p)!}{(g-l-p-k)!}\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l \\
&= 2^l(1-\beta t^2)^k t^{g+l-k} exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p \frac{(g-l-p)!}{(g-l-p-k)!}\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l.
\end{aligned}
$$

When $k=0$,

$$
\begin{aligned}
G(t)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] &= 2^l t^{g+l} exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p \frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \\
&= 2^l t^{g+l}\sigma_l.
\end{aligned} \tag{0}
$$

When $k=1$,

$$
\begin{aligned}
&G(t)A\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= 2^l t^{g+l-1}(1-\beta t^2) exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (g-l-p)\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \\
&= 2^l t^{g+l-1}(g-l-(g-l)\beta t^2 - 2\gamma t^3)\sigma_l.
\end{aligned} \tag{1}
$$

When $k=2$,

(2)

$$
\begin{aligned}
&G(t)A^2\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= 2^l t^{g+l-2}(1-\beta t^2)^2 exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (g-l-p)(g-l-p-1)\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \\
&= 2^l t^{g+l-2}((g-l)(g-l-1) - 2(g-l)(g-l-1)\beta t^2 - (4g-4l-4)\gamma t^3 \\
&+ (g-l)(g-l-1)\beta^2 t^4 + (4g-4l-4)\beta\gamma t^5 + 4\gamma^2 t^6)\sigma_l.
\end{aligned}
$$

Recall that c_n was defined to be the n-th coefficient of $\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = (1-\beta t^2)^{-\frac{1}{2}} exp(\alpha t + \xi \sum_{k\geq 1} \frac{\beta^{k-1}t^{2k+1}}{2k+1}) = (1-\beta t^2)^{-\frac{1}{2}} e^{-\frac{2\gamma t}{\beta}} (\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t})^{\frac{\xi}{2\beta\sqrt{\beta}}}$. One can check that the sequence $\{c_n\}$ is determined by the following recursion formula;

$$nc_n = \alpha c_{n-1} + (n-1)\beta c_{n-2} + 2\gamma c_{n-3}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \frac{\alpha^2+\beta}{2}$, $c_3 = \frac{\alpha^3+5\alpha\beta+4\gamma}{3!}$, etc.

Let I_g be the ideal of $\mathbb{Q}[\alpha,\beta,\gamma]$ generated by c_i for $i \geq g$. Then by the above recursion formula, I_g is in fact generated by just three elements, c_g, c_{g+1}, c_{g+2}.

By the formula (0) above,

$$c(f_!U)_{-2t}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = 2^l t^{g+l}\Phi(t)\sigma_l$$

and thus $c_n\sigma_l$ is a relation for $n \geq g-l$. Hence, $Prim_l \otimes I_{g-l}$ is a set of relations. One can compute the Poincare series as in [Ki] to show that $\oplus_{l=0}^{g} Prim_l \otimes I_{g-l}$ is in fact the complete set of relations. Therefore, we obtained a new proof of the following structure theorem [ST][KN][Z][Ba].

Theorem. $H^*(M_L) \cong \oplus_{l=0}^{g} Prim_l \otimes \mathbb{Q}[\alpha,\beta,\gamma]/I_{g-l}$.

3.Proof of the main theorm

Note that we used just the formula (0) to prove the structure theorem. In this section, we use the other two formulas to prove our main theorem. As a by-product of the proof, we will see that the $\mathbb{Q}[\alpha,\beta,\gamma]$-module in $\mathbb{Q}[\alpha,\beta] \otimes \wedge^*(\psi_i)$ generated by the Mumford relations from $c_{2g}(f_!U)$ is a proper subspace of the relation ideal. (Remark 2.)

From the structure theorem above, we note that it suffices to show that $c_{g-l}\sigma_l$, $c_{g-l+1}\sigma_l$ and $c_{g-l+2}\sigma_l$ belong to the ideal J generated by the Mumford relations for $r = 2g$ only, i.e.

$$\{c_{2g}(f_!U)A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \mid 0 \leq k \leq g-l,\ 0 \leq l \leq g,\ \tilde{\sigma}_l \in Prim_l(d_i)\}.$$

By (0), $c_{g-l}\sigma_l$ is, up to constant,

$$\begin{aligned} c_{2g}(f_!U)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] &= Coeff_{t^{2g}}(\Phi(t)G(t)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]) \\ &= Coeff_{t^{2g}}(\Phi(t)2^l t^{g+l}\sigma_l) = 2^l c_{g-l}\sigma_l \end{aligned}$$

and so $c_{g-l}\sigma_l \in J$. By (1) and the recursion forumla, $c_{2g}(f_!U)A\tilde{\sigma}_l/[\prod_{i=1}^g d_i d_{i+g}]$ is, up to constant,

$$((g-l)c_{g-l+1} - (g-l)\beta c_{g-l-1} - 2\gamma c_{g-l-2})\sigma_l = (-c_{g-l+1} + \alpha c_{g-l})\sigma_l$$

and thus $c_{g-l+1}\sigma_l \in J$ unless $l = g$. Similarly, by (2), $c_{2g}(f_!U)A^2\tilde{\sigma}_l/[\prod_{i=1}^g d_i d_{i+g}]$ is, up to constant,

$$[(-2g+2l+2)c_{g-l+2} - 2\alpha c_{g-l+1} + ((2g-2l-1)\beta + \alpha^2)c_{g-l})]\sigma_l$$

and thus $c_{g-l+2}\sigma_l \in J$ unless $l \geq g-1$.

In particular, c_g, c_{g+1}, c_{g+2} are in J and $c_{g-l}\sigma_l \in J$. As any $\sigma_l \in Prim_l$ is a sum of elements of the form $\sigma_{l-1}\sigma_1$ for some $\sigma_{l-1} \in Prim_{l-1}$ and $\sigma_1 \in Prim_1$,

$$c_{g-l+1}\sigma_l = (c_{g-l+1}\sigma_{l-1})\sigma_1 \in J$$

and similarly

$$c_{g-l+2}\sigma_l = (c_{g-l+2}\sigma_{l-2})\sigma_1'\sigma_1 \in J.$$

Note that for $l = 0, 1$, the same follows from the fact that c_g, c_{g+1}, c_{g+2} are in J. Therefore, we proved our main theorem.

Remark 1. Technically, Zagier assumed $x = 0$ in the expression for $c_1(U)$, which amounts to saying that $\Phi(t)G(t) = (1-2vt)^{2g-1}c(f_!U)_{\frac{-2t}{1-2vt}}$ for a class $v \in H^2(Jac)$. But the coefficient of t^{2g} in $c(f_!U)_{-2t}$ and that in $\Phi(t)G(t)$ are same up to constant as one can check by expanding both in t. Therefore, for our purpose, we can take $\Phi(t)G(t) = c(f_!U)_{-2t}$.

Remark 2. One may ask whether the Mumford relations, for $r = 2g$ only, generate the whole relations as a $\mathbb{Q}[\alpha, \beta, \gamma]$ module. The answer is negative because the module does not contain $c_3\sigma_{g-1}$. Actually, those elements are the only missing piece and the $\mathbb{Q}[\alpha, \beta, \gamma]$-module generated by the relations from c_{2g+1} as well as those from c_{2g} is the whole set of relations as one can deduce from the above formulas. Therefore, even as a $\mathbb{Q}[\alpha, \beta, \gamma]$-module, $c_r(f_!U)$ for $r > 2g+1$ are all redundant.

Remark 3. In the even degree case, there is no universal bundle and the moduli space is singular. However, there exits a (topological) universal bundle on the homotopy quotient

$$U \to (\mathcal{C}^{ss} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}) \times X$$

where $\mathcal{C}^{ss}$ is the space of semistable holomorphic structures on a complex vector bundle and $\overline{\mathcal{G}}$ is the based gauge group. Note that the moduli space is the Mumford quotient of $\mathcal{C}^{ss}$ by the complexified gauge group.

The bundle U is holomorphic in X-direction and one can consider the push-forward bundle and the Mumford relations. In [Ki], the ring structure for the

equivariant cohomology was completely determined and an obvious analogue of Mumford's conjecture was proved. One may ask whether the results of this paper hold for the equivariant cohomology. It turns out to be affirmative when $g > 3$. One can modify the computation in this paper to prove it.

Acknowledgement. I would like to thank Professor Ronnie Lee for constant support and helpful advices, and Professor Michael Thaddeus for valuable comments and suggestions. I am also grateful to the referee for careful reading and comments.

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