COHOMOLOGY OF QUOTIENTS AND MODULI SPACES OF VECTOR BUNDLES

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Abstract

Cohomology of quotients and moduli spaces of vector bundles

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Let $M(r,d)$ be (the compactification of) the moduli space of holomorphic vector bundles of rank r and degree d, which can be constructed as a geometric invariant theory (GIT) quotient of a smooth quasi-projective variety. Because GIT quotients are symplectic reductions, we can apply the techniques of symplectic geometry. The most powerful tools in computing the cohomology ring of a symplectic reduction are the equivariant Morse theory and the nonabelian localization principle. We review these and generalize to K-groups and Chow groups.

Many important GIT quotients are singular and hence intersection cohomology is an interesting invariant. Kirwan invented a way to desingularize such quotients and used this procedure to define a map from the equivariant cohomology of the set of semistable points to the intersection cohomology of the quotient. We construct a splitting (i.e. right inverse) of this map and hence the intersection cohomology groups are embedded in the equivariant cohomology groups explicitly. Several applications are discussed.

The cohomology ring of $M(2,1)$ is by now well-understood. We prove the structure theorem and the Mumford conjecture in one stroke by improving a technique of Zagier. We also prove the strong Mumford conjecture which was open before this thesis.

For the singular moduli space $M(2,0)$, we first determine the equivariant cohomology ring. Then we apply the splitting theorem to describe the intersection cohomology of $M(2,0)$. This enables us to compute the intersection Betti numbers, the intersection pairing and the mapping class group action.

A brief summary on higher rank case is included to complete this work.

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Contents

Chapter 1

Introduction

1.1 Cohomology of quotients

Algebraic geometry has been enriched by various fields of mathematics from commutative algebra to Riemannian geometry to complex analysis. The most recent and important influx came from symplectic geometry and mathematical physics. During the past two decades, many difficult problems were successfully attacked with ideas from symplectic geometry. The key link has been the *moment map* or angular momentum. A linear action of a reductive group on a projective space with Fubini-Study Kähler form is Hamiltonian, i.e. it carries a moment map. Therefore, any linear action on a smooth projective variety carries a moment map which has many interesting properties.

Perhaps one of the most interesting problems in algebraic geometry is the study of cohomology of moduli spaces.[1] For many moduli problems, the moduli spaces are naturally constructed by geometric invariant theory which often provides natural compactifications. Often the parameter space (Hilbert scheme) is constructed in a natural way and the moduli space is obtained as the quotient of such parameter space.[2] Geometric invariant theory is a recipe for forming the quotient of a variety by a reductive group action.

An orbit by a reductive group action can behave badly because of the noncompact "imaginary direction". So, one has to delete some points whose orbits cannot be controlled.

[1] Every cohomology group in this thesis has rational or complex coeffcient.

[2] Following fairly standard linguistic convention, we call the solution to a moduli problem the *parameter space* if the problem is not "intrinsic". We call it the *moduli space* otherwise.

Even after deleting these bad points, we need to identify some orbits to get a Hausdorff quotient since some orbits get arbitrarily close to one another. The purpose of geometric invariant theory is precisely to deal with these problems.

Given a Hamiltonian action, the reduced space is defined as the quotient of the zero set of the moment map. This is called the symplectic reduction which may be singular if the action on the zero set is not free. In case the action is free, the symplectic reduction is again a symplectic manifold and many examples of symplectic manifolds are constructed in this way.[3] One fundamental result joining symplectic geometry with geometric invariant theory is the principle:

GIT quotients are symplectic reductions.

For symplectic reductions, there are by now many ways to compute the cohomology rings. Among them the most powerful tools have been the *equivariant Morse theory* and the *nonabelian localization theorems*. In [24], F. Kirwan developed the Morse theory for the norm square of the moment map for a flow-closed Hamiltonian space. The main result is that the Morse stratification is equivariantly perfect and the strata are described precisely. Since the bottom strata retracts onto the zero set of the moment map, the Kirwan-Morse theory provides a way to compute the equivariant cohomology of the zero set in terms of the Gysin maps of the other strata. If the group action on the zero set is (locally) free, the equivariant cohomology is canonically isomorphic to the ordinary cohomology of the symplectic reduction. This technique turns out to be very powerful for the computation of Betti numbers and sometimes cup product structures as well [31, 50]. Brion [8] used this idea to prove the abelian localization theorem for equivariant Chow groups, defined by Edidin and Graham [10]. In Chapter 3 of this thesis, we will apply the Kirwan-Morse theory to study the equivariant K-groups and equivariant Chow groups and show the perfectness of the Morse stratification and the abelian localization theorem for each theory.

In principle, the Kirwan-Morse theory determines the cup product structure completely

[3] To show that a manifold is symplectic one may wish to demonstrate that it is a reduction of a symplectic manifold.

and thus the intersection pairings can be computed, if the quotient is smooth and compact. However, it is extremely difficult to work out this "exercise" in practice. Nonabelian localization is perhaps the strongest tool for this purpose. Notice that by Poincaré duality the cup product structure is uniquely determined by the intersection pairing in case of a smooth compact quotient. The nonabelian localization principle enables us to compute the intersection pairings from some local contributions on the fixed point components by the maximal torus. These local contributions often can be computed by induction.

Witten described his version of nonabelian localization [52] which expresses the intersection pairing in terms of certain integral together with some terms of exponential decay corresponding to the critical sets of the norm square of the moment map. Jeffrey and Kirwan [19] used Witten's integral to prove their nonabelian localization theorem by using abelian localization in place of the physics arguments and hence the contributions are from the torus fixed point components. Still they had to rely on heavy analysis which was replaced by topological arguments of S. Martin [34, 35] and Guillemin-Kalkman [17]. In Chapter 4 below, we will use a theorem due to Martin and the Kirwan-Morse theory for equivariant Chow groups to prove a theorem of Ellingsrud-Stromme [12] concerning the Chow ring of a smooth GIT quotient.

In this way, the Kirwan-Morse theory and the nonabelian localization often give us a satisfactory description of the cohomology ring of a symplectic reduction if 0 is a regular value of the moment map. However, if 0 is not regular then the quotient can have serious singularities and the above tools do not apply for this case. Fortunately, Kirwan in [25] invented a systematic way to desingularize GIT quotients partially[4] and this process was generalized to symplectic reductions by Meinrenken-Sjamaar [36] by using the technique of symplectic cutting and the local normal form theorem. The idea is to blow up the set of points in the zero set of the moment map with maximal dimensional infinitesimal stabilizers. The upshot here is that the reduction of this blow-up is strictly "less singular". So, by keeping blowing up in this fashion, we get an orbifold reduction, which is called the

[4] We cannot resolve orbifold singularities by this process.

partial desingularization.[5]

For a singular pseudomanifold, intersection cohomology is an interesting invariant. By using the partial desingularization, Kirwan defined a map, called the *Kirwan map* from the equivariant cohomology of the set of semistable points to the intersection cohomology[6] of the GIT quotient. This was generalized to the symplectic case by J. Woolf in [53].

One main result of this thesis is on the construction of an embedding of the intersection cohomology into the equivariant cohomology. Namely, a subspace V of the equivariant cohomology of the set of semistable points was defined by truncating along each stratum in an appropriate degree where the stratification is by infinitesimal orbit types. This is shown to be a splitting of the Kirwan map. We prove this theorem, under an assumption, named weakly balanced action, which is satisfied by many interesting spaces.[7]

This theorem, called the splitting theorem, turns out to be very useful. First, it simplifies the intersection Betti number computation. It gives the correct Poincaré series without going through the partial desingularization process. Second, it enables us to compute the intersection pairing in terms of the cup product structure of the equivariant cohomology. Or, we can apply nonabelian localization after perturbing the moment map slightly and then compute the intersection pairings using the splitting theorem. Third, we can assign a Hodge structure on the intersection cohomology explicitly.

1.2 Cohomology of moduli spaces of vector bundles

Ever since the construction of the moduli spaces $M(r,d)$ of rank r holomorphic vector bundles of degree d over a Riemann surface as geometric invariant theory quotients $X/\!/G$ of nonsingular quasi-projective varieties, their cohomology groups have been studied intensively by many authors.

For the simplest nontrivial case where the rank r is 2 and the degree d is odd, the Betti numbers and a set of generators for the cup product structure were given by Newstead in

[5] We assume that there is at least one regular point in the zero set of the moment map.

[6] Every intersection cohomology in this thesis has middle perversity.

[7] This theorem is valid also for symplectic reductions. See [22].

1970s. A major breakthrough was the "magisterial" work of Atiyah and Bott [3], where they generalized Newstead's results to the case where the degree is coprime to the rank. If r is coprime to d, then the space of stable holomorphic structures $\mathcal{C}^s$ coincides with that of semistable holomorphic structures $\mathcal{C}^{ss}$ on a fixed Hermitian vector bundle and the moduli space is smooth. Moreover, the equivariant cohomology $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ is isomorphic to the ordinary cohomology $H^*(M(r,d))$ which is in turn isomorphic to $IH^*(M(r,d))$ where $\overline{\mathcal{G}} = \mathcal{G}/U(1)$ and $\mathcal{G}$ is the $U(2)$-gauge group. They showed that the Morse stratification for the Yang-Mills functional is equivariantly perfect and hence, the equivariant cohomology $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ can be computed in terms of the equivariant cohomology of the unstable strata and the classifying space of the gauge group. This Morse theoretic argument in principle determines the cohomology rings as in [31], where Kirwan proved the Mumford conjecture and the Newstead conjecture for $r = 2$, $d = 1$.

If the moduli space is nonsingular, then the cup product structure is completely determined by the intersection pairing by Poincaré duality. In early 1990s, Thaddeus [48] used the twisted $SU(2)$-Verlinde formula and Riemann-Roch to compute the intersection pairing for the case where $r = 2$, $d = 1$, which was also computed by Donaldson [9] in a different way. Shortly later, several authors, Baranovsky [5], King-Newstead [23], Siebert-Tian [45], and Zagier [54], proved the structure theorem for the cohomology ring, which gives us a finite number of relations which generate all the others. In this thesis, we prove this structure theorem and the Mumford conjecture in a purely combinatorial way by improving Zagier's method. Perhaps, the only remaining open problem in this case was the strong Mumford conjecture which is settled in Chapter 7. This says the Mumford relations from the first vanishing Chern class *only*, generate the relation ideal. So all the others are redundant.

The formula for the intersection pairing for the above case where $r = 2$, $d = 1$ was generalized by Witten to the case where r is coprime to d. His formulas were proved mathematically by Jeffrey and Kirwan [20] by applying nonabelian localization principle to the extended moduli space [21].

When r is *not* coprime to d, the moduli space $M(r,d)$ is singular and little has been known for this case. For years, it has been on the focus of interest whether Witten's formulas can be generalized to the singular moduli spaces. Unfortunately, all the above arguments fail when the quotient is singular. On the other hand, for many purposes, the singular moduli spaces are more important. For example, Casson's invariant and its generalizations are defined as the "intersection numbers" of Lagrangian subvarieties on the singular moduli spaces.

Atiyah-Bott's equivariant Morse theory provides a way to compute the Betti numbers of the equivariant cohomology $H^*_{\mathcal{G}}(\mathcal{C}^{ss})$ but the natural map

$$H^*(M(r,d)) \to H^*_{\mathcal{G}}(\mathcal{C}^{ss})$$

is neither injective nor surjective even for $M(2,0)$. Also, the natural map from the ordinary cohomology to the intersection cohomology is neither injective nor surjective.

For the rank 2 case, Kirwan in [27] used the partial desingularization process to compute the Betti numbers of the intersection cohomology of $M(2,0)$ and Cappell-Lee-Miller in [43] computed the Betti numbers of the ordinary cohomology based on the gauge theoretic model of Atiyah-Bott. However, these methods do not provide a strong insight on the cup product or intersection pairing.

In this thesis, our focus is laid on the intersection cohomology. We first prove (analogues of) the structure theorem and the Mumford conjecture for the equivariant cohomology ring for the rank 2 case. These determine the cup product structure completely and moreover we get a Gröbner basis for the relation ideal. Next, we apply the splitting theorem to compute the intersection cohomology of $M(2,0)$. We get a closed expression for the Poincaré polynomial of $IH^*(M(2,0))$ which is equivalent to Kirwan's [27]. Also we can describe the mapping class group action as well as the Hodge structure. Moreover, the intersection pairings can be computed, directly for low genus case in Chapter 8 and by nonabelian localization in general in Chapter 9. Our method works well for the higher rank case which will be investigated in [32].

1.3 Contents of this thesis

In Part 1, we study the cohomology of symplectic and GIT quotients.

Chapter 2 contains a summary of basic results on symplectic reduction: We recall the results of Sjamaar-Lerman [47] which give us a nice local description near an orbit and show that the quotients are stratified symplectic spaces. We also recall the partial desingularization process from [36, 25] and the "GIT quotients are symplectic reductions" principle from [24].

Chapter 3 consists of the Kirwan-Morse theory: We first summarize facts about several equivariant theories and then recall Kirwan's results on equivariant cohomology from [24]. Then we apply the Morse theory to equivariant K-groups and prove that the stratification is perfect for equivariant K-theory and that an analogue of the abelian localization theorem is valid. Next, we apply it to equivariant Chow rings and show similar results. Namely, the Morse stratification is perfect for equivariant Chow rings and the analogues of the abelian localization and related facts are established. Though straightforward, our applications for equivariant K-groups and Chow groups seem new (except for the abelian localization theorem for equivariant Chow groups [8]).

Chapter 4 is for nonabelian localization principle: Martin's trick and his theorems comparing T-quotients and K-quotients are reviewed. Also, the wall crossing formulas for torus quotients are recalled and we use them to establish the Jeffrey-Kirwan localization theorem. Finally, from Martin's theorem and the Kirwan-Morse theory, we provide a new proof of a theorem of Ellingsrud and Stromme [12], which describes the Chow rings of smooth quotients of projective spaces.

Chaper 5 is devoted to the splitting theorem: We recall the Kirwan map and then define the weakly balanced action and the splitting. After proving the theorem we discuss applications. The results in this chapter are new.

In part 2, we study the cohomology of moduli spaces of vector bundles.

Chapter 6 contains a survey of basics on the moduli spaces of vector bundles over a

Riemann surface: Coarse and fine moduli spaces are defined. Various facts and constructions of these moduli spaces are discussed.

Chapter 7 deals with the cohomology ring of $M(2,1)$: We prove the structure theorem and the Mumford conjecture. Furthermore, we prove the strong Mumford conjecture. The proof of the structure theorem is based on Zagier's technique but it is clarified and improved. The strong Mumford conjecture was an open problem [49, 23] but it is proved here.

Chapter 8 is for $M(2,0)$: After establishing necessary geometric facts, we use the combinatorial method of Chapter 7 to prove the structure theorem of the equivariant cohomology. Then using the splitting theorem, we compute the intersection cohomology of the moduli space. The results in this chapter are original.

Chapter 9 is for higher rank case: We recall Jeffrey and Kirwan's proof of Witten's formulas. Then we briefly show how their results together with the splitting theorem can be used to compute the intersection pairing of the intersection cohomology of the singular moduli spaces. This is new and will be part of [32].

Part I

Cohomology of quotients

Chapter 2

Symplectic reductions and GIT quotients

We review some basic materials on symplectic reductions and GIT quotients. Everything in this chapter was borrowed from [47, 24, 36, 37, 25].

2.1 Symplectic reduction

Hamiltonian group action

Let (M, ω) be a symplectic manifold. As ω is nondegenerate, it defines an isomorphism $TM \to TM^*$ by $X \to -\imath_X \omega$. Suppose a compact Lie group K acts smoothly on M, preserving the nondegenerate closed 2-form ω. For each $\xi \in \mathfrak{k} = Lie(K)$, let ξ_M denote the vector field generated by the infinitesimal action of ξ. Then the corresponding 1-form $-\imath_{\xi_M}\omega$ is closed since $d\imath_{\xi_M}\omega = L_{\xi_M}\omega - \imath_{\xi_M} d\omega = 0$. Hence, we get the following diagram

$$\begin{array}{ccc} \mathfrak{k} & \longrightarrow & \{\text{closed 1-forms on } M\} \\ & \searrow & \uparrow d \\ & & \{\text{functions on } M\} \end{array}$$

A moment map is a lifting $\mathfrak{k} \to \{\text{functions on } M\}$ in the above diagram. To be precise, a *moment map* $\mu : M \to \mathfrak{k}^*$ is a K-equivariant smooth map such that

$$< d\mu_m(\xi), a >= \omega_m(\xi, a_m)$$

for all $m \in M$, $\xi \in T_m M$, $a \in \mathfrak{k}$. We say the action is *Hamiltonian* if there is a moment map for the action. If K is semisimple, or if K is a torus and $H^1(X) = 0$, then a moment map always exists.[1]

The standard unitary group $U(n+1)$-action on $\mathbb{P}^n$ (together with the Fubini-Study symplectic form) is Hamiltonian since

$$< \mu_{\mathbb{P}^n}(x), a >= \frac{\overline{x}^* a \overline{x}}{2\pi i |\overline{x}|^2}$$

is a moment map where $\overline{x}$ is a point in $\mathbb{C}^{n+1} \setminus 0$ representing x. In geometric invariant theory, the typical situation is the case where $M \subset \mathbb{P}^n$ is a quasi-projective variety on which a reductive group G acts via a homomorphism $G \to GL(n+1)$. Let K be a maximal compact subgroup. After conjugation if necessary, we may assume that K acts unitarily. Then if M is smooth, we always have a moment map

$$\mu_M : M \hookrightarrow \mathbb{P}^n \xrightarrow{\mu_{\mathbb{P}^n}} u(n+1)^* \longrightarrow \mathfrak{k}^*.$$

Symplectic reduction

Let M be a Hamiltonian K-space with moment map $\mu : M \to \mathfrak{k}^*$. The *symplectic reduction* is defined as the quotient

$$M/\!\!/K = \mu^{-1}(0)/K$$

of the zero set of the moment map by K. If 0 is a regular value of μ and K acts freely on $\mu^{-1}(0)$, then the reduced space is again a symplectic manifold. But if 0 is not a regular value, then the reduced space is a complicated singular space. In the subsequent subsections, we will recall the results in [47] which show that the reduced space is a stratified space each of whose pieces is symplectic in a compatible fashion.

Local normal form

[1]In general, for every compact connected Lie group K, if we denote $S = [K, K]$, H =the identity component of the center of K, $D = S \cap H$, then D is a finite group, S semisimple and the map $H \times S \to K$ is a finite central extension such that $H/D \times S/D \cong K/D$. Hence, a moment map for the K-action exists if there is a moment map for the H-action.

Let p be a point in the zero set $Z = \mu^{-1}(0)$ of the moment map $\mu : M \to \mathfrak{k}^*$. Let H be the stabilizer of p and V denote the symplectic complement of $T_p(K \cdot p)$ in T_pM, which we call the *symplectic slice*.

Consider the (coisotropic) vector bundle

$$Y = K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times V) \to K/H = K \cdot p.$$

First, we observe that Y is the symplectic reduction of $T^*K \times V$ by the H action (right action for T^*K and left action for V) with moment map

$$\Phi : T^*K \times V \cong K \times \mathfrak{k}^* \times V \to \mathfrak{h}^*$$

given by $\Phi(g, \xi, v) = \Phi_V(v) - \xi|_{\mathfrak{h}}$ where $\Phi_V : V \to \mathfrak{h}^*$ is the moment map for the linear action of H on V. So, Y is a symplectic manifold containing K/H as the zero section and V is also the symplectic slice in Y of the orbit $K/H \cong K \cdot p$.

Theorem 2.1 *(Constant rank embedding theorem) Let B be a K-manifold with a closed K-invariant 2-form τ of constant rank. Then there is a one-to-one correspondence between*

1. *symplectic K-vector bundles over B and*

2. *K-equivariant embeddings $\imath$ of B into higher dimensional symplectic manifold (A, σ) such that $\imath^*\sigma = \tau$.*

Therefore, there is a neighborhood U_0 of the zero section of Y which is symplectomorphic to a neighborhood U in M of the orbit $K \cdot p$. So, Y serves as a local model for M.

The left K-action of T^*K commutes with the (right) H-action. Thus the moment map for the left action

$$\Psi : T^*K \times V \cong K \times \mathfrak{k}^* \times V \to \mathfrak{k}^*$$

with $\Psi(g, \xi, v) = Ad^*(g)\xi$, reduces to the moment map $\mu : Y \to \mathfrak{k}^*$ given by $\mu(g, \xi, v) = Ad^*(g)(\xi + \Phi_V(v))$. Hence, we get the following.

Theorem 2.2 *Let H be the stabilizer of $p \in \mu^{-1}(0)$ and V be the symplectic slice of the orbit $K \cdot p$. Then a neighborhood of the orbit is equivariantly symplectomorphic to a neighborhood of the zero section of*

$$Y = K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times V)$$

with the moment map

$$\mu(g, \xi, v) = Ad^*(g)(\xi + \Phi_V(v)).$$

Decomposition by orbit types

Let (M, ω) be a Hamiltonian K-space with moment map $\mu : M \to \mathfrak{k}^*$. For a subgroup H of K, let (H) denote the conjugacy class of H and define

$$M_H = \{x \in M | \mathrm{Stab} x = H\}$$

$$M_{(H)} = \{x \in M | \mathrm{Stab} x = gHg^{-1} \text{ for some } g \in K\}.$$

Then M is the disjoint union of the locally closed submanifolds $M_{(H)}$ which satisfy

$$M_{(H)} \cap \overline{M}_{(L)} \neq \emptyset \iff M_{(H)} \subset \overline{M}_{(L)} \iff H \supset gLg^{-1}.$$

This decomposition descends to the symplectic reduction

$$M/\!\!/K = \mu^{-1}(0)/K = \cup_{(H)} (\mu^{-1}(0) \cap M_{(H)})/K.$$

To see that each piece $\mu^{-1}(0) \cap M_{(H)}/K$ is a smooth manifold, we use the local normal form $Y = K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times V)$ near a point $p \in M_H \cap \mu^{-1}(0)$. It is easy to check that $Y_{(H)} \cap \mu^{-1}(0)$ is $K/H \times V_H$ where V_H is the subspace of V fixed by H. Thus the piece

$$(Y_{(H)} \cap \mu^{-1}(0))/K = V_H$$

is symplectic and smooth.

Reduction in stages

To describe the normal to each stratum, we need the technique of reduction in stages. Let K_1, K_2 be two groups acting on (M, ω) with moment maps $\mu_1 : M \to \mathfrak{k}_1^*$, $\mu_2 : M \to \mathfrak{k}_2^*$.

We assume that the actions commute so that we get an action of $K_1 \times K_2$ with moment map $\mu = \mu_1 \times \mu_2$.

Let $X_1 = \mu_1^{-1}(0)/K_1$. The action of K_2 on M descends to X_1 and this preserves the decomposition discussed before by the K_1 orbit types. The moment map $\mu_2 : M \to \mathfrak{k}_2^*$ also descends to a map $\mu_2' : X_1 \to \mathfrak{k}_2^*$ which restricts to a moment map on each stratum. We define the reduced space of X_1 to be

$$X_{12} = (\mu_2')^{-1}(0)/K_2.$$

By construction, X_{12} is homeomorphic to

$$X = (\mu_1 \times \mu_2)^{-1}(0,0)/K_1 \times K_2 = M/\!\!/K_1 \times K_2.$$

We can assign a Poisson structure on reduced spaces by declaring

$$C^\infty(M/\!\!/K) = C^\infty(M)^K/I^K$$

where I is the ideal of smooth functions vanishing on $\mu^{-1}(0)$ and similarly assign a Poisson structure on X_{12}. It turns out that these match up and thus $X_{12} = X$ as a Poisson space. Hence, we can reverse the order of reduction, i.e. $X_{12} = X = X_{21}$.

Local structure of the decomposition

We describe the normal cone of each stratum. Recall that $p \in \mu^{-1}(0) \cap M_H$ has a neighborhood symplectomorphic to a neighborhood of the zero section of $Y = K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times V)$ which is the reduction of $T^*K \times V$ by the H-action. The reduction of Y is homeomorphic to the reduction of $T^*K \times V$ by the $K \times H$ action. We first reduce by the K-action to get V and then reduce by the H action to get finally

$$V/\!\!/H = \Phi_V^{-1}(0)/H.$$

Therefore, near the point $x \in M/\!\!/K$ corresponding to p, $M/\!\!/K$ is homeomorphic to $V/\!\!/H$.

Write $V = V_H \times W$ where V_H is the linear subspace fixed by H. Then $V/\!\!/H = W/\!\!/H \times V_H$. V_H is tangent to the stratum $\mu^{-1}(0) \cap M_{(H)}/K$ while the cone $W/\!\!/H$ is the normal cone of the stratum at x. Hence, the decomposition by orbit types is in fact a stratification. All pieces are symplectic manifolds as they are reductions $M_H /\!\!/ N^H$ where N^H is the normalizer of H in K and the Poisson structure of $M/\!\!/K$ defined above is compatible with the symplectic structure of each stratum. So, we get the following.

Theorem 2.3 *$X = M/\!\!/K$ is a* stratified symplectic space, *i.e. X is a stratified space with the above smooth structure $C^\infty(X)$ such that*

1. *each stratum is a symplectic manifold*
2. *$C^\infty(X)$ is a Poisson algebra*
3. *the embedding of each stratum is Poisson.*

Moreover, the normal cones above form a fiber bundle over the stratum $X_H = \mu^{-1}(0) \cap M_{(H)}/K$ with typical fiber $W/\!\!/H$.

Theorem 2.4 *There exists a principal $L = N_{K\times U}(H)/H$-bundle Q over X_H where U is the unitary group of W for a compatible Hermitian structure such that a neighborhood of $M_{(H)}$ in M is symplectically diffeomorphic to a neighborhood of the zero section of the bundle*

$$Q \times_L (K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times W)) \to X_H.$$

Hence, a neighborhood of X_H in X is diffeomorphic to a neighborhood of the vertex section of

$$Q \times_L W/\!\!/H \to X_H$$

as a stratified symplectic space.

2.2 Partial desingularization

Symplectic cutting

Suppose S^1 acts on a Hamiltonian K-space M in a Hamiltonian fashion with moment map $\psi : M \to \mathbb{R}$ and that the action commutes with the K-action. Consider the diagonal action of S^1 on $M \times \mathbb{C}$ where the moment map is $\tilde{\psi}(x,z) = \psi(x) - \frac{1}{2}|z|^2 - \epsilon$. The *symplectic cut* is defined in [33] as the symplectic reduction

$$M_{\geq\epsilon} = M \times \mathbb{C}/\!\!/S^1.$$

Proposition 2.5 *[33] The canonical embeddings*

$$\imath_\epsilon : M_\epsilon := \psi^{-1}(\epsilon)/S^1 \hookrightarrow M_{\geq\epsilon}, \quad \imath_{>\epsilon} : M_{>\epsilon} := \{x \in M | \psi(x) > \epsilon\} \hookrightarrow M_{\geq\epsilon}$$

are symplectic embeddings. If ϵ is a regular value of ψ, $M_{\geq\epsilon}$ is a symplectic orbifold. The lifted K-action $g\cdot(x,z) = (gx,z)$ on $M\times\mathbb{C}$ induces a Hamiltonian K-action on $M_{\geq\epsilon}$ whose moment map is the map $\mu_{\geq\epsilon}$ induced by the S^1-invariant map $(x,e^{i\theta}) \mapsto \mu(x)$. In particular, the original S^1 action on M induces a Hamiltonian S^1 action on $M_{\geq\epsilon}$. The image of its moment map $\psi_{\geq\epsilon}$ is $\psi(M) \cap \mathbb{R}_{\geq\epsilon}$.

Symplectic blow-up

Let S be a closed K-invariant symplectic submanifold of a Hamiltonian K-space M. Then we can blow up M along S.

Let N denote the normal bundle of S in M. This is a symplectic vector bundle and by choosing a compatible K-invariant complex structure j, N becomes a complex vector bundle with a Hermitian metric τ. Consider the natural S^1-action on this complex vector bundle which commutes with with the K-action on N. Hence, we can perform symplectic cut on N for sufficiently small $\epsilon > 0$ to get $N_{\geq\epsilon}$. Recall

$$N_\epsilon \hookrightarrow N_{\geq\epsilon} \hookleftarrow N_{>\epsilon}.$$

We call N_ϵ the exceptional divisor. Now, by the (equivariant) constant rank embedding theorem, S has a K-invariant neighborhood in M which is K-equivariantly symplectomorphic to a neighborhood U of the zero section of N. By taking ϵ small enough, we may assume that U contains 2ϵ-neighborhood of S. So, we can paste $U_{\geq\epsilon}$ with the complement of the ϵ-neighborhood of S to obtain the *symplectic blow-up* $\mathrm{Bl}(M,S,j,\epsilon)$.

Partial desingularization

Suppose 0 is not a regular value of proper moment map $\mu : M \to \mathfrak{k}^*$. Then as we have seen, the symplectic reduction $M/\!\!/K$ is not an orbifold but a complicated stratified singular space. Kirwan invented a process to resolve the singularities of GIT quotients to obtain an orbiforld [25] and this was generalized to symplectic case by Meinrenken and Sjamaar [47].

Choose a maximal dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ among the infinitesimal stabilizers of points in $Z = \mu^{-1}(0)$ and let $H = \exp\mathfrak{h}$. Consider the fixed point set Z_H by H in Z and $Z_{(H)} = KZ_H \cong K \times_{N^H} Z_H$. By refining the arguments in the previous section, Meinrenken and Sjamaar proved in [36] the following.

Theorem 2.6 *There exists a principal orbibundle Q with structure group $L = N_{K\times U}(H)/H$ over $X_H = Z_{(H)}/K$ such that a K-invariant open neighborhood $U_{(H)}$ of $Z_{(H)}$ in*

$$\nu_{(H)} = Q \times_L (K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times W))$$

is isomorphic, as a Hamiltonian K-space, to a neighborhood of $Z_{(H)}$ in M.

Let $S_{(H)} = Q \times_L (K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times 0)) \subset \nu_{(H)}$. Then $U_{(H)} \cap S_{(H)}$ is a locally closed K-invariant symplectic submanifold and we can choose a sufficiently small K-invariant neighborhood U of Z in M such that $S = U \cap S_{(H)}$ is closed in U. So, we can blow up U along S to get

$$U' = \mathrm{Bl}(U, S, j, \epsilon)$$

for some complex structure j and small $\epsilon > 0$.

What we achieved here by this blow-up is that the reduced space $U'/\!\!/K$ is strictly less singular.

Lemma 2.7 *[25, 36] Let H be a connected Lie group acting on a vector space W unitarily. Let $w \neq 0$ be a point in $\mu_W^{-1}(0)$ and $[w]$ denote the ray through w in $\mathbb{P}W$. Then w and $[w]$ have the same infinitesimal stabilizer.*

Proof Let $\exp(\eta)w = e^{i<\sigma,\eta>}w$ for some infinitesimal character σ of $\mathrm{stab}_H w$ and $\eta \in \mathrm{stab}_H w$. Then since $w \in \mu_W^{-1}(0)$,

$$0 =< \mu_W(w), \eta >= \omega(\eta \cdot w, w) = - < \sigma, \eta > |w|^2$$

and thus $\sigma = 0$. The statement follows. □

By the lemma, conjugates of $\mathfrak{h}$ cannot appear as an infinitesimal stabilizer of any point in U' and therefore the number of conjugacy classes of infinitesimal stabilizers in the zero level set has strictly decreased. We can successively blow up in the above fashion till we obtain a Hamiltonian K-space $\tilde{U}$ whose zero level set $\tilde{Z}$ of the moment map has only finite stabilizers. Therefore, the reduced space

$$\tilde{X} = \tilde{U} /\!\!/ K$$

has at worst orbifold singularities and we call it the *partial desingularization*.

The construction of $\tilde{X}$ involved several choices. However, the final result is independent of the choices up to deformation equivalence, which means roughly that any two such $\tilde{U}_1$ and $\tilde{U}_2$ are diffeomorphic and that the symplectic forms and moment maps can be deformed smoothly from one to the other, fixing the zero level. In particular, the quotient $\tilde{X}$ is essentially canonical. See [36] for details of this discussion.

2.3 Geometric invariant theory quotients

Geometric and categorical quotients

We wish to construct quotients of algebraic varieties by algebraic group actions. First we make precise what we mean by quotients.

Definition 2.8 *[37] Given an action $\sigma : G \times X \to X$ of G on X, a G-invariant morphism $\phi : X \to Y$, (i.e.*

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{\phi} & Y \end{array}$$

is commutative) is called a categorical quotient *if given any G-invariant $\psi : X \to Z$, there is a unique $\zeta : Y \to Z$ such that $\psi = \zeta \circ \phi$.*

Definition 2.9 *[37] Given an action σ of G on X, $\phi : X \to Y$ is called a* geometric quotient *if*

1. $\phi \circ \sigma = \phi \circ pr_2$
2. *ϕ is surjective and the image of $\Psi : G \times X \to X \times X$ is $X \times_Y X$*
3. *$U \subset Y$ is open if and only if $\phi^{-1}(U)$ is open in X (universally submersive)*
4. *$\mathcal{O}_Y = \phi_* \mathcal{O}_X^G$, the G-invariant part.*

Intuitively, each geometric fiber of ϕ is the orbit of a geometric point of X and thus we can think of Y as the orbit space.

Given a reductive group action on an *affine* variety (over $\mathbb{C}$) there always exists a categorical quotient.

Proposition 2.10 *[37] Let X be an affine scheme over $\mathbb{C}$ and G be a complex reductive group acting on X. Then a categorical quotient exists $\phi : X \to Y$ such that Y is affine and ϕ is universally submersive.*

In fact, if $X = \mathrm{Spec} A$ for a commutative algebra A, $Y = \mathrm{Spec} A^G$ where A^G is the G-invariant part of A.

Semistability and categorical quotients

In general, given a reductive group action on a quasiprojective variety we wish to construct a quotient by patching up the above local quotients. But these local quotients do not necessarily patch together to produce a variety. For example, there is the "jump phenomenon" that makes it impossible. The cure to this problem is to get rid of bad points.

Definition 2.11 *[37] Let $X \subset \mathbb{P}^n$ be a quasiprojective variety on which a reductive group G acts via a homomorphism $G \to GL(n+1)$. A point $x \in X$ is* semistable *if there is a* *nonconstant homogeneous polynomial f such that the complement X_f of the zero set of f in X is affine and contains x. A point x is* stable *if moreover the G-action on X_f is closed and the stabilizer is a finite group. If a point is not semistable it is called* unstable. *If a point is semistable but not stable, then we call it* strictly semistable.

Let X^{ss} denote the set of semistable points and X^s denote the set of stable points.

We remove unstable points from a variety with a linearized reductive group action and then it is possible to patch the affine quotients to get a categorical quotient.

Theorem 2.12 *[37] There exists a categorical quotient $\phi : X^{ss} \to Y$ of X^{ss} by G such that*

1. *ϕ is affine and universally submersive*
2. *Y is quasiprojective*
3. *the restriction of ϕ to X^s whose image is open and dense is a geometric quotient.*

We denote the quotient by $X/\!\!/G$. It should be noted that $X/\!\!/G$ is not the orbit space of X^{ss} unless $X^{ss} = X^s$. In fact, two points in X^{ss} are identified if their orbit closures meet. Clearly, this is necessary to make the quotient Hausdorff.

It is not always easy to determine (semi)stability directly from the definition. When X is projective the following criterion is especially useful. Let λ be a 1-parameter subgroup $\mathbb{C}^* \to G$. This action can be diagonalized, i.e. we can choose coordinates $(x_0 : x_1 : \cdots : x_n)$ of $\mathbb{P}^n$ such that λ acts with weight r_i on the i-th component. For each $x \in X$, define

$$q(x, \lambda) = \max\{-r_i | x_i \neq 0\}.$$

Theorem 2.13 *[37] Suppose X is projective. Then $x \in X$ is semistable if and only if $q(x, \lambda) \geq 0$ for every 1-parameter subgroup λ. x is stable if and only if $q(x, \lambda) > 0$ for each λ.*

GIT quotients are symplectic reductions

We wish to identify the GIT quotient of a smooth projective variety by a reductive group with the symplectic reduction by the maximal compact subgroup. For details, see Chapter 8 of [24].

Let $X \subset \mathbb{P}^n$ be a smooth projective variety with a linear action of a reductive group G which is the complexification of K. As remarked in the first section, we may assume that the K-action is unitary and thus Hamiltonian with moment map μ induced from the canonical moment map of the projective space. Let X^{min} denote the set of points in X whose gradient flows for $f = -|\mu|^2$ have limit points in $\mu^{-1}(0)$.

Kirwan shows $X^{ss} = X^{min}$. This can be achieved in three steps. First, observe from the above criterion for semistability that $x \in X$ is semistable for G if and only if it is semistable for any 1-parameter subgroup. Hence,

$$X^{ss} = \cap_\lambda X^{ss}_\lambda$$

where X^{ss}_λ is the set of semistable points for a 1-parameter subgroup λ. Second, when $G = \mathbb{C}^*$, it is easy to describe the moment map explicitly and check that $X^{ss} = X^{min}$. Third, one shows

$$X^{min} = \cap_\lambda X^{min}_\lambda$$

to deduce $X^{ss} = X^{min}$. (This requires a bit of Morse theory that we will review in the next chapter.) In particular, X^{ss} retracts onto $\mu^{-1}(0)$.

Consider the diagram

$$\begin{array}{ccccc} \mu^{-1}(0) & \longrightarrow & X^{min} = X^{ss} & \longrightarrow & X/\!\!/G \\ & \searrow & & \nearrow & \\ & & \mu^{-1}(0)/K & & \end{array}$$

The symplectic reduction $\mu^{-1}(0)/K$ is compact and the map to $X/\!\!/G$ is injective. Therefore, the induced map on $\mu^{-1}(0)/K$ is a homeomorphism since $X/\!\!/G$ is Hausdorff. In summary, we get the identification as desired.

Theorem 2.14 *$X/\!\!/G$ is homeomorphic to $\mu^{-1}(0)/K$.*

Partial desingularization of a GIT quotient

We will use the following definitions.

Definition 2.15 *[25]*

- *Let $\mathcal{R}(X)$ be a set of representatives of the conjugacy classes of identity components of all reductive subgroups of G which appear as stabilizers of points $x \in X^{ss}$ such that Gx is closed in X^{ss} and that $R \cap K$ is a maximal compact subgroup of R where K is a fixed maximal compact subgroup for G.*
- *Let Z_R^{ss} denote the set of those $x \in X^{ss}$ fixed by $R \in \mathcal{R}(X)$.*
- *Let $r(X) = max\{dim\, R |\, R \in \mathcal{R}(X)\}$.*

Suppose the set of stable points X^s is nonempty. Let $\pi : Y \to X^{ss}$ be the blowup of X^{ss} along $\cup_{dim R = r(X)} G Z_R^{ss}$. The action of G lifts to a linear action on Y via $\pi^* \mathcal{O}(k) \otimes \mathcal{O}(-E)$ for large k.

Kirwan in [25] proved that $\mathcal{R}(Y) = \{R \in \mathcal{R}(X) | dim R \leq r(X) - 1\}$ and hence $r(Y) < r(X)$. Moreover, $Y/\!\!/G$ is the blow-up of $X/\!\!/G$ along $\cup_{\dim R = r(X)} Z_R /\!\!/ N^R$. We may keep blowing up till we reach a variety $\tilde{Y}$ such that $r(\tilde{Y}) = 0$. $\tilde{Y}/\!\!/G$ has at worst finite quotient singularites and we call it again the partial desingularization.

This is homeomorphic to the symplectic partial desingularization described in the previous section.

Chapter 3

Equivariant Morse Theory

Equivariant Morse theory is one of the most powerful tools in computing cohomology of symplectic quotients. In this chapter we review the Kirwan-Morse theory for equivariant cohomology from [24]. Then we apply it for equivariant K-groups and equivariant Chow groups. Though straightforward, this application for the latter two seems to be new.

3.1 Equivariant theories

In this section, we recall various facts about equivariant cohomology, equivariant K-groups, and equivariant Chow groups with *complex* or *rational* coefficients.

Definition and basic lemmas

Let G be any topological group. Then there is a contractible G-space EG on which G acts freely. Let $BG = EG/G$ denote the classifying space for G. If G acts on a space M, we consider the following Borel diagram

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BG & \longleftarrow & EG \times_G M & \longrightarrow & M/G. \end{array}$$

The G-equivariant cohomology of M is defined to be the cohomology of the *homotopy quotient* $EG \times_G M$, i.e.

$$H_G^*(M) = H^*(EG \times_G M).$$

If M is just a point, we often denote $H^*_G(pt) = H^*(BG)$ by H^*_G. This is a polynomial ring for any compact connected Lie group G.

On the one extreme, if the G action is trivial, then $EG \times_G M = BG \times M$ and thus $H^*_G(M) = H^*_G \otimes H^*(M)$. On the other extreme, if the G action on M is free, then the fibers of the map $EG \times_G M \to M/G$ are contractible and hence $H^*(M/G) \cong H^*_G(M)$. In general, the equivariant cohomology may be computed by the Leray spectral sequence for the map $EG \times_G M \to M/G$ whose fiber over a point in M/G is the classifying space of the stabilizer of a point in the corresponding orbit on M. For example, if the stabilizers are all finite groups then the natural map $H^*(M/G) \to H^*_G(M)$ is an isomorphism since we are using complex or rational coefficients.

We collect here some useful lemmas. The first is about "quotient in stages".

Lemma 3.1 *Let H be a closed normal subgroup of G and M be a G-space on which H acts freely. Then the quotient group $S = G/H$ acts on $N = M/H$ and*

$$H^*_G(M) = H^*_S(N).$$

Proof Consider the fibration $EG \times_G M \cong (EG \times ES) \times_G M \to ES \times_G M \cong ES \times_S N$ whose fibers are contractible. □

Next we consider extending a group action to a larger group.

Lemma 3.2 *Let H be a subgroup of G and N be an H-space. Define a G-space $M = G \times_H N$. Then*

$$H^*_G(M) = H^*_H(N).$$

Proof $EG \times_G M = EG \times_G (G \times_H N) \cong EG \times_H Y \cong EH \times_H N$. □

Let K be a compact connected Lie group acting on M and T be a maximal torus of K. Then it turns out that the fibration

$$K/T \to EK \times_T M \to EK \times_K M$$

is cohomologically trivial. Hence, we get

Lemma 3.3 *$H_T^*(M) = H_K^*(M) \otimes H^*(K/T)$ and $H_K^*(M) = [H_T^*(M)]^W$, the invariant part with respect to the action of the Weyl group W.*

If the group is not connected, the following lemma is useful.

Lemma 3.4 *Let K_0 denote the identity component of K. Then the equivariant cohomology $H_K^*(M)$ is just the invariant part $[H_{K_0}^*(M)]^{\pi_0 K}$ of $H_{K_0}^*(M)$.*

Proof Notice that $EK = EK_0$ and thus $EK \times_K M = EK_0 \times_{K_0} M/\pi_0 K$. Hence the lemma follows since we are using complex or rational coefficients. □

De Rham model for equivariant cohomology

This treatment is borrowed from [4].

Let $\mathfrak{k}$ denote the Lie algebra of a compact connected Lie group K. Then we define the Weil algebra as the tensor product of the exterior algebra and the symmetric algebra $W(\mathfrak{k}) = \wedge\mathfrak{k}^* \otimes S\mathfrak{k}^*$. Let $\{\theta^i\}$ be a basis of $\mathfrak{k}^*$ in $\wedge\mathfrak{k}^*$ and $\{u_i\}$ denote the corresponding basis of $\mathfrak{k}^*$ in $S\mathfrak{k}^*$. We set the degree of θ^i to be 1 while that of u_i to be 2. Elements in $\mathfrak{k}^*$ can be thought of as left invariant 1-forms on K and the exterior differential defines the structure constants c^i_{jk} by $d\theta^i + \frac{1}{2}\sum c^i_{jk}\theta^j\theta^k = 0$. We define the differential D on the Weil algebra by

$$D\theta^i = d\theta^i - u_i, \quad Du_i = \sum c^i_{jk} u_j \theta_k.$$

The Jacobi identity implies $D^2 = 0$ in $W(\mathfrak{k})$. Moreover, it turns out that the cohomology is trivial, i.e. $H_D^*(W(\mathfrak{k})) = \mathbb{C}$. For example, if K is a torus, all the structure constants are zero and the Weil complex is just the Koszul complex which is a resolution of $\mathbb{C}$. Hence, the Weil algebra serves as the de Rham model for the contractible space EK.

If $P \to M$ is a principal K-bundle, then the de Rham complex $\Omega^*(M)$ for M is a subcomplex of $\Omega^*(P)$ of basic elements. A differential form ϕ on P is called basic if it is K-invariant and the interior product $\imath_X\phi$ vanishes for any vertical vector field X. In the Weil algebra, for the dual basis $\{e_i\}$ of $\mathfrak{k}$, $\imath_{e_i}\theta^j = \delta^j_i$, $\imath_{e_i}u_j = 0$ and the Lie derivative is defined by $L_{e_i} = \imath_{e_i}D + D\imath_{e_i}$. Hence, the basic subcomplex of the Weil algebra is the K-invariant part of the symmetric algebra $[S\mathfrak{k}^*]^K$ with the trivial differential. This serves as the de Rham model for the classifying space BK. Indeed, for a compact connected Lie group K, one can apply the Chern-Weil theory after approximating $EK \to BK$ by geometric fibrations to establish an isomorphism

$$H_K^* \xleftarrow{\cong} [S\mathfrak{k}^*]^K.$$

More generally, if M is a manifold on which K acts smoothly, then the basic subcomplex of $\Omega^*(M) \otimes W(\mathfrak{k})$ is the K-invariant part $\Omega_K^*(M) := [\Omega^*(M) \otimes S\mathfrak{k}^*]^K$. The differential is given by

$$(D\eta)(X) = d(\eta(X)) - \imath_X(\eta(X))$$

for $X \in \mathfrak{k}$ and $\eta \in \Omega_K^*(M)$. Again, the Chern-Weil theory tells us the following.

Proposition 3.5 $H_K^*(M) \cong H^*(\Omega_K^*(M))$.

Equivariant K-groups

We recall some facts from [44] on equivariant K-groups.

When M is a compact G-space, the equivariant K-group $K_G(M)$ with rational coefficients is the $\mathbb{Q}$-vector space associated to the semigroup of G-equivariant complex vector bundles over M. Equivariant K-groups satisfy many properties of the equivariant cohomology. We record some of them whose proofs are elementary.

1. $K_G(pt) = R(G)$, the representation ring.

2. $K_G(M)$ is an algebra over $K_G := K_G(pt)$.

3. $K_G(G/H) \cong R(H)$.

4. $K_G(G \times_H M) = K_H(M)$.

5. $K(M/G) \cong K_G(M)$ if G acts freely on M.

6. $K_{G/H}(M/H) \cong K_G(M)$ if a normal subgroup H acts freely.

7. $K_G(M) \cong R(G) \otimes K(M)$ if M is a trivial G-space.

Moreover, if M is locally compact and A is a closed subset, then we get a long exact sequence

$$\cdots \to K_G^{-q}(M, A) \to K_G^{-q}(M) \to K_G^{-q}(A) \to K_G^{-q+1}(M, A) \to \cdots$$

where $K_G^{-q}(M) = K_G(M \times \mathbb{R}^q)$. We also have the isomorphism

$$K_G^{-q}(M \setminus A) \cong K_G^{-q}(M, A)$$

as well as the Bott periodicity

$$K_G^{-q}(M) = K_G^{-q-2}(M).$$

If $E \to M$ is a G-vector bundle with the zero section $\phi : M \to E$ then there is the Thom isomorphism $\phi_* : K_G(M) \to K_G(E)$ such that we have the self-intersection formula

$$\phi^* \phi_*(\xi) = \xi \cdot e_G(E)$$

where $e_G(E)$ is the Euler class $\sum(-1)^i \wedge^i E$.

For a compact connected Lie group G with a maximal torus T and a locally compact G-space M, we have $K_G(M) = [K_T(M)]^W$, the Weyl group invariant part.

Equivariant characteristic classes

Let $E \to M$ be a G-equivariant complex vector bundle so that $EG \times_G E \to EG \times_G M$ is a vector bundle. We define the equivariant Chern classes $c_i^G(E)$ to be the Chern classes of the latter vector bundle over the homotopy quotient.

In [3], Atiyah and Bott proved that the Morse stratification of the space of connections on a principal bundle over a Riemann surface given by the norm square of curvature is equivariantly perfect. One key observation for their proof was the following.

Proposition 3.6 *Let E be a complex vector bundle over a connected space M on which a compact group K acts as a group of bundle automorphisms of E. If there is a subtorus T_0 in K acting trivially on M such that the representation of T_0 on the fiber of E at any point of M has no nonzero fixed point, then the equivariant Euler class is not a zero divisor.*

Equivariant Chow groups

If the Lie group G is reductive algebraic, the universal principal bundle $EG \to BG$ can be approximated by smooth algebraic varieties. In this section, we review the equivariant Chow groups obtained from such approximations.

Let G be a reductive algebraic group. For each $i > 0$, there exists a representation V of G, an open subset U of V such that $V \setminus U$ has codimension $> i$ and such that a principal bundle quotient $U \to U/G$ exists. This quotient is our approximation for the universal quotient. For example, if G is a torus $T = (\mathbb{C}^*)^g$, then we can take $V = (\mathbb{C}^{q+1})^g$, $U = (\mathbb{C}^{q+1} \setminus 0)^g$ for $q > \frac{1}{2}i$ on which the torus acts with weight 1 on each component. The quotient U/G in this case is $(\mathbb{P}^q)^g$. If $G = GL_n$ then we can take V to be the space of $n \times p$ matrices and U to be the subspace of matrices of maximal rank. The codimension of U is $p - n + 1$ and the quotient U/G is the Grassmannian $Gr(n,p)$. In general, we can embed G into GL_n and use the above to construct a quotient. (See [10], [29] for details.)

Now, let M be an n-dimensional smooth quasi-projective variety on which a g-dimensional algebraic group G acts (linearly). Then we can approximate the homotopy quotient $M_G := EG \times_G M$ by $U \times_G M$. The advantage of this approach is that U is smooth and algebraic. The equivariant Chow groups, with complex or rational coefficients, are defined as

$$A_j^G(M) = A_{j+l-g}(U \times_G M).$$

Bogomolov's double fibration argument shows that they are independent of the choice of the representation as long as the codimension of U in V is big enough. Since M is smooth, the equivariant operational Chow group $A^i_G(M)$ is isomorphic to $A^G_{n-i}(M)$ and hence the Chow groups are equipped with the intersection product $A^i_G(M) \times A^j_G(M) \to A^{i+j}_G(M)$. Moreover, if $M' \subset M$ is invariant open, then we have an exact sequence

$$A_i^G(M \setminus M') \to A_i^G(M) \to A_i^G(M') \to 0.$$

If the G-action on M is locally free and locally proper and the geometric quotient $M \to M/G$ exists, then we still have the isomorphism $A_*^G(M) \cong A_*(M/G)$. (See [10].) On the other extreme, if the action is trivial, we get $A_*^G(M) \cong A_*(M) \otimes A_G^*$ where $A_G^* = A_G^*(pt)$.

Characteristic classes can be similarly defined. If $E \to M$ is an equivariant vector bundle, then $U \times_G E \to U \times_G M$ is a vector bundle and the equivariant Chern classes c_i^G are the Chern classes of the latter vector bundle. By using the Chern classes, the self-intersection formula can be written as follows: if $\imath : M' \to M$ is a regular embedding of a smooth invariant subvariety of codimension d into a smooth variety M, then $\imath^* \imath_*(\alpha) = c_d^G(\nu_M(M')) \cap \alpha$ for $\alpha \in A_*^G(M')$ where $\nu_M(M')$ denotes the normal bundle.

We conclude this section with the following lemma.

Lemma 3.7 *Let H be a connected normal subgroup of a connected compact Lie group K. Set $S = K/H$. Let M be a K-space on which H acts trivially. Then there is an isomorphism of equivariant cohomology*

$$H_K^*(M) \cong H_S^*(M) \otimes H_H^*.$$

Proof Consider the fibration

$$BH \to (EK \times ES) \times_K M \to ES \times_S M.$$

The spectral sequence for this fibration degenerates by Deligne's criterion because BH can be approximated by nonsigular projective varieties. □

3.2 Equivariant cohomology

In this section, we review how the Morse theory can be applied to compute the cohomology of symplectic reductions, from [24].

Morse stratification

Let M be a compact symplectic manifold acted on by a compact Lie group with a moment map $\mu : M \to \mathfrak{k}^*$. Fix an invariant norm on $\mathfrak{k}$ by which we identify $\mathfrak{k}$ with $\mathfrak{k}^*$ and a K-invariant Riemannian metric on M. Let $f = |\mu|^2 : M \to \mathbb{R}$ denote the norm square of the moment map.

Consider the maximal torus T of K. Its action on M has a moment map, namely the composite

$$\mu_T : M \xrightarrow{\mu} \mathfrak{k}^* \longrightarrow \mathfrak{t}^*$$

The image of μ_T is the convex hull of the finite set $\mu_T(M^T)$, the image of the fixed point set M^T, by the convexity theorem due to Atiyah [2] and Gullemin-Sternberg [18]. Let $\mathcal{B}$ denote the set of the elements β in the Weyl chamber $\mathfrak{t}_+$ which are closest from the origin to the convex hulls of some subsets of $\mu_T(M^T)$, together with 0. Then M has a smooth stratification $\{S_\beta\}$ indexed by $\mathcal{B}$, by the following result of Kirwan [24].

Theorem 3.8 *There exists a smooth stratification $\{S_\beta | \beta \in \mathcal{B}\}$ of M such that $x \in S_\beta$ if and only if the path of steepest descent for f has a limit point in the critical set $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ where Z_β is the fixed point set with respect to the action of $T_\beta = \overline{\exp \mathbb{R}\beta}$ such that $< \mu(Z_\beta), \beta >= |\beta|^2$. Moreover, S_β retracts onto C_β which is diffeomorphic to $K \times_{\mathrm{Stab}\beta} (Z_\beta \cap \mu^{-1}(\beta))$.*

For simplicity, we assume that each Z_β is connected. The general case no more difficult except for more careful indexing. Let $U_\beta = \cup_{\beta' \leq \beta} S_{\beta'}$ and consider the Gysin sequence

$$\cdots \to H_K^{*-2d(\beta)}(S_\beta) \to H_K^*(U_\beta) \to H_K^*(U_\beta \setminus S_\beta) \to \cdots$$

where $2d(\beta)$ is the real codimension of S_β in M. An argument due to Atiyah and Bott [3] shows that this sequence breaks into short exact sequences.

Theorem 3.9 *[24] $0 \to H_K^{*-2d(\beta)}(S_\beta) \to H_K^*(U_\beta) \to H_K^*(U_\beta \setminus S_\beta) \to 0$ is exact.*

Proof Notice that since S_β retracts onto $C_\beta = K \times_{\mathrm{Stab}\beta} (Z_\beta \cap \mu^{-1}(\beta))$,

$$H_K^*(S_\beta) \cong H_{\mathrm{Stab}\beta}^*(Z_\beta \cap \mu^{-1}(\beta)) \subset H_T^*(Z_\beta \cap \mu^{-1}(\beta))$$

since T is a maximal torus of Stabβ. Set $T = T_\beta \times T_1$. Then

$$H_T^*(Z_\beta \cap \mu^{-1}(\beta)) \cong H_{T_1}^*(Z_\beta \cap \mu^{-1}(\beta)) \otimes H_{T_\beta}^*$$

because T_β acts trivially on Z_β. The Euler class of the restriction of the normal bundle of S_β to $Z_\beta \cap \mu^{-1}(\beta)$ is not a zero divisor since the action of T_β on any fiber has no nonzero fixed point. Therefore, the composite

$$H_K^{*-2d(\beta)}(S_\beta) \to H_K^*(U_\beta) \to H_K^*(S_\beta)$$

of the Gysin map with the restriction map is injective. So, we are done. $\square$

Obviously, the minimal critical set for f is $\mu^{-1}(0)$ and S_0 retracts onto $\mu^{-1}(0)$.

Corollary 3.10 *The restriction $\kappa : H_K^*(M) \to H_K^*(\mu^{-1}(0))$ is surjective and*

$$H_K^*(M) = H_K^*(\mu^{-1}(0)) \oplus \bigoplus_{\beta \neq 0} H_{\mathrm{Stab}\beta}^{*-2d(\beta)}(Z_\beta \cap \mu^{-1}(\beta)).$$

If 0 is a regular value of the moment map μ,

$$H^*(\mu^{-1}(0)/K) \cong H_K^*(\mu^{-1}(0)) \cong H_K^*(M)/ \oplus_{\beta \neq 0} H_{\mathrm{Stab}\beta}^{*-2d(\beta)}(Z_\beta \cap \mu^{-1}(\beta)).$$

Remark 3.11 $H_{\mathrm{Stab}\beta}^*(Z_\beta \cap \mu^{-1}(\beta))$ can be computed in a similar manner since the action of Stabβ on the smooth compact Hamiltonian space Z_β has a moment map $\mu - \beta : Z_\beta \to (\mathrm{stab}\beta)^*$.

A stratification for torus case

Let M be a compact symplectic manifold acted on by a compact torus T with a moment map $\mu : M \to \mathfrak{t}^*$. Let $\alpha \in \mathfrak{t}$ be a generic element such that $\overline{\exp \mathbb{R}\alpha} = T$. Then the function $f_\alpha =< \mu, \alpha >$ is a Bott-Morse function and hence we get a smooth Morse stratification $M = \cup_\gamma S_\gamma$ by locally closed smooth submanifolds S_γ that retract onto the fixed point components F_γ. Let $U_\gamma = \cup_{\gamma' \leq \gamma} S_{\gamma'}$ and consider the Gysin sequence

$$\cdots \to H_T^{*-2d(\gamma)}(S_\gamma) \to H_T^*(U_\gamma) \to H_T^*(U_\gamma \setminus S_\gamma) \to \cdots .$$

Theorem 3.12 *The Gysin sequence breaks into short exact sequences*

$$0 \to H_T^{*-2d(\gamma)}(S_\gamma) \to H_T^*(U_\gamma) \to H_T^*(U_\gamma \setminus S_\gamma) \to 0.$$

Proof As in the previous subsection, one can show that the composite $H_T^{*-2d(\gamma)}(S_\gamma) \to H_T^*(U_\gamma) \to H_T^*(S_\gamma)$ is injective by observing the Euler class of the restriction of the normal bundle of S_γ to F_γ is not a zero-divisor. $\square$

Hence,

$$H_T^*(M) \cong \oplus_\gamma H_T^{*-2d(\gamma)}(F_\gamma) \cong \oplus_\gamma H^*(F_\gamma) \otimes H_T^*$$

and thus the Poincaré series $P^T(M) = \sum t^i \dim H_T^i(M)$ satiesfies

$$P^T(M) = \sum t^{2d(\gamma)} P(F_\gamma) P(BT) \geq P(M)P(BT)$$

by the Morse inequality. Therefore, the spectral sequence for the fibration $M \to ET \times_T M \to BT$ degenerates at E_2 terms. More generally, we have the following.

Proposition 3.13 *Let M be a compact symplectic manifold acted on by a compact connected Lie group K such that a moment map $\mu : M \to \mathfrak{k}^*$ exists. Then the spectral sequence for the fibration $M \to EK \times_K M \to BK$ degenerates at E_2 terms, and hence $H_K^*(M) \cong H^*(M) \otimes H_K^*$.*

Proof For a connected compact Lie group K with a maximal torus T, we have the fibration $K/T \to ET \times_T M \to EK \times_K M$ by taking $ET = EK$. As is well-known, the spectral sequence for this fibration degenerates at E_2 terms since the map $H_T^*(M) \to H^*(K/T)$ has a right inverse. Therefore, $P^T(M) = P^K(M)P(K/T)$. Similarly, $P(BT) = P(BK)P(K/T)$. Hence, $P^K(M) = P(M)P(BK)$ and the spectral sequence degenerates. $\square$

Hence, together with the result of the previous subsection, we get a formula for the Betti numbers of symplectic reductions when 0 is a regular value of the moment map. However, the isomorphism in the above proposition is in general not a ring homomorphism. To understand the ring structure, we need the following localization results.

Abelian localization and cup product structure

Recall that for a compact connected Lie group K with a maximal torus T we have $H_K^*(M) \cong [H_T^*(M)]^W$ as a ring for any K-space M. Hence, we may compute the equivariant cohomology ring $H_K^*(M)$, once we know $H_T^*(M)$, by considering the Weyl group invariant part. So, we focus on the abelian case.

Let M be a compact Hamiltonian T-space. Consider the restriction map $\imath^* : H_T^*(M) \to H_T^*(M^T)$ to the fixed point set $M^T = \cup_\gamma F_\gamma$.

Theorem 3.14 *[31, 14, 51] The map $i^* : H_T^*(M) \to H_T^*(M^T)$ is injective and the image is the same as the image of the restriction $H_T^*(\cup_{T_1} M^{T_1}) \to H_T^*(M^T)$ where T_1 runs through all subtori of codimension 1.*

Proof We use induction on γ (with respect to the ordering by the values of f_α). Let $U_\gamma = \cup_{\gamma' \leq \gamma} S_{\gamma'}$ be open where $S_{\gamma'}$ is the stratum that retracts onto $F_{\gamma'}$. First, we want to show that the restriction $H_T^*(U_\gamma) \to H_T^*(U_\gamma \cap M^T)$ is injective. For minimal γ, it is trivial. Assume that it is true for $U_\gamma \setminus S_\gamma$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^{*-2d(\gamma)}(F_\gamma) & \longrightarrow & H_T^*(U_\gamma) & \longrightarrow & H_T^*(U_\gamma \setminus S_\gamma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_T^*(F_\gamma) & \longrightarrow & \oplus_{\gamma' \leq \gamma} H_T^*(F_{\gamma'}) & \longrightarrow & \oplus_{\gamma' < \gamma} H_T^*(F_{\gamma'}) & \longrightarrow & 0. \end{array}$$

The first and last verticals are injective and hence the middle is also injective.

Now we consider the image of the restriction. Obviously, the image of $H_T^*(M)$ is contained in the image of $H_T^*(M_1)$ where $M_1 = \cup_{T_1} M^{T^1}$ and we show the converse by induction.

Again, the base case is trivial and suppose the images of $H_T^*(U_\gamma \setminus S_\gamma)$ and $H_T^*(U_\gamma \setminus S_\gamma \cap M_1)$ are same in $\oplus_{\gamma' < \gamma} H_T^*(F_{\gamma'})$. Let η be an element in $H_T^*(U_\gamma \cap M_1)$ that becomes zero when restricted to $H_T^*(U_\gamma \setminus S_\gamma \cap M_1)$.[1] We claim that the image of η in $H_T^*(F_\gamma)$ is a multiple of the Euler class of the restriction N_γ of the normal to S_γ. This will finish the proof since every such element is from $H_T^*(U_\gamma)$.

The T action on a fiber of the normal to S_γ at a point in F_γ splits into one dimensional representations with weights α_j and each of these gives us a stabilizer subgroup T_j of codimension 1. Consider the fixed point set M^{T_j} by T_j. The tangent directions at F_γ to this set give us a subbundle N_j of the normal to S_γ and hence the normal decomposes into a direct sum of the subbundles N_j. By the assumption on η, the restriction of η on N_j is a multiple of the top Chern class of N_j, which has the form

$$f_j \otimes 1 + (\text{terms with lower degree polynomials})$$

where f_j is a homogeneous polynomial. These are coprime for any two distinct N_j's.

[1] It is easy to see by diagram chase that we have to only to consider these elements.

Now we deduce from the following lemma that η is divisible by the product of the top Chern classes of all N_j's, which equals to the Euler class of the normal bundle. So, we are done. $\square$

Lemma 3.15 *Let A be a finite dimensional $\mathbb{Q}$-algebra with unity 1 such that there is a decomposition $A = \mathbb{Q}1 \oplus \tilde{A}$ and $\tilde{A}$ is nilpotent. Let S be a polynomial ring over $\mathbb{Q}$ and set $B = S \otimes_{\mathbb{Q}} A = \oplus_i S^i \otimes A$ where S^i is the degree i part of S. Let f, g, h be elements in B such that the top degree part with respect to the grading of S are of the form*

$$f_0 \otimes 1, \;\; g_0 \otimes 1, \;\; h_0 \otimes 1$$

for nontrivial homogeneous polynomials f_0, g_0, h_0. Suppose g and h divide f in B and g_0 and h_0 are coprime. Then gh divides f in B.

Proof By the assumption on nilpotency, g/g_0 has an inverse in $B_{g_0} = S_{g_0} \otimes_{\mathbb{Q}} A$, the localized B by the multiplicative system $\{g_0^m | m \in \mathbb{Z}\}$ and h/h_0 has an inverse in B_{h_0}. Let $k = \frac{f}{g_0 h_0}(\frac{g}{g_0})^{-1}(\frac{h}{h_0})^{-1}$ in $B_{g_0 h_0}$. Now it suffices to show that k lies in fact in B.

Since g divides f, $f = g\tilde{g}$ for some $\tilde{g} \in B$. Since $f = g(hk)$ in $B_{g_0 h_0}$, $\tilde{g} = hk \in B \subset B_{g_0 h_0}$ and thus $k = \tilde{g}(\frac{h}{h_0})^{-1}\frac{1}{h_0} \in B_{h_0}$. Similarly, $k \in B_{g_0}$. But $B_{g_0} \cap B_{h_0} = B$ because g_0 and h_0 are coprime. So we are done. $\square$

An immediate consequence of the theorem is the following integration formula.

Corollary 3.16 *For each $\xi \in H_T^*(M)$, $\xi = \sum_\gamma \imath_{F_\gamma *} \frac{\imath_{F_\gamma}^* \xi}{e^T(\nu_X(F_\gamma))}$ and hence*

$$\int_M \xi = \sum_\gamma \int_{F_\gamma} \frac{\imath_{F_\gamma}^* \xi}{e^T(\nu_X(F_\gamma))}$$

where $\imath_{F_\gamma} : F_\gamma \hookrightarrow X$ is the embedding and $e^T(\nu_X(F_\gamma))$ is the equivariant Euler class of the normal bundle to F_γ.

Proof By the above theorem, $\sum_\gamma \imath_{F_\gamma}^*$ is injective. Since

$$\imath_{F_\gamma}^* \imath_{F_\gamma *} \imath_{F_\gamma}^* \xi = e^T(\nu_X(F_\gamma)) \imath_{F_\gamma}^* \xi$$

we deduce that $\xi = \sum_\gamma \imath_{F_\gamma *} \frac{\imath^*_{F_\gamma}\xi}{e^T(\nu_X(F_\gamma))}$. The integration now follows by applying π_* on both sides where $\pi : M \to pt$ is the constant map. □

Since $\imath^*$ is a ring homomorphism, the above theorem completely determines the cup product structure of $H^*_T(M)$. Now, the equivariant cohomology ring $H^*_T(\mu^{-1}(0))$ can be determined by the surjective ring homomorphism $H^*_T(M) \to H^*_T(\mu^{-1}(0))$ in terms of the Gysin maps as seen before, at least in principle. For example, Kirwan proved the Mumford conjecture and the Newstead conjecture by using this idea in [31].

The following theorem gives a nice description of the kernel.

Theorem 3.17 *[50] Suppose 0 is a regular value of the moment map μ. Then the kernel of the surjective map $\kappa : H^*_T(M) \to H^*_T(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/T)$ is given by*

$$K = \sum_{\xi \in \mathfrak{t}} \{\alpha \in H^*_T(M) \mid \alpha|_{M^T \cap M_\xi} = 0\}$$

where $M_\xi = \{x \in M \mid <\mu(x), \xi> \le 0\}$.

Proof Consider the Morse stratification $\{S_\beta\}$ by the norm square of the moment map for the torus action. Recall that β's are the closest points from the origin to the convex hulls of some points in $\mu(M^T)$ and that S_β retracts onto $Z_\beta \cap \mu^{-1}(\beta)$. More precisely, in this torus case, the stratum S_β is an open subset of the Morse stratum Y_β with the critical set Z_β with respect to the function $f_\beta =< \mu, \beta >$.

Let U_β be the union of the strata for f_β less than or equal to Y_β and consider the image via the Gysin map $H_T^{*-2d(\beta)}(Y_\beta) \to H^*_T(U_\beta)$. Let I_β denote any lifting of the image in $H^*_T(M)$. Then the kernel of κ is $\sum_{\beta \neq 0} I_\beta$.

Clearly, the subspace I_β becomes zero when restricted to $M_\beta = \{x \in M \mid <\mu(x), \beta> \le 0\}$ which is a subset of $U_\beta \setminus Y_\beta$. Conversely, if $\eta|_{M_\beta} = 0$, then η lies in

$$\sum \{I_{\beta'} | \beta' \text{ is parallel to } \beta \text{ and } <\beta', \beta>> 0\}$$

and hence in the kernel of κ. Therefore, the kernel of κ is $\sum_{\beta \neq 0} \{\alpha \in H^*_T(M) | \, \alpha|_{M_\beta} = 0\}$.

From the proof of the previous theorem, $H_T^*(M_\beta)$ injects into $H_T^*(M_\beta \cap M^T)$. Hence, the theorem now follows from the observation that for any ξ there is a β such that $M_\xi = M_\beta$, which is an easy exercise. $\square$

3.3 Equivariant K-groups

In this section, we apply the Morse theory to compute the rational K-groups of symplectic reductions.

Perfectness of the Morse stratification

Let G be a compact connected Lie group acting on a compact symplectic manifold M in a Hamiltonian fashion. From [24], we know that M has the Morse stratification $M = \cup_\beta S_\beta$ as described before. Give a linear ordering on the index set compatible with the partial ordering by length. Let D_β be a small invariant neighborhood of S_β and set $U_\beta = M \setminus \cup_{\beta' > \beta} D_{\beta'}$. So, we get a compact filtration $\{U_\beta\}$. Consider the long exact sequence

$$\cdots \to K_G^{-q}(U_\beta, U_\beta \setminus D_\beta) \to K_G^{-q}(U_\beta) \to K_G^{-q}(U_\beta \setminus D_\beta) \to \cdots$$

We have the isomorphism

$$K_G^{-q}(S_\beta \cap U_\beta) \cong K_G^{-q}(D_\beta \cap U_\beta) \cong K_G^{-q}(U_\beta, U_\beta \setminus D_\beta).$$

The composition of this isomorphism with the natural map $K_G^{-q}(U_\beta, U_\beta \setminus D_\beta) \to K_G^{-q}(U_\beta)$ in the above long exact sequence and the restriction map $K_G^{-q}(U_\beta) \to K_G^{-q}(S_\beta \cap U_\beta)$ is equal to the composite

$$K_G^{-q}(S_\beta \cap U_\beta) \to K_G^{-q}(D_\beta \cap U_\beta) \to K_G^{-q}(S_\beta \cap U_\beta)$$

where the first is the Thom homomorphism. By the self-intersection formula, this composite is given by multiplying the Euler class of the normal bundle.

Now observe that as before $K_G^{-q}(S_\beta \cap U_\beta) \cong K_{\mathrm{Stab}\beta}^{-q}(Z_\beta \cap \mu^{-1}(\beta)) \subset K_{T_1}^{-q}(Z_\beta \cap \mu^{-1}(\beta)) \otimes R(T_\beta)$ where the maximal torus splits as $T = T_\beta \times T_1$.[2] Since the action of T_β on a fiber

[2] $S_\beta \cap U_\beta$ retracts onto $K(Z_\beta \cap \mu^{-1}(\beta))$ by the gradient flow.

of the normal bundle has no nonzero fixed points, the restriction of the Euler class is not a zero divisor.[3] Therefore, the Gysin map $K_G^{-q}(S_\beta \cap U_\beta) \to K_G^{-q}(U_\beta)$ is injective and thus the long exact sequence breaks into short exact sequences. So, we proved the following.

Theorem 3.18 $0 \to K_G(S_\beta \cap U_\beta) \to K_G(U_\beta) \to K_G(U_\beta \setminus D_\beta) \to 0$ *is exact.*

Corollary 3.19 *The restriction* $K_G(M) \to K_G(\mu^{-1}(0))$ *is surjective and*

$$K_G(M) = K_G(\mu^{-1}(0)) \oplus \bigoplus_{\beta \neq 0} K_{\mathrm{Stab}\,\beta}(Z_\beta \cap \mu^{-1}(\beta)).$$

In particular, if G acts freely on $\mu^{-1}(0)$ then we get

$$K(\mu^{-1}(0)/G) \cong K_G(\mu^{-1}(0)) \cong K_G(M)/\oplus_{\beta \neq 0} K_{\mathrm{Stab}\,\beta}(Z_\beta \cap \mu^{-1}(\beta)).$$

Hence, we can compute the K-groups of the symplectic reduction inductively.

Torus action and abelian localization

Let M be a compact Hamiltonian T-space. Consider the stratification $M = \cup_\gamma S_\gamma$ for the torus case in the previous section, given by the Morse function $f_\alpha =< \mu, \alpha >$ for some generic $\alpha \in \mathfrak{t}$. Let D_γ be a small invariant neighborhood of S_γ and let $U_\gamma = M \setminus \cup_{\gamma' > \gamma} D_{\gamma'}$. As above, it is easy to adapt the proof of the previous section to show the following.

Proposition 3.20 *The long exact sequence for the equivariant K-groups of the pair* $(U_\gamma, U_\gamma \setminus D_\gamma)$ *breaks into short exact sequences*

$$0 \to K_T(S_\gamma \cap U_\gamma) \to K_T(U_\gamma) \to K_T(U_\gamma \setminus D_\gamma) \to 0$$

and hence $K_T(M) \cong \oplus_\gamma K(F_\gamma) \otimes R(T)$.

Let $\imath : M^T \to M$ denote the embedding of the T fixed point set. Then the following can be proved by an easy modification of the proof in the previous section.[4]

[3] Though K-groups do not have a natural grading, $R(T_\beta)$ does. The part of the Euler class with highest degree on the $R(T_\beta)$ direction is (nonzero polynomial)$\otimes 1$ and thus it cannot be a zero divisor.

[4] Notice that $K(F_\gamma) = \mathbb{Q}1 \oplus \tilde{K}(F_\gamma)$ and $\tilde{K}(F_\gamma)$ is nilpotent. So, the lemma in the previous section applies.

Theorem 3.21 *The map $\imath^* : K_T(M) \to K_T(M^T)$ is injective and the image is the same as the image of the restriction $K_T(\cup_{T_1} M^{T_1}) \to K_T(M^T)$ where T_1 runs through all subtori of codimension 1.*

This version of abelian localization theorem was proved in [42] in a different way.

3.4 Equivariant Chow groups

In this section, we apply the Kirwan-Morse theory to compute the rational Chow groups of GIT quotients.

Algebraic stratification

Let M be a smooth projective variety in $\mathbb{P}^n$ acted on by a complex connected reductive group G via a homomorphism $G \to GL(n+1)$. After conjugation if necessary we may assume that there is a maximal compact subgroup K of G such that the homomorphism restricts to $K \to U(n+1)$. Then the Morse stratification in §2 is a smooth stratification ([24])

$$M = \cup_{\beta \in \mathcal{B}} S_\beta$$

by locally closed smooth subvarieties where $\mathcal{B}$ is the finite subset, containing 0, of the Weyl chamber $\mathfrak{t}_+$ corresponding to a choice of a Borel subgroup B of G. The minimal stratum S_0 is the set of semistable points M^{ss}.

Let

$$Z_\beta = \{(x_0 : x_1 : \cdots : x_n) \in M | x_j = 0 \text{ if } \alpha_j \cdot \beta \neq |\beta|^2\}$$

$$Y_\beta = \{(x_0 : x_1 : \cdots : x_n) \in M | x_j = 0 \text{ if } \alpha_j \cdot \beta < |\beta|^2 \text{ , } x_j \neq 0 \text{ for some } j \text{ such that } \alpha_j \cdot \beta = |\beta|^2\}$$

where α_j's are the weights of the action of the maximal torus for the action $G \to GL(n+1)$ and x_j's are the coordinates of the eigenvectors. Then Z_β is a union of fixed point components by the action of $T_\beta = \overline{\exp \mathbb{R}\beta}_{\mathbb{C}}$ whose values by the function $f_\beta = <\mu, \beta>$ are $|\beta|^2$, while Y_β is the Morse stratum by f_β that retracts onto Z_β. In fact, the retraction $p_\beta : Y_\beta \to Z_\beta$ is an algebraic vector bundle.

The stabilizer Stabβ acts on Z_β and consider the set of semistable points Z_β^{ss}. Let $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$. It was also proved in [24] that

$$S_\beta = GY_\beta^{ss} \cong G \times_{P_\beta} Y_\beta^{ss}$$

for the reductive subgroup $P_\beta = B \cdot \mathrm{Stab}\beta$.

The set $\mathcal{B}$ has a natural partial ordering by the norm and we give a total ordering compatible with the partial ordering. Let $U_\beta = \cup_{\beta' \leq \beta} S_{\beta'}$. Then we have the following exact sequence of equivariant Chow groups

$$A_G^{*-d(\beta)}(S_\beta) \to A_G^*(U_\beta) \to A_G^*(U_\beta \setminus S_\beta) \to 0.$$

As in the previous section we can show the following theorem by using the argument of Atiyah and Bott [3].

Theorem 3.22 *$0 \to A_G^{*-d(\beta)}(S_\beta) \to A_G^*(U_\beta) \to A_G^*(U_\beta \setminus S_\beta) \to 0$ is exact.*

Proof The idea is basically the same as in the previous section. So we just sketch the proof.

Notice that $A_G^*(S_\beta) \cong A_{P_\beta}^*(Y_\beta^{ss})$. Moreover, since Stab$\beta$ is the Levi part of the parabolic group P_β, the fibration

$$P_\beta/\mathrm{Stab}\beta \to EG_n \times_{P_\beta} Y_\beta^{ss} \to EG_n \times_{\mathrm{Stab}\beta} Y_\beta^{ss}$$

is an algebraic vector bundle. Hence, $A_{P_\beta}^*(Y_\beta^{ss}) \cong A_{\mathrm{Stab}\beta}^*(Y_\beta^{ss})$ which in turn must be isomorphic to $A_{\mathrm{Stab}\beta}^*(Z_\beta^{ss}) \subset A_T^*(Z_\beta^{ss})$ possibly after shifting the degree because p_β is a vector bundle. Since T_β acts trivially on Z_β, $A_T^*(Z_\beta^{ss}) \cong A_{T_1}^*(Z_\beta^{ss}) \otimes A_{T_\beta}^*$. Here, $A_{T_\beta}^*$ is the polynomial ring on $\mathrm{Lie}(T_\beta) = \mathfrak{t}_\beta$.

Now, consider the composition

$$A_G^{*-d(\beta)}(S_\beta) \to A_G^*(U_\beta) \to A_G^*(S_\beta)$$

of the Gysin map and the restriction map. Since the T_β action on the fiber of the restriction of the normal bundle of S_β to Z_β^{ss} has no nonzero fixed point, it is clear that the Euler class of the normal bundle is not a zero divisor. Hence the composite is injective and the result follows. □

Corollary 3.23 *The restriction $A^*_G(M) \to A^*_G(M^{ss})$ is surjective and $A^*_G(M) \cong A^*_G(M^{ss}) \oplus \bigoplus_{\beta \neq 0} A^{*-d(\beta)}_{\mathrm{Stab}\beta}(Z^{ss}_\beta)$.*

In particular, if the G-action on M^{ss} is free, then it is easy to see from the definition that $A^*(M//G) \cong A^*_G(M^{ss})$. Hence we can compute the Chow ring of the GIT quotient by the surjective ring homomorphism $A^*_G(M) \to A^*_G(M^{ss})$ whose kernel, isomorphic to $\bigoplus_{\beta \neq 0} A^{*-d(\beta)}_{\mathrm{Stab}\beta}(Z^{ss}_\beta)$, is given by the Gysin maps. If we have only $M^{ss} = M^s$, then by a deep result due to Edidin and Graham [10] $A_*(M//G) \cong A^G_*(M^{ss})$ and hence we can still compute the Chow groups of the GIT quotient by the above Morse theory.

Abelian localization for Chow groups

In this subsection, we determine the Chow ring $A^*_G(M)$ by abelian localization.

Edidin and Graham [10, 11] proved that for a maximal (complex) torus T of G, we have $A^*_G(M) \cong [A^*_T(M)]^W$ the Weyl group invariant part. Hence, in principle, we need to consider only the abelian group action.

For a linear algebraic torus action on a smooth projective variety M, there is a stratification $M = \cup_r Y_r$, by locally closed smooth subvarieties, called Bialynicki-Birula decomposition of M: Let α be a generic element in the Lie algebra of the torus T so that the closed subtorus generated by α is the whole T. Consider the function f_α given by pairing the moment map μ and α. The Bialynicki-Birula decomposition is the same as the Morse stratification of the function f_α.

The critical set is the fixed point set $M^T = \cup_r F_r$ where the indices r's have a linear ordering, by the values of f_α. Each stratum S_r retracts onto F_r and the retraction is in fact a vector bundle.

Now, the arguments in the previous subsection proves that for each r,

$$0 \to A^{*-d(r)}_T(S_r) \to A^*_T(U_r) \to A^*_T(U_r \setminus S_r) \to 0$$

is exact where $U_r = \cup_{r' \leq r} S_{r'}$ and $d(r)$ is the complex codimension of S_r. The arguments used in the proof of the localization theorem for equivariant cohomology in §2 prove also the following [11, 8].

Theorem 3.24 *The restriction $\imath^* : A_T^*(M) \to A_T^*(M^T)$ is injective. The image of $\imath^*$ is the intersection of the images of $\jmath_{T_1}^*$ for all subtori T_1 of codimension 1, where $\jmath_{T_1} : M^T \to M^{T_1}$ is the embedding into the T_1 fixed point set.*

Chapter 4

Nonabelian localization and intersection pairings

4.1 Reduction to abelian case

Let K be a compact connected Lie group, acting on a compact connected symplectic manifold with a moment map $\mu : M \to \mathfrak{k}^*$. We assume that the action of K on $\mu^{-1}(0)$ is free and so is the action of the maximal torus T on $\mu_T^{-1}(0)$ where $\mu_T : M \to \mathfrak{k}^* \to \mathfrak{t}^*$ is the moment map for the T-action. The purpose of this section is to recall S. Martin's trick that relates the pairings on $M/\!\!/K := \mu^{-1}(0)/K$ with those on $M/\!\!/T = \mu_T^{-1}(0)/T$.

Key observation

For each weight α of T, we have a 1-dimensional representation $\mathbb{C}_{(\alpha)}$ and a line bundle

$$L_\alpha := \mu_T^{-1}(0) \times_T \mathbb{C}_{(\alpha)} \to \mu_T^{-1}(0)/T = M/\!\!/T$$

Let Δ, Δ_+ and Δ_- denote the sets of roots, positive roots, and negative roots, respectively. Let $E_\pm = \oplus_{\alpha \in \Delta_\pm} L_\alpha$.

The main idea is captured in the following diagram:

$$\begin{array}{ccccc} & & \mu^{-1}(0)/T & \overset{i}{\hookrightarrow} & \mu_T^{-1}(0)/T = M/\!\!/T \\ & & \downarrow{\scriptstyle \pi} & & \\ M/\!\!/K & = & \mu^{-1}(0)/K & & \end{array}$$

Proposition 4.1 *[34] The vector bundle E_- has a section s which is transverse to the zero section and such that the zero set of s is the submanifold $\mu^{-1}(0)/T$. Hence the normal bundle to $\mu^{-1}(0)/T$ in $M/\!\!/T$ is $E_-|_{\mu^{-1}(0)/T}$. Moreover, the vector bundle $vert(\pi)$ tangent to the fibers of π is E_+ and the orientations match up.*

Proof The restriction of the moment map

$$\mu : \mu_T^{-1}(0) \to (\mathfrak{k}/\mathfrak{t})^* \cong \oplus_{\alpha \in \Delta_-} \mathbb{C}_{(\alpha)}$$

is T-equivariant and thus it defines a section s whose zero set is precisely $\mu^{-1}(0)/T$. The remainder of the proof is a rather straightforward check and we omit the details. See [34]. □

Integration formula

Let $e_\pm$, e denote the equivariant Euler classes of $E_\pm$, $E = E_+ \oplus E_-$ respectively. Then the above theorem tells us that e_- is the Poincaré dual to $\mu^{-1}(0)/T$ and thus for $\tilde{a} \in H^*(M/\!\!/T)$

$$\int_{\mu^{-1}(0)/T} i^*\tilde{a} = \int_{M/\!\!/T} \tilde{a} \cup e_-.$$

On the other hand, the Euler class of $vert(\pi)$ is the restriction i^*e_+ and thus for $a \in H^*(M/\!\!/K)$

$$\int_{\mu^{-1}(0)/T} \pi^*a \cup i^*e_+ = \int_{M/\!\!/K} a \cup \pi_*(i^*e_+)$$

while $\pi_*(i^*e_+)$ is the Euler characteristic $|W|$ of a fiber K/T. Hence, we get the integration formula (Theorem B of [34]).

Theorem 4.2 *For $\pi^*a = i^*\tilde{a}$,*

$$\int_{M/\!\!/K} a = \frac{1}{|W|} \int_{M/\!\!/T} \tilde{a} \cup e.$$

Cohomology rings

Martin relates the cohomology rings of the K-quotient and the T-quotient. Indeed, we consider the restriction map $\phi : H^*(M/\!\!/T)^W \to H^*(\mu^{-1}(0)/T)^W \cong H^*(M/\!\!/K)$ which is surjective from the commutative diagram

$$\begin{array}{ccccc} H_T^*(M)^W & \twoheadrightarrow & H_T^*(\mu_T^{-1}(0))^W & \xrightarrow{\cong} & H^*(M/\!\!/T)^W \\ \downarrow \cong & & & & \downarrow \\ H_K^*(M) & \twoheadrightarrow & H_K^*(\mu^{-1}(0)) & \xrightarrow{\cong} & H^*(M/\!\!/K). \end{array}$$

Let κ_T be the composite $H_T^*(M) \to H_T^*(\mu_T^{-1}(0)) \cong H^*(M/\!\!/T)$ of the restriction and the isomorphism. Similarly, let $\kappa : H_K^*(M) \to H_K^*(\mu^{-1}(0)) \cong H^*(M/\!\!/K)$ be the composite of restriction and isomorphism. The integration formula enables us to identify the kernel of the surjection. (Theorem A of [34].)

Theorem 4.3 *The kernel is the annihilator $ann(e)$ of e. Therefore,*

$$H^*(M/\!\!/K) \cong H^*(M/\!\!/T)^W / ann(e).$$

Proof By Poincaré duality, $\phi(\kappa_T(a)) = 0$ iff $\int_{M/\!\!/K} \kappa(a) \cup \kappa(b) = 0$ for all b of complementary degree since $\phi(\kappa_T(a)) = \kappa(a)$. But $\pi^*\kappa(a) = i^*\kappa_T(a)$ and hence the integral is same as $\int_{M/\!\!/T} \kappa_T(a) \cup e \cup \kappa_T(b)$. Therefore, $\kappa_T(a) \cup e = 0$, i.e. $\kappa_T(a) \in ann(e)$. $\square$

In fact, Theorems 4.2, 4.3 above imply each other.

Now suppose 0 is not a regular value of the T-moment map μ_T. Then we can use the same trick by perturbing the quotient slightly. Namely, choose a regular value ϵ close enough to 0. A small neighborhood of $\mu^{-1}(0)/T$ in $M/\!\!/T$ is diffeomorphic to an open subset of $\mu_T^{-1}(\epsilon)/T$ and the above arguments are easily modified to prove the theorems after replacing $M/\!\!/T$ by $\mu_T^{-1}(\epsilon)/T =: M/\!\!/T(\epsilon)$.

4.2 Intersection pairings on torus quotients

In this section, we review the wall crossing formulas for torus quotients due to Gullemin-Kalkman and Martin.

By the results of the previous section, we can focus on torus quotients. Let T be a compact torus. We choose a metric so that we can identify $\mathfrak{t}$ with $\mathfrak{t}^*$. Let M be a compact Hamiltonian T-space with a moment map $\mu : M \to \mathfrak{t}^*$. Let $X(p)$ denote the quotient

$\mu^{-1}(p)/T$ for $p \in \mathfrak{t}$. If p is regular then the Kirwan map $\kappa : H_T^*(M) \to H_T^*(\mu^{-1}(p)) \cong H^*(X(p))$ is surjective.

According to the convexity theorems due to Atiyah [2] and Guillemin-Sternberg [18], the image of the moment map is the convex hull of the image of the fixed point set $\mu(M^T)$. This convex polyhedron is subdivided into chambers by the walls which are the images of the fixed point sets of the 1-dimensional subgroups. Let p be a general point in the polyhedron (not on any walls). For $a \in H_T^*(M)$ we wish to understand the variation of the integral $\int_{X(p)} \kappa(a)$ as p varies in $\mathfrak{t}$.

First notice that inside a given connected open chamber C of the polyhedron, the integral is constant. To see this, consider the locally free T-action on $U = \mu^{-1}(C)$ which is homeomorphic to $\mu^{-1}(p) \times C$ for $p \in C$. The Kirwan map can be factored as follows:

$$H_T^*(M) \to H_T^*(U) \cong H^*(U/T) \cong H^*(X(p)).$$

Any class of degree $\dim_{\mathbb{R}} X(p)$ is mapped to a constant multiple of the fundamental class in $H^*(X(p))$. This comes from a class in $H^*(U/T)$ which is independent of p.

Next, we consider wall crossing. Consider two points p, q near a general point r of a wall, the image of some components of the fixed point set M^H of a 1-dimensional subtorus H in T, lying on the opposite sides of the wall. Choose a path l from p to q meeting the wall transversely only at r. In this case, by Proposition 1.5 of [35], μ is transverse to l and the stabilizer subgroup of any $x \in \mu^{-1}(l)$ is either a finite group or a finite extension of H. And the submanifold M^H is transverse to $\mu^{-1}(l)$. Let

$$W = \mu^{-1}(l) \setminus N_\epsilon(M^H) \cap \mu^{-1}(l)$$

where N_ϵ is the small neighborhood of the submanifold M^H. The *wall crossing cobordism* is defined as the quotient W/T whose boundary is by Theorem A of [35]

$$\partial(W/T) = -X(p) \cup X(q) \cup P_r$$

where P_r is the weighted projective bundle $S(N(M^H))|_{M^H \cap \mu^{-1}(r)}/T$ over $X(r)$, the quotient of the sphere bundle in the normal bundle to M^H restricted to $M^H \cap \mu^{-1}(r)$.

Now we apply the Stokes theorem on W/T to the image of a class $a \in H_T^{\dim X(p)}(M)$ via $H_T^*(M) \to H_T^*(W) \cong H^*(W/T)$. Then we get

$$0 = -\int_{X(p)} \kappa(a) + \int_{X(q)} \kappa(a) + \int_{X(r)} \pi_* \kappa(a)$$

where $\pi : P_r \to X(r)$ is the bundle projection. Therefore, we get the wall crossing formula. (Theorem B [35].)

Theorem 4.4 *There is a map $\lambda : H_T^*(M) \to H_{T/H}^*(M^H)$ such that for $a \in H_T^*(M)$*

$$\int_{X(p)} \kappa(a) - \int_{X(q)} \kappa(a) = \int_{X(r)} \kappa(\lambda(a)).$$

If we split the torus $T = H \times T_1$, the map λ turns out to be the residue operator [17]

$$\lambda(a) = \mathrm{res}_{t=0} \frac{i^* a}{e_T(N)}$$

where t is the basis in $\mathfrak{h}$ and $i : M^H \to M$ the embedding and $e_T(N)$ is the T-equivariant Euler class of the normal bundle N to the components of M^H. This amounts to taking integral along the fiber which is a weighted projective space.

The theorem is true even when the wall is the boundary of the polyhedron. Hence, by choosing a path from 0 to a point outside the polyhedron, transverse to the walls, we can compute the integral by adding up the contributions from the walls crossed. For example, if $T = S^1$ is the circle group, then we get the following theorem of Kalkman and Wu.

Theorem 4.5 *Let $\eta \in H_{S^1}^*(M)$ and $\mathcal{F}_+$ be the set of fixed point components whose values by the moment map are positive. Suppose 0 is a regular value of the moment map. Then*

$$\int_{M/\!/S^1} \kappa(\eta) = -n_0 \mathrm{res}_{t=0} \Big(\sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^* \eta(t)}{e_F(t)} \Big)$$

where n_0 is the order of the stabilizer of a generic point in $\mu^{-1}(0)$ and e_F is the Euler class of the normal bundle to F in M.

In general, since the T/H-space M^H is Hamiltonian, the contributions from the walls can be computed inductively. It is easy to see that eventually, the integral reduces to some integrals over the fixed point set M^T. Guillemin-Kalkman [17] explains this sum of the integrals over the fixed point components in terms of "dendrite" while Martin uses some abelian group to keep track of the contributions from each fixed point component [35].

4.3 Jeffrey-Kirwan localization theorem

The results of the above sections can be combined to "prove" the Jeffrey-Kirwan localization theorem [19].

As usual, we assume that M is a compact Hamiltonian K-space with a moment map $\mu : M \to \mathfrak{k}^*$ and $M/\!\!/K = \mu^{-1}(0)/K$ is smooth. Recall that the Kirwan map

$$\kappa : H_K^*(M) \to H^*(M/\!\!/K)$$

is the surjection induced from the restriction to $\mu^{-1}(0)$. Let $\tilde{e}$ be the product of all roots of K.

Theorem 4.6 *Given* $\eta \in H_K^*(M)$,

$$\int_{M/\!\!/K} \kappa(\eta) = \frac{n_0}{|W|} \mathrm{Res}(\tilde{e} \sum_{F \in \mathcal{F}} \int_F \frac{i_F^* \eta(t)}{e_F(t)}).$$

Here Res is the device to filter the contributions from fixed point components F. For example, when $K = SU(2)$, the theorem reads as follows.

Corollary 4.7 *Let* $\eta \in H_{SU(2)}^*(M)$. *Then*

$$\int_{M/\!\!/SU(2)} \kappa(\eta) = 2n_0 \mathrm{res}_{t=0}(t^2 \sum_{F \in \mathcal{F}_+} \frac{i_F^* \eta(t)}{e_F(t)})$$

where res *is the ordinary residue.*

Proof We deduce it from the results of the previous sections. First by Martin's trick,

$$\int_{M/\!\!/K} \kappa(\eta) = \frac{1}{|W|} \int_{M/\!\!/T} \kappa_T(\eta) \cup e = \frac{1}{2} \int_{M/\!\!/T} \kappa_T(\eta \cup \tilde{e}).$$

Here $\tilde{e} = (-2t)(2t) = -4t^2$ and by the wall crossing formula we get

$$\int_{M/\!\!/T} \kappa_T(\eta \cup \tilde{e}) = -2n_0 \mathrm{res}_{t=0}(t^2 \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^* \eta}{e_F}).$$

So, we proved the formula. □

In fact, Jeffrey and Kirwan proved the localization theorem by studying Witten's integral

$$\mathcal{I}^{\epsilon}(\zeta) = \int_{t\in\mathfrak{k}} e^{-\epsilon<t,t>/2} \int_M \zeta(t).$$

The intersection pairing is shown to be obtained by taking the limit $\epsilon \to 0$. On the other hand, one can apply the abelian localization theorem on the right hand side to get local contributions. The details of the proof involve quite an amount of analysis and we skip the definition of the Residue operator in the theorem ([19]). We can dispense with this Res operator by using the "dendrite" of Guillemin-Sternberg or some abelian group of Martin [17, 35] as mentioned.

4.4 Chow rings of smooth quotients

In this section, we try to understand the Chow ring of a nonabelian geometric quotient. We provide a new proof of a theorem of Ellingsrud and Stromme [12] about Chow rings of geometric quotients.

Let M be a smooth projective variety acted on linearly by a connected reductive group G. Let $T_{\mathbb{C}}$ denote its maximal torus. Let $M^s(T), M^{ss}(T)$ denote the set of stable and semistable points respectively, with respect to the $T_{\mathbb{C}}$ action.

One may wish to apply Martin's trick to prove analogues of his theorems for Chow rings. The obstacle is that the trick cannot be applied in the algebraic context since the submanifold $\mu^{-1}(0)/T$ in Martin's diagram is not a holomorphic subvariety. To see it, just use the local normal form theorem to deduce that locally the diagram looks like

$$\begin{array}{ccc} K/T \times W & \overset{i}{\hookrightarrow} & T^*(K/T) \times W \\ \downarrow{\scriptstyle \pi} & & \\ W & & \end{array}$$

where W is a Hermitian vector space. K/T is a real Lagrangian submanifold of $T^*(K/T) \cong G/T_{\mathbb{C}}$, but not a holomorphic subvariety.

However, we can still define a surjective map

$$A^*_{T_{\mathbb{C}}}(M^{ss}(T))^W \to A^*_G(M^{ss})$$

by composing the always surjective restriction map $A^*_{T_{\mathbb{C}}}(M^{ss}(T))^W \to A^*_{T_{\mathbb{C}}}(M^{ss})^W$ with the isomorphism $A^*_{T_{\mathbb{C}}}(M^{ss})^W \cong A^*_G(M^{ss})$. Notice here that M^{ss} is an open subvariety of $M^{ss}(T)$ by Mumford's criterion since every 1-parameter subgroup of $T_{\mathbb{C}}$ is a 1-parameter subgroup for G as well. In case the $T_{\mathbb{C}}$ action on $M^{ss}(T)$ and the G action on M^{ss} are free, we get a surjection $A^*(M/\!/T_{\mathbb{C}})^W \to A^*(M/\!/G)$.

We focus on a particular case considered by Ellingsrud and Stromme in [12]. Let $M = \mathbb{P}^n$ be the whole projective space on which G acts linearly. Then the natural maps

$$A^*_T(M) \cong A^*(M) \otimes S[\mathfrak{t}^*] \to H^*(M) \otimes S[\mathfrak{t}^*] \cong H^*_T(M)$$

and similarly $A^*_G(M) \to H^*_G(M)$ are isomorphisms. Using the notations of the previous chapter on the Morse stratification, the sets Z_β are projective subspaces of smaller dimensions and thus we may inductively assume that $A^*_T(Z^{ss}_\beta(T)) \cong H^*_T(Z^{ss}_\beta(T))$ and $A^*_{\mathrm{Stab}\beta}(Z^{ss}_\beta) \cong H^*_{\mathrm{Stab}\beta}(Z^{ss}_\beta)$ where $Z^{ss}_\beta(T)$ means the semistable part with respect to the $T_{\mathbb{C}}$ action. Now, the Morse theory of the previous chapter tells us that

$$A^*_T(M^{ss}(T)) \cong H^*_T(M^{ss}(T)) \quad A^*_G(M^{ss}) \cong H^*_G(M^{ss}).$$

Suppose furthermore that the $T_{\mathbb{C}}$ action on $M^{ss}(T)$ is free and so is the G action on M^{ss}, so that we get

$$A^*(M/\!/T_{\mathbb{C}}) \cong H^*(M/\!/T_{\mathbb{C}}) \quad A^*(M/\!/G) \cong H^*(M/\!/G).$$

In this situation, Martin's theorem on the cohomology rings is equivalent to the following theorem.

Theorem 4.8 *The kernel of the surjective restriction $A^*(M/\!/T_{\mathbb{C}})^W \to A^*(M/\!/G)$ is the annihilator of e where e is the image of the product of all roots of G via $[A^*_T]^W \to A^*_T(M^{ss}(T))^W \cong A^*(M/\!/T_{\mathbb{C}})^W$.*

This appears as a corollary in [12] which is equivalent to the main theorem of the paper.

We suspect that the above theorem may be proved directly by comparing the Gysin maps (for Chow rings) of the Morse stratifications by the $T_{\mathbb{C}}$ action and by the G action. This may lead us to a generalization of the theorem to a fairly general situation. Notice that Ellingsrud and Stromme used Poincaré duality in an essential way but we cannot expect it in general since homological equivalence is different from rational equivalence. So, it seems improbable to use their arguments to generalize the theorem. However the Morse theory in the previous chapter is valid in a very general situation and this might be used to extend the result.

Chapter 5

Intersection cohomology of singular quotients

In this chapter, we show that for a weakly balanced quotient the intersection cohomology $IH^*(X/\!\!/G)$ embeds into the equivariant cohomology $H^*_G(X^{ss})$.[1]

5.1 The Kirwan map

Let $X \subset \mathbb{P}^n$ be a connected nonsingular quasi-projective variety acted on linearly by a connected reductive group G via a homomorphism $G \to GL(n+1)$. We may assume that the maximal compact subgroup K of G acts unitarily possibly after conjugation. Furthermore, after resolving singularities if necessary, we may assume that X has a nonsingular closure in $\mathbb{P}^n$. Let $\mu : \mathbb{P}^n \to u(n+1)^* \to k^*$ be the moment map for the action of K. Throughout this chapter, we will assume the following as in [24].

Definition 5.1 *We say X is* flow-closed *if every path of steepest descent under $f = |\mu|^2$ is contained in some compact subset of X^{ss}.*

For example the assumption is automatically satisfied when X is projective or when X is a complex vector space with a linear action (See Example 2.3 of [46]).

Under this assumption, most of the statements in §§3, 4, 5 of [24] are still valid. Let X^{ss} denote $X \cap (\mathbb{P}^n)^{ss}$. Then it retracts onto $\mu^{-1}(0) \cap X$ by the gradient flow of $-f$ and the Kähler quotient $X/\!\!/G$ is homeomorphic to the symplectic quotient $X \cap \mu^{-1}(0)/K$, which is

[1]See [7, 30] for basic facts about intersection cohomology. For the decomposition theorem, see [6, 15].

homeomorphic to the GIT quotient when X is projective. Moreover, the stabilizer groups of points in X^{ss} whose orbits are closed are all reductive. Recall the following definitions.

Definition 5.2 *[25]*

- *Let $\mathcal{R}(X)$ be a set of representatives of the conjugacy classes of identity components of all reductive subgroups of G which appear as stabilizers of points $x \in X^{ss}$ such that Gx is closed in X^{ss} and that $R \cap K$ is a maximal compact subgroup of R where K is a fixed maximal compact subgroup for G.*
- *Let Z_R^{ss} denote the set of those $x \in X^{ss}$ fixed by $R \in \mathcal{R}(X)$.*
- *Let $r(X) = max\{\dim_{\mathbb{C}} R |\, R \in \mathcal{R}(X)\}$.*

The definition of the Kirwan map is by induction on $r(X)$. Let $\pi : Y \to X^{ss}$ be the blow-up of X^{ss} along $\cup_{\dim_{\mathbb{C}} R = r(X)} GZ_R^{ss}$ in the partial desingularization process. It is easy to check from the local normal form theorem that Y is again flow-closed. Since $r(Y) < r(X)$ from [25], suppose that we have a surjection $\kappa_Y : H_G^*(Y^{ss}) \to IH^*(Y/\!\!/G)$ and hence a surjection $H_G^*(Y) \to IH^*(Y/\!\!/G)$ by composing it with the restriction map.

From [16] we have

$$H_G^*(Y) \cong H_G^*(X^{ss}) \oplus H_G^*(E)/H_G^*(\mathcal{N}) \tag{5.1}$$

The projection $Y \to X^{ss}$ induces an embedding $H_G^*(X^{ss}) \hookrightarrow H_G^*(Y)$. On the other hand, by the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [6], there is a decomposition

$$IH^*(Y/\!\!/G) \cong IH^*(X/\!\!/G) \oplus IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) \tag{5.2}$$

which comes from a decomposition on the sheaf complex level. Hence we get a surjection $p : IH^*(Y/\!\!/G) \to IH^*(X/\!\!/G)$. The decomposition can be made canonical by considering the Lefschetz decomposition and the primitive part.

By composing the above maps, we get the Kirwan map for X as follows:

$$\begin{array}{ccc} H^*_G(Y) & \longleftarrow & H^*_G(X^{ss}) \\ \downarrow & & \vdots \\ IH^*(Y/\!\!/G) & \longrightarrow & IH^*(X/\!\!/G) \end{array} \tag{5.3}$$

Our goal in the next section is to define a useful splitting of the map.

Remark 5.3 The Kirwan maps behave functorially with respect to restrictions to G-invariant open subsets if flow-closed, once the decompositions of intersection homology groups are chosen compatibly.

5.2 Weakly balanced action and a splitting of the Kirwan map

Let X be a connected nonsingular quasi-projective variety on which a connected complex reductive group G acts linearly. Then there is a quotient $\phi : X^{ss} \to X/\!\!/G$ of the open dense stratum of the Morse stratification for the norm square of the moment map. Assume that the set of stable points $X^s = X \cap (\mathbb{P}^n)^s$ with respect to this action is nonempty. Then we have the Kirwan map $\kappa_X : H^*_G(X^{ss}) \to IH^*(X/\!\!/G)$ as explained above. In this section, we first define *weakly balanced* action and then construct a splitting of κ_X for such an action.

Definition 5.4 *Suppose a nontrivial connected reductive group R acts on a vector space $\mathbb{C}^k$ linearly. Let $\mathcal{B}$ be the set of the closest points from the origin to the convex hulls of some weights of the action. For each $\beta \in \mathcal{B}$, denote by $n(\beta)$ the number of weights α such that $\alpha \cdot \beta < \beta \cdot \beta$. The action is said to be* linearly balanced *if $2n(\beta) \geq k$ for every $\beta \in \mathcal{B}$. The action is said to be* weakly linearly balanced *if $2n(\beta) - 2\dim_{\mathbb{C}} R/B \cdot \mathrm{Stab}\,\beta > k - \dim_{\mathbb{C}} R$ for every $\beta \in \mathcal{B}$ where B is a Borel subgroup of R.*

Definition 5.5 *Let G be a connected reductive group acting linearly on a connected nonsingular quasi-projective variety X. The G-action is said to be* balanced *(resp.* weakly balanced*) if for each $R \in \mathcal{R}(X)$ the linear action of R on the normal space $\mathcal{N}_x$ at a generic*

$x \in Z_R$ to GZ_R^{ss} is linearly balanced (resp. weakly linearly balanced) and so is the action of each $R \cap N^{gR'g^{-1}}/gR'g^{-1}$ on the linear subspace $Z^{ss}_{gR'g^{-1}} \cap \mathcal{N}_x$, for $R' \in \mathcal{R}(X)$, $gR'g^{-1} \subset R$.

Remark 5.6 Because $2\dim_{\mathbb{C}} B > \dim_{\mathbb{C}} R$, a balanced action is weakly balanced.

Example 5.7 Let $G = \mathbb{C}^*$ act on $\mathbb{P}^n$ via a representation $\mathbb{C}^* \to GL(n+1)$. Let n_+, n_0, n_- denote the number of positive, zero, and negative weights, respectively. Then the action is (weakly) balanced if and only if $n_+ = n_-$. □

For any connected reductive subgroup R of G, let Z_R^{ss} be the set of points in X^{ss} fixed by R. Consider the "blow-up" map

$$G \times_{N^R} Z_R^{ss} \to GZ_R^{ss} \tag{5.4}$$

and the corresponding map on the cohomology ([27][Lemma 1.21])

$$H^*_G(GZ_R^{ss}) \to H^*_G(G \times_{N^R} Z_R^{ss}) = H^*_{N^R}(Z_R^{ss}) \cong [H^*_{N_0^R/R}(Z_R^{ss}) \otimes H^*_R]^{\pi_0 N^R} \tag{5.5}$$

where N^R is the normalizer of R in G and N_0^R is the identity component of N^R. For any $\zeta \in H^*_G(X^{ss})$ let $\zeta|_{G\times_{N^R} Z_R^{ss}}$ denote the image of ζ by the composition of the above map and the restriction map $H^*_G(X^{ss}) \to H^*_G(GZ_R^{ss})$. Now, we can define our splitting.

Definition 5.8 *Let X be a nonsingular quasi-projective variety acted on linearly by G. Let*

$$V_X = \{\zeta \in H^*_G(X^{ss}) |\ \zeta|_{G\times_{N^R} Z_R^{ss}} \in [\oplus_{i<n_R} H^*_{N_0^R/R}(Z_R^{ss}) \otimes H^i_R]^{\pi_0 N^R} \text{ for each } R \in \mathcal{R}\} \tag{5.6}$$

where $\mathcal{R} = \mathcal{R}(X)$ is as in the previous section and $n_R = \dim_{\mathbb{C}} \mathcal{N}_x - \dim_{\mathbb{C}} R$, $\mathcal{N}_x$ is the normal space to GZ_R^{ss} at any $x \in Z_R^s$.

The definition of V_X is independent of the choices of R's in the conjugacy classes and the tensor product expressions in (5.5): The former is easy to check by translating by g if R is replaced by gRg^{-1}. The latter can be immediately seen by considering the gradation of the degenerating spectral sequence for the cohomology of the fibration

$$\begin{array}{ccc} BR & \longrightarrow & (EN_0^R \times EN_0^R/R) \times_{N_0^R} Z_R^{ss} \\ & & \downarrow \\ & & EN_0^R/R \times_{N_0^R/R} Z_R^{ss} \end{array} \tag{5.7}$$

In view of (5.5), V_X can be thought of as a subset of $H_G^*(X^{ss})$, "truncated locally".

Theorem 5.9 *Suppose the action of G is weakly balanced and flow-closed. Then the restriction of the Kirwan map $\kappa_X : V_X \to IH^*(X/\!\!/G)$ is an isomorphism. Moreover, the splitting is compatible with the blow-ups of the inductive procedure for the definition of the map.*

We call this *the splitting theorem.*

Remark 5.10 Note that $H_G^*(X^{ss}) = H_K^*(X^{ss})$ and $H_G^*(GZ_R^{ss}) = H_K^*(KZ_R^{ss})$ because GZ_R^{ss} retracts onto $KZ_R^{ss} \cap \mu^{-1}(0)$ and so does KZ_R^{ss}. Furthermore, the map (5.5) in the definition of V_X is equivalent to

$$H_K^*(KZ_R^{ss}) \to H_K^*(K \times_{N^H} Z_R^{ss}) = H_{N^H}^*(Z_R^{ss}) \cong [H_{N_0^H/H}^*(Z_R^{ss}) \otimes H_H^*]^{\pi_0 N^H} \tag{5.8}$$

where H is the real form of R and N^H is the real form of N^R. Hence, we get an equivalent definition as follows:

$$V_X = \{\zeta \in H_K^*(X^{ss}) |\ \zeta|_{K\times_{N^H} Z_R^{ss}} \in [\oplus_{i<n_R} H_{N_0^H/H}^*(Z_R^{ss}) \otimes H_H^i]^{\pi_0 N^H} \text{ for each } R \in \mathcal{R}\}. \tag{5.9}$$

The splitting theorem is valid even if X is not a variety but just a compactifiable flow-closed symplectic manifold, with necessary modifications. See [22].

Remark 5.11 The choice of our splitting V_X is functorial with respect to restrictions to G-invariant open subsets: For every G-invariant open subset U of X^{ss}, the restriction $H_G^*(X^{ss}) \to H_G^*(U)$ induces $V_X \to V_U$. Moreover, V_X is preserved by any G-equivariant isomorphism of X.

Remark 5.12 The splitting V_X can be characterized by requiring functorial behaviours with respect to

1. restrictions to invariant open subsets and

2. blow-ups in the partial desingularization process.

Notice that for any R, after blow-ups, $G\hat{Z}_R^{ss} = G\times_{N^R}\hat{Z}_R^{ss}$ and $H_G^*(G\hat{Z}_R^{ss}) \cong [H^*(\hat{Z}_R/\!\!/N_0^R)\otimes H_R^*]^{\pi_o N^R}$. Because $IH^*(\hat{\mathcal{N}}/\!\!/G) \cong [H^*(\hat{Z}_R/\!\!/N_0^R)\otimes IH^*(\mathcal{N}_x/\!\!/R)]^{\pi_o N^R}$ and $IH^i(\mathcal{N}_x/\!\!/R) = 0$ for $i \geq n_R$, the definition makes sense by noting that $G\hat{Z}_R^{ss} = G\times_{N^R}\hat{Z}_R^{ss} \to GZ_R^{ss}$ factors through $G\times_{N^R} Z_R^{ss}$.

5.3 Proof of the splitting theorem

This section is devoted to a proof of the theorem. Recall that

$$r = r(X) = max\{\dim_{\mathbb{C}} R \,|\, R \in \mathcal{R}(X)\}. \tag{5.10}$$

When $r = 0$, we have nothing to prove since $V_X = H_G^*(X^{ss}) \cong IH^*(X/\!\!/G)$. We use an induction on r. Suppose the theorem is true for all W with $r(W) \leq r-1$. Let $r > 0$ and Y be the blowup of X^{ss} along $\cup_{\dim_{\mathbb{C}} R=r} GZ_R^{ss}$. Then $\mathcal{R}(Y) = \{R \in \mathcal{R}(X) \,|\, \dim_{\mathbb{C}} R \leq r-1\}$ and thus $r(Y) \leq r-1$.[2] Obviously, the G-action on Y^{ss} is also weakly balanced as the actions of the reductive subgroups on the normal spaces remain the same. Hence, by the induction hypothesis, the restriction of the Kirwan map $\kappa_Y : V_Y \to IH^*(Y/\!\!/G)$ is an isomorphism.

Similarly, if we let E be the exceptional divisor of Y, then $r(E) \leq r(Y) \leq r-1$ and the G action is weakly balanced. Thus, by the induction hypothesis, the composition $\kappa_E : V_E \hookrightarrow H_G^*(E^{ss}) \to IH^*(E/\!\!/G)$ is an isomorphism.

For simplicity, we assume from now on that there's only one R such that $\dim_{\mathbb{C}} R = r$ and E is a projective bundle over GZ_R^{ss}. (The general case is no more difficult except for repetition. See [25], Cor.8.3.) We fix this R once and for all till the end of this section.

[2] See [25]. To be precise, one should compactify X and then apply Kirwan's results.

According to [25] and [28], $GZ_R^{ss} = G \times_{N^R} Z_R^{ss}$ and

$$H_G^*(E) \cong [H^*(Z_R /\!\!/ N_0^R) \otimes H_R^*(\mathbb{P}\mathcal{N}_x)]^{\pi_0 N^R} \tag{5.11}$$

$$H_G^*(E^{ss}) \cong [H^*(Z_R /\!\!/ N_0^R) \otimes H_R^*(\mathbb{P}\mathcal{N}_x^{ss})]^{\pi_0 N^R} \tag{5.12}$$

$$IH^*(E /\!\!/ G) \cong [H^*(Z_R /\!\!/ N_0^R) \otimes IH^*(\mathbb{P}\mathcal{N}_x /\!\!/ R)]^{\pi_0 N^R} \tag{5.13}$$

$$IH^*(\mathcal{N} /\!\!/ G) \cong [H^*(Z_R /\!\!/ N_0^R) \otimes IH^*(\mathcal{N}_x /\!\!/ R)]^{\pi_0 N^R} \tag{5.14}$$

where $\mathcal{N}$ is the normal bundle of GZ_R^{ss}, $x \in Z_R^{ss}$.

Lemma 5.13 *The restriction to V_X of $H_G^*(X^{ss}) \hookrightarrow H_G^*(Y) \to H_G^*(Y^{ss})$ factors through $V_X \to V_Y$ and is injective.*

Proof Let ζ be a nonzero element in V_X. Then

$$\zeta|_{G\times_{N^{R'}} Z_{R'}^{ss}} \in [\oplus_{i<n_{R'}} H^*_{N_0^{R'}/R'}(Z_{R'}^{ss}) \otimes H_{R'}^i]^{\pi_0 N^{R'}} \text{ for each } R' \in \mathcal{R}(X).$$

Its image in $H_G^*(Y^{ss})$ satisfies

$$\begin{gathered} \zeta|_{G\times_{N^{R'}} \hat{Z}_{R'}^{ss}} \in [\oplus_{i<n_{R'}} H^*_{N_0^{R'}/R'}(\hat{Z}_{R'}^{ss}) \otimes H_{R'}^i]^{\pi_0 N^{R'}} \\ \text{for each } R' \in \mathcal{R}(Y) = \{R' \in \mathcal{R}(X) | \dim_{\mathbb{C}} R' < r(X)\} \end{gathered} \tag{5.15}$$

where $\hat{Z}_{R'}^{ss}$ is the proper transform of $Z_{R'}^{ss}$ in Y^{ss}. It follows from the commutative diagram

$$\begin{array}{ccccc} Y^{ss} & \longleftarrow & G\hat{Z}_{R'}^{ss} & \longleftarrow & G \times_{N^{R'}} \hat{Z}_{R'}^{ss} \\ \downarrow & & \downarrow & & \downarrow \\ X^{ss} & \longleftarrow & GZ_{R'}^{ss} & \longleftarrow & G \times_{N^{R'}} Z_{R'}^{ss}. \end{array}$$

Therefore, ζ is mapped to an element in V_Y.

Because each unstable stratum in Y retracts onto its intersection with E ([25, 27]), we have an isomorphism coming from restriction:

$$ker(H_G^*(Y) \to H_G^*(Y^{ss})) \cong ker(H_G^*(E) \to H_G^*(E^{ss}))$$

Therefore, if $\zeta|_{GZ_R^{ss}} = 0$ i.e. $\zeta|_E = 0$, then $\zeta|_{Y^{ss}} \neq 0$. So, we consider the case when $\zeta|_{GZ_R^{ss}} \neq 0$.

By the definition of V_X, $\zeta|_{GZ_R^{ss}}$ is in $[\oplus_{i<n_R} H^*(Z_R/\!\!/N_0^R) \otimes H_R^i]^{\pi_0 N^R}$ and $H_G^*(E) = [H^*(Z_R/\!\!/N_0^R) \otimes H_R^*(\mathbb{P}\mathcal{N}_x)]^{\pi_0 N^R}$. From [27, 24], the real codimension of each unstable stratum S_β of $\mathbb{P}\mathcal{N}_x$ with respect to the R action is $2n(\beta) - 2\dim_{\mathbb{C}} R/B \cdot \mathrm{Stab}\,\beta > \dim_{\mathbb{C}} \mathcal{N}_x - \dim_{\mathbb{C}} R = n_R$ where $n(\beta)$ is the number of weights α such that $\alpha \cdot \beta < |\beta|^2$ and B is a Borel subgroup of R because the G action is weakly balanced. Therefore, $\zeta|_{E^{ss}} \neq 0$ and thus $\zeta|_{Y^{ss}} \neq 0$. □

As mentioned above, the normal bundle $\mathcal{N}$ has an induced action of G and clearly it is weakly balanced.

Lemma 5.14 *The linear action of R on $\mathcal{N}_x$ is weakly balanced and the theorem is true for $\mathcal{N}_x$ with respect to the action of R.*

Proof For this proof only, let $Z_{R'}$ denote the points in $\mathcal{N}$ (not X) fixed by R'. Let R' be a connected reductive subgroup of R. By Lemma 5.19 below,

$$GZ_{R'}^{ss} \cap \mathcal{N}_x = A^{-1} Z_{R'}^{ss} \cap \mathcal{N}_x = \cup_{1 \le j \le l} RZ_{g_j^{-1} R' g_j}^{ss} \cap \mathcal{N}_x$$

where $A = \{g \in G | g^{-1} R' g \subset R\}$. Let y be a point in $\mathcal{N}_x$ such that the identity component of $\mathrm{Stab}\, y$ is R'. By the above equalities, the normal space to $GZ_{R'}^{ss}$ at y in X^{ss} is the normal space $\mathcal{N}_y$ to $RZ_{R'}^{ss} \cap \mathcal{N}_x$ in $\mathcal{N}_x$ because $GZ_R^{ss} \subset GZ_{R'}^{ss}$. Therefore, the R-action on $\mathcal{N}_x$ is weakly balanced as the G-action on $\mathcal{N}$ is so. Moreover,

$$\dim_{\mathbb{C}}(RZ_{R'}^{ss} \cap \mathcal{N}_x) + \dim_{\mathbb{C}} \mathcal{N}_y = \dim_{\mathbb{C}} \mathcal{N}_x.$$

As $\mathcal{N}_x/\!\!/R$ is a cone, $IH^i(\mathcal{N}_x/\!\!/R) = 0$ for $i \geq n_R$. If $i < n_R$, then $IH^i(\mathcal{N}_x/\!\!/R) = IH^i(\mathcal{N}_x/\!\!/R - x) = IH^i(\mathcal{N}_x - \phi_x^{-1}(x)/\!\!/R)$ where $\phi_x : \mathcal{N}_x \to \mathcal{N}_x/\!\!/R$ is the quotient map and the following diagram commutes:

$$\begin{array}{ccc} H_R^i(\mathcal{N}_x) & \longrightarrow & H_R^i(\mathcal{N}_x - \phi_x^{-1}(x)) \\ \downarrow & & \downarrow \\ IH^i(\mathcal{N}_x/\!\!/R) & \longrightarrow & IH^i(\mathcal{N}_x - \phi_x^{-1}(x)/\!\!/R) \end{array} \tag{5.16}$$

As $\phi_x^{-1}(x)$ is union of the complex cones over the unstable strata of $\mathbb{P}\mathcal{N}_x$ and the real codimension of each unstable stratum is greater than n_R as seen in the proof of the previous lemma, the real codimension of $\phi_x^{-1}(x)$ is greater than n_R. Hence, the top horizontal map is an isomorphism and the vertical maps are the Kirwan maps.

Now, because $r(\mathcal{N}_x - \phi_x^{-1}(x)) \le r-1$, the Kirwan map restricts to an isomorphism $\kappa_{\mathcal{N}_x-\phi_x^{-1}(x)} : V_{\mathcal{N}_x-\phi_x^{-1}(x)} \to IH^*(\mathcal{N}_x - \phi_x^{-1}(x)/\!\!/R)$. From the above commutative diagram, we get an isomorphism $\kappa_{\mathcal{N}_x} : V_{\mathcal{N}_x} \to IH^*(\mathcal{N}_x/\!\!/R)$ because $V^i_{\mathcal{N}_x}$ is $V^i_{\mathcal{N}_x-\phi_x^{-1}(x)}$ for $i < n_R$ and 0 otherwise. Here, V^i means the degree i part of V. To see the last claim, we note once again that for $i < n_R$, $H^i_R(\mathcal{N}_x) \to H^i_R(\mathcal{N}_x - \phi_x^{-1}(x))$ is an isomorphism and the same is true for $H^j_{R\cap N_0^{R'}/R'}(Z^{ss}_{R'} \cap \mathcal{N}_x) \otimes H^k_{R'} \to H^j_{R\cap N_0^{R'}/R'}(Z^{ss}_{R'} \cap \mathcal{N}_x - \phi_x^{-1}(x)) \otimes H^k_{R'}$ if $k \ge n_{R'}$, $j < n_R - k \le n_R - n_{R'}$, because the real codimension of $\phi_x^{-1}(x) \cap Z^{ss}_{R'} \cap \mathcal{N}_x$ in $Z^{ss}_{R'} \cap \mathcal{N}_x$ is greater than $n_R - n_{R'}$. For by the weakly balanced assumption, it is greater than $\dim_{\mathbb{C}}(Z^{ss}_{R'} \cap \mathcal{N}_x) - \dim_{\mathbb{C}}(R \cap N^{R'}/R') \ge \dim_{\mathbb{C}}(RZ^{ss}_{R'} \cap \mathcal{N}_x) - \dim_{\mathbb{C}} R + \dim_{\mathbb{C}} R' = \dim_{\mathbb{C}} \mathcal{N}_x - \dim_{\mathbb{C}} \mathcal{N}_y - \dim_{\mathbb{C}} R + \dim_{\mathbb{C}} R' = n_R - n_{R'}$.

If we let $\hat{\mathcal{N}}_x$ be the blow-up of $\mathcal{N}_x$ at x, then $\hat{\mathcal{N}}_x^{ss} \supset \mathcal{N}_x - \phi_x^{-1}(x)$ and we have a commutative diagram by restriction

$$\begin{array}{ccc} V^i_{\mathbb{P}\mathcal{N}_x} = V^i_{\hat{\mathcal{N}}_x} & \longrightarrow & V^i_{\mathcal{N}_x-\phi_x^{-1}(x)} = V^i_{\mathcal{N}_x} \\ \downarrow & & \downarrow \\ IH^i(\mathbb{P}\mathcal{N}_x/\!\!/R) & \longrightarrow & IH^i(\mathcal{N}_x - \phi_x^{-1}(x)/\!\!/R) = IH^i(\mathcal{N}_x/\!\!/R) \end{array} \tag{5.17}$$

for $i < n_R$ where the vertical maps are the Kirwan maps and the horizontal maps are the restrictions to the complement of $\phi_x^{-1}(x)$. Moreover, by Lemma 5.13, $H^*_R(\mathcal{N}_x) \hookrightarrow H^*_R(\hat{\mathcal{N}}_x) \to H^*_R(\hat{\mathcal{N}}_x^{ss})$ induces an embedding $V_{\mathcal{N}_x} \to V_{\hat{\mathcal{N}}_x}$, which is a splitting of the top row of the above diagram. So, the choice of $V_{\mathcal{N}_x}$ is compatible with the (first) blowup of the partial desingularization process. □

As one can expect from (5.11), (5.12), (5.13), (5.14), we have

$$V_E \cong [H^*(Z_R/\!\!/N_0^R) \otimes V_{\mathbb{P}\mathcal{N}_x}]^{\pi_0 N^R} \tag{5.18}$$

$$V_{\mathcal{N}} \cong [H^*(Z_R/\!\!/N_0^R) \otimes V_{\mathcal{N}_x}]^{\pi_0 N^R}. \tag{5.19}$$

We postpone the proof of the above statements till we finish the proof of the splitting theorem.

By (5.17), (5.18) and (5.19), we get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_E/V_{\mathcal{N}} & \longrightarrow & V_E & \longrightarrow & V_{\mathcal{N}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) & \longrightarrow & IH^*(E/\!\!/G) & \longrightarrow & IH^*(\mathcal{N}/\!\!/G) & \longrightarrow & 0 \end{array} \tag{5.20}$$

As the last two vertical maps are isomorphisms, the first vertical is also an isomorphism $\kappa_{E/\mathcal{N}} : V_E/V_{\mathcal{N}} \to IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G)$. Notice that as the maps in the diagram (5.17) were defined by restriction, the bottom surjection above is the projection from the decomposition (5.2) as it comes from a decomposition on the sheaf complex level.

By Lemma 5.13, we have a splitting $V_{\mathcal{N}} \to V_E$ of the top right surjection in the above diagram. We identify $IH^*(\mathcal{N}/\!\!/G)$ with the image of $V_{\mathcal{N}}$ in $IH^*(E/\!\!/G)$. Then we can reverse the arrows of the above diagram:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & V_E/V_{\mathcal{N}} & \longleftarrow & V_E & \longleftarrow & V_{\mathcal{N}} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) & \longleftarrow & IH^*(E/\!\!/G) & \longleftarrow & IH^*(\mathcal{N}/\!\!/G) & \longleftarrow & 0 \end{array} \tag{5.21}$$

Lemma 5.15 *The restriction of the Kirwan map $\kappa_X : V_X \to IH^*(X/\!\!/G)$ is injective.*

Proof We first claim that V_X as a subset of V_Y is mapped into $V_{\mathcal{N}}$ when restricted to E^{ss}. It follows from the description of V_X, $V_{\mathcal{N}}$ in terms of K-equivariant cohomology in Remark 5.10: By identifying the normal bundle $\mathcal{N}$ with a K-invariant tubular neighborhood of GZ_R^{ss}, the embedding $\mathcal{N} \hookrightarrow X^{ss}$ induces $H_K^*(X^{ss}) \to H_K^*(\mathcal{N})$ and thus it is easy to check that V_X is mapped into $V_{\mathcal{N}} \subset H_K^*(\mathcal{N})$, from the following diagram.

$$\begin{array}{ccccccc} V_X & \longrightarrow & H_K^*(X^{ss}) & \longrightarrow & H_K^*(Y) & \longrightarrow & H_K^*(Y^{ss}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V_{\mathcal{N}} & \longrightarrow & H_K^*(\mathcal{N}) & \longrightarrow & H_K^*(E) & \longrightarrow & H_K^*(E^{ss}). \end{array} \tag{5.22}$$

Therefore, the composition $V_X \hookrightarrow V_Y \to V_E \to V_E/V_{\mathcal{N}}$ is zero. On the other hand, we have the following commutative diagram by restriction and (5.21):

$$\begin{array}{ccccc} V_Y & \longrightarrow & V_E & \longrightarrow & V_E/V_{\mathcal{N}} \\ \downarrow & & \downarrow & & \downarrow \\ IH^*(Y/\!\!/G) & \longrightarrow & IH^*(E/\!\!/G) & \longrightarrow & IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) \end{array} \tag{5.23}$$

We claim that the composition of the bottom arrows is surjective. As the map $IH^*(E/\!\!/G) \to IH^*(\mathcal{N}/\!\!/G)$ is from the restriction (5.17), it fits in the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & IH^*(Y/\!\!/G, E/\!\!/G) & \longrightarrow & IH^*(X/\!\!/G, \mathcal{N}/\!\!/G) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IH^*(Y/\!\!/G)/IH^*(X/\!\!/G) & \longrightarrow & IH^*(Y/\!\!/G) & \longrightarrow & IH^*(X/\!\!/G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) & \longrightarrow & IH^*(E/\!\!/G) & \longrightarrow & IH^*(\mathcal{N}/\!\!/G) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & IH^{*+1}(Y/\!\!/G, E/\!\!/G) & \longrightarrow & IH^{*+1}(X/\!\!/G, \mathcal{N}/\!\!/G) & & \end{array} \tag{5.24}$$

The second row is from the decomposition (5.2). As the top and bottom horizontal maps are isomorphisms by excision, by a diagram chase the left vertical arrow is an isomorphism. So, $IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G)$ lies in the image of $IH^*(Y/\!\!/G) \to IH^*(E/\!\!/G)$. Therefore, $IH^*(Y/\!\!/G) \to IH^*(E/\!\!/G) \to IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G)$ is surjective.

Moreover, the diagram (5.24) gives an embedding $IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) \hookrightarrow IH^*(Y/\!\!/G)$ and the kernel of the composite of the bottom arrows of (5.23) in this proof is complementary to the embedding. Therefore, the kernel is mapped isomorphically onto $IH^*(X/\!\!/G)$ by the projection $p : IH^*(Y/\!\!/G) \to IH^*(X/\!\!/G)$ from the decomposition (5.2). Hence, $IH^*(X/\!\!/G)$ can be identified with the kernel of the composition of the maps in the bottom row of (5.23). Therefore, the restriction of κ_Y to V_X factors through $\kappa_X : V_X \to IH^*(X/\!\!/G)$. Because κ_Y is injective, κ_X is also injective. □

Lemma 5.16 *The Kirwan map $\kappa_X : V_X \to IH^*(X/\!\!/G)$ is surjective.*

Proof In view of (5.23), it suffices to show that V_X is the full kernel of the composition $V_Y \to V_E \to V_E/V_{\mathcal{N}}$, which is already surjective by the proof of the previous lemma.

Suppose $\zeta \in V_Y \subset H^*_G(Y^{ss})$ such that $\zeta|_{E^{ss}} \in V_{\mathcal{N}}$. We first claim that there is $\xi \in H^*_G(X^{ss})$ such that ξ is pulled back to ζ via $Y^{ss} \to Y \to X^{ss}$. Let $\xi' \in H^*_G(\mathcal{N})$ be the element corresponding to $\zeta|_{E^{ss}}$. As $H^*_G(Y) \to H^*_G(Y^{ss})$ is surjective, we can choose an element $\zeta' \in H^*_G(Y)$ that restricts to ζ. Then $\xi'|_{E^{ss}} = \zeta'|_{E^{ss}}$ and hence $\xi' - \zeta'|_E = \zeta''|_E$ for some $\zeta'' \in ker(H^*_G(Y) \to H^*_G(Y^{ss}))$ since $ker(H^*_G(Y) \to H^*_G(Y^{ss})) \cong ker(H^*_G(E) \to H^*_G(E^{ss}))$. Let $\zeta_1 = \zeta' + \zeta'' \in H^*_G(Y)$. Then $\zeta_1|_{Y^{ss}} = \zeta$ and $\zeta_1|_E = \xi' \in H^*_G(\mathcal{N}) \subset H^*_G(E)$. Thus ζ_1 is the pullback of a class $\xi \in H^*_G(X^{ss})$ via $Y \to X^{ss}$ and the claim is proved.

Now, consider the $N_0^{R'}/R'$ action on $Z^{ss}_{R'}$ and its proper transform $\hat{Z}^{ss}_{R'}$ which is the intersection of Y^{ss} with the blow-up $\hat{Z}_{R'}$ of $Z^{ss}_{R'}$ along $GZ^{ss}_R \cap Z^{ss}_{R'}$. As each unstable stratum retracts onto its intersection with E,

$$ker(H^*_{N_0^{R'}/R'}(\hat{Z}_{R'}) \to H^*_{N_0^{R'}/R'}(\hat{Z}^{ss}_{R'})) \cong ker(H^*_{N_0^{R'}/R'}(\hat{Z}_{R'} \cap E) \to H^*_{N_0^{R'}/R'}(\hat{Z}_{R'} \cap E^{ss})).$$

Hence, a nonzero class $\tau \in H^*_{N_0^{R'}/R'}(\hat{Z}_{R'})$ such that $\tau|_{\hat{Z}_{R'} \cap E} = 0$ does not vanish when restricted to $\hat{Z}^{ss}_{R'}$.

Suppose $\xi|_{G\times_{N^{R'}} Z^{ss}_{R'}} \notin \oplus_{i<n_{R'}} H^*_{N_0^{R'}/R'}(Z^{ss}_{R'}) \otimes H^i_{R'}$. As

$$\xi|_{K\times_{N^{H'}} Z^{ss}_{R'} \cap \mathcal{N}} \in \oplus_{i<n_{R'}} H^*_{N_0^{H'}/H'}(Z^{ss}_{R'} \cap \mathcal{N}) \otimes H^i_{R'}$$

by assumption, we can deduce from the previous paragraph that

$$\zeta|_{G\times_{N^{R'}} \hat{Z}^{ss}_{R'}} \notin \oplus_{i<n_{R'}} H^*_{N_0^{R'}/R'}(\hat{Z}^{ss}_{R'}) \otimes H^i_{R'}.$$

This is a contradiction. Therefore, $\xi \in V_X$ and the proof is complete. $\square$

Proof [The splitting theorem] The statement now follows from Lemmas 5.15, 5.16. Though we used an embedding $IH^*(X/\!\!/G) \hookrightarrow IH^*(Y/\!\!/G)$ different from Kirwan's choice, we use the same projection $p : IH^*(Y/\!\!/G) \to IH^*(X/\!\!/G)$. Therefore, V_X gives a splitting of the Kirwan map:

$$\begin{array}{ccc} V_X & \longrightarrow & V_Y \\ \vdots & & \downarrow \\ IH^*(X/\!\!/G) & \longleftarrow & IH^*(Y/\!\!/G) \end{array}$$

□

Moreover, we have the following commutative diagram of exact sequences with respect to the identification of $IH^*(X/\!\!/G)$ in the proof of Lemma 5.15:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_X & \longrightarrow & V_Y & \longrightarrow & V_E/V_{\mathcal{N}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & IH^*(X/\!\!/G) & \longrightarrow & IH^*(Y/\!\!/G) & \longrightarrow & IH^*(E/\!\!/G)/IH^*(\mathcal{N}/\!\!/G) & \longrightarrow & 0 \end{array} \tag{5.25}$$

The vertical maps are all isomorphisms, induced from the Kirwan maps.

Now we complete the proof by showing (5.18) and (5.19). We recall some facts from [25].

Lemma 5.17 *Let $S' \subset S$ be compact subgroups of a compact Lie group K. Let $N_K(S')$ denote the normalizer of S' in K. Then there exist $k_1, \cdots, k_m \in K$ such that*

$$\{k \in K |\, k^{-1}S'k \subset S\} = \cup_{1 \le i \le m} N_K(S')k_i S.$$

Proof This is essentially Lemma 8.10 in [25]. □

In particular, we take $S' = K \cap R'$ and $S = K \cap R$ where $R' \subset R$ are in $\mathcal{R}$ and K is a fixed maximal compact subgroup of G.

Lemma 5.18 *Let $R' \subset R$ be reductive subgroups of G in $\mathcal{R}$. Then for any $x \in Z_R^{ss}$, there is $n \in M := N^R \cap (\cap_{1 \le i \le m} N^{k_i^{-1}R'k_i})$ such that $nx \in \mu^{-1}(0) \cap Z_R$, where μ is the moment map associated to the induced action of K.*

Proof One can systematically replace $N^R \cap N^{R'}$ in Lemma 8.9 of [25] by M. □

Let $A = \{g \in G |\, g^{-1}R'g \subset R\}$ for $R' \subset R$. Then $N^{R'}$ acts on the left and N^R acts on the right.

Lemma 5.19 *There exist $g_1, \cdots, g_l \in G$ so that*

$$A = \cup_{1 \le j \le l} N_0^{R'} g_j R.$$

Proof Let $g \in A$ and $x \in Z_R^{ss}$. Then $gx \in Z_{gRg^{-1}}^{ss}$. By Lemma 5.18, there exists $n_1 \in M$ such that $n_1 x \in \mu^{-1}(0)$ and $n_2 \in N^{gRg^{-1}} \cap N^{R'}$ such that $n_2 gx \in \mu^{-1}(0)$. By [25](2.4), $n_2 g n_1^{-1} \in K$. Therefore, every double $(N^{R'}, M)$-coset meets $A \cap K$. Hence, by Lemma 5.17,

$$\begin{aligned} A &= N^{R'}(T \cap K)M = \cup N^{R'} N_K(S')k_i SM = \cup N^{R'} k_i MS \\ &= \cup N^{R'} k_i M k_i^{-1} k_i S \subset \cup N^{R'} k_i S \subset \cup N^{R'} k_i R \end{aligned} \tag{5.26}$$

and thus there are only finitely many double $(N^{R'}, R)$-cosets in A. Since $N^{R'}/N_0^{R'}$ is finite, the double $(N_0^{R'}, R)$-cosets in A are also finitely many. $\square$

Now, we can prove the isomorphisms (5.18) and (5.19). First, observe that $GZ_R^{ss} \cap Z_{R'}^{ss} = AZ_R^{ss}$ and that $GZ_{R'}^{ss} \cap \mathcal{N}_x = A^{-1} Z_{R'}^{ss} \cap \mathcal{N}_x$. Also, recall that $H_G^*(\mathcal{N}) \cong [H^*(Z_R /\!\!/ N_0^R) \otimes H_R^*(\mathcal{N}_x)]^{\pi_0 N^R}$. For (5.19), the conditions for the left hand side are given by the following maps:

$$\begin{aligned} H_G^*(\mathcal{N}) &\to H_G^*(GZ_{R'}^{ss} \cap \mathcal{N}) \to H_G^*(G \times_{N^{R'}} Z_{R'}^{ss} \cap \mathcal{N}) \\ &\cong [H^*_{N_0^{R'}/R'}(Z_{R'}^{ss} \cap \mathcal{N}) \otimes H^*_{R'}]^{\pi_0 N^{R'}} \subset H^*_{N_0^{R'}/R'}(Z_{R'}^{ss} \cap \mathcal{N}) \otimes H^*_{R'} \\ &= H^*_{N_0^{R'}/R'}(AZ_R^{ss}) \otimes H^*_{R'} \subset \oplus_j H^*_{N_0^{R'}/R'}(N_0^{R'} g_j Z_R^{ss}) \otimes H^*_{R'} \\ &= \oplus_j H^*_{N_0^{R'}/R'}(N_0^{R'} Z^{ss}_{g_j R g_j^{-1}}) \otimes H^*_{R'} \\ &\subset \oplus H^*_{N_0^{R'} \cap N_0^{g_j R g_j^{-1}} / N_0^{R'} \cap g_j R g_j^{-1}}(Z^{ss}_{g_j R g_j^{-1}}) \otimes H^*(B(N_0^{R'} \cap g_j R g_j^{-1}/R')) \otimes H^*_{R'} \\ &\cong \oplus H^*_{N_0^R \cap N_0^{g_j^{-1} R' g_j} / R \cap N_0^{g_j^{-1} R' g_j}}(Z_R^{ss}) \otimes H^*(B(N_0^{g_j^{-1} R' g_j} \cap R / g_j^{-1} R' g_j)) \otimes H^*_{g_j^{-1} R' g_j} \\ &\cong \oplus H^*_{N_0^R/R}(Z_R^{ss}) \otimes H^*(B(N_0^{g_j^{-1} R' g_j} \cap R / g_j^{-1} R' g_j)) \otimes H^*_{g_j^{-1} R' g_j} \\ &\cong \oplus H^*(Z_R /\!\!/ N_0^R) \otimes H^*(B(N_0^{g_j^{-1} R' g_j} \cap R / g_j^{-1} R' g_j)) \otimes H^*_{g_j^{-1} R' g_j} \end{aligned} \tag{5.27}$$

For the second to the last equality, we consider the following injection:

$$(N_0^R \cap N_0^{g_j^{-1} R' g_j})/(R \cap N_0^{g_j^{-1} R' g_j}) \hookrightarrow N_0^R/R.$$

This is of finite index because of Lemma 5.19 and the fact that $N_0^R \subset A$. [3] Hence, since N_0^R/R is connected, the injection is in fact an isomorphism and $N_0^R = (N_0^R \cap N_0^{g_j^{-1} R' g_j})R$. The above computation essentially shows that the spectral sequence for $EN_0^{R'} \times_{N_0^{R'}} (Z_{R'}^{ss} \cap \mathcal{N}) \to Z_R /\!\!/ N_0^R$ degenerates while the cohomology of the fiber is naturally isomorphic to $\oplus_j H^*_{R \cap N^{g_j^{-1} R' g_j}}(Z^{ss}_{g_j^{-1} R' g_j} \cap \mathcal{N}_x)$.

[3]Note also that we may assume $g_k \in N_0^R$ whenever $N_0^{g_j^{-1} R' g_j} g_k R \cap N_0^R$ is nonempty.

One the other hand, the conditions for the right hand side of (5.19) can be given by the following maps:

$$\begin{aligned}
[H^*(Z_R/\!/N_0^R) \otimes H_R^*(\mathcal{N}_x)]^{\pi_0 N^R} &\to [\oplus_j H^*(Z_R/\!/N_0^R) \otimes H_R^*(RZ^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x)]^{\pi_0 N^R} \\
&\subset \oplus_j H^*(Z_R/\!/N_0^R) \otimes H_R^*(RZ^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x) \\
&\to \oplus_j H^*(Z_R/\!/N_0^R) \otimes H_R^*(R \times_{R\cap N^{g_j^{-1}R'g_j}} Z^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x) \\
&\cong \oplus H^*(Z_R/\!/N_0^R) \otimes H^*_{N_0^{g_j^{-1}R'g_j} \cap R/g_j^{-1}R'g_j}(Z^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x) \otimes H^*_{g_j^{-1}R'g_j} \\
&\cong \oplus H^*(Z_R/\!/N_0^R) \otimes H^*(B(N_0^{g_j^{-1}R'g_j} \cap R/g_j^{-1}R'g_j)) \otimes H^*_{g_j^{-1}R'g_j}
\end{aligned} \tag{5.28}$$

Notice that we used the fact that $Z^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x$ is contractible.

The fact that (5.27) and (5.28) are compatible follows from the following diagram

$$\begin{array}{ccc}
\cup R \times_{R\cap N^{g_j^{-1}R'g_j}} Z^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x & \longrightarrow & G \times_{N^{R'}} Z^{ss}_{R'} \cap \mathcal{N} \\
\downarrow & & \downarrow \\
\cup R Z^{ss}_{g_j^{-1}R'g_j} \cap \mathcal{N}_x & \longrightarrow & GZ^{ss}_{R'} \cap \mathcal{N} \\
 & & \downarrow \\
 & & GZ^{ss}_{R}
\end{array}$$

where the top horizontal map is given by $(r, g_j^{-1}y) \to (rg_j^{-1}, y)$.

As observed in the proof of Lemma 5.14, $n_{R'}$ are same in both truncations since the normal spaces are same. Therefore, the conditions for the left hand side and the right hand side are equivalent and we get the isomorphism (5.19). A similar computation proves (5.18).

5.4 Intersection Betti numbers

As the first application of the splitting theorem, we compute intersetion Betti numbers. Let

$$P_t^G(W) = \sum_{i\geq 0} t^i \dim H_G^i(W)$$

$$IP_t(W) = \sum_{i\geq 0} t^i \dim IH^i(W)$$

be the Poincaré series.

Weakly balanced $\mathbb{C}^*$-action on projective space

Consider $\mathbb{C}^*$-action on $\mathbb{P}^n$ via a representation $\mathbb{C}^* \to GL(n+1)$. Let n_+, n_0, n_- be the number of positive, zero, negative weights. Suppose the action is weakly balanced, i.e. $n_+ = n_-$. In this case, we can easily compute the intersection Betti nubmers by the splitting theorem.

From the equivariant Morse theory [24],

$$\begin{aligned} P_t^{\mathbb{C}^*}((\mathbb{P}^n)^{ss}) &= \frac{1+t^2+\cdots+t^{2n}}{1-t^2} - \frac{t^{2n_0+2n_+}+\cdots+t^{2n}}{1-t^2} - \frac{t^{2n_0+2n_-}+\cdots+t^{2n}}{1-t^2} \\ &= \frac{1+t^2+\cdots+t^{2n_0+2n_- -2} - t^{2n_0+2n_+} - \cdots - t^{2n}}{1-t^2}. \end{aligned} \tag{5.29}$$

In this case, $\mathcal{R} = \{\mathbb{C}^*\}$ and $Z_R = \mathbb{P}^{n_0-1}$, $n_R = n_+ + n_- - 1 = 2n_- - 1$.

As $H^*_{\mathbb{C}^*}(\mathbb{P}^n) \to H^*_{\mathbb{C}^*}((\mathbb{P}^n)^{ss}) \to H^*_{\mathbb{C}^*}(\mathbb{P}^{n_0-1})$ is surjective, we have only to subtract out the Poincaré series of $\oplus_{i\geq 2n_-} H^*(\mathbb{P}^{n_0-1}) \otimes H^i_{\mathbb{C}^*}$, which is precisely

$$\frac{t^{2n_-}(1+t^2+\cdots+t^{2n_0-2})}{1-t^2}.$$

Therefore,

$$IP_t(\mathbb{P}^n/\!\!/\mathbb{C}^*) = \frac{1+t^2+\cdots+t^{2n_- -2} - t^{2n_0+2n_+} - t^{2n_0+2n_+ +2}\cdots - t^{2n}}{1-t^2}$$

which is a palindromic polynomial of degree $2n-2$.

Ordered $2n$-tuples of points of $\mathbb{P}^1$

Let us consider $G = SL(2)$ action on the set $X = (\mathbb{P}^1)^{2n}$ of ordered $2n$-tuples of points in $\mathbb{P}^1$ as Möbius transformations. Then the semistable $2n$-tuples are those containing no point of $\mathbb{P}^1$ strictly more than n times and the stable points are those containing no point at least n times. Let R be the maximal torus of G. Then

$$\mathcal{R}(X) = \{R\}$$

$$Z_R^{ss} = \{q_I | I \subset \{1,2,3,\cdots,2n\}, |I| = n\}$$

where $q_I \in X$, the j-th component of which is ∞ if $j \in I$, 0 otherwise. Therefore, the action is balanced.

The normalizer $N = N^R$ satisfies $N/R = \mathbb{Z}/2$ and $N_0 = R$. Hence,

$$H^*_{N^R}(Z^{ss}_R) = [\oplus_{|I|=n} H^*_R]^{\mathbb{Z}/2}.$$

The $\mathbb{Z}/2$ action interchanges ∞ and 0, i.e. q_I and q_{I^c}, and therefore, from now on, we only think of those I's that contain 1 so that we can forget the $\mathbb{Z}/2$ action.

$H^*_G(X) = H^*_G((\mathbb{P}^1)^{2n})$ has generators $\xi_1, \xi_2, \cdots, \xi_{2n}$ of degree 2 and ρ^2 of degree 4, subject to the relations $\xi_j^2 = \rho^2$ for $1 \le j \le 2n$. The I-th component of the restriction map

$$H^*_G(X) \to H^*_{N^R}(Z^{ss}_R) = \oplus_I H^*_R$$

maps ρ^2 to ρ^2 and ξ_j to ρ if $j \in I$, $-\rho$ otherwise. From [24, 28],

$$P^G_t(X^{ss}) = \frac{(1+t^2)^{2n}}{1-t^4} - \sum_{n<r\le 2n} \binom{2n}{r} \frac{t^{2(r-1)}}{1-t^2}.$$

Proposition 5.20

$$IP_t(X/\!\!/G) = P^G_t(X^{ss}) - \frac{1}{2}\binom{2n}{n}\frac{t^{2n-2}}{1-t^2}$$

Proof By the splitting theorem, we have only to count the dimension of

$$Im\{H^*_G(X^{ss}) \to \oplus_I H^*_R\} \;\cap\; \{\oplus_I \oplus_{i\ge n_R} H^i_R\}$$

where n_R is in this case $2n-3$. By the lemma below, which is essentially combinatorial, the image contains $\oplus_I \oplus_{i\ge 2n-3} H^i_R$ and thus the intersection is $\oplus_I \oplus_{i>2n-3} H^i_R$, whose Poincaré series is precisely

$$\frac{1}{2}\binom{2n}{n}\frac{t^{2n-2}}{1-t^2}.$$

So we are done. $\square$

Lemma 5.21 *The restriction map $H^{2k}_G(X^{ss}) \to \oplus_I H^{2k}_R$ is surjective for $k \ge n-1$.*

Proof It is equivalent to show that $H^{2k}_G(X) \to \oplus_I H^{2k}_R$ is surjective. Let $\xi = \xi_2 + \cdots + \xi_{2n}$ and consider, for each $I = (1, i_2, \cdots, i_n)$,

$$\eta_I = (\xi - \xi_{i_2})^{k_2} \cdots (\xi - \xi_{i_n})^{k_n}.$$

Then since $\xi|_{q_J} = -\rho$ for all J, $\eta_I|_{q_J} = (-2\rho)^k$ if $J = I$ and 0 otherwise, where $k = k_2 + \cdots + k_n$, $k_i \geq 1$. Therefore, the images of those η_I span $\oplus_I H_R^{2k}$ for any $k \geq n-1$ and thus the restriction is surjective. $\square$

The lemma is also a consequence of a pleasant combinatorial problem about the non-degeneracy of a matrix of $+1, -1$, whose sign is determined by the parity of incidence of I's.

5.5 Intersection Pairing

In this section we determine the intersection pairing of the intersection cohomology of GIT quotients under the assumptions of §2. For that purpose, in this section we assume that X is projective so that the quotient $X/\!\!/G$ is compact.

We first characterize the top dimensional class of $IH^*(X/\!\!/G)$. Let $X^{sss} = X^{ss} - X^s$ [4] and m be the real dimension of $X/\!\!/G$. Consider the following commutative diagram

$$\begin{array}{ccccc} H^m(X/\!\!/G, X^{sss}/\!\!/G) & \longrightarrow & H^m(X/\!\!/G) & \longrightarrow & H^m(X^{sss}/\!\!/G) \\ \downarrow & & \downarrow & & \downarrow \\ H^m_G(X^{ss}, X^{sss}) & \longrightarrow & H^m_G(X^{ss}) & \longrightarrow & H^m_G(X^{sss}) \\ & & \downarrow & & \\ & & IH^m(X/\!\!/G) & & \end{array}$$

Obviously, $H^m_G(X^{ss}, X^{sss}) \cong H^m(X/\!\!/G, X^{sss}/\!\!/G) \cong H^m(X/\!\!/G) \cong \mathbb{C}$.

Proposition 5.22 *The map $H^m_G(X^{ss}, X^{sss}) \to H^m_G(X^{ss})$ factors through V^m_X and*

$$H^m_G(X^{ss}, X^{sss}) \cong V^m_X \cong IH^m(X/\!\!/G).$$

Proof By definition,

$$V_X \supset ker(H^*_G(X^{ss}) \to H^*_G(X^{sss})) = im(H^*_G(X^{ss}, X^{sss}) \to H^*_G(X^{ss}))$$

Let $\xi \neq 0$ be a nonzero class in $H^m(X/\!\!/G)$. Then ξ is mapped to a nonzero class ξ'' in $IH^m(X/\!\!/G)$ by $H^m(X/\!\!/G) \to H^m_G(X^{ss}) \to IH^*(X/\!\!/G)$. As $\xi|_{X^{sss}/\!\!/G} = 0$ by dimension, ξ

[4] sss=strictly semistable

is the image of a class η in $H^m(X/\!\!/G, X^{sss}/\!\!/G) \cong H^m_G(X^{ss}, X^{sss})$. The image ξ' of η in $H^m_G(X^{ss})$ is the image of ξ by $H^m(X/\!\!/G) \to H^m_G(X^{ss})$ and thus must be mapped to ξ'' by the Kirwan map. Therefore, the composition $H^m_G(X^{ss}, X^{sss}) \to V^m_X \cong IH^m(X/\!\!/G)$ is an isomorphism because they are 1 dimensional. □

Theorem 5.23 *The intersection pairing in $IH^*(X/\!\!/G)$ is given by the cup product structure via the isomorphism of the splitting theorem.*

Proof We prove it again by induction on $r = r(X)$. Let Y be the same as in section 3. Assume that it is true for Y^{ss}, i.e. for any two intersection classes of complementary dimension, the cup product of the corresponding elements in the equivariant cohomology lies in V_Y and the coefficient of its image with respect to the top degree fundamental class of the intersection cohomology is the number given by the intersection pairing.

First, we observe that the product of two elements in V_X of complementary dimensions lies in V^m_X. We observe that the cup product should be mapped to a constant multiple of the top degree class in V_Y, by the induction hypothesis. And the top degree class in V_X is mapped to the top degree class in V_Y. Therefore, the product minus some constant multiple of the top degree class in V_X, (we denote this difference by z) should lie in the kernel of the ring homomorphism $H^*_G(X^{ss}) \to H^*_G(Y^{ss})$.

We also observe that the restriction of the product to the blow-up center is zero because it lies in $[H^k(Z_R/\!\!/N^R_0) \otimes H^j_R]^{\pi_0 N^R}$ where $j < 2n_R$ and thus $k > 2\dim_{\mathbb{C}}(Z_R/\!\!/N^R_0)$. Similarly, the top degree class in V_X also restricts to zero. Therefore, z restricts to zero. However, the intersection of the kernel of $H^*_G(X^{ss}) \to H^*_G(Y^{ss})$ with the kernel of the restriction to the blow-up center is $\{0\}$ since each unstable stratum retracts onto its intersection with the exceptional divisor. Hence, $z = 0$ and thus the product is a constant multiple of the top degree class in V_X.

From §3, we have

$$\begin{array}{ccccc} 0 & \longrightarrow & V_X & \longrightarrow & V_Y \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & IH^*(X/\!\!/G) & \longrightarrow & IH^*(Y/\!\!/G) \end{array}$$

As the top row is just a restriction of a ring homomorphism $H_G^*(X^{ss}) \to H_G^*(Y^{ss})$, it suffices to show that the bottom inclusion preserves the intersection pairing.

Now, we show that the bottom embedding comes from a decomposition on the sheaf complex level. For simplicity, we assume $\pi_0 N^R = 1$. Let $p : \mathcal{N}/\!\!/G \to Z_R/\!\!/N^R$, $q : E/\!\!/G \to Z_R/\!\!/N^R$, $\pi : Y/\!\!/G \to X/\!\!/G$, $\phi : X^{ss} \to X/\!\!/G$, $i : Z_R/\!\!/N^R \to X/\!\!/G$ be the obvious maps. By the local models,[5] $q_*\mathbf{IC}^{\cdot}(E/\!\!/G) = Q \times_L IC^*(\mathbb{P}\mathcal{N}_x/\!\!/R)$ and $p_*\mathbf{IC}^{\cdot}(\mathcal{N}/\!\!/G) = Q \times_L IC^*(\mathcal{N}_x/\!\!/R)$. Choose a surjective quasi-isomorphism of chain complexes $IC^*(\mathbb{P}\mathcal{N}_x/\!\!/R) \to IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)$. Then it induces $Q \times_L IC^*(\mathbb{P}\mathcal{N}_x/\!\!/R) \to Q \times_L IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)$ i.e.

$$q_*\mathbf{IC}^{\cdot}(E/\!\!/G) \to \mathbf{IC}^{\cdot}(Z_R/\!\!/N^R; IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)).$$

Fixing an embedding $IH^*(\mathcal{N}_x/\!\!/R) \hookrightarrow IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)$ as in §3 gives an embedding

$$\mathbf{IC}^{\cdot}(Z_R/\!\!/N^R; IH^*(\mathcal{N}_x/\!\!/R)) \hookrightarrow \mathbf{IC}^{\cdot}(Z_R/\!\!/N^R; IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)).$$

Let $\mathcal{I}_X$ be the kernel of the composite

$$\pi_*\mathbf{IC}^{\cdot}(Y/\!\!/G) \to i_*q_*\mathbf{IC}^{\cdot}(E/\!\!/G) \to i_*\mathbf{IC}^{\cdot}(Z_R/\!\!/N^R; IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)/IH^*(\mathcal{N}_x/\!\!/R))$$

where the first map is by restriction and the second is the projection by the embedding. Then we get a decomposition

$$\pi_*\mathbf{IC}^{\cdot}(Y/\!\!/G) = \mathcal{I}_X \oplus i_*\mathbf{IC}^{\cdot}(Z_R/\!\!/N^R; IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)/IH^*(\mathcal{N}_x/\!\!/R)).$$

The second factor is from the decomposition theorem [6], which tells us

$$\pi_*\mathbf{IC}^{\cdot}(Y/\!\!/G) = \mathbf{IC}^{\cdot}(X/\!\!/G) \oplus i_*\mathbf{IC}^{\cdot}(Z_R/\!\!/N^R, IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)/IH^*(\mathcal{N}_x/\!\!/R)).$$

Hence, $\mathcal{I}_X = \mathbf{IC}^{\cdot}(X/\!\!/G)$ in the derived category. By construction, taking hypercohomology of gives the decomposition of intersection cohomology.

[5] See Chapter 2 §1.

In this setting, the intersection pairing is given by the Poincaré duality isomorphism. The duality isomorphism for $Y/\!\!/G$ induces an isomorphism for $\mathcal{I}_X$ which coincides with the Poincaré duality isomorphism on the smooth part. By uniqueness of such isomorphism [7], §9, we deduce that the inclusion preserves the pairing.

In case when $\pi_0 N^R \neq 1$, one can deduce the same by replacing the sheaf $\overline{x} \to IH^*(\mathbb{P}\mathcal{N}_x/\!\!/R)$ (resp. $IH^*(\mathcal{N}_x/\!\!/R)$) by $\overline{x} \to [\oplus_{\epsilon(\overline{y})=\overline{x}} IH^*(\mathbb{P}\mathcal{N}_y/\!\!/R)]^{\pi_0 N^R}$ (resp. $[\oplus_{\epsilon(\overline{y})=\overline{x}} IH^*(\mathcal{N}_y/\!\!/R)]^{\pi_0 N^R}$) where $\overline{x} = \phi(x)$, $\overline{y} = \phi(y)$ and $\epsilon : Z_R/\!\!/N_0^R \to Z_R/\!\!/N^R$. $\square$

The fact that our embedding $IH^*(X/\!\!/G) \hookrightarrow IH^*(Y/\!\!/G)$ preserves the intersection pairing can be also proved as follows: It is shown that the decomposition of intersection cohomology is orthogonal. Any two classes α, β of complementary dimensions in $IH^*(X/\!\!/G)$ is mapped to $\alpha + \zeta a$, $\beta + \zeta b$ for $a, b \in IH^*(E/\!\!/G)$, where ζ is the first Chern class of the normal to the exceptional divisor. By orthogonality, it suffices to show that $< \zeta a, \zeta b >= 0$, or equivalently, $< \zeta a, b >_{E/\!\!/G} = 0$. By induction, we can assume that the conclusion is true for $E/\!\!/G$, i.e. the pairing in $IH^*(E/\!\!/G)$ is given by the cup product structure in $V_E \subset H_G^*(E^{ss})$. As the classes come from $IH^*(X/\!\!/G) \cong V_X$, the equivariant classes corresponding to ζa, ζb are in $\oplus_{i<n_R} H^*(Z_R/\!\!/N^R) \otimes H_R^i(\mathbb{P}\mathcal{N}_x^{ss})$ and thus their product divided by ζ lies in $\oplus_{i<2n_R-2} H^*(Z_R/\!\!/N^R) \otimes H_R^i(\mathbb{P}\mathcal{N}_x^{ss})$, which must be zero by dimension counting. So, we are done.

Example 5.24 Consider the $\mathbb{C}^*$ action on $\mathbb{P}^7$ by a representation with weights $+1, 0, -1$ of multiplicity $3, 2, 3$ respectively. Hence, $n_+ = n_- = 3$, $n_0 = 2$. Then

$$H^*_{\mathbb{C}^*}(\mathbb{P}^7) = \mathbb{C}[\xi, \rho]/ < \xi^2(\xi - \rho)^3(\xi + \rho)^3 >$$

where ξ is a generator in $H^2(\mathbb{P}^7)$ and ρ is a generator in $H^2_{\mathbb{C}^*}$.

The equivariant Euler classes for the two unstable strata are $\xi^2(\xi - \rho)^3$, $\xi^2(\xi + \rho)^3$ respectively. Therefore,

$$H^*_{\mathbb{C}^*}((\mathbb{P}^7)^{ss}) = \mathbb{C}[\xi, \rho]/ < \xi^2(\xi - \rho)^3, \xi^2(\xi + \rho)^3 >$$

A Gröbner basis for the relation ideal is

$$\{\xi^5 + 3\xi^3\rho^2, \xi^4\rho + \frac{1}{3}\xi^2\rho^3, \xi^3\rho^3, \xi^2\rho^5\}$$

where $\xi > \rho$. Hence as a vector space,

$$H^*_{\mathbb{C}^*}((\mathbb{P}^7)^{ss}) = \mathbb{C}\{\xi^i\rho^j | i = 0, 1,\ j \geq 0\} \oplus \mathbb{C}\{\xi^i\rho^j | 2i + j < 9, i \geq 2, j \geq 0\}$$

By definition, as $n_R = 5$, we remove $\mathbb{C}\{\xi^i\rho^j | i = 0, 1,\ j \geq 3\}$ to get

$$V = \oplus_{0 \leq i \leq 6} V^{2i}$$

$$V^0 = \mathbb{C}, \quad V^2 = \mathbb{C}\{\rho, \xi\}, \quad V^4 = \mathbb{C}\{\rho^2, \xi\rho, \xi^2\},$$

$$V^6 = \mathbb{C}\{\xi\rho^2, \xi^2\rho, \xi^3\}, \quad V^8 = \mathbb{C}\{\xi^2\rho^2, \xi^3\rho, \xi^4\},$$

$$V^{10} = \mathbb{C}\{\xi^2\rho^3, \xi^3\rho^2\}, \quad V^{12} = \mathbb{C}\{\xi^2\rho^4\}.$$

First, consider the pairing $V^2 \otimes V^{10} \to V^{12}$. As $\rho(\xi^2\rho^3) = \xi^2\rho^4$, $\rho(\xi^3\rho^2) = \xi^3\rho^3 = 0$, $\xi(\xi^2\rho^3) = \xi^3\rho^3 = 0$, $\xi(\xi^3\rho^2) = \xi^4\rho^2 = -\frac{1}{3}\xi^2\rho^4$, the pairing matrix is up to constant

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

The determinant is $-\frac{1}{3} \neq 0$ and the signature is 0.

Next, consider the pairing $V^4 \otimes V^8 \to V^{12}$. One can similarly use the Gröbner basis to compute the pairing as above. The pairing matrix is up to constant

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

The determinant is $-\frac{8}{27} \neq 0$ and the signature is 1.

Similarly, the intersection pairing matrix for $V^6 \otimes V^6 \to V^{12}$ is up to constant

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

The determinant is $-\frac{8}{27} \neq 0$ and the signature is 1.

In this way, one can compute the intersection pairing for any $n_0, n_- = n_+$. □

5.6 Hodge Structure

It is a classical theorem of Hodge that every compact Kähler manifold admits a Hodge structure. Deligne introduced the concept of mixed Hodge structure and proved that there is a functorial mixed Hodge structure on every separated algebraic scheme, which coincides with the classical Hodge structure when the scheme is a compact Kähler manifold. It is also a theorem of Deligne that for any linear algebraic group G, H_G^* has a Hodge structure so that every class is of type (n,n) for some n.

Let X be a nonsingular projective variety acted on linearly by a reductive algebraic group G such that X^s is nonempty. Then by Deligne's criterion for degeneration of spectral sequences,

$$H_G^*(X) \cong H_G^* \otimes H^*(X).$$

As X is a compact Kähler manifold, it has a Hodge structure and thus $H_G^*(X)$ has a Hodge structure by Deligne's theorem. Now, as explained in [24], it induces a Hodge structure on $H_G^*(X^{ss})$ as follows. The norm square of the moment map with respect to the symplectic action of a maximal compact subgroup in G gives us a Morse stratification

$$X = X^{ss} \cup [\cup_\delta S_\delta]$$

and it is equivariantly perfect, i.e. the following is exact

$$0 \to H_G^{*-2d(\delta)}(S_\delta) \to H_G^*(\cup_{\gamma\le\delta} S_\gamma) \to H_G^*(\cup_{\gamma\le\delta} S_\gamma - S_\delta) \to 0$$

where $d(\delta)$ is the complex codimension of the stratum. As the maps in the above exact sequence are morphisms of mixed Hodge structures, inductively we get a Hodge structure of $H_G^*(X^{ss})$.

Proposition 5.25 *V_X carries a Hodge structure induced from that of $H_G^*(X^{ss})$ and hence so does $IH^*(X/\!\!/G)$ via the Kirwan map.*

Proof Just notice that the maps that we used in defining V_X are all morphisms of mixed Hodge structures by Deligne's theorem as they are from geometric maps. So, V_X has an induced mixed Hodge structure which must be pure because that for $H_G^*(X^{ss})$ is pure. □

Example 5.26 Consider the $G = SL(2)$ action on $X = (\mathbb{P}^1)^{2n}$ as in §4. As $H^*(\mathbb{P}^1)$ has a Hodge structure such that $h^{p,q} = 0$ for $p \neq q$, so does $H_G^*(X^{ss})$. Therefore, the intersection cohomology has a Hodge structure such that $h^{p,q} = 0$ for $p \neq q$. □

Part II

Cohomology of moduli spaces of vector bundles

Chapter 6

Moduli spaces of vector bundles

We review various facts about moduli spaces of holomorphic vector bundles over a (smooth closed) Riemann surface. Everything in this chapter is standard and borrowed from [40, 37, 3, 49, 27].

6.1 The concept of moduli

Many of the fundamental problems in mathematics are to classify a collection, say A, of objects with respect to some equivalence relation $\sim$. This would be achieved if one can find a nice space that is in one to one correspondence with the set of the equivalence classes of the collection. In algebraic geometry, we wish this space furthermore to be an object in algebraic geometry as Newstead put in [40]: "Almost always there exist 'continuous families' of objects of A, and we would like to give $A/\sim$ some algebro-geometric structure to reflect this fact. This is the object of the theory of moduli".

Let A be a collection of objects in algebraic geometry with an equivalence relation $\sim$. A *family* of objects in A parametrized by a variety S is a morphism $\phi : X \to S$ such that, for any inclusion $\imath : \{s\} \hookrightarrow S$ of a point into S, the pull-back (or the fibred product) X_s is an object in A. Suppose there is a notion of equivalence of families parametrized by any given variety S, which reduces to the given equivalence relation on A when restricted to a point. Let $\mathcal{F}(S)$ denote the set of equivalence classes of families parametrized by S. Then $S \to \mathcal{F}(S)$ is a contravariant functor by pull-back.

Definition 6.1 *A* fine moduli space *is a pair of a variety M and a natural transformation $\Phi : \mathcal{F} \to \mathrm{Mor}(-, M)$, which represents the functor $\mathcal{F}$. Equivalently, M is a fine moduli space if there is a family U parametrized by M such that every family $X \to S$ is the pull-back of U by a morphism $S \to M$. We call U a* universal family.

But in many practical moduli problems, there do not exist fine moduli spaces. So we need a weaker concept of moduli.

Definition 6.2 *A* coarse moduli space *is a pair of a variety M and a natural transformation $\Phi : \mathcal{F} \to \mathrm{Mor}(-, M)$ such that*

1. *$\Phi(pt)$ is bijective*

2. *for any N and any natural transformation $\Psi : \mathcal{F} \to \mathrm{Mor}(-, N)$, there exists a unique natural transformation*

$$\Omega : \mathrm{Mor}(-, M) \to \mathrm{Mor}(-, N)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Phi} & \mathrm{Mor}(-, M) \\ & \searrow^{\Psi} & \vdots\,\Omega \\ & & \mathrm{Mor}(-, N) \end{array}$$

commutes.

By the usual abstract nonsense, two coarse moduli spaces of a given moduli problem are isomorphic.

6.2 Jump phenomenon and stability

The moduli problem we are interested in is the moduli space of holomorphic vector bundles, of rank r and degree d, over a Riemann surface Σ of genus $g \geq 2$. A family of vector bundles[1] parametrized by S is a vector bundle over $S \times \Sigma$ and the equivalence relation is the isomorphism of vector bundles over $S \times \Sigma$.

[1] Unless stated otherwise, vector bundles in this chapter mean holomorphic vector bundles.

Jump phenomenon

The problem here is that there is no *Hausdorff* moduli space of all holomorphic vector bundles due to the jump phenomenon that we are going to explain.

Let $L \to \Sigma$ be a line bundle of positive degree. Then by Riemann-Roch, $H^1(\Sigma, L^{-1}) \neq 0$. Choose any 1-dimensional subspace S of $H^1(\Sigma, L^{-1})$. Then the dual space $S^* \subset H^0(S, \mathcal{O})$. Consider the identity

$$I \in S^* \otimes S \subset H^0(S, \mathcal{O}) \otimes H^1(\Sigma, L^{-1}) \subset H^1(S \times \Sigma, pr_2^* L^{-1})$$

where pr_2 is the projection onto the second component. I defines an extension, a continuous family

$$0 \to \mathcal{O} \to \mathcal{E} \to pr_2^* L \to 0$$

over $S \times \Sigma$. For $t \in S$, the restriction to $\{t\} \times \Sigma$ is

$$0 \to \mathcal{O} \to E_t \to L \to 0$$

the extension of L by $\mathcal{O}$, prescribed by $t \in H^1(\Sigma, L^{-1})$. If t, t' are nonzero elements in S, $E_t \cong E_{t'}$ while $E_0 = \mathcal{O} \oplus L$ is not isomorphic to E_t for any $t \neq 0$. So, a "jump" in isomorphism classes can take place even in a continuous family. This makes it impossible to construct any *Hausdorff* moduli space of all vector bundles.

Stability of bundles

In order to construct the moduli space as a *Hausdorff* variety, Mumford suggested to get rid of "bad" objects from the collection. Let $\mu(F) = \deg(F)/\text{rank}(F)$ for any vector bundle F over Σ. We call it the *slope* of the bundle.

Definition 6.3 *A vector bundle E over Σ is* stable (semistable) *if for every nonzero proper subbundle $F < E$,*

$$\mu(F) < \mu(E) \quad (\mu(F) \leq \mu(E)).$$

The following two lemmas are important for the construction of the moduli spaces of vector bundles.

Lemma 6.4 *[40]*

1. *Every line bundle is stable.*

2. *If F is (semi)stable, then $F \otimes L$ is (semi)stable for any line bundle L.*

3. *If E and F are stable with the same slope, then $H^0(\Sigma, \mathrm{Hom}(E, F))$ is either $\mathbb{C}$ if $E \cong F$ or 0 otherwise.*

Proof The first statement is obvious and the second follows from the fact that

$$\deg(E \otimes L) = \deg(E) + \mathrm{rank}(E)\deg(L).$$

For the last statement, observe that if $h : E \to F$ is a nonzero homomorphism, the kernel and cokernel should be trivial due to stability. □

Lemma 6.5 *[40] Let E be a semistable vector bundle over Σ of rank r and degree d.*

1. *If $d > r(2g-2)$, then $H^1(\Sigma, E) = 0$.*

2. *If $d > r(2g-1)$, then E is generated by its sections.*

Proof The first statement follows from the Serre duality. For the second, given $x \in \Sigma$, consider the short exact sequence

$$0 \to m_x E \to E \to \mathcal{E}_x \to 0$$

where m_x is the sheaf of ideals of x and $\mathcal{E}_x$ is the skyscraper sheaf at x. We need to show that $H^0(E) \to H^0(\mathcal{E}_x) = E_x$ is a surjection, or equivalently that $H^1(m_x E) = 0$ from the long exact sequence of sheaf cohomology.

Notice that the sheaf m_x is the sheaf of the line bundle L_x of the divisor $-x$, whose degree is -1. Thus $\deg(E \otimes L_x) = d - r > r(2g-2)$ and by the first statement $H^1(m_x E) = H^1(E \otimes L_x) = 0$. □

Filtrations of vector bundles

Let $E \to \Sigma$ be any vector bundle over a Riemann surface. Choose a maximal subbundle E_1 whose slope is the largest among the slopes of the subbundles of E. Choose a maximal subbundle E_2' of E/E_1 whose slope is the largest and let E_2 denote the inverse image of E_2' by the projection $E \to E/E_1$. We can repeat this process till we get a filtration, called the Harder-Narasimhan filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E.$$

By construction, all $F_i = E_i/E_{i-1}$ are semistable and $\mu(F_i) > \mu(F_{i+1})$.

Now suppose F is a semistable vector bundle. Choose a minimal nonzero subbundle F_1 of F whose slope is same as $\mu(F)$. It is easy to check that F/F_1 is also semistable. Choose a minimal nonzero subbundle F_2' of F/F_1 whose slope is same as $\mu(F)$ and let F_2 denote the inverse image of F_2' by the projection $F \to F/F_1$. Continue this process till we get a filtration, called a Seshadri filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F.$$

By construction, F_i/F_{i-1} is stable with slope $\mu(F)$. Let

$$gr(F) = \oplus F_i/F_{i-1}.$$

It turns out that $gr(F)$ is independent of the filtration. We say two semistable bundles E, F are *s-equivalent* if

$$gr(E) \cong gr(F).$$

6.3 First construction of the moduli spaces

We construct the moduli space of holomorphic vector bundles of rank r and degree d over a (smooth closed) Riemann surface Σ of genus $g \geq 2$, as a geometric invariant theory quotient.

Theorem 6.6 *For fixed r, d, there exists a connected coarse moduli space $M^s(r,d)$ of stable bundles of rank r and degree d over Σ. Moreover, $M^s(r,d)$ has a natural compactification to a projective variety $M(r,d)$ which parametrizes all s-equivalence classes of semistable bundles.*

Proof [Sketch of the proof] [40] By tensoring with a line bundle, we may assume that $d > r(2g-1)$. Each semistable vector bundle E of rank r and degree d over Σ is generated by its global sections, i.e. there is a surjective bundle homomorphism $\mathcal{O}^{\oplus p} \to E$ where $p = d - r(g-1)$. Hence, we get a morphism $\Sigma \to Gr(r,p)$ of Σ into the Grassmannian. Let $A(r,d)$ denote the smooth quasi-projective variety of all holomorphic maps h such that the pull-back $E(h)$ of the universal quotient bundle has degree d and the map on sections $\mathbb{C}^p \to H^0(E(h))$ induced from the quotient bundle map $\mathbb{C}^p \times \Sigma \to E(h)$ is an isomorphism. It turns out that the natural $GL(p)$ action on $A(r,d)$ can be linearized in such a way that the (semi)stability of $h \in A(r,d)$ in Chapter 2 coincides with that of $E(h)$ defined above. Furthermore, two maps $h, h' \in A(r,d)$ induce isomorphic bundles $E(h) \cong E(h')$ if and only if h and h' lie in the same $GL(p)$ orbit. Therefore, the moduli space of stable bundles is identified with

$$M(r,d) = A(r,d)^s / GL(p).$$

Moreover, $h, h' \in A(r,d)^{ss}$ represent the same point of the geometric invariant theory quotient $A(r,d)/\!\!/GL(p)$ if and only if $gr(E(h)) \cong gr(E(h'))$. Hence,

$$M(r,d) = A(r,d)/\!\!/GL(p)$$

is the projective variety parametrizing the s-equivalence classes of semistable bundles, which compactifies $M^s(r,d)$. □

The complex dimension of $M(r,d)$ is $r^2(g-1)+1$. When r is coprime to d, clearly semistability coincides with stability and the moduli space $M(r,d)$ is smooth.

Given a vector bundle $E \to \Sigma$ of rank r, we can associate its determinant line bundle $det(E) = \wedge^r E$. It depends only on the s-equivalence class of the vector bundle and we have a morphism

$$det : M(r,d) \to Jac_d$$

onto the Jacobian of degree d over Σ. It is a fiber bundle whose fiber over L is the moduli space $N(r,d)$ of bundles of fixed determinant L.

Since we constructed the moduli space $M(r,d)$, the natural question is whether it is *fine* or not.

Theorem 6.7 *If the rank r is coprime to the degree d, then there is a universal bundle $U \to M(r,d) \times \Sigma$, i.e. it is a fine moduli space.*

Proof This bundle is in fact the descent of the pull-back W of the universal quotient bundle over the Grassmannian via the evaluation map

$$ev : A(r,d)^s \times \Sigma \to Gr(r,p).$$

To show that there is a descent of this bundle, by Kempf's descent lemma, we have to check that the stabilizer $\mathbb{C}^*$ in $GL(p)$ of each point acts trivially on the fiber of W.

Let $pr_1 : A(r,d)^s \times \Sigma \to A(r,d)^s$ denote the projection. Consider the line bundles $det((pr_1)_*W)$ and $det(W|_{A(r,d)^s \times pt})$ over $A(r,d)^s$. The stabilizer $\mathbb{C}^*$ acts on a fiber of these by weights p and r respectively. As $p = d - r(g-1)$ and r are coprime by assumption, we can combine them to get a line bundle L where $\mathbb{C}^*$ acts with weight -1. If we replace W by $W \otimes (pr_1)^*L$, then $\mathbb{C}^*$ acts trivially on the fibers and thus it descends to the quotient. □

On the other hand, if r and d are not coprime, then there is no universal bundle even if we restrict to any Zariski open subset of the moduli space by a theorem due to Ramanan [41].

6.4 Second construction of the moduli spaces

We review the gauge theoretic construction due to Atiyah and Bott [3].

Fix a Hermitian vector bundle $\mathcal{E} \to \Sigma$ of rank r, degree d. Let $\mathcal{C} = \mathcal{C}(r,d)$ be the space of holomorphic structures on $\mathcal{E}$. Then $\mathcal{C}$ is an infinite dimensional affine space based on $\Omega^{0,1}(\mathfrak{gl}(r))$. The group of automorphisms $Aut(\mathcal{E})$ can be identified with the complexification $\mathcal{G}_c$ of the $U(r)$ gauge group $\mathcal{G}$.

Let $\mathcal{C}^s$ ($\mathcal{C}^{ss}$) be the open subset of stable (semistable) holomorphic structures on $\mathcal{E}$. In fact, $\mathcal{C}^{ss}$ is the unique open dense stratum in the Shatz stratification: Any holomorphic bundle E has the canonical Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_s = E$$

such that $D_i = E_i/E_{i-1}$ are semistable and $\mu(D_i) > \mu(D_{i+1})$. Let $\mathrm{rank}(D_i) = r_i$ and $\mathrm{degree}(D_i) = d_i$. We assign to E an r-vector, the type

$$\mu = (\frac{d_1}{r_1}, \cdots, \frac{d_s}{r_s})$$

in such a way that each $\frac{d_i}{r_i}$ is repeated r_i times. For each μ, we define $\mathcal{C}_\mu$ as the set of holomorphic structures whose types are μ.

On the other hand, we consider the space $\mathcal{A}$ of all connections on the $U(2)$ principal bundle P associated to $\mathcal{E}$. It is an affine space based on the space of Lie algebra $u(2)$ valued 1-forms $\Omega^1(u(2))$. Each connection A defines a $u(2)$ valued 2-form, namely the curvature $F(A)$. This can be interpreted as the moment map for the gauge group action with respect to the symplectic form

$$\omega(a, b) = \int_\Sigma \mathrm{tr}(a \wedge b)$$

for $a, b \in \Omega^1(u(2))$. The Yang-Mills functional is defined as the norm square of the curvature $\int_\Sigma |F(A)|^2$.

Notice that taking $(0, 1)$ part of the covariant derivative of a connection defines an affine linear isomorphism

$$\mathcal{A} \to \mathcal{C}$$

based on $\Omega^1(u(r)) \cong \Omega^{0,1}(\mathfrak{gl}(r))$. Hence, we can import the Shatz stratification of $\mathcal{C}$ to $\mathcal{A}$ via this isomorphism. One fascinating result of Atiyah and Bott [3] is the following.

Theorem 6.8 *The Shatz stratification of $\mathcal{A}$ coincides with the Morse stratification by the Yang-Mills functional.*

In particular, $\mathcal{C}^{ss}$ retracts onto the space of flat connections $\mathcal{A}_{flat}$ by the gradient flow of the Yang-Mills functional. Moreover, any two semistable holomorphic structures retract to the same point in $\mathcal{A}_{flat}$ if and only if they are s-equivalent. Therefore, we can identify the moduli space $M(r,d)$ with

$$\mathcal{C}^{ss}/\!\!/\mathcal{G}_c := \mathcal{A}_{flat}/\mathcal{G}.$$

From this together with the holonomy theorem, we deduce the celebrated Narasimhan-Seshadri theorem.

Theorem 6.9 *$M(r,d)$ is homeomorphic to*

$$\{(a_i) \in U(2)^{2g} | \prod_i [a_i, a_{i+g}] = e^{\frac{2\pi i d}{r}} I\}.$$

It turns out that this is a stratified symplectic space [47]. The symplectic structure (on each piece) is the descent of the symplectic structure on $\mathcal{A}$. Heuristically, the moduli space is a symplectic reduction of the infinite dimensional symplectic space $\mathcal{A}$ by the gauge group and the theorem of Marsden and Weinstein (or Sjamaar and Lerman) tells us that there is an induced symplectic structure.

The unstable strata and its normal bundles have a nice homotopical description.

Proposition 6.10 *[3] Let $\mu = (\frac{d_1}{r_1}, \cdots, \frac{d_s}{r_s})$. The homotopy quotient $(\mathcal{C}_\mu)_{\mathcal{G}} = \mathcal{C}_\mu \times_{\mathcal{G}} E\mathcal{G}$ is homotopically equivalent to the product of homotopy quotients $\prod_{i=1}^s (\mathcal{C}(r_i,d_i)^{ss})_{\mathcal{G}(r_i,d_i)}$. The normal bundle of $\mathcal{C}_\mu$ restricted to $\prod_{i=1}^s \mathcal{C}(r_i,d_i)^{ss}$ is*

$$R^1\pi_*(\oplus_{i<j}\mathrm{Hom}(W_i,W_j))$$

where W_i is the pull-back to $(\prod_{i=1}^s \mathcal{C}(r_i,d_i)^{ss}) \times \Sigma$ of the universal bundle over $\mathcal{C}(r_i,d_i)^{ss} \times \Sigma$ and π is the projection onto $\prod_{i=1}^s \mathcal{C}(r_i,d_i)^{ss}$.

6.5 Cohomology of the moduli spaces

The Morse stratification of the previous section provides a powerful method in computing the cohomology of moduli spaces. First we give a partial ordering on the index set for the stratification: Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ and $\mu = (\mu_1, \cdots, \mu_r)$ be two r-vectors. Then

$$\lambda \geq \mu \quad \text{if} \quad \sum_{j\leq i} \lambda_j \geq \sum_{j \leq i} \mu_j$$

for each i. With this ordering, we have the "frontier condition" [3]

$$\overline{\mathcal{C}}_\mu \subset \cup_{\lambda \geq \mu} \mathcal{C}_\lambda.$$

Let $\mathcal{U}_\mu$ be an open union of strata containing $\mathcal{C}_\mu$ as a closed subset. Consider the Gysin sequence

$$\cdots \to H_{\mathcal{G}}^{*-2d_\mu}(\mathcal{C}_\mu) \to H_{\mathcal{G}}^*(\mathcal{U}_\mu) \to H_{\mathcal{G}}^*(\mathcal{U}_\mu \setminus \mathcal{C}_\mu) \to \cdots$$

where $2d_\mu$ is the real codimension of the stratum $\mathcal{C}_\mu$. Notice that $\mathcal{C}_\mu$ retracts onto the product $\prod_{i=1}^s \mathcal{C}(r_i, d_i)^{ss}$ where the product of s copies of the circle group acts trvially. From the description of the normal bundle in the previous section, it is easy to check that the torus action on the fiber of the normal bundle has no nonzero fixed point. Hence, Atiyah and Bott's criterion [3] tells us that the Gysin map is injective and hence the Gysin sequence breaks into short exact sequences:

$$0 \to H_{\mathcal{G}}^{*-2d_\mu}(\mathcal{C}_\mu) \to H_{\mathcal{G}}^*(\mathcal{U}_\mu) \to H_{\mathcal{G}}^*(\mathcal{U}_\mu \setminus \mathcal{C}_\mu) \to 0.$$

Therefore, the restriction map $H_{\mathcal{G}}^*(\mathcal{C}) \to H_{\mathcal{G}}^*(\mathcal{C}^{ss})$ is surjective and

$$H_{\mathcal{G}}^*(\mathcal{C}) = H_{\mathcal{G}}^*(\mathcal{C}^{ss}) \oplus \bigoplus_\mu H_{\mathcal{G}}^{*-2d_\mu}(\mathcal{C}_\mu).$$

Here, $H_{\mathcal{G}}^*(\mathcal{C}) = H^*(B\mathcal{G})$ since $\mathcal{C}$ is contractible. Now, the classifying space of the gauge group $B\mathcal{G}$ is homotopically equivalent to

$$\mathrm{Map}_P(\Sigma, BU(2)).$$

Consider the evaluation map

$$ev : \mathrm{Map}_P(\Sigma, BU(2)) \times \Sigma \to BU(2).$$

The Chern classes of the pull-back V of the universal bundle over $BU(2)$ decompose as follows by the Künneth theorem,

$$c_i(V) = a_i \otimes 1 + \sum_{j=1}^{2g} b_i^j \otimes e_j + f_i \otimes [\Sigma]$$

where $\{e_j\}$ is a symplectic basis of $H^1(\Sigma)$ such that $e_j e_{j+g} = [\Sigma] \in H^2(\Sigma)$.

Proposition 6.11 *[3] The elements a_i, b_i^j, f_i freely generate the cohomology ring of the classifying space of the gauge group.*

Hence, since the restriction to $\mathcal{C}^{ss}$ is surjective, the classes above generate the cohomology ring $H^*_{\mathcal{G}}(\mathcal{C}^{ss})$.

On the other hand, the cohomology ring $H^*_{\mathcal{G}}(\mathcal{C}_\mu)$ is isomorphic to $\otimes_{i=1}^s H^*_{\mathcal{G}(r_i,d_i)}(\mathcal{C}(r_i,d_i)^{ss})$. Thus we can compute the cohomology ring $H^*_{\mathcal{G}}(\mathcal{C}^{ss})$ inductively.

Let $\overline{\mathcal{G}}$ denote the quotient of the gauge group $\mathcal{G}$ by the constant central $U(1)$ subgroup. Then we have

$$H^*_{\mathcal{G}}(\mathcal{C}^{ss}) \cong H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}) \otimes H^*_{U(1)}.$$

Finally, $\overline{\mathcal{G}}$ acts freely on $\mathcal{C}^s$ and hence $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^s) \cong H^*(\mathcal{C}^s/\mathcal{G}_c)$. If r is coprime to d, then $\mathcal{C}^s = \mathcal{C}^{ss}$ and thus $H^*(M(r,d)) \cong H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ which can be computed inductively.

Example 6.12 We compute the Poincaré series of $H^*(M(2,1))$. The Poincaré series of the classifying space of $U(2)$ gauge group is

$$\frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)}.$$

For each unstable stratum of type $(i+1,-i)$, the Poincaré series of its equivariant cohomology is

$$(\frac{(1+t)^{2g}}{(1-t^2)^2})^2$$

and the codimension of the stratum is $4i+2g$ by Riemann-Roch. Hence, the $\mathcal{G}$ equivariant cohomology of $\mathcal{C}^{ss}$ has the Poincaré series

$$\frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - (\frac{(1+t)^{2g}}{(1-t^2)^2})^2 \sum_{i\geq 0} t^{4i+2g}$$

and thus the Poincaré series of the moduli space $M(2,1)$ is

$$\frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)(1-t^4)}.$$

□

Similarly, the Poincaré series for $H^*_{\mathcal{G}}(\mathcal{C}(2,0)^{ss})$ is

$$\frac{(1+t)^{2g}((1+t^3)^{2g}-t^{2g+2}(1+t)^{2g})}{(1-t^2)(1-t^4)}.$$

6.6 Stratifications of the singular moduli spaces and partial desingularizations

Suppose r, d are not coprime. Then the moduli space $M(r,d)$ is singular but it can be decomposed into a union of locally closed smooth subvarieties in a nice way, known as the Kirwan stratification [27].

Let I denote the set of finite sequences $\rho = \{(m_i, r_i)\}$ of pairs of natural numbers such that $\sum_{i=1}^s m_i r_i = r$, $r_1 \geq r_2 \geq \cdots \geq r_s$. For each ρ, we assign a subgroup R^ρ of $GL(p)$, isomorphic to $\prod_{i=1}^s GL(m_i)$ as follows: Fix an isomorphism

$$\prod_{i=1}^s \mathbb{C}^{m_i} \otimes \mathbb{C}^{p_i} \cong \mathbb{C}^p$$

for $p_i = d_i + r_i(1-g)$, $d_i = r_i \frac{d}{r}$. $GL(m_i)$ acts on the first factor $\mathbb{C}^{m_i}$ in the standard way and thus we get an embedding $\prod GL(m_i) \subset GL(p)$.

Let $\mathcal{R} = \{R^\rho | \rho \in I\}$. Then $\mathcal{R}$ is a set of representatives of the conjugacy classes of the identity components of all reductive subgroups of $GL(p)$ which appear as stabilizers of points in $A(r,d)^{ss}$ whose orbits are closed in $A(r,d)^{ss}$.

Let Z^ρ denote the subset of points in $A(r,d)^{ss}$ whose stabilizers have the identity component R^ρ.

Proposition 6.13 *[27] If E is semistable and $gr(E) = m_1E_1 \oplus \cdots \oplus m_sE_s$ where $E_1, \cdots, E_s$ are nonisomorphic stable bundles with $\mu(E_i) = \mu(E)$ for all i, then*

$$\dim \mathrm{Aut} E \leq \dim \prod_{i=1}^s GL(m_i)$$

with equality if and only if $E \cong gr(E)$, in which case

$$\mathrm{Aut} E \cong \prod_{i=1}^s GL(m_i).$$

Hence, $GL(p)Z^\rho$ is the set of $h \in A(r,d)^{ss}$ such that $E(h) \cong gr(E(h)) \cong m_1E_1 \oplus \cdots \oplus m_sE_s$ and $Z^\rho/N^\rho = GL(p)Z^\rho/GL(p)$ is a smooth locally closed subvariety of $A(r,d)^{ss}/\!\!/GL(p)$ where $N^\rho = \prod_{i=1}^s (GL(m_i) \times GL(p_i))/\mathbb{C}^*$ is the normalizer of R^ρ in $GL(p)$.

It is not difficult to see ([27] Lemma 3.11) that

$$Z^\rho \cong \prod_{i=1}^s A(r_i,d_i)^s \setminus \Delta_\rho^\#$$

where $\Delta_\rho^\#$ is the "diagonal" part, i.e. $\{(E_i)|E_i \cong E_j \text{ for some } i \neq j\}$. Therefore,

$$Z^\rho/N^\rho \cong \prod_{i=1}^s M(r_i,d_i)^s \setminus \Delta_\rho$$

where Δ_ρ denotes (the quotient of) the "diagonal" part. Since every point in the moduli space $M(r,d)$ is represented by a "split" bundle, i.e. $E \cong gr(E)$, we get the decomposition

$$M(r,d) = \cup_{\rho \in I}(\prod_{i=1}^s M(r_i,d_i)^s \setminus \Delta_\rho)$$

into smooth locally closed subvarieties.

This decomposition is in fact a Whitney stratification. Here, we just check the normal structure to each stratum. Let $h \in A(r,d)^{ss}$ be a split element, i.e. $E(h) \cong gr(E(h)) \cong m_1E_1 \oplus \cdots \oplus m_sE_s$. Then the normal space to GZ^ρ, for $\rho = \{(m_i,r_i)\}$ with $r_i = \text{rank}(E_i)$, is known to be isomorphic to [3]

$$\nu^\rho := [\oplus_{i\neq j} H^1(\text{Hom}(E_i,E_j)) \otimes \text{Hom}(\mathbb{C}^{m_i},\mathbb{C}^{m_j})] \oplus [\oplus_i H^1(End(E_i)) \otimes sl(m_i)].$$

The local normal form theorem (or the slice theorem) now tells us that the normal to the stratum Z^ρ/N^ρ in $M(r,d)$ is isomorphic to the quotient of the vector space

$$\nu^\rho/\!\!/R^\rho$$

which is a complex cone.

Example 6.14 Consider $M(2,0)$. There are only three elements in I, namely,

$$\{(1,2)\},\{(1,1),(1,1)\},\{(2,1)\}.$$

The smallest stratum in $M(2,0)$ parametrizes bundles of the form $E \cong L \oplus L$ for some line bundle L and it is isomorphic to the Jacobian $Jac_{d/2}$. The normal space to the stratum is

$$H^1(End(L)) \otimes sl(2) /\!\!/ GL(2) \cong \mathbb{C}^g \otimes sl(2) /\!\!/ GL(2)$$

where $GL(2)$ acts by conjugation on the second factor.

The next stratum parametrizes bundles of the form $E \cong L_1 \oplus L_2$ for nonisomorphic line bundles L_1, L_2 and it is isomorphic to $Jac_{d/2} \times Jac_{d/2} \setminus \Delta$ where Δ is the diagonal. The normal space to this stratum is

$$H^1(L_1^* \otimes L_2) \oplus H^1(L_2^* \otimes L_1) /\!\!/ \mathbb{C}^* \times \mathbb{C}^* \cong \mathbb{C}^{g-1} \times \mathbb{C}^{g-1} /\!\!/ \mathbb{C}^* \times \mathbb{C}^*$$

where the first $\mathbb{C}^*$ acts on L_1 while the second $\mathbb{C}^*$ acts on L_2 by weight 1. $\square$

Now, we consider the partial desingularization of the moduli space. The points with the largest stabilizers are those $h \in A(r,d)^{ss}$ such that $E(h) \cong L \oplus \cdots \oplus L$ for a line bundle L if there is any. We blow up $A(r,d)^{ss}$ along the set $GL(p)Z^{\rho_1}$ of those points for $\rho_1 = \{(r,1)\}$ and let $A_1(r,d)^{ss}$ denote the set of semistable points in this blow-up i.e. $A_1(r,d)$ minus the proper transform of $\{h | gr(E(h)) \cong L \oplus \cdots \oplus L\}$. Next we blow up along (the proper transform of) the set $GL(p)Z^{\rho_2} = \{h | E(h) \cong L \oplus \cdots \oplus L \oplus L',$ for nonisomorphic $L, L'\}$ for $\rho_2 = \{(r-1,1),(1,1)\}$ and remove the proper transform of $\{h | gr(E(h)) \cong L \oplus \cdots \oplus L \oplus L',$ for nonisomorphic $L, L'\}$ to get $A_2(r,d)$. We can keep blowing up till we get $\tilde{A}(r,d)^{ss}$ on which $SL(p)$ acts locally freely. The partial desingularization of $M(r,d)$ is then

$$\tilde{M}(r,d) = \tilde{A}(r,d)^{ss}/SL(p).$$

This is the same as the space obtained from $M(r,d)$ by blowing up along each stratum Z^ρ/N^ρ one by one.

Chapter 7

Vector bundles of rank 2 and odd degree

In this chapter, we prove the structure theorem and the Mumford conjecture for $H^*(M(2,1))$ which determine the cup product structure completely. The proof is based on Zagier's technique but it is improved and clarified. We also prove the strong Mumford conjecture which was open before this work [23, 49].

7.1 Cohomology ring of $M(2,1)$

Let $M = M(2,d)$ be the moduli space of rank 2 stable holomorphic vector bundles of odd degree over a Riemann surface Σ of genus $g \geq 2$. Tensoring by a line bundle of degree 1 gives us an isomorphism

$$M(2,d) \to M(2,d+2)$$

and thus we may take $d = 4g-3$ without loss of generality. Fix a line bundle L of degree $4g-3$ over Σ and let N be the moduli space of stable vector bundles with determinant L. Clearly, N is a subspace of M and is the fiber, over L, of the determinant map

$$\begin{array}{ccc} N & \hookrightarrow & M \\ & & \downarrow{\scriptstyle det} \\ & & Jac_{4g-3}. \end{array}$$

Since every semistable rank 2 bundles of odd degree are stable and stable bundles have only constant automorphisms (i.e. simple), the moduli spaces M and N are smooth projective varieties of complex dimension $4g-3$ and $3g-3$ respectively. These are fine moduli spaces. In fact, the universal bundle $U \to M \times \Sigma$ is the descent of the pull back of the universal bundle of the Grassmannian via the evaluation map

$$ev : A(2,d)^s \times \Sigma \subset \mathrm{Hol}(\Sigma, Gr(2,p)) \times \Sigma \to Gr(2,p)$$

as explained in the previous chapter. The restriction of U to N then is the universal bundle over $N \times \Sigma$.

Consider the Künneth decompositions of the Chern classes of the universal bundle:

$$c_1(U) = (4g-3) \otimes [\Sigma] + \sum_{i=1}^{2g} d_i \otimes e_i + x \otimes 1$$

$$c_2(End(U)) = 2\alpha \otimes [\Sigma] + 4 \sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1$$

where e_i are symplectic basis of $H^1(\Sigma)$ so that $e_i e_{i+g} = [\Sigma]$. Newstead [40] and Atiyah-Bott [3] showed that α, β, ψ_i, d_i generate $H^*(M)$ while (restricted) α, β, ψ_i generate $H^*(N)$.

By Narasimhan-Seshadri theorem, N is diffeomorphic to the quotient of $\Phi^{-1}(-1)$ by $SU(2)$ where $\Phi : SU(2)^{2g} \to SU(2)$ is the *group valued moment map* given by $\Phi((a_i)) = \prod [a_i, a_{i+g}]$. Let Σ' denote Σ minus a small disk and let $\tilde{\Sigma}'$ be the universal cover of Σ' and consider the vector bundle

$$(\Phi^{-1}(-1) \times End(\mathbb{C}^2)) \times_{\pi_1(\Sigma')} \tilde{\Sigma}' \to \Phi^{-1}(-1) \times \Sigma'$$

which $g \in \pi_1(\Sigma')$ maps $(\zeta, v) \in \Phi^{-1}(-1) \times End(\mathbb{C}^2)$ to $(\zeta, Ad\zeta(g)v)$. If we glue a trivial bundle over the deleted small disk, then we get a vector bundle which descends to $End(U) \to N \times \Sigma$ since the stabilizers act trivially on the fiber. It is easy to see that the $(\mathbb{Z}/2)^{2g}$ action on N by multiplication on each component (a_i) by ± 1 preserves the bundle $End(U)$. Therefore, this finite group action preserves the generators α, β, ψ_i since $c_2(End(U))$ are unchanged and thus $H^*(N)$ remains fixed.

Tensoring gives us a finite covering $N \times Jac_0 \to M$ with the structure group $(\mathbb{Z}/2)^{2g}$. As the finite group acts trivially on the cohomology $H^*(N)$ as well as $H^*(Jac)$, we get an isomorphism of rings

$$H^*(M) \cong H^*(N) \otimes H^*(Jac).$$

The goal of this chapter is to describe the cohomology ring $H^*(N)$.

Since α, β, ψ_i generate $H^*(N)$, we have a surjection

$$\mathbb{Q}[\alpha, \beta] \otimes \wedge(\psi_i) \to H^*(N)$$

and so it suffices to understand the kernel ideal.

Let $f : M \times \Sigma \to M$ denote the projection onto the first component. The pushforward $f_!(U)$ is a vector bundle since $H^1(\Sigma, E) = 0$ by semistability due to a lemma in the previous chapter. By Riemann-Roch, the rank of the bundle is $(4g-3)-2(g-1) = 2g-1$. Therefore,

$$c_r(f_!(U)) = 0 \in H^*(M) = H^*(N) \otimes H^*(Jac)$$

for $r > 2g - 1$. Mumford conjectured that the Künneth components of $c_r(f_!(U))$ form a complete system of relations for $H^*(N)$. This was proved by Kirwan in [31] and by Zagier [54]. But it turns out that a lot of the relations are redundant. We will prove a stronger version of this conjecture in §3. Namely, the Mumford relations from only the first vanishing Chern class $c_{2g}(f_!U)$ generate the relation ideal in $\mathbb{Q}[\alpha, \beta] \otimes \wedge(\psi_i)$.

Thaddeus used the Verlinde formula and Riemann-Roch to compute the intersection pairing [48] and left it as a "number theoretic exercise" to deduce the ring structure. Zagier [54] picked up this problem and proved the structure theorem of $H^*(N)$ which was also first conjectured by Mumford. We will develop his technique further to prove the theorem in a purely combinatorial way in the next section. This theorem not only describes the cup product structure but also the mapping class group action precisely. This was also proved by Newstead-King [23], Baranovsky [5] and Siebert-Tian [45].

7.2 The structure theorem

In this section, we recall and generalize Zagier's technique in [54], to prove the "structure theorem" for $H^*(N)$.

Let α, β, ψ_i, d_i are as in the previous section and let $\gamma = -2\sum_{i=1}^{g} \psi_i\psi_{i+g}$ and $\xi = \alpha\beta + 2\gamma$. We first compute the Chern classes of $f_! U$. As observed by Zagier ([54], p557), for our purpose, we may assume that $x = 0$. Let $\gamma = -2\sum_{i=1}^{g} \psi_i\psi_{i+g}$ and $\xi = \alpha\beta + 2\gamma$. Then α, β, γ generate the invariant part of $\mathbb{Q}[\alpha, \beta] \otimes \wedge(\psi_i)$ with respect to the $Sp(2g)$ action on ψ_i's and so do α, β, ξ.

Let

$$\lambda_j = (2g - \frac{3}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}}) \otimes [\Sigma] + \sum_{i=1}^{2g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i + (-1)^j \frac{\sqrt{\beta}}{2} \otimes 1.$$

Then one can check that $\lambda_1 + \lambda_2 = c_1(\mathcal{U})$ and $\lambda_1 \lambda_2 = c_2(\mathcal{U})$. Hence,

$$\begin{aligned}
&ch(U) = e^{\lambda_1} + e^{\lambda_2} \\
&= \sum_{j=1,2} (1 + (2g - \frac{3}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}}) \otimes [\Sigma]) \\
&(1 + \sum_{i=1}^{2g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i + \frac{1}{2}(\sum_{i=1}^{2g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i)^2) exp((-1)^j \frac{\sqrt{\beta}}{2}) \\
&= \sum_{j=1,2} (1 + \sum_{i=1}^{2g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i \\
&+ (2g - \frac{3}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}})) \otimes [\Sigma]) exp((-1)^j \frac{\sqrt{\beta}}{2})
\end{aligned} \tag{7.1}$$

and by Grothendieck-Riemann-Roch

$$\begin{aligned}
&ch(f_!(U)) = f_*(ch(U)(1 - (g-1)[\Sigma])) \\
&= \sum_{j=1,2} (g - \frac{1}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}})) exp((-1)^j \frac{\sqrt{\beta}}{2}) \\
&= \sum_{j=1,2} \sum_{n \geq 0} \frac{(g - \frac{1}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}})((-1)^j \sqrt{\beta}/2)^n}{n!} \\
&- (n+1) \frac{(\sum_{i=1}^{g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}}))((-1)^j \sqrt{\beta}/2)^n}{(n+1)!}
\end{aligned} \tag{7.2}$$

Noticing

$$log \prod (1 + u_k) = \sum_{n \geq 1} \frac{(-1)^{n-1} \sum u_k^n}{n}$$

$$\sum e^{u_k} = \sum_{n\geq 0} \frac{\sum u_k^n}{n!},$$

we deduce that

$$log\, c(f_! U) = \sum_{j=1,2} (g - \frac{1}{2} - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}}) log(1 + (-1)^j \frac{\sqrt{\beta}}{2}) - \frac{\sum_{i=1}^{g} (\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}})}{1 + (-1)^j \frac{\sqrt{\beta}}{2}}.$$

Therefore,

$$\begin{aligned} c(f_! U)_{-2t} &= \sum_{r\geq 0} c_r(f_! U)(-2t)^r \\ &= (1-\beta t^2)^{g-\frac{1}{2}} (\frac{1+t\sqrt{\beta}}{1-t\sqrt{\beta}})^{\frac{\xi}{2\beta\sqrt{\beta}}} exp(\frac{At + 2Bt^2 - 2\gamma t/\beta}{1-\beta t^2}) \\ &= \Phi(t)\, G(t) \end{aligned} \tag{7.3}$$

where

$$G(t) = (1-\beta t^2)^g exp(\frac{At + 2Bt^2 - 2\gamma t^3}{1-\beta t^2}),$$

$$\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = (1-\beta t^2)^{-\frac{1}{2}} exp(\alpha t + \xi \sum_{k\geq 1} \frac{\beta^{k-1} t^{2k+1}}{2k+1}) = (1-\beta t^2)^{-\frac{1}{2}} e^{-\frac{2\gamma t}{\beta}} (\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t})^{\frac{\xi}{2\beta\sqrt{\beta}}},$$

$$A = \sum_{i=1}^{g} d_i d_{i+g}, \quad B = \sum_{i=1}^{g} -d_i \psi_{i+g} + d_{i+g}\psi_i.$$

From Riemann-Roch, $f_! U$ is a vector bundle of rank $2g-1$ and therefore $\Phi(t)\, G(t)$ is a polynomial of degree$\leq 2g-1$.

Now, the question is how to read the Mumford relations off the expression. Our strategy is the following: Let $\{u_n\}$ be a basis of $H^*(Jac)$ and let $\{u_n^*\}$ be the dual basis with respect to the top degree class $[\prod_{i=1}^{g} d_i d_{i+g}]$, i.e. $u_i^* u_j / [\prod_{i=1}^{g} d_i d_{i+g}] = \delta_{ij}$. (Poincaré duality! Here, $\cdot/[\prod_{i=1}^{g} d_i d_{i+g}]$ means the coefficient of the top degree class.) Then for any $z \in H^*(M) = H^*(N) \otimes H^*(Jac)$, the coefficient of u_n^* in z is $z u_n / [\prod_{i=1}^{g} d_i d_{i+g}]$. Therefore, $\{c(f_! U) u_n / [\prod_{i=1}^{g} d_i d_{i+g}]\}$ gives us all the Mumford relations.

We need to generalize a lemma of Zagier ([54], p559). Let $\wedge^* H^3 = \oplus_{l=0}^{g} \oplus_{k=0}^{g-l} \gamma^k Prim_l$ be the Lefshetz decomposition of the exterior algebra of $H^3(N) = \mathbb{Q}\{\psi_1, \cdots, \psi_{2g}\}$. Now, let $\sigma_l = \sum_{|I|=l} a_I \psi_I \in Prim_l\{\psi_1, \cdots, \psi_{2g}\}$ be a primitive element and put $\tilde{\sigma}_l = \sum_{|I|=l} a_I d_I \in Prim_l\{d_1, \cdots, d_{2g}\}$. Then we have the following

Lemma 7.1

$$\frac{A^{g-l-p}B^{l+2p}}{(g-l-p)!(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = \frac{(\frac{\gamma}{2})^p}{p!}\sigma_l.$$

$$\frac{A^iB^j}{i!j!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = 0 \ otherwise.$$

Proof The second statement is obvious.

Considering the Sp_{2g} action, we may assume that $\sigma_l = \psi_{g-l+1}\cdots\psi_g$ since $Prim_l$ is an irreducible module. Now, Zagier's original lemma claims

$$\frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!} = \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}/[\prod_{i=1}^{g-l} d_i d_{i+g}].$$

Thus,

$$\begin{aligned}
\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l &= \frac{(-\sum_{i=1}^{g-l}\psi_i\psi_{i+g})^p}{p!}\sigma_l \\
&= \frac{(\sum_{i=1}^{g-l} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g-l} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p}}{(g-l-p)!(2p)!}\sigma_l/[\prod_{i=1}^{g-l} d_i d_{i+g}] \qquad (7.4)\\
&= \frac{(\sum_{i=1}^{g} d_i d_{i+g})^{g-l-p}(\sum_{i=1}^{g} -d_i\psi_{i+g} + d_{i+g}\psi_i)^{2p+l}}{(g-l-p)!(2p+l)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]
\end{aligned}$$

as one can check directly. So we are done. □

Therefore, for $\tilde{\sigma}_l \in Prim_l(d_i)$,

$$\begin{aligned}
&G(t)A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= \sum_{r,s}(1-\beta t^2)^g \frac{A^r t^r}{r!(1-\beta t^2)^r}\frac{2^s B^s t^{2s}}{s!(1-\beta t^2)^s} exp(\frac{-2\gamma t^3}{1-\beta t^2})A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{k-p}2^{l+2p}t^{g+l+3p-k}\frac{A^{g-l-p}}{(g-l-p-k)!}\frac{B^{l+2p}}{(l+2p)!}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \\
&= exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p (1-\beta t^2)^{k-p}2^{l+2p}t^{g+l+3p-k}\frac{(g-l-p)!}{(g-l-p-k)!}\frac{(\frac{\gamma}{2})^p}{p!}\sigma_l \\
&= 2^l(1-\beta t^2)^k t^{g+l-k} exp(\frac{-2\gamma t^3}{1-\beta t^2})\sum_p \frac{(g-l-p)!}{(g-l-p-k)!}\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l.
\end{aligned} \qquad (7.5)$$

When $k = 0$,

$$G(t)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = 2^l t^{g+l} exp(\frac{-2\gamma t^3}{1-\beta t^2}) \sum_p \frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l$$
$$= 2^l t^{g+l}\sigma_l. \tag{7.6}$$

When $k = 1$,

$$G(t)A\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]$$
$$= 2^l t^{g+l-1}(1-\beta t^2) exp(\frac{-2\gamma t^3}{1-\beta t^2}) \sum_p (g-l-p)\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \tag{7.7}$$
$$= 2^l t^{g+l-1}(g-l-(g-l)\beta t^2 - 2\gamma t^3)\sigma_l.$$

When $k = 2$,

$$G(t)A^2\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]$$
$$= 2^l t^{g+l-2}(1-\beta t^2)^2 exp(\frac{-2\gamma t^3}{1-\beta t^2}) \sum_p (g-l-p)(g-l-p-1)\frac{(2\gamma t^3)^p}{p!(1-\beta t^2)^p}\sigma_l \tag{7.8}$$
$$= 2^l t^{g+l-2}((g-l)(g-l-1) - 2(g-l)(g-l-1)\beta t^2 - (4g-4l-4)\gamma t^3$$
$$+ (g-l)(g-l-1)\beta^2 t^4 + (4g-4l-4)\beta\gamma t^5 + 4\gamma^2 t^6)\sigma_l.$$

Recall that c_n was defined to be the n-th coefficient of $\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = (1-\beta t^2)^{-\frac{1}{2}} exp(\alpha t + \xi \sum_{k\geq 1} \frac{\beta^{k-1}t^{2k+1}}{2k+1}) = (1-\beta t^2)^{-\frac{1}{2}} e^{-\frac{2\gamma t}{\beta}} (\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t})^{\frac{\xi}{2\beta\sqrt{\beta}}}$. One can check that the sequence $\{c_n\}$ is determined by the following recursion formula;

$$nc_n = \alpha c_{n-1} + (n-1)\beta c_{n-2} + 2\gamma c_{n-3}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \frac{\alpha^2+\beta}{2}$, $c_3 = \frac{\alpha^3+5\alpha\beta+4\gamma}{3!}$, etc.

Let I_g be the ideal of $\mathbb{Q}[\alpha,\beta,\gamma]$ generated by c_i for $i \geq g$. Then by the above recursion formula, I_g is in fact generated by just three elements, c_g, c_{g+1}, c_{g+2}.

By the formula (7.6) above,

$$c(f_! U)_{-2t}\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] = 2^l t^{g+l}\Phi(t)\sigma_l$$

and thus $c_n\sigma_l$ is a relation for $n \geq g-l$. Hence, $Prim_l \otimes I_{g-l}$ is a set of relations. One can compute the Poincaré series [1] as in [54, 23] to show that $\oplus_{l=0}^{g} Prim_l \otimes I_{g-l}$ is in fact

[1] See the similar computation in the next chapter §2.

the complete set of relations. Therefore, we obtained a new proof of the following structure theorem [45, 23, 54, 5].

Theorem 7.2 $H^*(N) \cong \oplus_{l=0}^{g} Prim_l \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-l}$.

Since all the relations are from the Mumford relations, we deduce that the Mumford conjecture is true.

From Atiyah-Bott's gauge theory description, we can prove that the mapping class group action on $H^*(N)$ factors through the symplectic group action on ψ_i's. Hence the above structure theorem describes the action precisely, as $Prim_l$'s are irreducible $Sp(2g)$-modules.

7.3 The strong Mumford conjecture

Though we proved that the Mumford relations generate the relation ideal, many of them are redundant. At the end of his excellent survey article [49], Thaddeus raised the following question (see also [23]): "...... But there is considerable redundancy among the Mumford relations. In fact, the shape of Harder-Narasimhan formula suggests that the cohomology ring is isomorphic to the quotient of $\mathbb{Q}[\alpha, \beta] \otimes \wedge^*(\psi_i)$,......, by the ideal freely generated over $\mathbb{Q}[\alpha, \beta]$ by the Mumford relations for $r = 2g$ only,...... This question remains open." We provide an answer to this question.

It is not literally true in the sense that the Mumford relations are not independent over $\mathbb{Q}[\alpha, \beta]$. For example, the coefficient of $d_{g+1}d_{g+2}\cdots d_{2g-1}$ in $c_{2g}(f_!U)$ is up to constant $(\alpha^2 - \beta)\psi_1\psi_2\cdots\psi_{g-1}$ and the coefficient of $d_g d_{g+1} d_{g+2}\cdots d_{2g-1}d_{2g}$ is up to constant $\alpha\psi_1\psi_2\cdots\psi_{g-1}$ as one can deduce from the computation below. [2]

However, we can prove the following

[2] Another argument for $\mathbb{Q}[\alpha, \beta]$-dependence of the Mumford relations from $c_{2g}(f_!U)$ is as follows. Because of the Harder-Narasimhan formula, they are dependent over $\mathbb{Q}[\alpha, \beta]$ if and only if the $\mathbb{Q}[\alpha, \beta]$-module generated by them is a proper subspace of the relation ideal. Since $\dim N = 6g - 6$ and $\deg\gamma^g = 6g$, γ^g is a relation but clearly it is not contained in the $\mathbb{Q}[\alpha, \beta]$-module generated by the Mumford relations from $c_{2g}(f_!U)$.

Theorem 7.3 *The Mumford relations from only $c_{2g}(f_!U)$ generate the whole relation ideal in $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$.*

Therefore, the relations from $c_r(f_!U)$, for $r>2g$, are all redundant.

Note that we used just the formula (7.6) to prove the structure theorem. We now use the other two formulas to prove our main theorem. As a by-product of the proof, we will see that the $\mathbb{Q}[\alpha,\beta,\gamma]$-module in $\mathbb{Q}[\alpha,\beta]\otimes\wedge^*(\psi_i)$ generated by the Mumford relations from $c_{2g}(f_!U)$ is a proper subspace of the relation ideal.

Proof From the structure theorem above, we note that it suffices to show that $c_{g-l}\sigma_l$, $c_{g-l+1}\sigma_l$ and $c_{g-l+2}\sigma_l$ belong to the ideal J generated by the Mumford relations for $r=2g$ only, i.e.

$$\{c_{2g}(f_!U)A^k\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] \mid 0\le k\le g-l,\ 0\le l\le g,\ \tilde{\sigma}_l\in Prim_l(d_i)\}.$$

By (7.6), $c_{g-l}\sigma_l$ is, up to constant,

$$\begin{aligned} c_{2g}(f_!U)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}] &= \mathrm{Coeff}_{t^{2g}}(\Phi(t)G(t)\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]) \\ &= \mathrm{Coeff}_{t^{2g}}(\Phi(t)2^l t^{g+l}\sigma_l) = 2^l c_{g-l}\sigma_l \end{aligned} \tag{7.9}$$

and so $c_{g-l}\sigma_l\in J$. By (7.7) and the recursion forumla, $c_{2g}(f_!U)A\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]$ is, up to constant,

$$((g-l)c_{g-l+1}-(g-l)\beta c_{g-l-1}-2\gamma c_{g-l-2})\sigma_l = (-c_{g-l+1}+\alpha c_{g-l})\sigma_l$$

and thus $c_{g-l+1}\sigma_l\in J$ unless $l=g$. Similarly, by (7.8), $c_{2g}(f_!U)A^2\tilde{\sigma}_l/[\prod_{i=1}^{g} d_i d_{i+g}]$ is, up to constant,

$$[(-2g+2l+2)c_{g-l+2}-2\alpha c_{g-l+1}+((2g-2l-1)\beta+\alpha^2)c_{g-l})]\sigma_l$$

and thus $c_{g-l+2}\sigma_l\in J$ unless $l\ge g-1$.

In particular, c_g, c_{g+1}, c_{g+2} are in J and $c_{g-l}\sigma_l\in J$. As any $\sigma_l\in Prim_l$ is a sum of elements of the form $\sigma_{l-1}\sigma_1$ for some $\sigma_{l-1}\in Prim_{l-1}$ and $\sigma_1\in Prim_1$,

$$c_{g-l+1}\sigma_l = (c_{g-l+1}\sigma_{l-1})\sigma_1\in J$$

and similarly

$$c_{g-l+2}\sigma_l = (c_{g-l+2}\sigma_{l-2})\sigma_1'\sigma_1 \in J.$$

Note that for $l = 0, 1$, the same follows from the fact that c_g, c_{g+1}, c_{g+2} are in J. Therefore, we proved our theorem. $\square$

Remark 7.4 Technically, Zagier assumed $x = 0$ in the expression for $c_1(U)$, which amounts to saying that $\Phi(t)G(t) = (1 - 2vt)^{2g-1}c(f_!U)_{\frac{-2t}{1-2vt}}$ for a class $v \in H^2(Jac)$. But the coefficient of t^{2g} in $c(f_!U)_{-2t}$ and that in $\Phi(t)G(t)$ are same up to constant as one can check by expanding both in t. Therefore, for our purpose, we can take $\Phi(t)G(t) = c(f_!U)_{-2t}$.

Remark 7.5 One may ask whether the Mumford relations, for $r = 2g$ only, generate the whole relations as a $\mathbb{Q}[\alpha, \beta, \gamma]$ module. The answer is negative because the module does not contain $c_3\sigma_{g-1}$. Actually, those elements are the only missing piece and the $\mathbb{Q}[\alpha, \beta, \gamma]$-module generated by the relations from $c_{2g+1}(f_!U)$ as well as those from $c_{2g}(f_!U)$ is the whole set of relations as one can deduce from the above formulas. Therefore, even as a $\mathbb{Q}[\alpha, \beta, \gamma]$-module, the relations from $c_r(f_!U)$ for $r > 2g + 1$ are all redundant.

Chapter 8

Vector bundles of rank 2 and even degree

We study $M := M(2,0)$. First we will prove (analogues of) the structure theorem and the Mumford conjecture for the equivariant cohomology ring $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$. Then using the splitting theorem, we will compute the intersection cohomology $IH^*(M(2,0))$ and consider some applications.

8.1 The equivariant cohomology ring

In this section, we recall various facts about the equivariant cohomology of the moduli space of rank 2 semistable vector bundles over a Riemann surface Σ of genus $g > 2$.

Let $\mathcal{C}$ be the space of holomorphic structures on a fixed rank 2 complex Hermitian vetor bundle $\mathcal{E}$ of even degree, $4g-2$. It is an affine space of Cauchy-Riemann operators based on $\Omega^{0,1}(End\mathcal{E})$ and therefore contractible. If we denote by $\mathcal{C}^{ss}$ the subspace of semistable holomorphic structures, then the moduli space of semistable bundles can be thought of as the "infinite dimensional symplectic quotient" $\mathcal{C}^{ss}/\!\!/\mathcal{G}_c$ where $\mathcal{G}_c$ is the complexification of the gauge group $\mathcal{G}$ of the principal $U(2)$-bundle associated to $\mathcal{E}$. Obviously, constants commute with Cauchy-Riemann operators and thus act trivially on $\mathcal{C}^{ss}$. We put $\overline{\mathcal{G}}_c = \mathcal{G}_c/\mathbb{C}^*$. Then $\overline{\mathcal{G}}_c$ acts freely on the open dense subset $\mathcal{C}^s$ of stable holomorphic structures, but not on $\mathcal{C}^{ss}$. So, we consider the homotopy quotient

$$\mathcal{C}^{ss}_{\overline{\mathcal{G}}} = \mathcal{C}^{ss} \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}.$$

In this chapter, the equivariant cohomology means the cohomology of this homotopy quotient.

Even though there does not exist any (holomorphic) universal bundle for the moduli space $\mathcal{C}^{ss}/\!\!/\mathcal{G}_c$ (See [41], Thm 2), we do have a (topological) universal bundle for the homotopy quotient as follows (See [3], p579): Let W be the obvious universal vector bundle over $\mathcal{C}^{ss}\times\Sigma$. By taking quotient of the pullback of $\mathbb{P}W$, over $\mathcal{C}^{ss}\times E\overline{\mathcal{G}}\times\Sigma$, by $\overline{\mathcal{G}}$, we get a projective bundle $\mathbb{P}\mathcal{U}$ over $\mathcal{C}^{ss}_{\overline{\mathcal{G}}}\times\Sigma$. This bundle lifts to a vector bundle $\mathcal{U}$ because the obstruction for the lifting vanishes by the existence of a vector bundle W' over

$$\mathcal{C}^{ss}_{\mathcal{G}}\times\Sigma\subset\mathcal{C}_{\mathcal{G}}\times\Sigma=B\mathcal{G}\times\Sigma=Map_{4g-2}(\Sigma,BU(2))\times\Sigma$$

which is the pullback of the universal bundle over $BU(2)$ via the evaluation map

$$ev: Map_{4g-2}(\Sigma,BU(2))\times\Sigma\to BU(2).$$

The universal bundle $\mathcal{U}$ is continuous in the $\mathcal{C}^{ss}_{\overline{\mathcal{G}}}$ direction and holomorphic in the Σ direction.

The moduli space of vector bundles can be also thought of as the moduli space of flat connections as follows [3]: Let $\mathcal{A}$ be the space of connections on the principal $U(2)$-bundle associated to $\mathcal{E}$. It is an affine space based on the space of $u(2)$-valued 1-forms. By taking the $(0,1)$-part of a connection, we get an isomorphism $\mathcal{A}\to\mathcal{C}$. The Morse stratification on $\mathcal{A}$ with respect to the norm square of curvature is equivalent to the Shatz stratification of $\mathcal{C}$ and the map induces an identification

$$\mathcal{A}_{flat}/\mathcal{G}=\mathcal{C}^{ss}/\!\!/\mathcal{G}_c=M(2,0)$$

where $\mathcal{A}_{flat}$ is the subspace of flat connections. Moreover, if we let $\mathcal{G}_0=\{g:\Sigma\to U(2)|g(p)=id\}$ with $p\in\Sigma$, then $\mathcal{G}_0$ acts freely on $\mathcal{A}_{flat}$ and

$$\mathcal{A}_{flat}/\mathcal{G}_0=Hom(\pi_1(\Sigma),U(2))=:\mathbf{R}^{\#}_{U(2)}.$$

Therefore, $\mathcal{C}^{ss}_{\overline{\mathcal{G}}}=\mathcal{A}_{flat}\times_{\overline{\mathcal{G}}}E\overline{\mathcal{G}}=\mathbf{R}^{\#}_{U(2)}\times_{PU(2)}EPU(2)$ and thus

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})\cong H^*_{\overline{\mathcal{G}}}(\mathcal{A}_{flat})\cong H^*_{PU(2)}(\mathbf{R}^{\#}_{U(2)})\cong H^*_{SU(2)}(\mathbf{R}^{\#}_{U(2)})$$

With the above identification, we can construct

$$End\mathcal{U} \to (\mathbf{R}^{\#}_{U(2)} \times_{PU(2)} EPU(2)) \times \Sigma$$

as follows: Let $\tilde{\Sigma}$ be the universal cover of Σ. Consider

$$(\mathbf{R}^{\#}_{U(2)} \times End(\mathbb{C}^2)) \times_{\pi_1(\Sigma)} \tilde{\Sigma} \to \mathbf{R}^{\#}_{U(2)} \times \Sigma$$

where $g \in \pi_1(\Sigma)$ maps $(\phi, v) \in \mathbf{R}^{\#}_{U(2)} \times End(\mathbb{C}^2)$ to $(\phi, Ad\,\phi(g)\,v)$. It is a vector bundle of rank 4 which induces $End\mathcal{U}$ over $(\mathbf{R}^{\#}_{U(2)} \times_{PU(2)} EPU(2)) \times \Sigma$ by pulling back and taking quotient.

Now, we consider the special structure: Fix a line bundle L of degree $4g-2$ over Σ. Let $\mathcal{C}^{ss}_L$ denote the subspace of semistable holomorphic structures with determinant L. Then we have the following fibration by taking determinant

$$\begin{array}{ccc} \mathcal{C}^{ss}_L & \longrightarrow & \mathcal{C}^{ss} \\ & & \downarrow \\ & & Jac_{4g-2} \end{array}$$

By the same arguments for $\mathcal{C}^{ss}$, the homotopy quotient $(\mathcal{C}^{ss}_L)_{\overline{\mathcal{G}}} = \mathcal{C}^{ss}_L \times_{\overline{\mathcal{G}}} E\overline{\mathcal{G}}$ is homotopically equivalent to $\mathbf{R}^{\#}_{SU(2)} \times_{PU(2)} EPU(2)$ where $\mathbf{R}^{\#}_{SU(2)} = Hom(\pi_1(\Sigma), SU(2))$. Therefore,

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L) \cong H^*_{PU(2)}(\mathbf{R}^{\#}_{SU(2)}) \cong H^*_{SU(2)}(\mathbf{R}^{\#}_{SU(2)}).$$

As $U(2) = SU(2) \times_{\mathbb{Z}/2} U(1)$,

$$\mathbf{R}^{\#}_{U(2)} = \mathbf{R}^{\#}_{SU(2)} \times_{(\mathbb{Z}/2)^{2g}} Jac$$

and

$$\mathbf{R}^{\#}_{U(2)} \times_{PU(2)} EPU(2) = (\mathbf{R}^{\#}_{SU(2)} \times_{PU(2)} EPU(2)) \times_{(\mathbb{Z}/2)^{2g}} Jac.$$

According to [3], $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss})$ is generated by the classes α, β, ψ_i, d_i defined by the Künneth decompositions of the Chern classes as follows:

$$c_1(\mathcal{U}) = (4g-2) \otimes [\Sigma] + \sum_{i=1}^{2g} d_i \otimes e_i + x \otimes 1$$

$$c_2(End(\mathcal{U})) = 2\alpha \otimes [\Sigma] + 4\sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1$$

where e_i are symplectic basis of $H^1(\Sigma)$ so that $e_i e_{i+g} = [\Sigma]$. Similarly, $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L)$ is generated by the (restricted) classes α, β, ψ_i.

Because $\mathbb{Z}/2$ is the center of $SU(2)$, $(\mathbb{Z}/2)^{2g}$ action preserves

$$End\mathcal{U} \to (\mathbf{R}^{\#}_{SU(2)} \times_{PU(2)} EPU(2)) \times \Sigma.$$

Hence, it acts trivially on α, β, ψ_i and it also acts trivially on the cohomology of the Jacobian. Therefore, we get

$$H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}) \cong H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L) \otimes H^*(Jac). \tag{8.1}$$

To understand the ring structure of the cohomology, we have only to understand the relations for $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L)$. Let

$$f : \mathcal{C}^{ss}_{\overline{\mathcal{G}}} \times \Sigma \to \mathcal{C}^{ss}_{\overline{\mathcal{G}}}$$

be the projection onto the first component. Then $f_!\mathcal{U}$ is a vector bundle of rank $2g$ by Riemann-Roch and semistability. Therefore, $c_r(f_!\mathcal{U})$ are relations for $r \geq 2g+1$. By choosing a basis of $H^*(Jac)$, we get 2^{2g} sequences of relations in $H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss}_L)$ via (8.1). Those relations are called the *Mumford relations*.

In the next section, we will prove that the Mumford relations generate the whole space of relations in $\mathbb{Q}[\alpha, \beta] \otimes \wedge^*(\psi_i)$. (Mumford's conjecture.) Moreover, we will find a finite number of classes that generate all the other relations and prove the "structure theorem".

8.2 The structure theorem

We first compute the generating function for the Chern classes of $f_!\mathcal{U}$ and then read off the Mumford relations in order to prove the structure theorem. The Mumford conjecture is a consequence of our proof.

Recall that

$$c_1(\mathcal{U}) = (4g-2) \otimes [\Sigma] + \sum_{i=1}^{2g} d_i \otimes e_i + x \otimes 1$$

$$c_2(End(\mathcal{U})) = 2\alpha \otimes [\Sigma] + 4\sum_{i=1}^{2g} \psi_i \otimes e_i - \beta \otimes 1.$$

As observed by Zagier ([54], p557), for our purpose, we may assume that $x = 0$. Let $\gamma = -2\sum_{i=1}^{g} \psi_i\psi_{i+g}$ and $\xi = \alpha\beta + 2\gamma$. Then α, β, γ generate the invariant part of $\mathbb{Q}[\alpha,\beta] \otimes \wedge(\psi_i)$ with respect to the $Sp(2g)$ action on ψ_i's and so do α, β, ξ.

Let

$$\lambda_j = (2g-1-(-1)^j \frac{\xi}{2\beta\sqrt{\beta}}) \otimes [\Sigma] + \sum_{i=1}^{2g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i + (-1)^j \frac{\sqrt{\beta}}{2} \otimes 1.$$

Then one can check that $\lambda_1 + \lambda_2 = c_1(\mathcal{U})$ and $\lambda_1 \lambda_2 = c_2(\mathcal{U})$. Hence,

$$\begin{aligned}
&ch(\mathcal{U}) = e^{\lambda_1} + e^{\lambda_2}\\
&= \sum_{j=1,2} (1 + (2g-1-(-1)^j \frac{\xi}{2\beta\sqrt{\beta}}) \otimes [\Sigma])\\
&(1 + \sum_{i=1}^{2g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i + \frac{1}{2}(\sum_{i=1}^{2g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i)^2) exp((-1)^j \frac{\sqrt{\beta}}{2})\\
&= \sum_{j=1,2} (1 + \sum_{i=1}^{2g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}}) \otimes e_i\\
&+ (2g-1-(-1)^j \frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}})) \otimes [\Sigma]) exp((-1)^j \frac{\sqrt{\beta}}{2})
\end{aligned} \tag{8.2}$$

and by Grothendieck-Riemann-Roch

$$\begin{aligned}
&ch(f_!(\mathcal{U})) = f_*(ch(\mathcal{U})(1-(g-1)[\Sigma])\\
&= \sum_{j=1,2} (g - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}} - \sum_{i=1}^{g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}})) exp((-1)^j \frac{\sqrt{\beta}}{2})\\
&= \sum_{j=1,2}\sum_{n\geq 0} \frac{(g - (-1)^j \frac{\xi}{2\beta\sqrt{\beta}})((-1)^j\sqrt{\beta}/2)^n}{n!}\\
&- (n+1)\frac{(\sum_{i=1}^{g}(\frac{d_i}{2} - (-1)^j \frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2} - (-1)^j \frac{\psi_{i+g}}{\sqrt{\beta}}))((-1)^j\sqrt{\beta}/2)^n}{(n+1)!}
\end{aligned} \tag{8.3}$$

Noticing

$$log \prod (1+u_k) = \sum_{n\geq 1} \frac{(-1)^{n-1}\sum u_k^n}{n}$$

$$\sum e^{u_k} = \sum_{n\geq 0} \frac{\sum u_k^n}{n!},$$

we deduce that

$$log\, c(f_!\mathcal{U}) = \sum_{j=1,2}(g-(-1)^j\frac{\xi}{2\beta\sqrt{\beta}})log(1+(-1)^j\frac{\sqrt{\beta}}{2}) - \frac{\sum_{i=1}^g(\frac{d_i}{2}-(-1)^j\frac{\psi_i}{\sqrt{\beta}})(\frac{d_{i+g}}{2}-(-1)^j\frac{\psi_{i+g}}{\sqrt{\beta}})}{1+(-1)^j\frac{\sqrt{\beta}}{2}}.$$

Therefore,

$$\begin{aligned} c(f_!\mathcal{U})_{-2t} &= \sum_{r\geq 0} c_r(f_!\mathcal{U})(-2t)^r \\ &= (1-\beta t^2)^g(\frac{1+t\sqrt{\beta}}{1-t\sqrt{\beta}})^{\frac{\xi}{2\beta\sqrt{\beta}}} exp(\frac{At+2Bt^2-2\gamma t/\beta}{1-\beta t^2}) \\ &= \Phi(t)\, G(t) \end{aligned} \tag{8.4}$$

where

$$G(t) = (1-\beta t^2)^g exp(\frac{At+2Bt^2-2\gamma t^3}{1-\beta t^2}),$$

$$\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = exp(\alpha t + \xi\sum_{k\geq 1}\frac{\beta^{k-1}t^{2k+1}}{2k+1}) = e^{-\frac{2\gamma t}{\beta}}(\frac{1+\sqrt{\beta}t}{1-\sqrt{\beta}t})^{\frac{\xi}{2\beta\sqrt{\beta}}},$$

$$A = \sum_{i=1}^g d_i d_{i+g}, \quad B = \sum_{i=1}^g -d_i\psi_{i+g} + d_{i+g}\psi_i.$$

From Riemann-Roch, $f_!\mathcal{U}$ is a vector bundle of rank $2g$ and therefore $\Phi(t)\, G(t)$ is a polynomial of degree$\leq 2g$.

Notice that $G(t)$ is the same as in Chapter 7 §2 though $\Phi(t)$ is different. As before let $\wedge(\psi_i) = \oplus_{l=0}^g \oplus_{k=0}^{g-l} \gamma^k Prim_l$ be the Lefshetz decomposition of the exterior algebra of $H^3_{\mathcal{G}}(\mathcal{C}_L^{ss})$. Let $\sigma_l = \sum_{|I|=l} a_I\psi_I \in Prim_l(\psi_i)$ be a primitive element and put $\tilde{\sigma}_l = \sum_{|I|=l} a_I d_I \in Prim_l(d_i)$. Then we proved that for $\tilde{\sigma}_l \in Prim_l(d_i)$,

$$G(t)\tilde{\sigma}_l/[\prod_{i=1}^g d_i d_{i+g}] = 2^l t^{g+l}\sigma_l. \tag{8.5}$$

As a consequence,

$$2^l t^{g+l}\Phi(t)\sigma_l = c(f_!(\mathcal{U}))_{-2t}\tilde{\sigma}_l/[\prod_{i=1}^g d_i d_{i+g}]$$

and thus $\Phi(t)\sigma_l$ is a polynomial of degree $\leq g-l$.

Proposition 8.1 *$\oplus_{l=0}^g Prim_l \otimes I_{g-l}$ is a subspace of the relation ideal, where I_n is the ideal of $\mathbb{Q}[\alpha,\beta,\xi]$ generated by $\{c_i | i \geq n+1\}$.*

Recall that c_n was defined to be the n-th coefficient of $\Phi(t) = \sum_{n=0}^{\infty} c_n t^n = exp(\alpha t + \xi \sum_{k\geq 1} \frac{\beta^{k-1}t^{2k+1}}{2k+1})$. One can readily check that the sequence $\{c_n\}$ is determined by the following recursion formula;

$$nc_n = \alpha c_{n-1} + (n-2)\beta c_{n-2} + 2\gamma c_{n-3}$$

where $c_0 = 1$, $c_1 = \alpha$, $c_2 = \frac{\alpha^2}{2}$, $c_3 = \frac{\alpha^3+2\xi}{3!}$, etc. Therefore, I_n is in fact generated by just three elements c_{n+1}, c_{n+2}, c_{n+3}.

Now, put $c_{n,k} = \sum_{i=0}^{k} \frac{1}{i!}\binom{n-1-i}{k-i}(2\gamma)^i \beta^{k-i} c_{n-k-i}$, for $0 \leq k < n$. Then by modifying a lemma of Zagier in [54], we get the following

Lemma 8.2

$$(-1)^k c_{n,k} = \sum_{i=0}^{\infty} (-1)^i (\binom{n-k+i}{i} + \binom{n-k+i-1}{i-1}) c_{k-i} c_{n+i}.$$

In particular, $c_{n,k}$ belongs to the ideal generated by c_n, c_{n+1}, c_{n+2}.

Proof One can check, as in [54], that both sides satisfy $kc_{n,k} = (n-1)\beta c_{n-1,k-1} + 2\gamma c_{n-2,k-1}$. □

Lemma 8.3 *$c_n \equiv \alpha^n/n!$ modulo the ideal generated by ξ.*

Proof It follows immediately from an induction with the recursion formula. □

It is also easy to check the following variation of a lemma in [23].

Lemma 8.4 *The leading term of $c_{n,k}$ is, up to constant, $\xi^k \alpha^{n-2k}$ for $0 \leq k \leq [\frac{n}{2}]$, and $\xi^{n-k}\beta^{2k-n}$ for $[\frac{n}{2}] < k < n$, with respect to the reverse lexicographical order where $\alpha > \xi > \beta$. If we put $c_{n,\frac{n}{2}} = 12\gamma c_{n,\frac{n-3}{2}} - \frac{n-1}{2}\alpha^2 c_{n,\frac{n-1}{2}}$, for odd n, then its leading term is up to constant $\xi^{\frac{n+1}{2}}$.*

Proof Obvious from the definition of $c_{n,k}$ and Lemma 8.3. □

From the above lemmas, we deduce that $\mathbb{Q}[\alpha, \beta, \xi]/I_g$ is a quotient of the vector space spanned by

$$\{\alpha^i \beta^j \xi^k \,|\, (1)\, i + 2k \leq g, \quad (2) \text{ if } k \geq 1 \text{ then } j + 2k \leq g\}. \tag{8.6}$$

Recall that deg$\alpha = 2$, deg$\beta = 4$, deg$\xi = 6$. We can compute the Poincaré series for this graded vector space.

Lemma 8.5

$$P_t(\mathbb{Q}\{\alpha^i\beta^j\xi^k \mid \textit{(1) } i+2k \le g, \quad \textit{(2) if } k \ge 1 \ \textit{ then } j+2k \le g\}) = \frac{\frac{1-t^{6g+6}}{1-t^6} - t^{2g+2}\frac{1-t^{2g+2}}{1-t^2}}{(1-t^2)(1-t^4)}.$$

Proof Combinatorial exercise. □

As a consequence,

$$P_t(\mathbb{Q}[\alpha,\beta,\xi]/I_g) \le \frac{\frac{1-t^{6g+6}}{1-t^6} - t^{2g+2}\frac{1-t^{2g+2}}{1-t^2}}{(1-t^2)(1-t^4)}.$$

Therefore, we have

Lemma 8.6

$$P_t(\oplus_{l=0}^{g} Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l}) \le \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Proof Combinatorial exercise. □

As $\oplus_{l=0}^{g} Prim_l \otimes I_{g-l}$ is a subspace of the relation ideal, there is a surjection $\oplus_{l=0}^{g} Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l} \to H^*_{\mathcal{G}}(\mathcal{C}_L^{ss})$. From Atiyah-Bott's equivariant Morse theoretic argument, it is well-known that

$$P_t(H^*_{\mathcal{G}}(\mathcal{C}_L^{ss})) = \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Therefore, together with the previous lemma, we deduce that $\oplus_{l=0}^{g} Prim_l \otimes I_{g-l}$ is, in fact, equal to the whole relation ideal. Since all the relations were derived from the Chern classes of the pushforward bundle, we conclude that the obvious analogue of the Mumford conjecture is true. In summary, we have the following "structure theorm".

Theorem 8.7

$$H^*_{\mathcal{G}}(\mathcal{C}_L^{ss}) \cong H^*_{SU(2)}(\mathbf{R}^{\#}_{SU(2)}) \cong \oplus_{l=0}^{g} Prim_l \otimes \mathbb{Q}[\alpha,\beta,\xi]/I_{g-l}.$$

Let $\mathbf{R}^{\#}_{red}$ be the set of homomorphisms of $\pi_1(\Sigma)$ into $SU(2)$ whose images are abelian. Then the inclusion $\mathbf{R}^{\#}_{red} \hookrightarrow \mathbf{R}^{\#} := \mathbf{R}^{\#}_{SU(2)}$ induces a homomorphism $H^*_{SU(2)}(\mathbf{R}^{\#}) \to H^*_{SU(2)}(\mathbf{R}^{\#}_{red})$. Here, according to [43], $H^*_{SU(2)}(\mathbf{R}^{\#}_{red})$ is the $\mathbb{Z}/2$-invariant part of the algebra freely generated by q_i and r, of degree 1 and 2 respectively. Moreover, α restricts to $-2w$, β to $4r^2$, ψ_i to $-2rq_i$ and γ to $4r^2w$ respectively, where $w = -2\sum_{i=1}^{g} q_i q_{i+g}$. We can now compute the kernel of the homomorphism.

Corollary 8.8 *The kernel of the homomorphism $H^*_{SU(2)}(\mathbf{R}^{\#}) \to H^*_{SU(2)}(\mathbf{R}^{\#}_{red})$ is generated by the single element $\xi = \alpha\beta + 2\gamma$.*

Proof Trivially, ξ is in the kernel. We have

$$H^*_{SU(2)}(\mathbf{R}^{\#}_{red}) = [\oplus_{l=0}^{g} Prim_l\{q_1, \cdots, q_{2g}\} \otimes \mathbb{Q}[r, w]/w^{g-l+1}]^{\mathbb{Z}/2}.$$

Because the homomorphism respects the symplectic action, we have only to consider the invariant part. From Lemma 8.3, one can easily deduce that

$$(\mathbb{Q}[\alpha, \beta, \xi]/I_{g-l})/(\xi) = \mathbb{Q}[\alpha, \beta]/\alpha^{g-l+1}.$$

The right hand side obviously injects into $[\mathbb{Q}[r, w]/w^{g-l+1}]^{\mathbb{Z}/2}$. This completes the proof. $\square$

Note that ξ is actually the V class in [43].

Let $\mathbf{R} = \mathbf{R}^{\#}/SU(2)$, which is homeomorphic to the moduli space N. Then the natural map $\mathbf{R}^{\#} \times_{SU(2)} ESU(2) \to \mathbf{R}$ induces a homomorphism $H^*(\mathbf{R}) \to H^*_{SU(2)}(\mathbf{R}^{\#})$.

Corollary 8.9 *The image in $H^{6g-6}_{SU(2)}(\mathbf{R}^{\#})$ of a top degree class in $H^{6g-6}(\mathbf{R})$ is a constant multiple of $\alpha^{g-2}\beta^{g-2}\xi$ where $\mathbf{R} = \mathbf{R}^{\#}/SU(2)$ and $\mathbf{R}_{red} = \mathbf{R}^{\#}_{red}/SU(2)$.*

Proof As the image of a top degree class by the composition map $H^{6g-6}(\mathbf{R}) \to H^{6g-6}(\mathbf{R}_{red}) = 0 \to H^{6g-6}_{SU(2)}(\mathbf{R}^{\#}_{red})$ is zero, the image of the class by the composition map $H^{6g-6}(\mathbf{R}) \to H^{6g-6}_{SU(2)}(\mathbf{R}^{\#}) \to H^{6g-6}_{SU(2)}(\mathbf{R}^{\#}_{red})$ is also zero. Thus, the image in $H^{6g-6}_{SU(2)}(\mathbf{R}^{\#})$ is in the kernel of the restriction map considered in Corollary 8.8. And from Theorem 8.7, Corollary 8.8, and our choice of the basis (8.6), we deduce that the kernel has only one generator in dimension $6g-6$, namely $\alpha^{g-2}\beta^{g-2}\xi$. $\square$

8.3 Intersection cohomology

In this section, we combine the splitting theorem of Chapter 5 and Theorem 8.7 above to deduce the intersection cohomology of the moduli space of rank 2 holomorphic vector bundles of even degree with a fixed determinant line bundle, over a Riemann surface Σ of genus $g > 2$.

Recall that the moduli space $N = M(2,0)_L$ can be constructed as a GIT quotient $N = A(2,d)_L^{ss}/\!/SL(p)$. To apply the splitting theorem, we need to check the weakly balanced condition.

Proposition 8.10 *The $SL(p)$-action on $A(r,d)^{ss}$ is balanced for any rank r and degree d.*

Proof Let E be a semistable vector bundle such that $E = m_1E_1 \oplus \cdots \oplus m_sE_s$ where E_i's are non-isomorphic stable bundles with the same slope. Then the identity component of Stab$E$ is $R = S(\prod_{i=1}^s GL(m_i))$ where S means the subset of elements whose determinant is 1. The normal space to GZ_R^{ss} at E is ([3], [27])

$$\begin{aligned} H^1(\Sigma, End'_{\oplus}E) &= H^1(\Sigma, \oplus_{i,j}(m_im_j - \delta_{ij})Hom(E_i,E_j)) \\ &= \oplus_{i,j}H^1(\Sigma,(m_im_j-\delta_{ij})E_i^*\otimes E_j) \end{aligned} \tag{8.7}$$

More precisely,

$$\begin{aligned} H^1(\Sigma, End'_{\oplus}E) = \oplus_{i<j}\, &[H^1(\Sigma, E_i^*\otimes E_j)\otimes Hom(\mathbb{C}^{m_i},\mathbb{C}^{m_j}) \oplus H^1(\Sigma, E_i\otimes E_j^*)\otimes Hom(\mathbb{C}^{m_j},\mathbb{C}^{m_i})] \\ &\oplus [\oplus_i H^1(\Sigma, End\,E_i)\otimes sl(m_i)] \end{aligned} \tag{8.8}$$

Because E_i is not isomorphic to E_j for $i \neq j$, $H^0(\Sigma, E_i^*\otimes E_j) = 0 = H^0(\Sigma, E_i\otimes E_j^*)$ and thus

$$\begin{aligned} \dim H^1(\Sigma, E_i^*\otimes E_j) &= -RR(\Sigma, E_i^*\otimes E_j) = (\text{rank}E_i)(\text{rank}E_j)(g-1) \\ &= -RR(\Sigma, E_i\otimes E_j^*) = \dim H^1(\Sigma, E_i\otimes E_j^*) \end{aligned} \tag{8.9}$$

Therefore, the weights of the representation of R on $H^1(\Sigma, End'_{\oplus}E)$ are symmetric with respect to the origin. Obviously, it implies that the action is linearly balanced. As each stabilizer subgroup $R' \subset R$ is conjugate to $S(\prod_{i=1}^s GL(m_i'))$ for a "subdivision" $(m_1', m_2', ...)$

of $(m_1, m_2, \ldots)$, it is easy to check that such $R \cap N^{R'}/R'$ action on the fixed point set by R' is also linearly balanced. $\Box$

By a theorem of Kirwan [26], the equivariant cohomology $H^*_{SL(p)}(A(2,d)^{ss}_L)$ is canonically isomorphic to $H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L) \cong H^*_{SU(2)}(Hom(\pi_1(\Sigma), SU(2)))$ up to a bound which goes to infinity as d goes to infinity. Since the intersection cohomology

$$IH^*(N) = IH^*(Hom(\pi_1(\Sigma), SU(2))/SU(2))$$

is finite dimensional, by taking d large enough, we can use

$$H^*_{\mathcal{G}}(\mathcal{C}^{ss}_L) \cong H^*_{SU(2)}(Hom(\pi_1(\Sigma), SU(2)))$$

instead of $H^*_{SL(p)}(A(2,d)^{ss}_L)$.

The splitting theorem now tells us that $V_{A(2,d)^{ss}_L}$ contains all the information about the intersection cohomology. We use the notations of Chapter 5. Recall from Chapter 6 that

$$\mathcal{R}(A(2,d)_L) = \{SL(2), \mathbb{C}^*\}.$$

It turns out that we do not have to think about truncation on the $SL(2)$ fixed point set because it is taken care of by that for $\mathbb{C}^*$. For that one has only to observe that $Z_{SL(2)} \subset Z_{\mathbb{C}^*}$ and that $n_{SL(2)} = 3g-3 > n_{\mathbb{C}^*} = 2g-3$.

We have to consider the following map

$$H^*_{SL(p)}(A(2,d)^{ss}_L) \to H^*_{SL(p)}(SL(p) \times_{N^{\mathbb{C}^*}} Z^{ss}_{\mathbb{C}^*}) = [H^*(Jac) \otimes \mathbb{C}[\rho]]^{\mathbb{Z}/2}$$

where Jac means the Jacobian variety and $\mathbb{Z}/2$ acts as -1 on both components. Let d_i be the basis of $H^1(Jac)$ defined as the Künneth coefficients of the first Chern class of a universal line bundle corresponding to e_i. Recall that α is mapped to $w = -2\sum_{i=1}^{g} d_i d_{i+g}$, β to $4\rho^2$, and ψ_i to $-2\rho d_i$.

Intersection Betti numbers

From the work of Atiyah and Bott, we know that the Poincaré series of the equivariant cohomology is

$$\frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

If one uses the Lefschetz decomposition of the exterior algebra in ψ_i's, it is an easy combinatorial exercise to show that the image of the restriction $H^*_{SL(p)}(A(2,d)^{ss}_L) \to [H^*(Jac) \otimes \mathbb{C}[\rho]]^{\mathbb{Z}/2}$ is precisely

$$[H^*(Jac) \otimes \rho^{g-1}\mathbb{C}[\rho]]^{\mathbb{Z}/2}$$

whose Poincaré series is

$$\frac{1}{2}\{\frac{(1+t)^{2g}(t^2)^{g-1}}{1-t^2} + \frac{(1-t)^{2g}(-t^2)^{g-1}}{1+t^2}\}.$$

Hence we get a closed formula for the intersection Poincaré series [27].

Theorem 8.11 *Let $IP_t(N) = \sum_{i\geq 0} t^i \dim IH^i(N)$. Then*

$$IP_t(N) = \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} - \frac{1}{2}\{\frac{(1+t)^{2g}(t^2)^{g-1}}{1-t^2} + \frac{(1-t)^{2g}(-t^2)^{g-1}}{1+t^2}\}.$$

Mapping class group action and Hodge structure

From the structure theorem and the Gröbner basis given in the proof in §2, we deduce the following

Theorem 8.12 *Let W_m be the vector space spanned by*

$$\{\alpha^i\beta^j\xi^k \mid (1)\ i+2k \leq m,\ (2)\ j+2k \leq m\ (3)\ \text{if } k=0 \text{ then } j < [\frac{m}{2}]\}.$$

Then

$$V_{A(2,d)^{ss}_L} = \oplus_{l=0}^{g} Prim_l(\psi_i) \otimes W_{g-l} \cong IH^*(N).$$

It is well known that the mapping class group action on

$$IH^*(N) = IH^*(Hom(\pi_1(\Sigma), SU(2))/SU(2))$$

factors through the symplectic group action on ψ_i's. As mentioned in Chapter 5, the splitting theorem stays valid for symplectic quotients. The action of the mapping class group Γ^1_g (say, on a small neighborhood of the zero set of the moment map in the extended moduli space[1]) commutes with the conjugation action of $SU(2)$ and thus preserves our

[1] See Chapter 9 §1.

splitting. Therefore, the above theorem describes the mapping class group action on the intersection cohomology precisely. The dimension of W_{g-l} should be thought of as the multiplicity of the irreducible representation $Prim_l(\psi_i)$. It is now a combinatorial exercise to compute the dimension of W_g whose Poincaré series equals

$$\frac{1}{(1-t^2)(1-t^4)}[\frac{1-t^{6g+6}}{1-t^6}-\frac{1-t^{2g+2}}{1-t^2}\{t^{2g+2}+\frac{(t^2)^{g-1}(1-t^4)+(-t^2)^{g-1}(1-t^2)^2}{2}\}].$$

This agrees with the computation of Nelson in [38] which is based on Kirwan's calculation [27].

The above theorem also describes the Hodge structure on $IH^*(N)$. Choose the basis e_i of $H^1(\Sigma)$ such that e_i are of type (1,0) and e_{i+g} are of type (0,1) for $1 \le i \le g$. Then α is a class of type (1,1), β is of type (2,2), ψ_i is of type (1,2) and ψ_{i+g} is of type (2,1) for $1 \le i \le g$. So, the above theorem gives a Hodge structure on $V_{A(2,d)^{ss}_L}$ and hence the intersection cohomology has the induced Hodge structure by the splitting theorem.

Casson's invariant

Casson's invariant is defined by counting intersection points algebraically of two Lagrangian (real) subvarieties in the moduli space N. Namely, given a Heegaard splitting of an integral homology 3-sphere, $M = H_1 \cup_\Sigma H_2$, we get two Lagrangian subvarieties $\mathrm{Hom}(\pi_1(H_i), SU(2))/SU(2)$ for $i = 1, 2$ in $\mathrm{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$ which is homeomorphic to the moduli space. Their common intersection with the singular part consists of a single element, the trivial representation, at which they intersect transversely. We can perturb one of the subvarieties while keeping their intersection at the trivial element transverse and then count the number of intersection points algebraically with respect to the naturally given orientations [1]. This is Casson's invariant. In terms of gauge theory, this invariant can be thought of as the "number" of irreducible flat connections on M.

For the past 15 years, various attempts have been made to generalize this to a larger class of 3 manifolds in the one direction (e.g. Walker's generalization to rational homology 3 spheres) and to Lie groups other than $SU(2)$ in the other (e.g. Cappell, Lee and Miller's generalized Casson invariants for $SU(n)$). In an attempt to generate a knot invariant using Casson's idea, Frohman and Nicas defined an *intersection homology invariant* for a 3 manifold M with a nontrivial element $\zeta \in H_2(M)$ [13]. In fact, they defined a cobordism functor for 3 manifolds with at least 2 boundary components each of which has genus > 1. The 3 manifold invariant is obtained by considering the cobordism

$$\overline{M \setminus \Sigma \times I}$$

obtained by removing a neighborhood of a closed surface Σ representing the homology class. The invariant is defined as the supertrace of the cobordism functor. The knot invariant is thus obtained by choosing a Seifert surface for any homologically trivial knot in a rational homology 3 sphere and then "capping off" the Seifert surface.

The cobordism functor of Frohman and Nicas is defined as follows: Let H be an oriented compact 3 manifold with boundaries $F, F_1, \cdots, F_l$ with $l \geq 1$. Consider the moduli space of $SU(r)$ connections on H whose restrictions to the boundary components have given degrees. This embeds into the product of the moduli spaces N_F, N_{F_i} of $SU(r)$ connections of given degrees over the boundary Riemann surfaces. It turns out that this subvariety intersects the singular part in a sufficiently nice way (s-allowable) so that we get a homomorphism

$$Q(H) : IH^*(N_F) \to \prod_{1 \leq i \leq l} IH^*(N_{F_i})$$

by intersecting an intersection cycle with the moduli for H. [2]

Since every cobordism with ≥ 2 boundary components is a product of cobordisms with only one "in" or "out" component as those in the previous paragraph, we get a functor assigning to each Riemann surface of genus > 1 the intersection homology of the moduli space of vector bundles, and to each cobordism a homomorphism between the intersection homology groups for the in and out components. In terms of gauge theory, one can think of the functor as propagation of flat connections along the cobordism. The 3 manifold (or knot) invariant defined by this cobordism functor can be thought of as the number of irreducible $SU(r)$ connections on the 3 manifold (or the knot complement).

[2] For a more gauge theoretic description, see [38].

From now on, we consider only rank 2 and even degree case. If the knot is *fibred*, i.e. the knot complement is a surface bundle over circle with monodromy diffeomorphism $h : \Sigma \to \Sigma$, the (polynomial) knot invariant is the Lefshetz polynomial of the induced map

$$h^* : IH^*(N) \to IH^*(N)$$

which can be expressed in terms of the Alexander polynomial because of the following proposition ([13], Proposition 5.4.)

Proposition 8.13 *The Lefschetz polynomial of the induced map*

$$h^* : H^*(Jac) \to H^*(Jac)$$

is the (un-normalized) Alexander polynomial $c(t)$ of the fibred knot K.

The Lefschetz polynomial of the equivariant cohomology $H^*_{SL(p)}(A(2,d)^{ss}_L)$ is just (up to sufficiently large degree)

$$\frac{c(t^3) - t^{2g+2}c(t)}{(1-t^2)(1-t^4)}.$$

The truncation for the definition of $V_{A(2,d)^{ss}_L}$ is equivariant for the $Sp(2g)$ action and hence we subtract out

$$\frac{1}{2}\{\frac{(t^2)^{g-1}c(t)}{1-t^2} + \frac{(-t^2)^{g-1}c(-t)}{1+t^2}\}$$

to get the Lefschetz polynomial for the intersection homology by the splitting theorem:

$$L(h) = \frac{c(t^3) - t^{2g+2}c(t)}{(1-t^2)(1-t^4)} - \frac{1}{2}\{\frac{(t^2)^{g-1}c(t)}{1-t^2} + \frac{(-t^2)^{g-1}c(-t)}{1+t^2}\}.$$

In particular, if we take $t = 1$, we get

$$\lambda_{2,0} = -\frac{1}{2}g(g-1)c(1) + \frac{1}{4}\{(-1)^g c(-1) - c(1)\} + \frac{1}{2}c''(1)$$

by applying L'Hospital's law twice and the fact $c'(1) = gc(1)$. Let $\tilde{c}(t) = t^{-g}c(t)$ be the normalized Alexander polynomial. Then by substitution we get the following theorem ([13], Theorem 6.4.)

Theorem 8.14 *The intersection Lefschetz number of the monodromy action on N is*

$$\lambda_{2,0} = \frac{1}{4}\{\tilde{c}(-1) - \tilde{c}(1)\} + \frac{1}{2}\tilde{c}''(1).$$

Intersection pairings

From Corollary 8.9, we can easily deduce the following

Corollary 8.15 *The fundamental class in $IH^{6g-6}(N)$ corresponds to a constant multiple of $\alpha^{g-2}\beta^{g-2}\xi$ in $V_{A(2,d)_L^{ss}}$.*

Because we can compute the cup product efficiently by using the structure theorem of the equivariant cohomology and the Gröbner basis, we can also compute the intersection pairings of $IH^*(N)$ by the splitting.

Example 8.16 Let $g = 4$. Then the fundamental class is $\alpha^2\beta^2\xi$.

1. From our choice of Gröbner basis, the degree 6 part of W_4 is $W_4^6 = \mathbb{C}\{\alpha^3, \alpha\beta, \xi\}$ and the degree 12 part is $W_4^{12} = \mathbb{C}\{\alpha^4\beta, \alpha\beta\xi, \xi^2\}$. One can check that the intersection matrix for these bases is
$$\begin{pmatrix} 56 & -4 & 0 \\ -20 & 1 & -1 \\ -4 & -1 & 3 \end{pmatrix}$$
whose determinant is $-144 \neq 0$.

2. $W_4^8 = \mathbb{C}\{\alpha^4, \alpha^2\beta, \alpha\xi\}$ and $W_4^{10} = \mathbb{C}\{\alpha^3\beta, \alpha^2\xi, \beta\xi\}$. The intersection matrix is
$$\begin{pmatrix} 56 & 24 & -4 \\ -20 & -4 & 1 \\ -4 & 0 & -1 \end{pmatrix}$$
whose determinant is $-288 \neq 0$.

□

Direct computation using the Gröbner basis shows

$$\alpha^m\beta^n = -m!b_{g-n-1}\frac{\alpha^{g-2}\beta^{g-2}\xi}{(g-2)!}$$

for $m+2n=3g-3$, $n<g-1$ and b_k are given by

$$\frac{t}{\tanh t}=\sum_{k\geq 0} b_k t^{2k}$$

at least for low genus. Hence, if we take $\tau=\frac{\alpha^{g-2}\beta^{g-2}\xi}{(g-2)!(-4)^{g-1}}$, then we get

$$<\kappa(\alpha^i\beta^j),\kappa(\alpha^k\beta^l)>=-(-4)^{g-1}m!b_{g-n-1}$$

for $i+k=m$, $j+l=n$, $m+2n=3g-3$, $n<g-1$. In principle, it is a number theoretic or combinatorial exercise to deduce the above formula from the structure theorem but it seems very difficult to achieve this in practice.

The above pairing formula, however, can be justified by using the computation of intersection pairings on the moduli spaces of parabolic bundles by Jeffrey and Kirwan [20] which will be reviewed in the next chapter.

Finally, let us point out that the formula can be derived by applying Thaddeus's technique in [48] formally or by using Donaldson theory over $\Sigma\times S^2$. This formal computation was carried out by some physicists.

Chapter 9

Vector bundles of higher rank

We complete this work with a brief survey of higher rank case. For smooth moduli spaces, i.e. when rank and degree are coprime, the cohomology ring is completely determined by intersection pairng by Poincaré duality. For rank 2 odd degree case, Thaddeus [48] and Donaldson [9] obtained formulas for the pairing, which were generalized to arbitrary rank case by Jeffrey and Kirwan [20] by using nonabelian localization principle described in Chapter 4. In fact, they also computed the pairings for the moduli spaces of parabolic bundles. For singular moduli spaces, i.e. noncoprime case, we can use their results to compute the pairing of the intersection homology. This will be part of [32] and it will be only sketched here.

9.1 Smooth moduli spaces

This section is from [20].

Let $N(r,d)$ be the moduli space of (semi)stable holomorphic vector bundles of rank r, degree d and fixed determinant on a compact Riemann surface Σ of genus $g \geq 2$. We assume in this section that the rank is coprime to the degree so that the moduli space is a smooth projective variety of real dimension $(2g-2)(r^2-1)$.

There is a universal bundle $U \to N(r,d) \times \Sigma$ and the Künneth components a_i, b_i^j, f_i of the Chern classes of U generate the cohomology ring $H^*(N(r,d))$ from [3]. These classes come from the equivariant cohomology classes $H^*_{\mathcal{G}}(\mathcal{A}_{flat})$ which we denote by the same notations. The Betti numbers can be computed by Morse theory inductively.

We wish to compute the intersection pairing by applying the nonabelian localization principle to the extended moduli space [21]: Let $c = exp(\frac{2\pi i d}{r})I \in SU(r)$ and consider the fibred product of the maps $\Phi : SU(r)^{2g} \to SU(r)$ and $e_c = c \cdot exp : su(r) \to SU(r)$

$$\begin{array}{ccc} M(c) & \xrightarrow{-\mu} & su(r) \\ \downarrow & & \downarrow e_c \\ SU(r)^{2g} & \xrightarrow{\Phi} & SU(r). \end{array}$$

Explicitly, let $P : SU(r)^{2g} \times su(r) \to SU(r)$ be the map given by

$$P((h_j), \Lambda) = \prod [h_j, h_{j+g}] c^{-1} exp(-\Lambda).$$

Then $M(c) = P^{-1}(I)$. The map $-\mu$ above is the projection to $su(r)$. In fact, μ is the moment map for the K-action on the smooth part. Moreover the extended moduli space has the following properties.

Proposition 9.1 *[21]*

- *$M(c)$ is smooth near $(h, \Lambda) \in SU(r)^{2g} \times su(r)$ for which $z(h) \cap ker(d(exp))_\Lambda \neq 0$ where $z(h)$ is the Lie algebra of the stabilizer of h.*

- *There is an invariant 2-form ω on $SU(r)^{2g} \times su(r)$ whose restriction to an open dense subset of $M(c)$ containing $M(c) \cap SU(r)^{2g} \times 0$ is symplectic.*

- *A moment map $\mu : M(c) \to su(r)^*$ is given by the restriction of (-1) times the projection to $su(r)$ with $su(r)$ identified with $su(r)^*$ by the invariant inner product.*

- *$M(c)$ is smooth in a neighborhood of $\mu^{-1}(0)$.*

- *The moduli space $N(r,d)$ is the symplectic quotient $M(c) \cap \mu^{-1}(0)/SU(r)$.*

- *The classes a_i, b_i^j, f_i lifts to classes $\tilde{a}_i, \tilde{b}_i^j, \tilde{f}_i \in H_K^*(M(c))$.*

The problem here is that $M(c)$ is singular and noncompact. Jeffrey and Kirwan avoided singularities by considering the equivariant Poincaré dual of $M(c)$ supported near $P^{-1}(B)$ for a small ball B near I and integrating along $P^{-1}(B)$. Let T denote the maximal torus of $SU(r)$.

Proposition 9.2 *[20] There is a T-equivariant closed differential form $\alpha \in \Omega_T^*(SU(r)^{2g} \times su(r))$ of degree r^2-1 with support contained in a neighborhood $P^{-1}(B)$ of $M(c)$ such that*

$$\int_{SU(r)^{2g}\times su(r)} \eta\alpha = \int_{M(c)} \eta|_{M(c)} \in H_T^*$$

for any T-equivariantly closed form $\eta \in \Omega_T^*(SU(r)^{2g} \times su(r))$ for which the intersection of $P^{-1}(\overline{B})$ with the support of η compact.

Then we can use Martin's trick to reduce the computation to torus quotients. Let

$$\kappa : H^*_{SU(r)}(M(c)) \to H^*_{SU(r)}(M(c)\cap\mu^{-1}(0)) \cong H^*(N(r,d))$$

$$\kappa_T : H^*_{SU(r)}(M(c)) \to H^*_T(M(c)) \to H^*_T(M(c)\cap\mu^{-1}(0)) \to H^*(M(c)\cap\mu^{-1}(0)/T)$$

be the restrictions. Then, we have the following.

Lemma 9.3 *For any $\eta \in H^*_{SU(r)}(M(c))$,*

$$\int_{N(r,d)} \kappa(\eta e^{\overline{\omega}}) = \frac{1}{n!}\int_{M(c)\cap\mu^{-1}(0)/T} \kappa_T(D\eta e^{\overline{\omega}}) = \frac{1}{n!}\int_{P^{-1}(B)\cap\mu^{-1}(0)/T} \kappa_T(D\eta e^{\overline{\omega}}\alpha)$$

where $\overline{\omega} = \omega + \mu$ and D is the product of positive roots of $SU(r)$.

We apply the localization theorem for the T_1-action on $P^{-1}(B)\cap\mu^{-1}(\mathfrak{t}_1)/T'$ where $\mathfrak{t}_1 = Lie(T_1)$, $T_1 = \{(t, t^{-1}, 1, \cdots, 1)\}$ and

$$T' = \{(t_1, t_1, t_3, \cdots, t_r) \in U(1)^r | (t_1)^2 \prod_{j=3}^r t_j = 1\}.$$

The integral can be computed from the local contributions from the T_1-fixed point components by Guillemin-Kalkman localization or by Martin's work. Each fixed point component turns out to be a finite quotient of the product of $M(\hat{c})\cap\mu_{\hat{r}}^{-1}(0)/T_{\hat{r}}$ for lower ranks $\hat{r}$ *periodically*. So, we can apply induction to complete the computation. (Proposition 8.4 of [20].)

Theorem 9.4 *Let $c = diag(c_1, \cdots, c_r) \in T$ be such that the product of no proper subset of $c_1, \cdots, c_r$ is 1. If η is a polynomial in $\tilde{a}_i$ and $\tilde{b}_i^j$ then*

$$\int_{M(c)\cap\mu^{-1}(0)/T} \kappa_T(D\eta e^{\overline{\omega}}) = (-1)^{n_+(g-1)}\mathrm{Res}_{Y_1=0}\cdots\mathrm{Res}_{Y_{r-1}=0}\left(\frac{\sum_{w\in W} e^{<[w\tilde{c}],X>}\int_{T^{2g}}\eta e^{\omega}}{D^{2g-2}\prod_{1\le j\le r-1}(\exp(Y_j)-1)}\right)$$

where $X \in \mathfrak{t}^*$, $Y_j = X_j - X_{j+1}$ for standard basis $\{X_j\}$ and $\tilde{c}$ is an element in the fundamental domain defined by the simple roots for the translation action on $\mathfrak{t}$ such that $exp(\tilde{c}) = c$. Here, n_+ is the number of positive roots, $n(n-1)/2$ and W is the Weyl group of $SU(r-1)$.

The pairing for $H^*(N(r,d))$ now follows from this theorem by Martin's trick, Lemma 9.3.

Corollary 9.5 *Let $c = exp(\frac{2\pi i d}{n})I$ and η as above. Then the pairing is given by*

$$\int_{N(r,d)} \kappa(\eta e^{\overline{\omega}}) = \frac{(-1)^{n_+(g-1)}}{n!}\mathrm{Res}_{Y_1=0}\cdots\mathrm{Res}_{Y_{r-1}=0}\left(\frac{\sum_{w\in W} e^{<[w\tilde{c}],X>}\int_{T^{2g}}\eta e^{\omega}}{D^{2g-2}\prod_{1\le j\le r-1}(\exp(Y_j)-1)}\right).$$

General pairings can be computed by the same process above after introducing formal variables and the result turns out to be equivalent to Witten's formulas in [52].

The Todd class of $N(r,d)$ is given by [39]

$$e^{rf_2}\kappa\left(\prod_{\gamma>0}\frac{\gamma(X)/2}{\sinh\gamma(X)/2}\right)^{2g-2}$$

where γ's are positive roots of $SU(r)$. It is well-known that the Picard group of the moduli space is $\mathbb{Z}$. Let $\mathcal{L}$ be the ample generator whose first Chern class is f_2. Now Riemann-Roch tells us that

$$\dim H^0(N(r,d),\mathcal{L}^k) = \int_{N(r,d)} e^{(k+r)f_2}\kappa\left(\prod_{\gamma>0}\frac{\gamma(X)/2}{\sinh\gamma(X)/2}\right)^{2g-2}.$$

The corollary above now computes the integral to prove the Verlinde formula:

Theorem 9.6

$$\begin{aligned}\dim H^0(N(r,d),\mathcal{L}^k) =& \frac{(-1)^{n_+(g-1)}}{n!}\sum_{w\in W}\mathrm{Res}_{Y_1=0}\cdots\mathrm{Res}_{Y_{r-1}=0}(e^{(k+r)<w\tilde{c},X>} \\ & \int_{T^{2g}} e^{(k+r)\omega}\prod_{\gamma>0}\left(\frac{\gamma(X)}{e^{\gamma(X)/2}-e^{-\gamma(X)/2}}\right)^{2g-2}\frac{1}{\prod(e^{(k+r)Y_j}-1)D^{2g-2}}).\end{aligned} \tag{9.1}$$

9.2 Intersection homology pairings for singular moduli spaces

For completeness, we add this section which will be part of [32].

Let $\mu : M \to \mathbf{k}^*$ be a proper moment map for a Hamiltonian K-space M. Suppose 0 is not a regular value of the moment map. Then we get a singular reduction $M_0 = \mu^{-1}(0)/K$. Let $n = \dim_{\mathbb{R}} M_0$. By the de Rham model for intersection cohomology, the fundamental class in $IH^n(M_0)$ can be represented by a differential form ξ, with compact support in the nonsingular part ${M_0}^s = \mu^{-1}(0)^s/K$ of M_0, whose integral is 1.

Suppose the action of K is free on the open dense subset $\mu^{-1}(0)^s$. Then we have the fibration

$$\begin{array}{ccc} K/T & \longrightarrow & \mu^{-1}(0)^s/T \\ & & \downarrow \pi \\ & & \mu^{-1}(0)^s/K = {M_0}^s \end{array}$$

Via the natural map $H_T^* \to H_T^*(\mu^{-1}(0))$, the product of positive roots D in $H_T^* = S(\mathfrak{t}^*)$ is mapped to a class $\mathcal{D}$ of degree $\dim K - \dim T$ in $H_T^*(\mu^{-1}(0))$. Let $m = n + \dim K - \dim T$. The product of $\mathcal{D}$ and $\pi^*[\xi]$ is in $H_T^m(\mu^{-1}(0))$ and it comes from $H_c^m(\mu^{-1}(0)^s/T)$. Its integral over the smooth part is the order of the Weyl group $|W|$.

Let $\epsilon \in \mathfrak{t}^*$ be a regular value sufficiently close to 0. Then there is a surjective map $f_\epsilon : M_\epsilon = \mu^{-1}(\epsilon)/T \to \mu^{-1}(0)/T$ induced from the gradient flow of $-|\mu|^2$. This is a diffeomorphism over the nonsingular part and thus $\pi^*[\xi] \cdot \mathcal{D}$, considered as an element in $H_c^m(\mu^{-1}(0)^s/T)$, pulls back to an element in $H^m(M_\epsilon)$. The integral of $(\pi f_\epsilon)^*[\xi] \cdot \mathcal{D}$ over M_ϵ is $|W|$.

In summary,

$$\int_{M_0} [\xi] = \frac{1}{|W|} \int_{M_\epsilon} (\pi f_\epsilon)^*[\xi] \cdot \mathcal{D}.$$

Let ζ be the image of $a[\xi]$ for $a \in \mathbb{C}$ by the natural map $\mathbb{C} \cong H_c^n({M_0}^s) \cong H^n(M_0, (M_0)_{sing}) \cong H_K^n(\mu^{-1}(0), \mu^{-1}(0)_{sing}) \to H_K^n(\mu^{-1}(0))$ where $(M_0)_{sing}$, $\mu^{-1}(0)_{sing}$ denote the singular parts. Then the image of ζ via the Kirwan map is the image of $a[\xi]$ via the natural map $H_c^n(M_0^s) \to IH^n(M_0)$ and thus $\kappa(\zeta)[M_0] = a$.

On the other hand, from the following commutative diagram (with an abuse of notations for the maps)

$$\begin{array}{ccccc} H^n_c({M_0}^s)\cong H^n(M_0) & \xrightarrow{\pi^*} & H^n_c(\mu^{-1}(0)^s/T) & \xrightarrow{f_\epsilon^*} & H^n(M_\epsilon) \\ \downarrow & & \downarrow & & \downarrow\cong \\ H^n_K(\mu^{-1}(0)) & \xrightarrow{\pi^*} & H^n_T(\mu^{-1}(0)) & \xrightarrow{f_\epsilon^*} & H^n_T(\mu^{-1}(\epsilon)) \end{array}$$

we deduce that

$$\kappa(\zeta)[M_0] = \frac{1}{|W|}\int_{M_\epsilon}(\pi f_\epsilon)^*\zeta\cdot\mathcal{D}.$$

Suppose the action of K on M is weakly balanced. Let η, ξ be two classes in V_M of complementary degrees with respect to n. Then it was shown in Chapter 5 that their product $\zeta=\eta\xi$ comes from a class in $H^n_c({M_0}^s)$ and that the intersection pairing $<\kappa(\eta),\kappa(\xi)>$ is equal to $\kappa(\zeta)[M_0]$. Therefore,

$$<\kappa(\eta),\kappa(\xi)>= \frac{1}{|W|}\int_{M_\epsilon}(\pi f_\epsilon)^*\zeta\cdot\mathcal{D}.$$

Finally, we return to the moduli space case. The right hand side of the above was computed [20] as in the previous section. Namely, the right hand side equals

$$(-1)^{n+(g-1)}\mathrm{Res}_{Y_1=0}\cdots\mathrm{Res}_{Y_{r-1}=0}\left(\frac{\sum_{w\in W}e^{<[w\tilde{\epsilon}],X>}\int_{T^{2g}}\zeta e^{\omega}}{D^{2g-2}\prod_{1\le j\le r-1}(\exp(Y_j)-1)}\right)$$

if $\zeta=\eta\xi$ is a polynomial of $\tilde{a}_i,\tilde{b}_i^j,\tilde{f}_2$ and $\eta,\xi\in V_{A(r,d)^{ss}_L}$.[1]

Hence,

$$<\kappa(\eta),\kappa(\xi)>= \frac{(-1)^{n+(g-1)}}{n!}\mathrm{Res}_{Y_1=0}\cdots\mathrm{Res}_{Y_{r-1}=0}\left(\frac{\sum_{w\in W}e^{<[w\tilde{\epsilon}],X>}\int_{T^{2g}}\zeta e^{\omega}}{D^{2g-2}\prod_{1\le j\le r-1}(\exp(Y_j)-1)}\right).$$

General pairings follow in a similar fashion.

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[1]The argument above also shows that there is no change in the integral by wall crossing near $\tilde{c}$ and thus we get the formula.

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