DESINGULARIZATIONS OF THE MODULI SPACE OF RANK 2 BUNDLES OVER A CURVE

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Abstract. Let X be a smooth projective curve of genus $g \geq 3$ and M_0 be the moduli space of rank 2 semistable bundles over X with trivial determinant. There are three desingularizations of this singular moduli space constructed by Narasimhan-Ramanan [NR78], Seshadri [Ses77] and Kirwan [Kir86b] respectively. The relationship between them has not been understood so far. The purpose of this paper is to show that there is a morphism from Kirwan's desingularization to Seshadri's, which turns out to be the composition of two blow-downs. In doing so, we will show that the singularities of M_0 are terminal and the plurigenera are all trivial. As an application, we compute the Betti numbers of the cohomology of Seshadri's desingularization in all degrees. This generalizes the result of [BS90] which computes the Betti numbers in low degrees. Another application is the computation of the stringy E-function (see [Bat98] for definition) of M_0 for any genus $g \geq 3$ which generalizes the result of [Kie03].

Dedicated to Professor Ronnie Lee.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 3$. Let M_0 be the moduli space of rank 2 semistable bundles over X with trivial determinant, which is a singular projective variety of dimension $3g-3$. There are three desingularizations of M_0.

(1) Seshadri's desingularization S : fine moduli space of parabolic bundles of rank 4 and degree zero such that the endomorphism algebra of the underlying vector bundle is isomorphic to a specialization of the matrix algebra $M(2)$. This is constructed in [Ses77].
(2) Narasimhan-Ramanan's desingularization N : moduli space of Hecke cycles, as an irreducible subvariety of the Hilbert scheme of conics. This is constructed in [NR78].
(3) Kirwan's desingularization K : the result of systematic blow-ups of M_0, constructed in [Kir86b].

For cohomological computation, K is most useful thanks to the Kirwan theory [Kir85, Kir86a, Kir86b]. On the other hand, S and N are moduli spaces themselves. The relationship between these desingularizations has not been understood.

The first main result of this paper is that there is a birational morphism (Theorem 4.1)

$$\rho : K \to S.$$

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Since both S and K contain the open subset M_0^s of stable bundles, there is a rational map $\rho' : K \dashrightarrow S$. By GAGA and Riemann's extension theorem [Mum76], it suffices to show that ρ' can be extended to a continuous map with respect to the usual complex topology. By Luna's slice theorem, for each point $x \in M_0 - M_0^s$, there is an analytic submanifold W of the Quot scheme whose quotient by the stabilizer H of a point in both W and the closed orbit represented by x is analytically equivalent to a neighborhood of x in M_0. Furthermore, Kirwan's desingularization $\tilde{W}/\!\!/H$ of $W/\!\!/H$ is a neighborhood of the preimage of x in K by construction. Our strategy is to construct a nice family of (parabolic) vector bundles of rank 4 parametrized by $\tilde{W}$, starting from the family of rank 2 bundles parametrized by W, which is induced from the universal bundle over the Quot scheme. This is achieved by successive applications of elementary modifications. Because S is the fine moduli space of such parabolic bundles of rank 4, we get a morphism $\tilde{W} \to S$. This is invariant under the action of H and hence we have a morphism $\tilde{W}/\!\!/H \to S$. Therefore, ρ' extends to a neighborhood of the preimage of x in K.

The second main result of this paper is that the above morphism ρ is in fact the consequence of two blow-downs which can be described quite explicitly (Theorem 5.6). To prove this theorem, we first show that Kirwan's desingularization K can be blown down twice by finding extremal rays. O'Grady in [OGr99] worked out such contractions for the moduli space of rank 2 sheaves on a K3 surface. Since the proofs are almost same as his case, we provide only the outline and necessary modifications in §5.1. Next, we show that ρ is constant along the fibers of the blow-downs and thus ρ factors through the blown-down of K. Finally, Zariski's main theorem tells us that S is isomorphic to the blown-down. Using this theorem, we can compute the discrepancy divisor of $\pi_K : K \to M_0$ (Proposition 5.3) and show that the singularities are terminal. This implies that the plurigenera of M_0 (or K, or S) are all trivial (Corollary 5.4). We conjecture that the intermediate variety between K and S is the desingularization N by Narasimhan and Ramanan.

Our third main result is the computation of the cohomology of S. In [Bal88, BS90], Balaji and Seshadri provides an algorithm for the Betti numbers of S for degrees up to $2g-4$. The cohomology of Kirwan's *partial* desingularization is computed in [Kir86b] and K is obtained as a single blow-up of this partial desingularization. Since it is well-known how to compare cohomology groups after blow-up (or blow-down) along a smooth submanifold of an orbifold ([GH78] p.605), we can compute the cohomology of S.

The last result of this paper is the computation of the stringy E-function of M_0. The stringy E-function is a new invariant of singular varieties, obtained as the measure of the arc space (see, for instance, [Bat98]). From the knowledge of the discrepancy divisor (Proposition 5.3) and explicit descriptions of the exceptional divisors of $\pi_K : K \to M_0$ (Proposition 5.1), we show that

$$E_{st}(M_0) = \frac{(1-u^2v)^g(1-uv^2)^g-(uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} - \frac{(uv)^{g-1}}{2}\Big(\frac{(1-u)^g(1-v)^g}{1-uv} - \frac{(1+u)^g(1+v)^g}{1+uv}\Big).$$

Surprisingly, this is equal to the E-polynomial of the intersection cohomology of M_0 when g is even. For g odd, $E_{st}(M_0)$ is not a polynomial. As a consequence, the stringy Euler number is

$$e_{st}(M_0) := \lim_{u,v\to 1} E_{st}(M_0) = 4^{g-1}.$$

If we denote by e_g the stringy Euler number of the moduli space M_0 for a genus g curve, then the equality

$$\sum_g e_g q^g = \frac{1}{4}\frac{1}{1-4q}$$

holds for degree ≥ 2. The coefficient $\frac{1}{4}$ might be related to the "mysterious" coefficient $\frac{1}{4}$ for the S-duality conjecture test in [VW94].

This paper is organized as follows. In sections 2 and 3, we review Seshadri's and Kirwan's desingularizations respectively. In section 4, we construct a morphism $\rho : K \to S$ by elementary modification. In section 5, we show that ρ is the composition of two blow-downs. In section 6, we compute the cohomology of S and the stringy E-function of M_0.

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2. Seshadri's desingularization

Let X be a compact Riemann surface of genus $g \geq 3$. Let $M_0 = M_X(2, \mathcal{O})$ denote the moduli space of semistable vector bundles over X of rank 2 with trivial determinant. Then M_0 is a *singular* normal projective variety of (complex) dimension $3g - 3$. In [Ses77], Seshadri constructed a desingularization

$$\pi_S : S \to M_0$$

which restricts to an isomorphism on $\rho_S^{-1}(M_0^s)$ where M_0^s denotes the open subset of stable bundles. In fact, this is constructed as the fine moduli space of a moduli problem which we recall in this section. The main reference is [Ses82] Chapter 5 and [BS90].

Fix a point $x_0 \in X$. Let E be a vector bundle of rank 4 and degree 0 on X and $0 \neq s \in E_{x_0}^*$ be a parabolic structure with parabolic weights $0 < a_1 < a_2 < 1$.

Lemma 2.1. *(*[Ses82] *5.III Lemma 5) There are real numbers a_1, a_2 such that for any semistable parabolic bundle (E, s) of rank 4 and degree 0, we have*

1. *(E, s) is stable*
2. *E is a semistable vector bundle.*

If we take sufficiently small a_1 and a_2, it is easy to see that the conditions of the lemma are satisfied. Let us fix such a_1, a_2.

It is well-known from [MS80] that the moduli functor

$$\mathcal{P} : \mathcal{V}ar \to \mathcal{S}ets \tag{2.1}$$

which assigns to each variety T the set of equivalence classes of families of stable parabolic bundles of rank 4 and degree 0 over X parameterized by T, is represented by a smooth projective variety, which we denote by P. It turns out that Seshadri's desingularization S is a closed subvariety of P.

We need a few more facts from [Ses82] (Chapter 5, Propositions 7, 8, 9).

Proposition 2.2. *Let E be a semistable vector bundle of rank 4 and degree 0 on X. There is $0 \neq s \in E_{x_0}^*$ such that the parabolic bundle (E, s) is stable if and only*

if for any line bundle L on X of degree 0 there is no injective homomorphism of vector bundles
$$L \oplus L \hookrightarrow E.$$

Proposition 2.3. *Let (E,s) be a stable parabolic bundle of rank 4 and degree 0. Then the algebra* $\mathrm{End}E$ *of endomorphisms of the underlying vector bundle E has dimension ≤ 4. Moreover, if the algebra* $\mathrm{End}E$ *is isomorphic to the matrix algebra $M(2)$ of 2×2 matrices, then $E \cong F \oplus F$ for a unique stable vector bundle F of rank 2 and degree 0.*

Proposition 2.4. *Let (E_1,s_1), (E_2,s_2) be two stable parabolic bundles of rank 4, degree 0 over X. Suppose* $\dim \mathrm{End}E_1 = \dim \mathrm{End}E_2 = 4$. *Then they are isomorphic as parabolic bundles if and only if the underlying vector bundles E_1 and E_2 are isomorphic.*

Let S' be the subset of P consisting of stable parabolic bundles (E,s) such that $\mathrm{End}E \cong M(2)$ and $\det E$ is trivial. Then Proposition 2.3 says we have a map $S' \to M_0^s$ from S' to the set of stable vector bundles. By Proposition 2.4, this map is injective. By Proposition 2.2, it is surjective as well. Seshadri's desingularization S of M_0 is defined as the closure of S' in P which is nonsingular by [BS90] Proposition 1. Furthermore, the morphism $S' \to M_0^s$ extends to a morphism $\pi_S : S \to M_0$ such that for each $(E,s) \in S$, $\mathrm{gr}E \cong F \oplus F$ where F is the polystable bundle representing the image of (E,s) in M_0.

Fix a nonzero element $e_0 \in \mathbb{C}^4$. Let $\mathcal{A}(2)$ be the set of elements in
$$\mathrm{Hom}(\mathbb{C}^4 \otimes \mathbb{C}^4, \mathbb{C}^4)$$
which gives us an algebra structure on $\mathbb{C}^4$ with the identity element e_0. There is a subset of $\mathcal{A}(2)$ which consists of algebra structures on $\mathbb{C}^4$, isomorphic to the matrix algebra $M(2)$. Let $\mathcal{A}_2$ be the closure of this subset. An element of $\mathcal{A}_2$ is called a *specialization* of $M(2)$. Obviously, there is a locally free sheaf W of $\mathcal{O}_{\mathcal{A}_2}$-algebras on $\mathcal{A}_2$ such that for every $z \in \mathcal{A}_2$, $W_z \otimes \mathbb{C}$ is the specialization of $M(2)$ represented by z.

Let $\mathcal{F}$ be the subfunctor of the functor $\mathcal{P}$ (2.1) defined as follows. For each variety T, $\mathcal{F}(T)$ is the set of equivalence classes of families $\mathcal{E} \to T \times X$ of stable parabolic bundles on X of rank 4 and degree 0 that satisfies the following property (*):
for any $t \in T$ there is a neighborhood T_1 of t in T and a morphism $f : T_1 \to \mathcal{A}_2$ such that $f^*W \cong (p_T)_*(\mathcal{E}nd\mathcal{E})|_{T_1}$ as $\mathcal{O}_{T_1}$-algebras where $p_T : T \times X \to T$ is the projection to T.

Theorem 2.5. *(*[Ses82] *Chapter 5, Theorem 15) The functor $\mathcal{F}$ is represented by S.*

The condition (*) can be weakened slightly by the following proposition.

Proposition 2.6. *(*[Ses82] *Chapter 5, Proposition 1) Let T be a complex manifold and B be a holomorphic vector bundle of rank 4 equipped with an $\mathcal{O}_T$ algebra structure. Suppose there is an open dense subset T' of T such that for each $t \in T'$, $B_t \otimes \mathbb{C}$ is a specialization of $M(2)$. Then for every $t \in T$, there is a neighborhood T_1 of t and a morphism $f : T_1 \to \mathcal{A}_2$ such that $f^*W \cong B|_{T_1}$.*

To prove this, it suffices to consider any open set of T over which B is trivial. But in this trivial case, the proposition is obvious.

The singular locus of M_0 is the Kummer variety $\mathfrak{K}$ or the complement of M_0^s, isomorphic to the quotient $Jac_0/\mathbb{Z}_2$ of the Jacobian of degree 0 line bundles over X by the involution $L \to L^{-1}$. There are 2^{2g} fixed points $\mathbb{Z}_2^{2g} = \{[L \oplus L^{-1}] : L \cong L^{-1}\}$ and we have a stratification

$$M_0 = M_0^s \sqcup (\mathfrak{K} - \mathbb{Z}_2^{2g}) \sqcup \mathbb{Z}_2^{2g}. \tag{2.2}$$

On the other hand, Seshadri's desingularization S is stratified by the rank of the natural conic bundle on S ([Bal88] §3) and thus we have a filtration by closed subvarieties

$$S \supset S_1 \supset S_2 \supset S_3 \tag{2.3}$$

such that $S - S_1 = \pi_S^{-1}(M_0^s) \cong M_0^s$.

Proposition 2.7. *([BS90])*

1. *The image $\pi_S(S_1 - S_2)$ is precisely the middle stratum $\mathfrak{K} - \mathbb{Z}_2^{2g}$. In fact, $S_1 - S_2$ is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ bundle over $\mathfrak{K} - \mathbb{Z}_2^{2g}$.*
2. *The image of S_2 is precisely the deepest strata $\mathbb{Z}_2^{2g}$ and $S_2 - S_3$ is the disjoint union of 2^{2g} copies of a vector bundle of rank $g-2$ over the Grassmannian $Gr(2, g)$ while S_3 is the disjoint union of 2^{2g} copies of the Grassmannian $Gr(3, g)$.*

We end this section with the following proposition which is the key for our construction of the morphism from Kirwan's desingularization to Seshadri's desingularization.

Proposition 2.8. (1) *Let $\mathcal{E} \to T \times X$ be a family of semistable holomorphic vector bundles of rank 4 and degree 0 on X parameterized by a complex manifold T. Assume the following:*

  (a) *for any $t \in T$ and any line bundle L of degree 0 on X, $L \oplus L$ is not isomorphic to a subbundle of $\mathcal{E}|_{t \times X}$*

  (b) *there is an open dense subset T' of T such that $\mathrm{End}(\mathcal{E}|_{t\times X}) \cong M(2)$ for any $t \in T'$.*

Then we have a holomorphic map $\tau : T \to S$.

(2) *Suppose a holomorphic map $\tau : T \to S$ is given. Suppose T is an open subset of a nonsingular quasi-projective variety W on which a reductive group G acts such that every point in W is stable and the (smooth) geometric quotient W/G exists. Furthermore, assume that there is an open dense subset W' of W such that whenever $t_1, t_2 \in T \cap W'$ are in the same orbit, we have $\tau(t_1) = \tau(t_2)$. Then τ factors through the (smooth) image $\overline{T}$ of T in the quotient W/G, i.e. we have a continuous map $\overline{T} \to S$ such that the diagram*

$$\begin{array}{ccc} T & \xrightarrow{\quad\tau\quad} & S \\ & \searrow \quad \nearrow & \\ & \overline{T} & \end{array}$$

commutes.

Proof. (1) Let $E_t = \mathcal{E}|_{t \times X}$. For each $t \in T$, there is a parabolic structure $0 \neq s_t \in (E_t)^*_{x_0}$ such that (E_t, s_t) is a stable parabolic bundle by (a) and Proposition 2.2. Hence we get a set-theoretic map $\tau : T \to P$. Moreover, by (b), a dense open subset of T is mapped to S' and thus τ is actually a map into S. We show that this is in fact holomorphic.

By Proposition 2.3, $\dim \mathrm{End} E_t \le 4$. Since $\dim \mathrm{End} E_t$ is an upper semi-continuous function of t, $\{t \in T \mid \dim \mathrm{End} E_t = 4\}$ is a closed subset of T. But there is a dense open subset in T where $\dim \mathrm{End} E_t = 4$ by (b). Hence, $\dim \mathrm{End} E_t = 4$ for all $t \in T$. Consequently, $(p_T)_* \mathcal{E}nd(\mathcal{E})$ is a locally free sheaf of $\mathcal{O}_T$-algebras of rank 4.

Since stability is an open property, there is a neighborhood T_1 of t and $s \in \mathcal{E}|_{T_1 \times x_0}$ such that $(E_{t'}, s_{t'})$ is a stable parabolic bundle for every $t' \in T_1$. Therefore $(\mathcal{E}|_{T_1 \times X}, s)$ is a family of stable parabolic bundles and $(p_{T_1})_* \mathcal{E}nd(\mathcal{E}|_{T_1 \times X})$ is a locally free sheaf of $\mathcal{O}_{T_1}$-algebras. Hence by assumption (b) and Proposition 2.6, we see that $(\mathcal{E}|_{T_1 \times X}, s)$ is a family of stable parabolic bundles satisfying (*) above. By deformation theory, we have a linear map from the tangent space of T_1 at t' to the deformation space of $(E_{t'}, s_{t'})$ which is isomorphic to the tangent space of P. This is the derivative of τ at t'. So we see that τ is a holomorphic map from T_1 to S. Because we can find a covering of T by such open sets T_1, we deduce that τ is holomorphic.

(2) This is an easy consequence of the étale slice theorem. In particular, the image $\overline{T}$ is an open subset of W/G in the usual complex topology. □

3. Kirwan's desingularization

In this section we recall Kirwan's desingularization from [Kir86b]. We refer to [Kie03] for a very explicit description of this desingularization process for the genus 3 case.

Note that we have the decomposition (2.2). The idea is to blow up M_0 along the deepest strata $\mathbb{Z}_2^{2g}$ and then along the proper transform of the middle stratum $\mathfrak{K}$. Let M_1 denote the result of the first blow-up and M_2 the second blow-up. Kirwan's *partial* desingularization is the projective variety M_2 which we have to blow up one more time to get the *full* desingularization K.

The moduli space M_0 is constructed as the GIT quotient of a smooth quasi-projective variety $\mathfrak{R}$, which is a subset of the space of holomorphic maps from the Riemann surface to the Grassmannian $Gr(2,p)$ of 2-dimensional quotients of $\mathbb{C}^p$ where p is a large even number, by the action of $G = SL(p)$. Over each point in the deepest strata $\mathbb{Z}_2^{2g}$ there is a unique closed orbit in $\mathfrak{R}^{ss}$. By deformation theory, the normal space of the orbit at a point h, which represents $L \oplus L^{-1}$ where $L \cong L^{-1}$, is

$$H^1(End_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \otimes sl(2) \tag{3.1}$$

where the subscript 0 denotes the trace-free part. According to Luna's slice theorem, there is a neighborhood of the point $[L \oplus L^{-1}]$ with $L \cong L^{-1}$, homeomorphic to $H^1(\mathcal{O}) \otimes sl(2) /\!\!/ SL(2)$ since the stabilizer of the point h is $SL(2)$ ([Kir86b] (3.3)). More precisely, there is an $SL(2)$-invariant locally closed subvariety W in $\mathfrak{R}^{ss}$ containing h and an $SL(2)$-equivariant morphism $W \to H^1(\mathcal{O}) \otimes sl(2)$, étale at h, such that we have a commutative diagram

$$\begin{array}{ccccc} G \times_{SL(2)} \big(H^1(\mathcal{O}) \otimes sl(2)\big) & \longleftarrow & G \times_{SL(2)} W & \longrightarrow & \mathfrak{R}^{ss} \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathcal{O}) \otimes sl(2) /\!\!/ SL(2) & \longleftarrow & W /\!\!/ SL(2) & \longrightarrow & M_0 \end{array} \tag{3.2}$$

whose horizontal morphisms are all étale.

Next, we consider the middle stratum $\mathfrak{K} - \mathbb{Z}_2^{2g}$. For each point, the normal space to the unique closed orbit over it at a point h representing $L \oplus L^{-1}$ with $L \neq L^{-1}$, is isomorphic to

$$H^1(End_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \oplus H^1(L^2) \oplus H^1(L^{-2}). \tag{3.3}$$

The stabilizer $\mathbb{C}^*$ acts with weights $0, 2, -2$ respectively on the components. Hence, there is a neighborhood of the point $[L \oplus L^{-1}] \in \mathfrak{K} - \mathbb{Z}_2^{2g}$ in M_0, homeomorphic to

$$H^1(\mathcal{O}) \bigoplus \left(H^1(L^2) \oplus H^1(L^{-2}) /\!\!/ \mathbb{C}^*\right).$$

Notice that $H^1(\mathcal{O})$ is the tangent space to $\mathfrak{K}$ and hence

$$H^1(L^2) \oplus H^1(L^{-2}) /\!\!/ \mathbb{C}^* \cong \mathbb{C}^{2g-2} /\!\!/ \mathbb{C}^*$$

is the normal cone. The GIT quotient of the projectivization $\mathbb{P}\mathbb{C}^{2g-2}$ by the induced $\mathbb{C}^*$ action is $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ and the normal cone $\mathbb{C}^{2g-2} /\!\!/ \mathbb{C}^*$ is obtained by collapsing the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}}(-1, -1)$.

Let H be a reductive subgroup of $G = SL(p)$ and define Z_H^{ss} as the set of semistable points in $\mathfrak{R}^{ss}$ fixed by H. Let $\mathfrak{R}_1$ be the blow-up of $\mathfrak{R}^{ss}$ along the smooth subvariety $GZ_{SL(2)}^{ss}$. Then by Lemma 3.11 in [Kir85], the GIT quotient $\mathfrak{R}_1^{ss} /\!\!/ G$ is the first blow-up M_1 of M_0 along $GZ_{SL(2)}^{ss} /\!\!/ G \cong \mathbb{Z}_2^{2g}$. The $\mathbb{C}^*$-fixed point set in $\mathfrak{R}_1^{ss}$ is the proper transform $\tilde{Z}_{\mathbb{C}^*}^{ss}$ of $Z_{\mathbb{C}^*}^{ss}$ and the quotient of $G\tilde{Z}_{\mathbb{C}^*}^{ss}$ by G is the blow-up $\tilde{\mathfrak{K}}$ of $\mathfrak{K}$ along $\mathbb{Z}_2^{2g}$. If we denote by $\mathfrak{R}_2$ the blow-up of $\mathfrak{R}_1^{ss}$ along the smooth subvariety $G\tilde{Z}_{\mathbb{C}^*}^{ss} = G \times_{N^{\mathbb{C}^*}} \tilde{Z}_{\mathbb{C}^*}^{ss}$ where $N^{\mathbb{C}^*}$ is the normalizer of $\mathbb{C}^*$, the GIT quotient $\mathfrak{R}_2^{ss} /\!\!/ G$ is the second blow-up M_2 again by Lemma 3.11 in [Kir85]. This is Kirwan's *partial* desingularization of M_0 (See §3 [Kir86b]).

The points with stabilizer greater than the center $\{\pm 1\}$ in $\mathfrak{R}_2^{ss}$ is precisely the exceptional divisor of the second blow-up and the proper transform $\tilde{\Delta}$ of the subset Δ of the exceptional divisor of the first blow-up, which corresponds, via Luna's slice theorem, to

$$SL(2)\mathbb{P}\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in H^1(\mathcal{O})\} \subset \mathbb{P}(H^1(\mathcal{O}) \otimes sl(2)).$$

This is a simple exercise. Hence, if we blow up M_2 along $\tilde{\Delta} /\!\!/ SL(2)$, we get a smooth variety K, Kirwan's desingularization.

4. Construction of the morphism

The goal of this section is to prove the following.

Theorem 4.1. *There is a birational morphism*

$$\rho : K \to S$$

from Kirwan's desingularization K to Seshadri's desingularization S.

Since the desingularization morphisms

$$\pi_K : K \to M_0, \quad \pi_S : S \to M_0$$

are both isomorphisms over M_0^s, we have a rational map

$$\rho' : K \dashrightarrow S.$$

By GAGA ([Har77] Appendix B, Ex.6.6), it suffices to find a holomorphic map $\rho : K \to S$ that extends ρ'. By Riemann's extension theorem [Mum76], it suffices to show that ρ' can be extended to a continuous map with respect to the usual complex topology.

4.1. **Points over the middle stratum.** Let us first extend to points over the middle stratum of M_0. Let $l = [L \oplus L^{-1}] \in \mathfrak{K} - \mathbb{Z}_2^{2g} \subset M_0$ and let W_l be the étale slice of the unique closed orbit in $\mathfrak{R}^{ss}$ over l. By Luna's slice theorem we have a commutative diagram

$$\begin{array}{ccccc} G \times_{\mathbb{C}^*} \mathcal{N}_l & \longleftarrow & G \times_{\mathbb{C}^*} W_l & \longrightarrow & \mathfrak{R}^{ss} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_l /\!\!/ \mathbb{C}^* & \longleftarrow & W_l /\!\!/ \mathbb{C}^* & \longrightarrow & M_0 \end{array} \tag{4.1}$$

whose horizontal morphisms are all étale where $G = SL(p)$ and

$$\mathcal{N}_l = H^1(\mathrm{End}(L \oplus L^{-1})_0) = H^1(\mathcal{O}) \oplus H^1(L^2) \oplus H^1(L^{-2}).$$

The slice W_l is a subvariety of $\mathfrak{R}^{ss}$ and the universal bundle over $\mathfrak{R}^{ss} \times X$ gives us a vector bundle over $W_l \times X$. Since $W_l \to \mathcal{N}_l$ is étale, this gives us a holomorphic family $\mathcal{F}$ of semistable vector bundles over X parametrized by a neighborhood U_l of 0 in $\mathcal{N}_l$. The idea now is to modify $\mathcal{F} \oplus \mathcal{F}$ to make it satisfy the assumptions of Proposition 2.8.

The restriction of $\mathcal{F}$ to $(U_l \cap H^1(\mathcal{O})) \times X$ is a direct sum

$$\mathcal{L} \oplus \mathcal{L}^{-1}$$

where $\mathcal{L}$ is a line bundle coming from an étale map between $H^1(\mathcal{O})$ and the slice in the Quot scheme for degree 0 line bundles.

To get Kirwan's desingularization, we blow up $\mathcal{N}_l$ along $H^1(\mathcal{O})$. Let $\pi_l : \tilde{\mathcal{N}}_l \to \mathcal{N}_l$ be the blow-up map. Let $\tilde{U}_l = \pi_l^{-1}(U_l) \cap \tilde{\mathcal{N}}_l^{ss}$ and D_l be the exceptional locus in $\tilde{U}_l$. Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{L}}$ denote the pull-backs of $\mathcal{F}$ and $\mathcal{L}$ to $\tilde{U}_l$ and D_l respectively. Then we have surjective morphisms

$$\tilde{\mathcal{F}}|_{D_l} \to \tilde{\mathcal{L}}, \quad \tilde{\mathcal{F}}|_{D_l} \to \tilde{\mathcal{L}}^{-1}.$$

Let $\tilde{\mathcal{F}}'$ and $\tilde{\mathcal{F}}''$ be the kernels of

$$\tilde{\mathcal{F}} \to \tilde{\mathcal{F}}|_{D_l} \to \tilde{\mathcal{L}}, \quad \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}|_{D_l} \to \tilde{\mathcal{L}}^{-1}$$

respectively. Define $\mathcal{E} = \tilde{\mathcal{F}}' \oplus \tilde{\mathcal{F}}''$ over $\tilde{U}_l \times X$.

Lemma 4.2. *The bundle $\mathcal{E}$ is a family of semistable vector bundles of rank 4 and degree 0 over X parametrized by $\tilde{U}_l$ such that the assumptions of Proposition 2.8 are satisfied, i.e.*

1. *For each $t \in \tilde{U}_l$ and $L' \in Pic^0(X)$, $L' \oplus L'$ is not isomorphic to any subbundle of $\mathcal{E}|_{t \times X}$.*
2. *$\mathcal{E}|_{(\tilde{U}_l - D_l) \times X} \cong (\tilde{\mathcal{F}} \oplus \tilde{\mathcal{F}})|_{(\tilde{U}_l - D_l) \times X}$ and there is an open dense subset of $\tilde{U}_l$ where $\mathrm{End}(\mathcal{E}|_{t \times X})$ is a specialization of $M(2)$.*
3. *With respect to the action of $\mathbb{C}^*$ on $\tilde{\mathcal{N}}_l - D_l$, if $t_1, t_2 \in \tilde{U}_l - D_l$ are in the same orbit, then $\mathcal{E}|_{t_1 \times X} \cong \mathcal{E}|_{t_2 \times X}$.*

Proof. Since D_l is a smooth divisor in $\tilde{U}_l$, $\mathcal{E}$ is locally free of rank 4. Let $(a,b,c) \in \mathcal{N}_l = H^1(\mathcal{O}) \oplus H^1(L^2) \oplus H^1(L^{-2})$. The weights of the $\mathbb{C}^*$ action are $0, 2, -2$ respectively. It is well-known (see [Kir86b, (2.5) (iv)]) that the bundle $\mathcal{F}|_{(a,b,c)\times X}$ is stable if and only if the image of (a,b,c) in $\mathfrak{R}^{ss}$ is a stable point. This is equivalent to saying that (a,b,c) is stable with respect to the $\mathbb{C}^*$ action. Hence $\mathcal{F}|_{(a,b,c)\times X}$ is stable if and only if $b \neq 0$ and $c \neq 0$.

Let $t_0 \in \tilde{U}_l - D_l$ and $\pi_l(t_0) = (a,b,c)$. This point has nothing to do with the blow-up and the Hecke modification. Hence $\tilde{\mathcal{E}}|_{t_0\times X} \cong \mathcal{F} \oplus \mathcal{F}|_{\pi_l(t_0)\times X}$. The unstable points in $\tilde{\mathcal{N}}_l$ are the proper transform of $\{(a,b,c)|b=0 \text{ or } c=0\}$. Since t_0 is (semi)stable, we have $b \neq 0$ and $c \neq 0$ which implies that $F = \mathcal{F}|_{\pi_l(t_0)\times X}$ is stable. Therefore, $\mathrm{End}(F \oplus F) \cong M(2)$ which proves (2).

For $t_1, t_2 \in \tilde{U}_l - D_l$, $\tilde{\mathcal{E}}|_{t_j\times X} \cong \mathcal{F} \oplus \mathcal{F}|_{\pi_l(t_j)\times X}$ $(j=1,2)$. But $\mathcal{F}|_{\pi_l(t_1)\times X} \cong \mathcal{F}|_{\pi_l(t_2)\times X}$ if and only if $\pi_l(t_1)$ and $\pi_l(t_2)$ are in the same orbit. This is equivalent to t_1 and t_2 being in the same orbit since $\tilde{U}_l - D_l$ is isomorphic to the stable part of $\mathcal{N}_l$. So we proved (3).

Let us prove (1). For $t \in \tilde{U}_l - D_l$, it is trivial since $\tilde{\mathcal{F}}'|_{t\times X} \cong \tilde{\mathcal{F}}|_{t\times X} \cong \mathcal{F}|_{\pi_l(t)\times X}$ which is stable and the same is true for $\tilde{\mathcal{F}}''$.

Let C be a line in $\mathcal{N}_l$ given by a map $\mathbb{C} \to \mathcal{N}_l$ with $z \to (a, zb, zc)$ for $a \in H^1(\mathcal{O}), 0 \neq b \in H^1(L^2), 0 \neq c \in H^1(L^{-2})$. Note that any point in D_l is represented by such a line. Let t be the point in D_l represented by C.

Let $C_0 = C \cap U_l$. By restricting U_l if necessary, we can find an open covering $\{V_i\}$ of X such that $\mathcal{F}|_{C_0\times V_i}$ are all trivial. Fix a trivialization for each i and let $L_a = \mathcal{L}|_{a\times X}$. Since $\mathcal{F}|_{0\times X} \cong L_a \oplus L_a^{-1}$, the transition matrices are of the form

$$\begin{pmatrix} \lambda_{ij} & zb_{ij} \\ zc_{ij} & \lambda_{ij}^{-1} \end{pmatrix}$$

where $\lambda_{ij}|_{z=0}$ is the transition for L_a. The cocycle condition tells us that

$$\{\lambda_{ij} b_{ij}|_{z=0}\}, \qquad \{\lambda_{ij}^{-1} c_{ij}|_{z=0}\}$$

are cocycles whose cohomology classes are nonzero because $\mathcal{F}|_{(a,zb,zc)\times X}$ is stable for $z \neq 0$. Let $\mathcal{F}'$ be the kernel of $\mathcal{F}|_{C_0\times X} \to \mathcal{F}|_{0\times X} \cong L_a \oplus L_a^{-1} \to L_a$ where the first morphism is the restriction and the last is the projection. Define $\mathcal{F}''$ as the kernel of $\mathcal{F}|_{C_0\times X} \to \mathcal{F}|_{0\times X} \cong L_a \oplus L_a^{-1} \to L_a^{-1}$. Let $F' = \mathcal{F}'|_{0\times X}$ and $F'' = \mathcal{F}''|_{0\times X}$. Then by construction, $\tilde{\mathcal{F}}'|_{t\times X} \cong F'$ and $\tilde{\mathcal{F}}''|_{t\times X} \cong F''$.

Any section of $\mathcal{F}'$ over $C_0 \times V_i$ is of the form (zs_1, s_2). Because

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} zs_1 \\ s_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_{ij} & zb_{ij} \\ zc_{ij} & \lambda_{ij}^{-1} \end{pmatrix} \begin{pmatrix} zs_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} z(\lambda_{ij}s_1 + b_{ij}s_2) \\ \lambda_{ij}^{-1}s_2 + z^2 c_{ij}s_1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \lambda_{ij}s_1 + b_{ij}s_2 \\ \lambda_{ij}^{-1}s_2 + z^2 c_{ij}s_1 \end{pmatrix}$$

the transition for $\mathcal{F}'$ is

$$\begin{pmatrix} \lambda_{ij} & b_{ij} \\ z^2 c_{ij} & \lambda_{ij}^{-1} \end{pmatrix}.$$

Hence F' fits into a short exact sequence

$$0 \to L_a \to F' \to L_a^{-1} \to 0$$

whose extension class is given by $\{\lambda_{ij} b_{ij}|_{z=0}\}$ which is nonzero. Hence, $F' = \mathcal{F}'|_{z=0}$ is a nonsplit extension of L_a^{-1} by L_a and similarly $F'' = \mathcal{F}''|_{z=0}$ is a nonsplit extension of L_a by L_a^{-1}. It is now an elementary exercise to show that $E = F' \oplus F''$

does not have a subbundle isomorphic to $L' \oplus L'$ for any $L' \in Pic^0(X)$. So we proved (1). □

By Proposition 2.8, we have a holomorphic map from the image of $\tilde{U}_l$ in $\tilde{\mathcal{N}}_l^{ss}/\mathbb{C}^*$ to S. Since the image is open in the usual complex topology by the slice theorem, this implies that ρ' extends continuously to a neighborhood of the points in K lying over l. Since ρ' is defined on an open dense subset, there is at most one continuous extension. Therefore, the extensions for various points l in the middle stratum $\mathfrak{K} - \mathbb{Z}_2^{2g}$ are compatible and so ρ' is extended to all the points in K except those over the deepest strata $\mathbb{Z}_2^{2g}$.

4.2. **Points over the deepest strata.** Let us next extend ρ' to the points over the deepest strata $\mathbb{Z}_2^{2g}$. The exactly same argument applies to all the points in $\mathbb{Z}_2^{2g}$, so we consider only the points in K over $0 = [\mathcal{O} \oplus \mathcal{O}]$. Let W be the étale slice of the unique closed orbit in $\mathfrak{R}^{ss}$ over $[\mathcal{O} \oplus \mathcal{O}] \in M_0$. Let

$$\mathcal{N} = H^1(\mathcal{O}) \otimes sl(2).$$

By Luna's slice theorem, a neighborhood of $[\mathcal{O} \oplus \mathcal{O}]$ in M_0 is analytically equivalent to a neighborhood of the vertex $\overline{0}$ in the cone $\mathcal{N}/\!\!/SL(2)$ from the diagram (3.2). Hence a neighborhood of the preimage of $[\mathcal{O} \oplus \mathcal{O}]$ in K is biholomorphic to an open set of the desingularization $\tilde{\mathcal{N}}/\!\!/SL(2)$, obtained as a result of three blow-ups from $\mathcal{N}/\!\!/SL(2)$, described below. Therefore it suffices to construct a holomorphic map from a neighborhood $\tilde{V}$ of the preimage of $\overline{0}$ in $\tilde{\mathcal{N}}/\!\!/SL(2)$ to S.

Let Σ be the subset of $\mathcal{N}$ defined by

$$SL(2)\{H^1(\mathcal{O}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}.$$

Let $\pi_1 : \mathcal{N}_1 \to \mathcal{N}$ be the first blow-up in the partial desingularization process, i.e. the blow-up at 0, and let $\mathcal{D}_1^{(1)}$ be the exceptional divisor. Recall that Δ is the subset of $\mathcal{D}_1^{(1)}$ defined as

$$SL(2)\mathbb{P}\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in H^1(\mathcal{O})\}.$$

Let $\tilde{\Sigma}$ be the proper transform of Σ in $\mathcal{N}_1$. Then the singular locus of $\mathcal{N}_1^{ss}/\!\!/SL(2)$ is the quotient of $\Delta \cup \tilde{\Sigma}$ by $SL(2)$. It is an elementary exercise to check that

$$\mathcal{D}_1^{(1)} \cap \tilde{\Sigma} = SL(2)\mathbb{P}\{H^1(\mathcal{O}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\} = \Delta \cap \tilde{\Sigma}. \tag{4.2}$$

Let $\pi_2 : \mathcal{N}_2 \to \mathcal{N}_1$ be the second blow-up, i.e. the blow-up along $\tilde{\Sigma}$ and let $\mathcal{D}_2^{(2)}$ be the exceptional divisor. Let $\mathcal{D}_2^{(1)}$ be the proper transform of $\mathcal{D}_1^{(1)}$. The singular locus of $\mathcal{N}_2/\!\!/SL(2)$ is the quotient of the proper transform $\tilde{\Delta}$ of Δ.

Finally let $\pi_3 : \tilde{\mathcal{N}} = \mathcal{N}_3 \to \mathcal{N}_2$ denote the blow-up of $\mathcal{N}_2$ along $\tilde{\Delta}$ and let $\tilde{\mathcal{D}}^{(3)} = \mathcal{D}_3^{(3)}$ be the exceptional divisor while $\tilde{\mathcal{D}}^{(1)} = \mathcal{D}_3^{(1)}$, $\tilde{\mathcal{D}}^{(2)} = \mathcal{D}_3^{(2)}$ are the proper transforms of $\mathcal{D}_2^{(1)}$ and $\mathcal{D}_2^{(2)}$ respectively. Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the composition of the three blow-ups. Also let $D_i^{(j)}$ be the quotient of $\mathcal{D}_i^{(j)}$ in $\mathcal{N}_i/\!\!/SL(2)$ for $1 \le i \le 3$ and $1 \le j \le i$.

As in the middle stratum case, the pull-back of the universal bundle over $\mathfrak{R}^{ss} \times X$ to $W \times X$ gives us a holomorphic family $\mathcal{F}$ of rank 2 semistable vector bundles over X parametrized by an open neighborhood U of 0 in $\mathcal{N}$. Let V be the image of U under the good quotient morphism $\mathcal{N} \to \mathcal{N}/\!\!/SL(2)$. Then V is an open neighborhood of $\overline{0}$. Let $U_1 = \pi_1^{-1}(U) \cap \mathcal{N}_1^{ss}$ and V_1 be the image of U_1 by the good quotient morphism $\mathcal{N}_1 \to \mathcal{N}_1/\!\!/SL(2)$. From the commutative diagram

$$\begin{array}{ccc} \mathcal{N}_1^{ss} & \longrightarrow & \mathcal{N}_1/\!\!/SL(2) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \overline{\pi}_1} \\ \mathcal{N} & \longrightarrow & \mathcal{N}/\!\!/SL(2) \end{array}$$

we see that $V_1 = \overline{\pi}_1^{-1}(V)$.

Let $U_2 = \pi_2^{-1}(U_1) \cap \mathcal{N}_2^{ss}$ and V_2 be the image of U_2 in the quotient of $\mathcal{N}_2$. Then we have $V_2 = \overline{\pi}_2^{-1}(V_1)$ where $\overline{\pi}_2 : \mathcal{N}_2/\!\!/SL(2) \to \mathcal{N}_1/\!\!/SL(2)$. Similarly, let $\tilde{U} = \pi_3^{-1}(U_2) \cap \tilde{\mathcal{N}}^{ss}$ and $\tilde{V}$ be the image of $\tilde{U}$ in the quotient of $\tilde{\mathcal{N}}$. By construction, $\tilde{V}$ is smooth with simple normal crossing divisors $\tilde{D}^{(1)}, \tilde{D}^{(2)}, \tilde{D}^{(3)}$ where $\tilde{D}^{(j)} = D_3^{(j)}$. To simplify our notation we denote the intersection of $\tilde{D}^{(2)}$ with $\tilde{V}$ again by $\tilde{D}^{(2)}$.

Since we already extended ρ' to the points over the middle stratum, we have a holomorphic map $\rho' : \tilde{V} - (\tilde{D}^{(1)} \cup \tilde{D}^{(3)}) \to S$ and we have to extend it to $\rho : \tilde{V} \to S$.

4.3. **Points in $\tilde{D}^{(1)} - (\tilde{D}^{(2)} \cup \tilde{D}^{(3)})$.** In this subsection, we extend ρ' to points in $\tilde{V}$ that lies over the quotient of $\mathcal{D}_1^{(1)} - \Delta$ via $\overline{\pi}_3 \circ \overline{\pi}_2$. Notice that $\mathcal{D}_1^{(1)} - \Delta$ does not intersect with the blow-up centers of the second and third blow-up and hence it remains unchanged.

Our strategy is again to modify the pull-back of $\mathcal{F} \oplus \mathcal{F}$ to $U_1 - \Delta \cup \tilde{\Sigma}$ so that ρ' extends to a holomorphic map near the quotient of $\mathcal{D}_1^{(1)} - \Delta$ by Proposition 2.8.

Let $\mathcal{F}_1$ be the pull-back of $\mathcal{F}$ to $U_1 \times X$ via $\pi_1 \times 1_X$. Then $\mathcal{F}_1|_{\mathcal{D}_1^{(1)} \times X} \cong \mathcal{O} \oplus \mathcal{O}$ since $\mathcal{F}|_{0 \times X}$ is trivial. Let $\mathcal{F}_1'$ be the kernel of

$$\mathcal{F}_1 \to \mathcal{F}_1|_{\mathcal{D}_1^{(1)} \times X} \cong \mathcal{O}_{\mathcal{D}_1^{(1)} \times X} \oplus \mathcal{O}_{\mathcal{D}_1^{(1)} \times X} \to \mathcal{O}_{\mathcal{D}_1^{(1)} \times X}$$

where the second arrow is the projection onto the first component. Let $\mathcal{F}_1''$ be defined similarly with the projection onto the second component. By computing transition matrices as in the proof of Lemma 4.2, it is immediate that $\mathcal{F}_1'|_{t_1 \times X}$ and $\mathcal{F}_1''|_{t_1 \times X}$ are nonsplit extensions of $\mathcal{O}$ by $\mathcal{O}$ if $t_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathbb{P}\mathcal{N} = \mathcal{D}_1^{(1)}$ with $b \neq 0$ and $c \neq 0$ in $H^1(\mathcal{O})$.

Suppose $t_1 \in \mathcal{D}_1^{(1)} - \Delta$. Then a, b, c are linearly independent because otherwise we can find $g \in SL(2)$ such that gt_1g^{-1} is of the form

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}. \tag{4.3}$$

The first case belongs to Δ while the second is unstable in $\tilde{\mathcal{N}}$ and is deleted after all. In particular, a, b, c are all nonzero and thus $\mathcal{F}_1'|_{t_1 \times X}$ and $\mathcal{F}_1''|_{t_1 \times X}$ are nonsplit extensions of $\mathcal{O}$ by $\mathcal{O}$ whose extension classes are b, c respectively.

The inclusion $\mathcal{F}_1' \hookrightarrow \mathcal{F}_1$ gives us a homomorphism $\mathcal{F}_1'|_{\mathcal{D}_1^{(1)} \times X} \to \mathcal{F}_1|_{\mathcal{D}_1^{(1)} \times X} \cong \mathcal{O} \oplus \mathcal{O}$ whose image is the second factor $\mathcal{O}$ and the kernel of this homomorphism is $\mathcal{O}$. Similarly, the trivial bundle $\mathcal{O}_{\mathcal{D}_1^{(1)} \times X}$ is a subbundle of $\mathcal{F}_1''|_{\mathcal{D}_1^{(1)} \times X}$ and we have a diagonal embedding of $\mathcal{O}_{\mathcal{D}_1^{(1)} \times X}$ into $\mathcal{F}_1' \oplus \mathcal{F}_1''|_{\mathcal{D}_1^{(1)} \times X}$. Let $\mathcal{E}_1$ be the kernel of

$$\mathcal{F}_1' \oplus \mathcal{F}_1'' \to \mathcal{F}_1' \oplus \mathcal{F}_1''|_{\mathcal{D}_1^{(1)} \times X} \to \mathcal{F}_1' \oplus \mathcal{F}_1''|_{\mathcal{D}_1^{(1)} \times X} / \mathcal{O}_{\mathcal{D}_1^{(1)} \times X}.$$

As in the proof of Lemma 4.2, introduce a local coordinate z of a suitable curve passing through t_0 and write the transition for $\mathcal{F}_1' \oplus \mathcal{F}_1''$ as

$$\begin{pmatrix} \lambda_{ij} & b_{ij} & 0 & 0 \\ z^2 c_{ij} & \lambda_{ij}^{-1} & 0 & 0 \\ 0 & 0 & \lambda_{ij} & z^2 b_{ij} \\ 0 & 0 & c_{ij} & \lambda_{ij}^{-1} \end{pmatrix} \tag{4.4}$$

where $\lambda_{ij} = 1 + z a_{ij}$. Note that, when restricted to $z = 0$, the cocycles $\{a_{ij}\}$, $\{b_{ij}\}$, $\{c_{ij}\}$ represent the classes $a, b, c \in H^1(\mathcal{O})$ respectively.

A local section of $\mathcal{E}_1$ as a subsheaf of $\mathcal{F}_1' \oplus \mathcal{F}_1''$ is of the form $(s_1, zs_2, zs_3, s_1 + zs_4)$. Because

(4.5)

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} s_1 \\ zs_2 \\ zs_3 \\ s_1 + zs_4 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_{ij} & b_{ij} & 0 & 0 \\ z^2 c_{ij} & \lambda_{ij}^{-1} & 0 & 0 \\ 0 & 0 & \lambda_{ij} & z^2 b_{ij} \\ 0 & 0 & c_{ij} & \lambda_{ij}^{-1} \end{pmatrix} \begin{pmatrix} s_1 \\ zs_2 \\ zs_3 \\ s_1 + zs_4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_{ij} s_1 + z b_{ij} s_2 \\ z^2 c_{ij} s_1 + z \lambda_{ij}^{-1} s_2 \\ z^2 b_{ij} s_1 + z \lambda_{ij} s_3 + z^3 b_{ij} s_4 \\ z c_{ij} s_3 + \lambda_{ij}^{-1} s_1 + z \lambda_{ij}^{-1} s_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} \lambda_{ij} s_1 + z b_{ij} s_2 \\ z c_{ij} s_1 + \lambda_{ij}^{-1} s_2 \\ z b_{ij} s_1 + \lambda_{ij} s_3 + z^2 b_{ij} s_4 \\ \frac{\lambda_{ij}^{-1} - \lambda_{ij}}{z} s_1 - b_{ij} s_2 + c_{ij} s_3 + \lambda_{ij}^{-1} s_4 \end{pmatrix},$$

the transition for $\mathcal{E}_1$ is

$$\begin{pmatrix} \lambda_{ij} & z b_{ij} & 0 & 0 \\ z c_{ij} & \lambda_{ij}^{-1} & 0 & 0 \\ z b_{ij} & 0 & \lambda_{ij} & z^2 b_{ij} \\ -2a_{ij} & -b_{ij} & c_{ij} & \lambda_{ij}^{-1} \end{pmatrix}. \tag{4.6}$$

Put $z = 0$ to see that the transition for $\mathcal{E}|_{t_1 \times X}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2a_{ij}|_{z=0} & -b_{ij}|_{z=0} & c_{ij}|_{z=0} & 1 \end{pmatrix}. \tag{4.7}$$

Hence we have a filtration by subbundles

$$\mathcal{E}|_{t_1 \times X} = E_4 \supset E_3 \supset E_2 \supset E_1 \supset E_0 = 0 \tag{4.8}$$

such that $E_{i+1}/E_i \cong \mathcal{O}_X$. The extension E_2 of $\mathcal{O}$ by $E_1 \cong \mathcal{O}$ is nontrivial since $c \neq 0$. An extension of $\mathcal{O}$ by E_2 is parameterized by $Ext^1(\mathcal{O}, E_2)$ which fits in the exact sequence

$$Hom(\mathcal{O}, \mathcal{O}) \xrightarrow{c} Ext^1(\mathcal{O}, \mathcal{O}) \longrightarrow Ext^1(\mathcal{O}, E_2) \to Ext^1(\mathcal{O}, \mathcal{O})$$

and E_3 is the image of $b \in Ext^1(\mathcal{O}, \mathcal{O}) \cong H^1(\mathcal{O})$ which is nonzero since b, c are linearly independent. Hence E_3 is a nonsplit extension. Similarly E_4 is a nonsplit extension since a, b, c are linearly independent. Hence (4.8) is the result of three nonsplit extensions. This certainly implies that the condition (a) of Proposition 2.8 is satisfied for points in $\tilde{U}$ over $\mathcal{D}_1^{(1)} - \Delta$. The other conditions of Proposition 2.8 (1), (2) are trivially satisfied and hence ρ' extends to the points over the quotient of the points over $\mathcal{D}_1^{(1)} - \Delta$ as desired.

4.4. **Points in $\tilde{D}^{(3)} - \tilde{D}^{(2)}$.** We use the notation of §4.3. Suppose now $t_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \Delta - \tilde{\Sigma}$. Then a, b, c span 2-dimensional subspace of $H^1(\mathcal{O})$. The bundle $\mathcal{E}_1|_{t_1 \times X}$ in the previous subsection has transition matrices of the form (4.7). The one dimensional space of linear relations of a, b, c gives rise to an embedding of $\mathcal{O}$ into $\mathcal{E}_1|_{t_1\times X}$. More generally, the family of linear relations of a, b, c gives us a line bundle over $\Delta - \tilde{\Sigma}$. Let $\mathcal{L}_1$ denote the pull-back of this line bundle to $(\Delta - \tilde{\Sigma}) \times X$. Then we have an embedding of $\mathcal{L}_1$ into $\mathcal{E}_1|_{(\Delta - \tilde{\Sigma})\times X}$. Let $\mathcal{E}_3$ (resp. $\mathcal{L}_3$) be the pull-back of $\mathcal{E}_1$ (resp. $\mathcal{L}_1$) to $\tilde{U} = U_3$ (resp. $\tilde{\mathcal{D}}^{(3)} - \tilde{\mathcal{D}}^{(2)}$).

Let $\tilde{\mathcal{E}}$ be the kernel of

$$\mathcal{E}_3| \to \mathcal{E}_3|_{(\tilde{\mathcal{D}}^{(3)} - \tilde{\mathcal{D}}^{(2)})\times X} \to \mathcal{E}_3|_{(\tilde{\mathcal{D}}^{(3)} - \tilde{\mathcal{D}}^{(2)})\times X}/\mathcal{L}_3.$$

We claim that $\tilde{\mathcal{E}}$ satisfies the conditions of Proposition 2.8 and hence ρ' extends to the quotient of $\tilde{\mathcal{D}}^{(3)} - \tilde{\mathcal{D}}^{(2)}$.

For simplicity, let t_1 be $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \in \Delta - \tilde{\Sigma}$ with b, c linearly independent. (The general case is obtained by conjugation.) Let $t_3 \in \tilde{\mathcal{D}}^{(3)} - \tilde{\mathcal{D}}^{(2)}$ be a (semi)stable point lying over t_1. Now we make local computations as in (4.5) and (4.6).

A point $t_3 \in \tilde{\mathcal{D}}^{(3)}$ represents a normal direction to Δ at t_1. Choose a local parameter z of the direction such that $z = 0$ represents t_1.

If t_3 represents a normal direction of Δ tangent to $\tilde{\mathcal{D}}^{(1)}$, then from (4.7), the transition of the restriction of $\mathcal{E}_3$ to the direction is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2zd_{ij} & -b_{ij} & c_{ij} & 1 \end{pmatrix} \tag{4.9}$$

for some cocycle $\{d_{ij}\}$ which gives rise to a nonzero class $d \in H^1(\mathcal{O})$ at $z = 0$ such that d, b, c are linearly independent. In this case, the transition for $\tilde{\mathcal{E}}|_{t_3\times X}$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2d_{ij}|_{z=0} & -b_{ij}|_{z=0} & c_{ij}|_{z=0} & 1 \end{pmatrix} \tag{4.10}$$

by a local computation. Hence, the condition (1) of Proposition 2.8 is satisfied because the bundle is obtained by three nonsplit extensions.

Suppose t_3 represents the direction normal to $\mathcal{D}^{(1)}$. Then we can use the same curve we used in §4.3 and the transition of $\mathcal{E}_3$ is given by (4.6). More generally, the transition of $\mathcal{E}_3$ restricted to the direction of any t_3, not tangent to $\mathcal{D}^{(1)}$, is of the form

$$\begin{pmatrix} 1 + za_{ij} & zb_{ij} & 0 & 0 \\ zc_{ij} & 1 - za_{ij} & 0 & 0 \\ zb_{ij} & 0 & 1 + za_{ij} & 0 \\ -2zd_{ij} & -b_{ij} & c_{ij} & 1 - za_{ij} \end{pmatrix} \tag{4.11}$$

mod z^2 for some cocycle $\{d_{ij}\}$. A local section of $\tilde{\mathcal{E}}$ is of the form (s_1, zs_2, zs_3, zs_4) and by computing as in (4.5) starting with (4.11), we deduce that the transition for $\tilde{\mathcal{E}}|_{t_3\times X}$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ c_{ij}|_{z=0} & 1 & 0 & 0 \\ b_{ij}|_{z=0} & 0 & 1 & 0 \\ -2d_{ij}|_{z=0} & -b_{ij}|_{z=0} & c_{ij}|_{z=0} & 1 \end{pmatrix}. \tag{4.12}$$

This implies that the bundle has a filtration by subbundles as in (4.8) obtained by three nonsplit extensions. Hence $\tilde{\mathcal{E}}|_{t_3\times X}$ satisfies the condition (1) of Proposition 2.8.

Because the other conditions of Proposition 2.8 are trivially satisfied on the stable part of U, we deduce that the holomorphic map ρ' extends to the quotient of $\tilde{U} - \tilde{\mathcal{D}}^{(2)}$. So far, we extended ρ' to the complement of the quotient of $\tilde{\mathcal{D}}^{(2)} \cap (\tilde{\mathcal{D}}^{(1)} \cup \tilde{\mathcal{D}}^{(3)})$ which consists of points lying over $\Delta \cap \tilde{\Sigma}$.

4.5. **Points in $\tilde{D}^{(2)} \cap (\tilde{D}^{(1)} \cup \tilde{D}^{(3)})$.** In this subsection, we finally extend ρ' to everywhere in K and finish the proof of Theorem 4.1. We use the notation of §4.2. By the slice theorem, we have a map $\tilde{V} \to K$, biholomorphic onto a neighborhood of the preimage of $[\mathcal{O} \oplus \mathcal{O}]$. So it suffices to construct a holomorphic map $\tilde{V} \to S$.

We have a commutative diagram

$$\begin{array}{ccc} \tilde{V} & \hookrightarrow & K \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ V_1 & \hookrightarrow & M_1 \end{array}$$

where the vertical maps are blow-ups. We already constructed a holomorphic map

$$\nu : \tilde{V} - \alpha^{-1}(\Delta \cap \tilde{\Sigma}/\!\!/SL(2)) \to S$$

Let x be any point in $\Delta \cap \tilde{\Sigma}/\!\!/SL(2)$. From (4.2), x is represented by the orbit of $\begin{bmatrix} a^0 & 0 \\ 0 & -a^0 \end{bmatrix}$ for some $[a^0] \in H^1(X, \mathcal{O})$. The stabilizer of the point in $SL(2)$ is $\mathbb{C}^*$ and the normal space Y to its orbit is isomorphic to $\mathbb{C}^g \oplus \mathbb{C}^{2g-2}$ where $\mathbb{C}^g$ is the tangent space of the blow-up $\widetilde{H^1(\mathcal{O})} = \mathrm{bl}_0 H^1(\mathcal{O})$ and $\mathbb{C}^{2g-2} \cong H^1(\mathcal{O})/\mathbb{C}a^0 \oplus H^1(\mathcal{O})/\mathbb{C}a^0$.

Obviously, a neighborhood Y_1 of 0 in Y is holomorphically embedded into U_1, perpendicular to the $SL(2)$-orbit of the point $[a^0]$ and the vector bundle $\mathcal{F}_1|_{Y_1\times X}$ has transition matrices of the form

$$\begin{pmatrix} 1 + z_1(a^0_{ij} + a_{ij}) & z_1 b_{ij} \\ z_1 c_{ij} & 1 - z_1(a^0_{ij} + a_{ij}) \end{pmatrix}. \tag{4.13}$$

Here $a = \{a_{ij}\}, b = \{b_{ij}\}, c = \{c_{ij}\}$ are classes in $H^1(\mathcal{O})$, not parallel to a^0 if nonzero and z_1 is the coordinate for the normal direction of $\mathbb{P}H^1(\mathcal{O})$ in $\widetilde{H^1(\mathcal{O})}$.

By Luna's étale slice theorem, a neighborhood of the vertex of the cone $Y/\!\!/\mathbb{C}^*$ is analytically equivalent to a neighborhood of x in V_1 or M_1. Let $\tilde{Y}$ denote the proper transform of Y_1 in $\tilde{U}$. Then the image of $\tilde{Y}$ in $\tilde{V}$ is biholomorphic to a neighborhood of $\alpha^{-1}(x)$. Our goal is to construct a family of rank 4 bundles on X parametrized by $\tilde{Y}$ satisfying the conditions of Proposition 2.8. Then we can conclude that ν extends to $\alpha^{-1}(x)$.

Recall that we have a rank 2 bundle $\mathcal{F}_1$ over $U_1 \times X$. Let $\mathcal{F}_{Y_1} = \mathcal{F}_1|_{Y_1\times X}$. Let $\mathcal{D}^{(1)}_{Y_1}$ be the divisor in Y_1 given by $z_1 = 0$. Then from (4.13) we see that

$$\mathcal{F}_{Y_1}|_{\mathcal{D}^{(1)}_{Y_1}\times X} \cong \mathcal{O}\oplus\mathcal{O}.$$

Let $\mathcal{F}'_{Y_1}$ (resp. $\mathcal{F}''_{Y_1}$) be the kernel of

$$\mathcal{F}_{Y_1} \to \mathcal{F}_{Y_1}|_{\mathcal{D}^{(1)}_{Y_1}\times X} \cong \mathcal{O}\oplus\mathcal{O}\to\mathcal{O}$$

where the last arrow is the projection onto the first (resp. second) component. From a local computation as in §4.1, the transition matrices of $\mathcal{F}'_{Y_1}$ and $\mathcal{F}''_{Y_1}$ are respectively

$$\begin{pmatrix} 1+z_1(a^0_{ij}+a_{ij}) & b_{ij} \\ z_1^2 c_{ij} & 1-z_1(a^0_{ij}+a_{ij})\end{pmatrix}, \qquad \begin{pmatrix} 1+z_1(a^0_{ij}+a_{ij}) & z_1^2 b_{ij} \\ c_{ij} & 1-z_1(a^0_{ij}+a_{ij})\end{pmatrix}$$

In particular, $\mathcal{F}'_{Y_1}$ and $\mathcal{F}''_{Y_1}$ restricted to

$$\tilde{\Sigma}_{Y_1} = Y_1\cap\{b=c=0\} = Y_1\cap(\mathbb{C}^g\oplus 0)\subset \mathbb{C}^g\oplus\mathbb{C}^{2g-2} = Y$$

are given by transition matrices

$$\begin{pmatrix} 1+z_1(a^0_{ij}+a_{ij}) & 0 \\ 0 & 1-z_1(a^0_{ij}+a_{ij})\end{pmatrix}$$

and thus

$$\mathcal{F}'_{Y_1}|_{\tilde{\Sigma}_{Y_1}\times X} \cong \mathcal{L}_{Y_1}\oplus\mathcal{L}^{-1}_{Y_1}$$

for some line bundle $\mathcal{L}_{Y_1}$ over $\tilde{\Sigma}_{Y_1}\times X$.

Let Y_2 be the proper transform of Y_1 in U_2 by the blow-up (and subtraction of unstable points) map $U_2\to U_1$. In other words, Y_2 is the blow-up of Y_1 along $\tilde{\Sigma}_{Y_1}$ with unstable points removed. Let z_2 be the coordinate of the normal direction of the exceptional divisor $\mathcal{D}^{(2)}_{Y_2}$ at a point $[b,c]$ over (z_1,a). Let $\mathcal{F}'_{2,0}$, $\mathcal{F}''_{2,0}$ be the pull-back of $\mathcal{F}'_{Y_1}$, $\mathcal{F}''_{Y_1}$ to $Y_2\times X$ respectively. Let $\mathcal{L}_{Y_2}$ denote the pull-back of $\mathcal{L}_{Y_1}$ to $\mathcal{D}^{(2)}_{Y_2}\times X$.

Let $\mathcal{F}'_{Y_2}$ be the kernel of

$$\mathcal{F}'_{2,0}\to\mathcal{F}'_{2,0}|_{\mathcal{D}^{(2)}_{Y_2}\times X}\cong\mathcal{L}_{Y_2}\oplus\mathcal{L}^{-1}_{Y_2}\to\mathcal{L}_{Y_2}$$

and $\mathcal{F}''_{Y_2}$ be the kernel of

$$\mathcal{F}''_{2,0}\to\mathcal{F}''_{2,0}|_{\mathcal{D}^{(2)}_{Y_2}\times X}\cong\mathcal{L}_{Y_2}\oplus\mathcal{L}^{-1}_{Y_2}\to\mathcal{L}^{-1}_{Y_2}.$$

Let $\mathcal{D}^{(1)}_{Y_2}$ be the proper transform of $\mathcal{D}^{(1)}_{Y_1}$. By a local computation, it is easy to see that the trivial bundle $\mathcal{O}$ is a subbundle of both $\mathcal{F}'_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}$ and $\mathcal{F}''_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}$ as in §4.3. Let $\mathcal{E}_{Y_2}$ be the kernel of

$$\mathcal{F}'_{Y_2}\oplus\mathcal{F}''_{Y_2}\to\mathcal{F}'_{Y_2}\oplus\mathcal{F}''_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}\to\mathcal{F}'_{Y_2}\oplus\mathcal{F}''_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}/\mathcal{O}.$$

The inclusion $\mathcal{E}_{Y_2}\hookrightarrow\mathcal{F}'_{Y_2}\oplus\mathcal{F}''_{Y_2}$ induces $\mathcal{E}_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}\to\mathcal{F}'_{Y_2}\oplus\mathcal{F}''_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}$ whose image is the diagonal $\mathcal{O}$. Hence $\mathcal{E}_{Y_2}|_{\mathcal{D}^{(1)}_{Y_2}\times X}$ is a family of extensions of a line bundle by rank 3 bundles. This extension splits along $\tilde{\Delta}\cap Y_2$ so that we have an embedding of $\mathcal{O}$ into $\mathcal{E}_{Y_2}|_{\tilde{\Delta}\cap Y_2\times X}$.

Note that $\tilde{Y}$ is the blow-up of Y_2 along $\tilde{\Delta}\cap Y_2$ with unstable points removed. Let $\mathcal{E}_{\tilde{Y}}$ be the pull-back of $\mathcal{E}_{Y_2}$ to $\tilde{Y}\times X$ and $\mathcal{D}^{(3)}_{\tilde{Y}}$ be the exceptional divisor while $\mathcal{D}^{(1)}_{\tilde{Y}}$ and $\mathcal{D}^{(2)}_{\tilde{Y}}$ denote the proper transforms of $\mathcal{D}^{(1)}_{Y_2}$ and $\mathcal{D}^{(2)}_{Y_2}$ respectively. Let $\tilde{\mathcal{E}}$ be the kernel of

$$\mathcal{E}_{\tilde{Y}}\to\mathcal{E}_{\tilde{Y}}|_{\mathcal{D}^{(3)}_{\tilde{Y}}\times X}\to\mathcal{E}_{\tilde{Y}}|_{\mathcal{D}^{(3)}_{\tilde{Y}}\times X}/\mathcal{O}$$

This is the desired family of semistable bundles of rank 4. Verifying that this satisfies the conditions of Proposition 2.8 is a repetition of the computations in the previous subsections and so we leave it to the reader.

5. Blowing down Kirwan's desingularization

In this section we show that the morphism

$$\rho:K\to S$$

constructed in section 4, is in fact the result of two contractions. In [OGr99], O'Grady worked out such contractions for the moduli space of sheaves on a *K3 surface*. We follow O'Grady's arguments to show that K can be contracted twice

$$f:\qquad K\xrightarrow{f_\sigma}K_\sigma\xrightarrow{f_\epsilon}K_\epsilon \tag{5.1}$$

and these contractions are actually blow-downs. Then we show that the map ρ factors through K_ϵ, i.e.

$$\begin{array}{ccc} K & \xrightarrow{\rho} & S \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \rho_\epsilon} & \\ & K_\epsilon & \end{array} \tag{5.2}$$

By Zariski's main theorem, we will conclude that $K_\epsilon\cong S$.

5.1. Contractions.

Since the details are almost identical to section 3 of [OGr99], we provide only the outline.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the tautological rank 2 (resp. rank 3) bundle over the Grassmannian $Gr(2,g)$ (resp. $Gr(3,g)$). Let $W=sl(2)^\vee$ be the dual vector space of $sl(2)$. Fix $B\in Gr(3,g)$. Then the variety of complete conics $\mathbf{CC}(B)$ is the blow-up

$$\mathbb{P}(S^2B)\xleftarrow{\Phi_B}\mathbf{CC}(B)\xrightarrow{\Phi_B^\vee}\mathbb{P}(S^2B^\vee)$$

of both of the spaces of conics in $\mathbb{P}B$ and $\mathbb{P}B^\vee$ along the locus of rank 1 conics.

Proposition 5.1. (1) *$\tilde{D}^{(1)}$ is the variety of complete conics $\mathbf{CC}(\mathcal{B})$ over $Gr(3,g)$. In other words, $\tilde{D}^{(1)}$ is the blow-up of the projective bundle $\mathbb{P}(S^2\mathcal{B})$ along the locus of rank 1 conics.*

(2) *There is an integer l such that*

$$\tilde{D}^{(3)}\cong\mathbb{P}(S^2\mathcal{A})\times_{Gr(2,g)}\mathbb{P}(\mathbb{C}^g/\mathcal{A}\oplus\mathcal{O}(l)).$$

Hence $\tilde{D}^{(3)}$ is a $\mathbb{P}^2\times\mathbb{P}^{g-2}$ bundle over $Gr(2,g)$.

(3) *The intersection* $\tilde{D}^{(1)} \cap \tilde{D}^{(3)}$ *is isomorphic to the fibred product*

$$\mathbb{P}(S^2\mathcal{A}) \times \mathbb{P}(\mathbb{C}^g/\mathcal{A})$$

over $Gr(2,g)$. *As a subvariety of* $\tilde{D}^{(1)}$, $\tilde{D}^{(1)} \cap \tilde{D}^{(3)}$ *is the exceptional divisor of the blow-up* $\mathbf{CC}(\mathcal{B}) \to \mathbb{P}(S^2\mathcal{B}^\vee)$.

(4) *The intersection* $\tilde{D}^{(1)} \cap \tilde{D}^{(2)} \cap \tilde{D}^{(3)}$ *is isomorphic to*

$$\mathbb{P}(S^2\mathcal{A})_1 \times \mathbb{P}(\mathbb{C}^g/\mathcal{A})$$

over $Gr(2,g)$ *where* $\mathbb{P}(S^2\mathcal{A})_1$ *denotes the locus of rank 1 quadratic forms.*

(5) *The intersection* $\tilde{D}^{(1)} \cap \tilde{D}^{(2)}$ *is the exceptional divisor of the blow-up* $\mathbf{CC}(\mathcal{B}) \to \mathbb{P}(S^2\mathcal{B})$.

Proof. The proofs are identical to (3.1.1), (3.5.1), and (3.5.4) in [OGr99]. □

Next, we consider some rational curves to be contracted. Define the following classes in $N_1(\tilde{D}^{(1)})$ (the group of numerical equivalence classes of 1-cycles)

$$\sigma := \text{ the class of lines in the fiber of } \Phi_B^\vee$$

$$\epsilon := \text{ the class of lines in the fiber of } \Phi_B$$

$$\gamma := \text{ the class of } \{\Phi_{B_t}^{-1}(q_t)\}_{t\in\Lambda}$$

where $\{B_t\}$ is a line Λ of 3-dimensional subspaces in $Gr(3,g)$ containing a fixed 2-dimensional space A with $q \in S^2A$ and q_t is the induced quadratic form on B_t.

To show that these form a basis of $N_1(\tilde{D}^{(1)})$ we consider the following diagram

$$\begin{array}{ccc} \tilde{D}^{(1)} & \xrightarrow{\theta} & \mathbb{P}(S^2\mathcal{B}) \\ & & \downarrow{\scriptstyle \phi} \\ & & Gr(3,g) \end{array}$$

where θ is the blow-up. Let $h = c_1(\mathcal{B}^\vee)$, $x = c_1(\mathcal{O}_{\mathbb{P}(S^2\mathcal{B})}(1))$ and e be the exceptional divisor of θ. Then obviously h, x, e form a basis of $N^1(\tilde{D}^{(1)})$ which is dual to $N_1(\tilde{D}^{(1)})$. By elementary computation, the intersection pairing is given by the table

	h	x	e
ϵ	0	0	-1
σ	0	1	2
γ	1	0	0

Hence, σ, ϵ, γ form a basis of $N_1(\tilde{D}^{(1)})$.

Lemma 5.2. (1) $[\tilde{D}^{(1)}]|_{\mathbf{CC}(B)} = -2x + e|_{\mathbf{CC}(B)}$ *for* $B \in Gr(3,g)$.

(2) $[\tilde{D}^{(2)}]|_{\tilde{D}^{(1)}} = e$

(3) $[\tilde{D}^{(3)}]|_{\tilde{D}^{(1)}} = 3x - 2h - 2e$

(4) $\Theta_{\tilde{D}^{(1)}} = -(g-4)h - 6x + 2e$ *where* $\Theta_{\tilde{D}^{(1)}}$ *denotes the canonical divisor of* $\tilde{D}^{(1)}$.

The proofs are identical to those of (3.2.3) - (3.2.5), (3.4.3) with obvious modifications.

Let $\hat{\sigma} = \imath_*\sigma$, $\hat{\epsilon} = \imath_*\epsilon$ and $\hat{\gamma} = \imath_*\gamma$ where $\imath$ is the inclusion of $\tilde{D}^{(1)}$ into K. By the above lemma, x, h, e are in the image of $N^1(K)$ by restriction. Hence,

$N^1(K) \to N^1(\tilde{D}^{(1)})$ is surjective and dually $\imath_*$ is injective. Consequently, $\hat{\sigma}, \hat{\epsilon}, \hat{\gamma}$ are linearly independent.

At this point, we can compute the discrepancy $\omega_K - \pi^*\omega_{M_0}$ of the canonical divisors ω_K and ω_{M_0}.

Proposition 5.3.

$$\omega_K - \pi^*\omega_{M_0} = (3g-1)\tilde{D}^{(1)} + (g-2)\tilde{D}^{(2)} + (2g-2)\tilde{D}^{(3)}$$

Proof. Obvious adaptation of the proof of (3.4.1) in [OGr99]. □

Corollary 5.4. *For $g \geq 3$, M_0 has terminal singularities and the plurigenera are all trivial.*

Proof. It is well-known that ω_{M_0} is anti-ample. Since the singularities are terminal, $\pi_*\omega_K = \omega_{M_0}$. It follows from spectral sequence and Kodaira's vanishing theorem that $H^0(K, \omega_K^{\otimes m}) \cong H^0(M_0, \omega_{M_0}^{\otimes m}) = 0$ for $m > 0$. □

Finally we can show that K can be blown-down twice.

Proposition 5.5. (1) *$\hat{\sigma}, \hat{\epsilon}$ are ω_K-negative extremal rays. For $g > 3$, $\hat{\gamma}$ is also ω_K-negative extremal.*

(2) *The contraction K_σ of the ray $\mathbb{R}^+\hat{\sigma}$ is a smooth projective desingularization of M_0. In fact, this is the contraction of the $\mathbb{P}(S^2\mathcal{A})$-direction of $\tilde{D}^{(3)}$. Since the normal bundle is $\mathcal{O}(-1)$ up to tensoring a line bundle on $\mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l))$, the contraction is a blow-down map.*

(3) *The image of $\hat{\epsilon}$ in $N_1(K_\sigma)$ is ω_{K_σ}-negative extremal ray and its contraction K_ϵ is a smooth projective desingularization of M_0. This is the contraction of the fiber direction of $\mathbb{P}(S^2\mathcal{B}^\vee) \to Gr(3,g)$ and is also a blow-down map.*

The proofs are same as those of (3.0.2)-(3.0.4) in [OGr99].

5.2. **Factorization of ρ.** Now we can show the following

Theorem 5.6. *ρ factors through K_ϵ and $K_\epsilon \cong S$.*

Proof. Let us consider the first contraction $f_\sigma : K \to K_\sigma$. We claim that there is a continuous map $\rho_\sigma : K_\sigma \to S$ such that $\rho_\sigma \circ f_\sigma = \rho$. (See the diagram (5.2).) By Riemann's extension theorem [Mum76], it suffices to show that ρ is constant on the fibers of f_σ. From Proposition 5.1, we know f_σ is the result of contracting the fibers $\mathbb{P}^2$ of

$$\tilde{D}^{(3)} = \mathbb{P}(S^2\mathcal{A}) \times \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l)) \to \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l))$$

which amounts to forgetting the choice of b, c in the 2-dimensional subspace of $H^1(\mathcal{O})$ spanned by b, c. We need only to check that the isomorphism classes of the vector bundles given by (4.12) and (4.10) depend *not* on the particular choice of b, c *but* only on the points in $\mathbb{P}^{g-2}$-bundle $\mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l)) \to \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l))$ over $Gr(2, g)$.

From [BS90] Proposition 5, the isomorphism classes of bundles given by (4.12) are parametrized by a vector bundle of rank $g-2$ over $Gr(2, g)$. In particular, the isomorphism classes are independent of the choice of b, c. Hence the bundles given by (4.12) are constant along the $\mathbb{P}(S^2\mathcal{A})$-direction. On the other hand, it is elementary to show that a similar statement holds for the bundles given by (4.10). Therefore, there exists a morphism $\rho_\sigma : K_\sigma \to S$ such that $\rho_\sigma \circ f_\sigma = \rho$.

Next we show that ρ_σ factors through K_ϵ. The morphism $f_\epsilon : K_\sigma \to K_\epsilon$ is the contraction of the fibers $\mathbb{P}^5$ of

$$\mathbb{P}(S^2\mathcal{B}) \to Gr(3, g)$$

and general points of a fiber give rise to a rank 4 bundle whose transition matrices are of the form (4.7). It is elementary to show that the isomorphism classes of the bundles given by (4.7) depend only on the 3-dimensional subspace spanned by a, b, c. Hence ρ_σ is constant along the fibers of f_ϵ. By Riemann's extension theorem again, we get a morphism $\rho_\epsilon : K_\epsilon \to S$ such that $\rho_\epsilon \circ f = \rho$.

From [Bal88, BS90], $\rho(\tilde{D}^{(2)} - \tilde{D}^{(1)} \cup \tilde{D}^{(3)})$ is a smooth divisor of $S - \rho(\tilde{D}^{(1)} \cup \tilde{D}^{(3)})$ that lies over $\mathfrak{K} - \mathbb{Z}_2^{2g}$. Hence, we have a morphism from $S - \rho(\tilde{D}^{(1)} \cup \tilde{D}^{(3)})$ to the blow-up of $M_0 - \mathbb{Z}_2^{2g}$ along $\mathfrak{K} - \mathbb{Z}_2^{2g}$ which is isomorphic to $K - \tilde{D}^{(1)} \cup \tilde{D}^{(3)} = K_\epsilon - f(\tilde{D}^{(1)} \cup \tilde{D}^{(3)})$ by construction. Hence, ρ_ϵ is an isomorphism in codimension one. Since K_ϵ and S are both smooth, Zariski's main theorem says K_ϵ is isomorphic to S. □

Conjecture 5.7. *The intermediate variety K_σ is the Narasimhan-Ramanan desingularization.*

We hope to get back to this conjecture in the future.

6. Cohomological consequences

6.1. **Cohomology of Seshadri's desingularization.** In [Bal88, BS90], Balaji and Seshadri show the Betti numbers of Seshadri's desingularization S can be computed, up to degree $\leq 2g - 4$. Thanks to the explicit description of S as the blow-down of K, we can compute the Betti numbers in all degrees.

For a variety T, let

$$P(T) = \sum_{k=0}^{\infty} t^k \dim H^k(T)$$

be the Poincaré series of T. In [Kir85], Kirwan described an algorithm for the Poincaré series of a partial desingularization of a good quotient of a smooth projective variety and in [Kir86b] the algorithm was applied to the moduli space without fixing the determinant. For $P(M_2)$ we use Kirwan's algorithm in [Kir85].

By [AB82] §11 and [Kir86a], it is well-known that the equivariant Poincaré series $P^G(\mathfrak{R}^{ss}) = \sum_{k\geq 0} t^k \dim H^k_G(\mathfrak{R}^{ss})$ is

$$P^G(\mathfrak{R}^{ss}) = \frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)}$$

up to degrees as high as we want. In order to get $\mathfrak{R}_1^{ss}$ we blow up $\mathfrak{R}^{ss}$ along $GZ^{ss}_{SL(2)}$ and delete the unstable strata. So we get

$$P^G(\mathfrak{R}_1^{ss}) = P^G(\mathfrak{R}^{ss}) + 2^{2g}\Big(\frac{t^2 + t^4 + \cdots + t^{6g-2}}{1-t^4} - \frac{t^{4g-2}(1 + t^2 + \cdots + t^{2g-2})}{1-t^2}\Big).$$

Now $\mathfrak{R}_2^{ss}$ is obtained by blowing up $\mathfrak{R}_1^{ss}$ along $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ and deleting the unstable strata. Thus we have

(6.1)

$$\begin{aligned} P^G(\mathfrak{R}_2^{ss}) = P^G(\mathfrak{R}_1^{ss}) \quad &+(t^2 + t^4 + \cdots + t^{4g-6})\Big(\tfrac{1}{2}\tfrac{(1+t)^{2g}}{1-t^2} + \tfrac{1}{2}\tfrac{(1-t)^{2g}}{1+t^2} + 2^{2g}\tfrac{t^2+\cdots+t^{2g-2}}{1-t^4}\Big) \\ &-\tfrac{t^{2g-2}(1+t^2+\cdots+t^{2g-4})}{1-t^2}\Big((1+t)^{2g} + 2^{2g}(t^2 + t^4 + \cdots + t^{2g-2})\Big). \end{aligned}$$

Because the stabilizers of the G action on $\mathfrak{R}_2^{ss}$ are all finite, we have

$$H_G^*(\mathfrak{R}_2^{ss}) \cong H^*(\mathfrak{R}_2^{ss}/G) = H^*(M_2)$$

and hence we deduce that

$$\begin{aligned} P(M_2) &= \tfrac{(1+t^3)^{2g}-t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} \\ &+2^{2g}\big(\tfrac{t^2+t^4+\cdots+t^{6g-2}}{1-t^4}-\tfrac{t^{4g-2}(1+t^2+\cdots+t^{2g-2})}{1-t^2}\big) \\ &+(t^2+t^4+\cdots+t^{4g-6})\big(\tfrac{1}{2}\tfrac{(1+t)^{2g}}{1-t^2}+\tfrac{1}{2}\tfrac{(1-t)^{2g}}{1+t^2}+2^{2g}\tfrac{t^2+\cdots+t^{2g-2}}{1-t^4}\big) \\ &-\tfrac{t^{2g-2}(1+t^2+\cdots+t^{2g-4})}{1-t^2}\big((1+t)^{2g}+2^{2g}(t^2+t^4+\cdots+t^{2g-2})\big). \end{aligned} \tag{6.2}$$

Kirwan's desingularization is the blow-up of M_2 along $\tilde{\Delta}/\!\!/SL(2)$ which is isomorphic to the 2^{2g} copies of $\mathbb{P}(S^2\mathcal{A})$ over $Gr(2,g)$. Hence,

$$P(K) = P(M_2) + 2^{2g}(1+t^2+t^4)P(Gr(2,g))(t^2+t^4+\cdots+t^{2g-4})$$

by [GH78] p. 605.[1]

On the other hand, K is the blow-up of K_σ along a $\mathbb{P}^{g-2}$-bundle over $Gr(2,g)$. Hence,

$$\begin{aligned} P(K_\sigma) &= P(K) - 2^{2g}(1+t^2+\cdots+t^{2g-4})P(Gr(2,g))(t^2+t^4) \\ &= P(M_2) + 2^{2g}P(Gr(2,g))\tfrac{t^6-t^{2g-2}}{1-t^2}. \end{aligned}$$

Similarly, K_σ is the blow-up of K_ϵ along a $Gr(3,g)$ and thus

$$\begin{aligned} P(K_\epsilon) &= P(K_\sigma) - 2^{2g}P(Gr(3,g))(t^2+\cdots+t^{10}) \\ &= P(M_2) + 2^{2g}P(Gr(2,g))\tfrac{t^6-t^{2g-2}}{1-t^2} - 2^{2g}P(Gr(3,g))(t^2+\cdots+t^{10}). \end{aligned}$$

Since K_ϵ is isomorphic to Seshadri's desingularization, we get

$$\begin{aligned} P(S) &= \tfrac{(1+t^3)^{2g}-t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} \\ &+2^{2g}\big(\tfrac{t^2+t^4+\cdots+t^{6g-2}}{1-t^4}-\tfrac{t^{4g-2}(1+t^2+\cdots+t^{2g-2})}{1-t^2}\big) \\ &+(t^2+t^4+\cdots+t^{4g-6})\big(\tfrac{1}{2}\tfrac{(1+t)^{2g}}{1-t^2}+\tfrac{1}{2}\tfrac{(1-t)^{2g}}{1+t^2}+2^{2g}\tfrac{t^2+\cdots+t^{2g-2}}{1-t^4}\big) \\ &-\tfrac{t^{2g-2}(1+t^2+\cdots+t^{2g-4})}{1-t^2}\big((1+t)^{2g}+2^{2g}(t^2+t^4+\cdots+t^{2g-2})\big) \\ &+2^{2g}P(Gr(2,g))\tfrac{t^6-t^{2g-2}}{1-t^2} - 2^{2g}P(Gr(3,g))(t^2+\cdots+t^{10}). \end{aligned}$$

By Schubert calculus [GH78], we have

$$P(Gr(2,g)) = \frac{(1-t^{2g})(1-t^{2g-2})}{(1-t^2)(1-t^4)}$$

$$P(Gr(3,g)) = \frac{(1-t^{2g})(1-t^{2g-2})(1-t^{2g-4})}{(1-t^2)(1-t^4)(1-t^6)}$$

and hence we obtained a closed formula for the Poincaré polynomial of S.

In [BS90], an algorithm for the Betti numbers only up to degree $2g-4$ is provided. It is an elementary exercise to check that in this range, their answer is identical to ours.

[1]The formula in [GH78] is stated for smooth manifolds. But the same Mayer-Vietoris argument gives us the same formula in our case (of orbifold M_2 blown up along a smooth subvariety). The only thing to be checked is that the pull-back homomorphism $H^*(M_2) \to H^*(K)$ is injective but this clearly holds by the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber.

6.2. **The stringy E-function.** The stringy E-function is an invariant of singular varieties introduced by Batyrev, Denef and Loeser, based on the suggestions by Kontsevich. In [Kie03], the stringy E-function of M_0 was computed for $g = 3$ by using the observation that the singularities are hypersurface singularities in this case.[2] In this subsection, we compute the stringy E-function of M_0 for arbitrary genus. For the definition and some basic facts on the stringy E-functions, see the introduction of [Kie03].

Since the discrepancy divisor is given by Proposition 5.3, our goal is to compute

$$\begin{aligned} E_{st}(M_0) &= E(M_0^s) + E(\tilde{D}_0^{(1)})\tfrac{uv-1}{(uv)^{3g}-1} + E(\tilde{D}_0^{(2)})\tfrac{uv-1}{(uv)^{g-1}-1} + E(\tilde{D}_0^{(3)})\tfrac{uv-1}{(uv)^{2g-1}-1} \\ &+E(\tilde{D}_0^{(1,2)})\tfrac{uv-1}{(uv)^{3g}-1}\tfrac{uv-1}{(uv)^{g-1}-1} + E(\tilde{D}_0^{(2,3)})\tfrac{uv-1}{(uv)^{g-1}-1}\tfrac{uv-1}{(uv)^{2g-1}-1} \\ &+E(\tilde{D}_0^{(1,3)})\tfrac{uv-1}{(uv)^{3g}-1}\tfrac{uv-1}{(uv)^{2g-1}-1} + E(\tilde{D}_0^{(1,2,3)})\tfrac{uv-1}{(uv)^{3g}-1}\tfrac{uv-1}{(uv)^{g-1}-1}\tfrac{uv-1}{(uv)^{2g-1}-1} \end{aligned}$$

where $\tilde{D}_0^{(I)} = \cap_{i\in I}\tilde{D}^{(i)} - \cup_{j\notin I}\tilde{D}^{(j)}$ for $I \subset \{1,2,3\}$ and E denotes the Hodge-Deligne polynomal.

The E-function of the smooth part is from [Kie03] §4,

$$\begin{aligned} E(M_0^s) &= E(M_2) - E(D_2^{(1)}) - E(D_2^{(2)} - D_2^{(1)}) \\ &= \tfrac{(1-u^2v)^g(1-uv^2)^g-(uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} \\ &-\tfrac{1}{2}\left(\tfrac{(1-u)^g(1-v)^g}{1-uv} + \tfrac{(1+u)^g(1+v)^g}{1+uv}\right). \end{aligned}$$

By Proposition 5.1, $\tilde{D}_0^{(1)} = \tilde{D}^{(1)} - (\tilde{D}^{(2)} \cup \tilde{D}^{(3)})$ is the union of 2^{2g} copies of $\mathbb{P}^5 - \mathbb{P}^2 \times_{\mathbb{Z}_2} \mathbb{P}^2$-bundle over $Gr(3,g)$ and thus

$$E(\tilde{D}_0^{(1)})\frac{uv-1}{(uv)^{3g}-1} = 2^{2g}((uv)^5 - (uv)^2)E(Gr(3,g))\frac{uv-1}{(uv)^{3g}-1}.$$

Since $\tilde{D}_0^{(2)}$ is the quotient of a $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$-bundle over $Jac_0 - \mathbb{Z}_2^{2g}$ by the action of $\mathbb{Z}_2$, the E-function of $\tilde{D}_0^{(2)}$ is

$$\begin{aligned} &E(\tilde{D}_0^{(2)})\tfrac{uv-1}{(uv)^{g-1}-1} \\ &= \left(\tfrac{1}{2}(1-u)^g(1-v)^g + \tfrac{1}{2}(1+u)^g(1+v)^g - 2^{2g}\right)E(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})^+\tfrac{uv-1}{(uv)^{g-1}-1} \\ &+\left(\tfrac{1}{2}(1-u)^g(1-v)^g - \tfrac{1}{2}(1+u)^g(1+v)^g\right)E(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})^-\tfrac{uv-1}{(uv)^{g-1}-1} \end{aligned}$$

where

$$E(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})^+ = \frac{((uv)^g-1)((uv)^{g-1}-1)}{(uv-1)((uv)^2-1)}$$

is the E-polynomial of the $\mathbb{Z}_2$-invariant part of $H^*(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})$ and

$$E(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})^- = uv\frac{((uv)^{g-1}-1)((uv)^{g-2}-1)}{(uv-1)((uv)^2-1)}$$

is the E-polynomial of the anti-invariant part.

By Proposition 5.1, $\tilde{D}_0^{(3)}$ is the union of 2^{2g} copies of a $(\mathbb{P}^2\times\mathbb{P}^{g-2} - \mathbb{P}^2\times\mathbb{P}^{g-3}\cup\mathbb{P}^1\times\mathbb{P}^{g-2})$-bundle over $Gr(2,g)$ and thus

$$E(\tilde{D}_0^{(3)})\frac{uv-1}{(uv)^{2g-1}-1} = 2^{2g}(uv)^gE(Gr(2,g))\frac{uv-1}{(uv)^{2g-1}-1}.$$

[2]There is a small error in [Kie03] page 1852. In line -3, α_1 should be replaced by α_7^2 and thus in line -1, the discrepancy divisor is $8D_1 + D_2 + 4D_3$ (cf. Proposition 5.3). The computation in [Kie03] §7 should be accordingly modified. The correct formula for any $g \geq 3$ is proved in this paper (Theorem 6.1).

Notice that $\tilde{D}_0^{(1,2)}$ is the disjoint union of 2^{2g} copies of a $(\mathbb{P}^2-\mathbb{P}^1)\times\mathbb{P}^2$-bundle over $Gr(3,g)$ and thus

$$E(\tilde{D}_0^{(1,2)})\frac{uv-1}{(uv)^{3g}-1}\frac{uv-1}{(uv)^{g-1}-1}=2^{2g}((uv)^2+(uv)^3+(uv)^4)E(Gr(3,g))\frac{uv-1}{(uv)^{3g}-1}\frac{uv-1}{(uv)^{g-1}-1}.$$

Also, $\tilde{D}_0^{(1,3)}$ is a $(\mathbb{P}^2-\mathbb{P}^1)\times\mathbb{P}^{g-3}$-bundle over $Gr(2,g)$ and thus

$$E(\tilde{D}_0^{(1,3)})\frac{uv-1}{(uv)^{3g}-1}\frac{uv-1}{(uv)^{2g-1}-1}=2^{2g}(uv)^2\frac{(uv)^{g-2}-1}{uv-1}E(Gr(2,g))\frac{uv-1}{(uv)^{3g}-1}\frac{uv-1}{(uv)^{2g-1}-1}.$$

Finally, a component of $\tilde{D}_0^{(2,3)}$ is a $\mathbb{P}^1\times(\mathbb{P}^{g-2}-\mathbb{P}^{g-3})$-bundle over $Gr(2,g)$ and a component of $\tilde{D}_0^{(1,2,3)}$ is a $\mathbb{P}^1\times\mathbb{P}^{g-3}$-bundle over $Gr(2,g)$. Therefore,

$$E(\tilde{D}_0^{(2,3)})\frac{uv-1}{(uv)^{g-1}-1}\frac{uv-1}{(uv)^{2g-1}-1}=2^{2g}(1+uv)(uv)^{g-2}E(Gr(2,g))\frac{uv-1}{(uv)^{g-1}-1}\frac{uv-1}{(uv)^{2g-1}-1}$$

and

$$\begin{aligned} E(\tilde{D}_0^{(1,2,3)}) & \tfrac{uv-1}{(uv)^{3g}-1}\tfrac{uv-1}{(uv)^{g-1}-1}\tfrac{uv-1}{(uv)^{2g-1}-1} \\ & =2^{2g}(1+uv)\tfrac{(uv)^{g-2}-1}{uv-1}E(Gr(2,g))\tfrac{uv-1}{(uv)^{3g}-1}\tfrac{uv-1}{(uv)^{g-1}-1}\tfrac{uv-1}{(uv)^{2g-1}-1}. \end{aligned}$$

Recall that

$$E(Gr(2,g))=\frac{((uv)^g-1)((uv)^{g-1}-1)}{(uv-1)((uv)^2-1)}$$

$$E(Gr(3,g))=\frac{((uv)^g-1)((uv)^{g-1}-1)((uv)^{g-2}-1)}{(uv-1)((uv)^2-1)((uv)^3-1)}.$$

Putting together all the pieces above, we get

Theorem 6.1.

$$\begin{aligned} E_{st}(M_0)= & \tfrac{(1-u^2v)^g(1-uv^2)^g-(uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} \\ & -\tfrac{(uv)^{g-1}}{2}\big(\tfrac{(1-u)^g(1-v)^g}{1-uv}-\tfrac{(1+u)^g(1+v)^g}{1+uv}\big). \end{aligned}$$

Remark 6.2. It is well-known that the middle perversity intersection cohomology of M_0 is equipped with a Hodge structure and hence it makes sense to think about the E-polynomial of the intersection cohomology. The computation of the Poincaré polynomial of $IH^*(M_0)$ in [Kir86b] can be easily refined as in [EK00] to give the E-polynomial of $IH^*(M_0)$

$$\begin{aligned} IE(M_0) & =\tfrac{(1-u^2v)^g(1-uv^2)^g-(uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} \\ & -\tfrac{(uv)^{g-1}}{2}\big(\tfrac{(1-u)^g(1-v)^g}{1-uv}+(-1)^{g-1}\tfrac{(1+u)^g(1+v)^g}{1+uv}\big). \end{aligned}$$

See also [Kiem]. Quite surprisingly, when g is even, $E_{st}(M_0)$ is identical to the E-polynomial of the middle perversity intersection cohomology of M_0. This indicates that there may be an unknown relation between the stringy E-function and the intersection cohomology. When g is odd, $E_{st}(M_0)$ is not a polynomial.

Corollary 6.3. *The stringy Euler number of M_0 is*

$$e_{st}(M_0):=\lim_{u,v\to 1}E_{st}(M_0)=4^{g-1}.$$

Let e_g be the stringy Euler number of the moduli space M_0 for a genus g curve. When $g = 2$, $M_0 \cong \mathbb{P}^3$ and so $e_2 = 4$. Therefore the equality

$$\sum_g e_g q^g = \frac{1}{4}\frac{1}{1-4q}$$

holds for degree ≥ 2. The coefficient $\frac{1}{4}$ might be related to the "mysterious" coefficient $\frac{1}{4}$ for the S-duality conjecture test in the K3 case in [VW94].

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