INTERSECTION COHOMOLOGY OF QUOTIENTS OF NONSINGULAR VARIETIES

YOUNG-HOON KIEM

1. Introduction

Let $M \subset \mathbb{P}^n$ be a nonsingular projective variety acted on by a connected complex reductive group $G = K^{\mathbb{C}}$ via a homomorphism $G \to GL(n+1)$ which is the complexification of a homomorphism $K \to U(n+1)$. Geometric invariant theory (GIT) gives us a recipe to form a quotient $\phi : M^{ss} \to M/\!\!/G$ of the set of semistable points and many interesting spaces in algebraic geometry are constructed in this manner [MFK94]. But often this quotient is singular and hence intersection cohomology with middle perversity is an important topological invariant. The purpose of this paper is to present a way to compute the middle perversity intersection cohomology of the singular quotients.

The choice of an embedding $M \subset \mathbb{P}^n$ provides us with a moment map for the K-action and the Morse stratification of M with respect to its norm square is K-equivariantly perfect, with the unique open dense stratum M^{ss} [Kir84]. Hence one can compute the Betti numbers for the equivariant cohomology $H^*_K(M^{ss})$ as well as the cup product, at least in principle. See also [Kir92].

When G acts locally freely on M^{ss}, we get an orbifold $M/\!\!/G = M^{ss}/G$ and

$$H^*_K(M^{ss}) \cong H^*(M/\!\!/G) \cong IH^*(M/\!\!/G).$$

Hence this equivariant Morse theory enables us to compute the cohomology ring of the orbifold quotient.

However, if the G action on M^{ss} is not locally free, then $M/\!\!/G$ has more serious singularities than finite quotient singularities. In general, the induced homomorphism $H^*(M/\!\!/G) \to H^*_K(M^{ss})$, which comes from the quotient map

$$\phi_K : EK \times_K M^{ss} \to M/\!\!/G,$$

is neither injective nor surjective. Furthermore, the natural map from the ordinary cohomology $H^*(M/\!\!/G)$ to the intersection cohomology $IH^*(M/\!\!/G)$ due to Goresky and MacPherson [GM80, GM83] is neither injective nor surjective. Hence, knowledge of the equivariant cohomology $H^*_K(M^{ss})$ does not directly enable us to compute the topological invariants for $M/\!\!/G$.

In [Kir85], Kirwan invented a method to partially desingularize $M/\!\!/G$ by blowing up M^{ss} systematically. When the set of stable points M^s is nonempty, she used this process to define a map, which we call the Kirwan map

$$\kappa^{ss}_M : H^*_K(M^{ss}) \to IH^*(M/\!\!/G)$$

and then to provide an algorithm for the computation of the Betti numbers of $IH^*(M/\!\!/G)$. See [Kir86a] for the Betti number computation of the moduli space of

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rank 2 holomorphic vector bundles over a Riemann surface. However, this method does not give us any information on the intersection pairing of $IH^*(M/\!\!/G)$ which is also an essential topological invariant.

In this paper, we construct a natural injection

$$\phi_K^* : IH^*(M/\!\!/G) \to H_K^*(M^{ss}) \tag{1.1}$$

under an assumption, named the *almost balanced* condition (Definition 5.1). This homomorphism is the right inverse of the Kirwan map above (Proposition 6.2).

It is well-known that intersection cohomology is contravariant only for some limited classes of maps. One of the most general known conditions for a subanalytic map

$$f : X \to Y$$

between subanalytic pseudo-manifolds to induce a contravariant homomorphism

$$f^* : IH^*(Y) \to IH^*(X)$$

by pulling back cycles is the *placid* condition introduced in [GM85]: The map f is placid if Y has a stratification such that for each stratum S of Y, we have

$$\text{codim } S \leq \text{codim } f^{-1}(S).$$

With this condition, the pull-back of an allowable cycle is again allowable and hence we get the desired homomorphism f^*.

Furthermore, when there is a compact Lie group K acting on X and f is invariant, we have a homomorphism

$$f_K^* : IH^*(Y) \to IH_K^*(X)$$

under the placid condition because $EK \times_K X$ can be approximated by a sequence of finite dimensional pseudo-manifolds (see §2.5).

More generally, if we have

$$q(\text{codim } S) \leq p(\text{codim } f^{-1}(S)) \tag{1.2}$$

for some perversities p and q, then the pull-back of cycles gives us a homomorphism

$$f^* : IH_q^*(Y) \to IH_p^*(X).$$

We call (1.2) the (p, q)-placid condition.

In our case, M^{ss} is smooth and hence $H_K^*(M^{ss})$ is isomorphic to the equivariant intersection cohomology

$$IH_{p,K}^*(M^{ss})$$

for any perversity p. Hence if ϕ is (p, m)-placid, we get a homomorphism

$$\phi_K^* : IH^*(M/\!\!/G) \to H_K^*(M^{ss}).$$

Obviously the most general condition is obtained when p is the top perversity t. Hence, when ϕ is (t, m)-placid, we obtain the natural homomorphism (1.1).

Because ϕ_K^* is defined by pulling back cycles, we can deduce that the intersection pairing is preserved in the following sense: for α, β of complementary degrees in $IH^*(M/\!\!/G)$, i.e. $\deg \alpha + \deg \beta = \dim M/\!\!/G$, we have

$$\phi_K^*(\alpha) \cup \phi_K^*(\beta) = \langle \alpha, \beta \rangle \phi_K^*(\tau) \tag{1.3}$$

where τ is the top degree class represented by a point. Hence, we can compute the intersection numbers in terms of the cup product structure of the equivariant cohomology.

By the local model theorem (Lemma 4.1 and Proposition 4.2) from [SL91], we can describe a topological stratification of M^{ss} and $M/\!\!/G$ explicitly and compute the codimensions of strata in terms of the weight distributions of the actions of the stabilizer subgroups on the symplectic slices. By the results of [Kir84], we will see in §5 that the (t,m)-placid condition is in fact a condition on the balancedness of the weight distributions. This is our almost balanced condition.

Suppose now the almost balanced condition is satisfied. Then by (1.1) and (1.3), the middle perversity intersection cohomology of $M/\!\!/G$ is completely determined as a graded vector space with non-degenerate intersection pairing if we can identify the image of ϕ_K^* in $H_K^*(M^{ss})$.

In the light of Deligne's construction of intersection cohomology sheaf, it seems reasonable to expect that the answer should be obtained by "truncating locally" along each stratum. But complications arise when various strata intersect. To control the complications, we require that the submanifolds of M fixed by subgroups of G also satisfy the almost balanced condition. We call it the *weakly balanced* condition (Definition 7.1). When this condition is satisfied, we can identify the image of ϕ_K^* as a subspace V_M^* of $H_K^*(M^{ss})$ obtained by truncation. See Definition 7.5 and (7.6). By computing the dimension of V_M^*, we get the intersection Betti numbers without going through the desingularization process as in [Kir86b, Kir86a]. One can furthermore compute the Hodge numbers since both ϕ_K^* and κ_M^{ss} preserve the Hodge structure.

The weakly balanced condition is satisfied by many interesting spaces including the moduli spaces of holomorphic vector bundles over a Riemann surface for any rank and degree. (See Proposition 7.4.) Also we demonstrate the computations of V_M^* for some standard examples in §9.

The layout of this paper is as follows. In section 2, we recall the sheaf theoretic definition of intersection cohomology while in section 3 we show that intersection cohomology is functorial with respect to (p,q)-placid maps. The stratification of a symplectic reduction is discussed in section 4 and the homomorphism ϕ_K^* is constructed in section 5 after defining the almost balanced condition. In section 6 we recall the Kirwan map and prove that ϕ_K^* is the right inverse of κ_M^{ss}. In section 7 we define V_M^* and state the theorem, which says

$$\phi_K^*(IH^*(M/\!\!/G)) = V_M^*. \tag{1.4}$$

This is proved in section 8 and several examples are computed in section 9.

The proof of (1.4) is unfortunately quite lengthy and so I briefly sketch the outline. Our proof is by induction on the maximum among the dimensions of stabilizers. We first show that

$$\phi_K^*(IH^*(M/\!\!/G)) \subset V_M^*.$$

This is best seen from the sheaf theoretic perspective and we need the full strength of the weakly balanced condition for the computation of stalks.

Next let $\hat{M}$ be the first blow-up in the partial desingularization process [Kir85]. By our induction hypothesis, we have

$$\hat{\phi}_K^*(IH^*(\hat{M}/\!\!/G)) = V_{\hat{M}}^*$$

where $\hat{\phi} : \hat{M}^{ss} \to \hat{M}/\!\!/G$ is the GIT quotient map. Then we show there is an embedding

$$V_M^* \hookrightarrow V_{\hat{M}}^* \cong IH^*(\hat{M}/\!\!/G).$$

Since the difference between $IH^*(\hat{M}/\!\!/G)$ and $IH^*(M/\!\!/G)$ comes from a sheaf complex supported at the exceptional divisor, we will see that the proof reduces to the computation for the normal bundle $\mathcal{N}$ to the blow-up center in M^{ss}. By spectral sequence, this further reduces to the computation for the normal space at a point. Upon checking this, we complete the proof.

The main application of the results of this paper is to generalize the theorem of Jeffrey and Kirwan [JK98] about the intersection numbers on the *smooth* moduli spaces of bundles over a Riemann surface. In a joint work with Jeffrey, Kirwan and Woolf, we will show that the intersection numbers of the intersection cohomology of the *singular* moduli spaces of bundles over a Riemann surface are given by a residue formula similar to that of [JK98].

The first version of this paper was written several years ago. There V_M^* was defined and it was shown that the Kirwan map κ_M^{ss} restricted to V_M^* is an isomorphism onto $IH^*(M/\!\!/G)$ under the weakly balanced condition (Definition 7.2). About a year later, in [KW], we generalized the Kirwan map to symplectic reductions and extended the results of this paper to the purely symplectic setting by interpreting the first condition of the weakly balanced action as the cosupport axiom for intersection homology sheaf. After finishing [KW], it occurred to us that the almost-balanced condition could be most naturally explained in terms of (p,q)-placid maps and hence the current paper was revised using this new observation. In [Kie], we showed that the extended moduli spaces defined by L. Jeffrey [Jef94] satisfy our assumptions and hence we can compute the intersection cohomology of representation spaces of surface groups in terms of the equivariant cohomology.

Every cohomology group in this paper has *complex coefficients*.

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2. Intersection cohomology

Intersection cohomology was introduced by Goresky and MacPherson in [GM80, GM83] as an invariant of singular spaces which retains useful properties like Poincaré duality and Lefschetz theorems. In this section we recall the definition and some properties that we will use.

2.1. **Topological stratification.** An even dimensional topological space X equipped with a filtration by even dimensional closed subsets

$$(\mathfrak{X}) \qquad X = X_n \supset X_{n-2} \supset \cdots \supset X_0 \supset \emptyset \tag{2.1}$$

is called a *stratified pseudo-manifold* if

- $X - X_{n-2}$ is dense
- for each i, $S_i = X_i - X_{i-2}$ is a topological manifold of dimension i or empty
- for each $x \in S_i$ there are a compact stratified pseudo-manifold

  $$L = L_{n-i-1} \supset \cdots \supset L_0 \supset \emptyset$$

  and a stratum-preserving homeomorphism of a neighborhood of x onto $\mathbb{R}^i \times cL$ where $cL = L \times [0,\infty)/L \times 0$.

In this case, $\mathfrak{X}$ is called a *topological stratification*. A pseudo-manifold X of dimension n is normal if $H_n(X, X-x) = 1$ for any $x \in X$.

2.2. **Deligne's construction of intersection cohomology sheaf.** Let

$$p : \{2, 3, \cdots, n\} \to \{0, 1, 2, \cdots, n-2\}$$

be an increasing function such that $q(i) = i - 2 - p(i)$ is also an increasing nonnegative function. We call such p a perversity. For instance,

$$m(i) = [\frac{i-2}{2}]$$

is the middle perversity and $t(i) = i - 2$ is the top perversity. Given a normal stratified pseudo-manifold X with stratification $(\mathfrak{X})$, put $U_{2k} = X - X_{n-2k}$ and let $j_{2k} : U_{2k} \hookrightarrow U_{2k+2}$ denote the inclusion. Then the perversity p intersection cohomology $IH_p^*(X)$ of X is the hypercohomology of the sheaf complex defined inductively by

$$\mathcal{IC}^{\cdot}_{p,X}|_{U_2} = \mathbb{C}_{U_2}$$

$$\mathcal{IC}^{\cdot}_{p,X}|_{U_{2k+2}} \cong \tau_{\leq p(2k)} Rj_{2k*}(\mathcal{IC}^{\cdot}_{p,X}|_{U_{2k}}).$$

This defines an object in the derived category $\mathbf{D}_c^+(X)$ of bounded below cohomologically constructible sheaves, which is independent of the choice of stratification, and the intersection cohomology is a homeomorphism invariant [GM83]. If p is the zero perversity, i.e. $p(i) = 0$, then $\mathcal{IC}^{\cdot}_{p,X} \cong \mathbb{C}$ and hence $IH_0^*(X)$ is isomorphic to the (singular) cohomology $H^*(X)$. When p is the top perversity t, $\mathcal{IC}^{\cdot}_{t,X}$ is isomorphic to the dualizing complex $D^{\cdot}_X$ and thus $IH_t^i(X)$ is isomorphic to the Borel-Moore homology $H^{BM}_{n-i}(X)$.

More generally, suppose we have a filtration

$$(\mathfrak{X}') \qquad X = X'_n \supset X'_{n-2} \supset \cdots \supset X'_0 \supset \emptyset$$

by even dimensional closed subsets (not necessarily a topological stratification) such that

(1) $X - X'_{n-2}$ is dense,
(2) $S'_i = X'_i - X'_{i-2}$ is a $\mathbb{C}$-homology manifold of dimension i or empty.

Let $U'_{2k} = X - X'_{n-2k}$ and $j'_{2k} : U'_{2k} \hookrightarrow U'_{2k+2}$. Define a sheaf complex $\mathcal{P}^{\cdot}_{p,X}$ inductively by

$$\mathcal{P}^{\cdot}_{p,X}|_{U'_2} = \mathbb{C}_{U'_2}$$

$$\mathcal{P}^{\cdot}_{p,X}|_{U'_{2k+2}} \cong \tau_{\leq p(2k)} Rj'_{2k*}(\mathcal{P}^{\cdot}_{p,X}|_{U'_{2k}}).$$

Definition/Lemma 2.1. *If $\mathcal{P}^{\cdot}_{p,X}$ is topologically constructible (see* [GM83] *p83) then $\mathcal{P}^{\cdot}_{p,X} \cong \mathcal{IC}^{\cdot}_{p,X}$. We call $\mathfrak{X}'$ a* nice *filtration if $\mathcal{P}^{\cdot}_{p,X}$ is topologically constructible.*

Proof. It suffices to show that $\mathcal{P}^{\cdot}_{p,X}$ satisfies [AX2] in [GM83] p107. The proof is exactly same as the proof of Lemma 2 in [GM83] p110. □

For a space X constructed as the GIT quotient of a smooth variety, the decomposition by orbit types gives us a topological stratification while the infinitesimal orbit types will give us only a nice filtration. See §4.

2.3. **Geometric chains.** Explicitly intersection cohomology can by described in terms of geometric chains. Suppose X is a subanalytic pseudo-manifold with subanalytic stratification $\mathfrak{X} = \{X_i\}$. For an open subset U let $C^*(U)$ be the chain complex defined by

$$C^i(U) = \{\text{locally finite chains in } U \text{ of dimension } n-i\}.$$

Note that our indexing scheme is "cohomology superscript" ([GM83] p98).

For an open subset U and a perversity p, define a subcomplex of $C^*(U)$ by

$$IC^i_p(U) = \{\xi \in C^i(U) \mid \begin{array}{l} \dim(|\xi| \cap X_{n-c} \cap U) \le n-i-c+p(c), \\ \dim(|\partial\xi| \cap X_{n-c} \cap U) \le n-i-1-c+p(c)\}. \end{array}$$

This gives rise to a *soft* sheaf complex which is isomorphic to $\mathcal{IC}^{\cdot}_{p,X}$. (See §2.1, §3.6 [GM83].) Hence the cohomology of $IC^*_p(X)$ is the perversity p intersection cohomology $IH^*_p(X)$.

2.4. **Intersection pairing.** If perversities p, q, r satisfy $p + q \le r$, then there is a unique morphism

$$\mathcal{IC}^{\cdot}_{p,X} \otimes^L \mathcal{IC}^{\cdot}_{q,X} \to \mathcal{IC}^{\cdot}_{r,X}$$

in $\mathbf{D}^+_c(X)$ extending $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ over U_2 ([GM83] p112). This morphism induces a homomorphism

$$IH^i_p(X) \otimes IH^j_q(X) \to IH^{i+j}_r(X).$$

For instance if $p = q = r = 0$ then this is just the cup product of cohomology classes.

For the middle perversity m, we have $m + m \le t$. Thus if X is normal compact connected oriented pseudo-manifold, we have the intersection pairing

$$IH^i_m(X) \otimes IH^j_m(X) \to IH^n_t(X) \cong \mathbb{C} \tag{2.2}$$

for $i + j = n$. In terms of geometric chains, this intersection pairing is defined as follows: For $\alpha, \beta \in IH^*_m(X)$ such that $\deg(\alpha) + \deg(\beta) = n$, we can find representative cycles ξ and σ such that they intersect only at finitely many points in the smooth part U_2, transversely. The intersection pairing $\langle \alpha, \beta \rangle$ of α and β is the number of intersection points counted with signs as usual. (See §2.3 [GM80].)

2.5. **Equivariant intersection cohomology.** Suppose a compact connected Lie group K acts on a pseudo-manifold preserving a topological stratification $\mathfrak{X} = \{X_i\}$ (§2.2). Let EK be a contractible space on which K acts freely and $BK = EK/K$. Then the closed subsets

$$EK \times_K X_i$$

form a filtration of $X_K := EK \times_K X$. For a perversity p, apply Deligne's construction (§2.2) to this filtration to obtain a sheaf complex $\mathcal{IC}^{\cdot}_{p,X_K}$. The equivariant intersection cohomology $IH^*_{p,K}(X)$ is defined as the hypercohomology of this sheaf complex. (See §5.2, §13.4 in [BL94].) From the fibration $X_K \to BK$ with fiber X, we get a spectral sequence

$$H^i(BK) \otimes IH^j(X) \Rightarrow IH^{i+j}_{p,K}(X).$$

More concretely, choose a smooth classifying sequence (§12 [BL94])

$$EK_0 \subset EK_1 \subset EK_2 \subset \cdots$$

where EK_k is a k-acyclic free K-manifold. Let $BK_k = EK_k/K$ so that $BK = \lim_{k\to\infty} BK_k$. The filtration

$$EK_k \times_K X_i$$

of $EK_k \times_K X$ is a topological stratification and thus Deligne's construction gives rise to $\mathcal{IC}^{\cdot}_{p,EK_k\times_K X}$. From the spectral sequences for the fibrations

$$\begin{array}{ccc} X \longrightarrow EK_k \times_K X & \qquad & X \longrightarrow EK \times_K X \\ \downarrow & & \downarrow \\ BK_k & & BK \end{array}$$

we see immediately that

$$IH^*_{p,K}(X) = \lim_{k\to\infty} IH^*_p(EK_k \times_K X) \tag{2.3}$$

because $H^{<k}(BK_k) \cong H^{<k}(BK)$ by Whitehead's theorem.

3. Placid maps

Unlike ordinary cohomology, intersection cohomology is contravariant only for a limited class of maps. In this section we generalize slightly the concept of placid maps due to Goresky and MacPherson and show the functoriality. This will give us the right inverse of the Kirwan map.

3.1. **Placid maps.** Let $f : X \to Y$ be a subanalytic map between subanalytic pseudo-manifolds. Suppose Y is compact.

Definition 3.1. *f is called* (p, q)-placid *if there is a subanalytic stratification of Y such that*

$$q(\operatorname{codim} S) \le p(\operatorname{codim} f^{-1}(S)) \tag{3.1}$$

for each stratum S of Y.

When $p = q$, we recover the original placid maps [GM85].

With this definition, Proposition 4.1 in [GM85] is modified as follows.

Proposition 3.2. *If f is (p, q)-placid, then the pull-back of generic chains induces a homomorphism on intersection cohomology*

$$f^* : IH^*_q(Y) \to IH^*_p(X). \tag{3.2}$$

Proof. See p373 in [GM85] for details. Since f is subanalytic, there is a stratification of X for which f is a stratified map ([GM83] §1.2). By McCrory's transversality, any cohomology class α in $IH^i_q(Y)$ can be represented by a chain ξ which is dimensionally transverse to any stratum in X and the cycle $f^{-1}(\xi)$ lies in $IC^i_p(X)$ because of (3.1). The class $f^*\alpha$ is represented by the cycle $f^{-1}(\xi)$. □

3.2. **Pull-back morphism.** The homomorphism (3.2) comes from a morphism

$$f^* : \mathcal{IC}^{\cdot}_{q,Y} \to Rf_*\mathcal{IC}^{\cdot}_{p,X} \tag{3.3}$$

in the derived category $\mathbf{D}^+(Y)$.

Suppose $\mathfrak{X} = \{X_i\}_{i=0}^n$ (resp. $\mathfrak{Y} = \{Y_i\}_{i=0}^l$) is a *nice* filtration of X (resp. Y). Let $f : X \to Y$ be a continuous map such that for every connected component S of any stratum $Y_i - Y_{i-2}$, $f^{-1}(S)$ is a union of some connected components of strata in X and (3.1) holds.

Over $U_2 = Y - Y_{l-2}$, let $f^*_{U_2}$ be the composition of the adjunction morphism $\mathbb{C}_{U_2} \to Rf_*f^*\mathbb{C}_{U_2} = Rf_*\mathbb{C}_{f^{-1}(U_2)}$ with the morphism $Rf_*\mathbb{C}_{f^{-1}(U_2)} \to Rf_*\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_2)}$ from §5.1 [GM83].

Proposition 3.3. *There is a unique morphism (3.3) in $\mathbf{D}^+_c(Y)$ extending $f^*_{U_2}$.*

We need the following simple lemma to prove the proposition.

Lemma 3.4. *For $\mathcal{A}^\cdot \in \mathbf{D}^+_c(X)$, $\tau_{\le q}Rf_*\mathcal{A}^\cdot \cong \tau_{\le q}Rf_*\tau_{\le p}\mathcal{A}^\cdot$ if $q \le p$.*

Proof. Choose an injective resolution $\mathcal{I}^\cdot$ of $\mathcal{A}^\cdot$. (See for instance [GeMa] p181.) It is elementary to find an injective resolution $\mathcal{J}^\cdot$ of $\tau_{\le p}\mathcal{A}^\cdot$ such that $\mathcal{I}^k = \mathcal{J}^k$ for $k \le p$. Then $Rf_*\mathcal{A}^\cdot = f_*\mathcal{I}^\cdot$ is equal to $Rf_*\tau_{\le p}\mathcal{A}^\cdot = f_*\mathcal{J}^\cdot$ up to degree p. So we proved the lemma. □

Proof of Proposition 3.3. Let $U_{2k} = Y - Y_{l-2k}$ and $j_{2k} : U_{2k} \hookrightarrow U_{2k+2}$. Recall from §2.2 that Deligne's construction applied to $\mathfrak{Y}$ with perversity q gives us the intersection cohomology sheaf $\mathcal{IC}^\cdot_{q,Y}$. Suppose we have constructed

$$f^*_{U_{2k}} : \mathcal{IC}^\cdot_{q,Y}|_{U_{2k}} \to Rf_*\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k})}.$$

This gives rise to

$$\begin{aligned}\mathcal{IC}^\cdot_{q,Y}|_{U_{2k+2}} &\cong \tau_{\le q(2k)}Rj_{2k*}\mathcal{IC}^\cdot_{q,Y}|_{U_{2k}} \to \tau_{\le q(2k)}Rj_{2k*}Rf_*\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k})}\\ &= \tau_{\le q(2k)}Rf_*Ri_{2k*}\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k})}\\ &\cong \tau_{\le q(2k)}Rf_*\tau_{\le q(2k)}Ri_{2k*}\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k})}\end{aligned}$$

by Lemma 3.4 where $i_{2k} : f^{-1}(U_{2k}) \hookrightarrow f^{-1}(U_{2k+2})$ since $j_{2k} \circ f = f \circ i_{2k}$. We claim

$$\tau_{\le q(2k)}Ri_{2k*}\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k})} \cong \tau_{\le q(2k)}\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k+2})}.$$

For simplicity, suppose $f^{-1}(U_{2k+2}) - f^{-1}(U_{2k})$ consists of only one connected stratum $\tilde{S}$. Then

$$\mathcal{IC}_{p,X}|_{f^{-1}(U_{2k+2})} \cong \tau^{\tilde{S}}_{\le p(\operatorname{codim}\tilde{S})}Ri_{2k*}\mathcal{IC}_{p,X}|_{f^{-1}(U_{2k})}$$

where $\tau^{\tilde{S}}_{\le p}$ is the "truncation over a closed subset functor" (see §1.14 [GM83]). In particular,

$$\begin{aligned}\tau_{\le q(2k)}\mathcal{IC}_{p,X}|_{f^{-1}(U_{2k+2})} &\cong \tau_{\le q(2k)}\tau^{\tilde{S}}_{\le p(\operatorname{codim}\tilde{S})}Ri_{2k*}\mathcal{IC}_{p,X}|_{f^{-1}(U_{2k})}\\ &\cong \tau_{\le q(2k)}Ri_{2k*}\mathcal{IC}_{p,X}|_{f^{-1}(U_{2k})}\end{aligned}$$

because $q(2k) \le p(\operatorname{codim}\tilde{S})$. When there are more than one strata in $f^{-1}(U_{2k+2}) - f^{-1}(U_{2k})$ we simply repeat the argument for each stratum in the order of increasing codimension. Hence we get a morphism

$$\begin{aligned}\mathcal{IC}^\cdot_{q,Y}|_{U_{2k+2}} &\to \tau_{\le q(2k)}Rf_*\tau_{\le q(2k)}\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k+2})}\\ &\cong \tau_{\le q(2k)}Rf_*\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k+2})} \to Rf_*\mathcal{IC}^\cdot_{p,X}|_{f^{-1}(U_{2k+2})}.\end{aligned}$$

Uniqueness is an elementary exercise. □

By taking hypercohomology, the morphism (3.3) induces a homomorphism

$$IH^*_q(Y) \to IH^*_p(X).$$

Suppose f is subanalytic and $\mathfrak{Y}$ is a subanalytic stratification. Note that an intersection cycle ξ is completely determined by its intersection with the open dense stratum ([B+84] p10) and over U_2 (3.3) is just the adjunction. If we use the sheaf complexes by geometric chains (§2.3) for $\mathcal{IC}^\cdot_{p,X}$ and $\mathcal{IC}^\cdot_{q,Y}$, the induced homomorphism sends the class $[\xi] \in IH^*_q(Y)$, which is represented by a chain ξ dimensionally transverse to any stratum of X, to $[f^{-1}(\xi)]$. In other words, the induced homomorphism is exactly the homomorphism (3.2).

3.3. **Equivariant case.** Suppose a compact connected Lie group K acts on X preserving a nice filtration $\mathfrak{X}=\{X_i\}_{i=0}^n$. Deligne's construction with the filtration $\{EK\times_K X_i\}$ and perversity p gives us $\mathcal{IC}^{\cdot}_{p,X_K}$ in $\mathbf{D}^+(X_K)$ whose hypercohomology is $IH^*_{p,K}(X)$.

Let Y be a compact pseudo-manifold with a nice filtration $\mathfrak{Y}=\{Y_i\}_{i=0}^m$. Let $f:X\to Y$ be an invariant continuous map such that for any connected component S of a stratum, $f^{-1}(S)$ is a union of some connected components of strata in X and (3.1) holds.

Then the proof of Proposition 3.3 gives us a morphism

$$\mathcal{IC}^{\cdot}_{q,Y}\to Rf_{K*}\mathcal{IC}^{\cdot}_{p,X_K} \tag{3.4}$$

where $f_K: X_K=EK\times_K X\to Y$ is the obvious map induced from f. So we have a homomorphism

$$f_K^*: IH^*_q(Y)\to IH^*_{p,K}(X). \tag{3.5}$$

Similarly, we have a morphism

$$\mathcal{IC}^{\cdot}_{q,Y}\to Rf_{k*}\mathcal{IC}^{\cdot}_{p,EK_k\times_K X}$$

where $f_k: EK_k\times_K X\to Y$ and a homomorphism

$$f_k^*: IH^*_q(Y)\to IH^*_p(EK_k\times_K X).$$

From the commutative diagram

$$\begin{tikzcd} EK_k\times_K X \arrow[r, "\imath_k"] \arrow[dr] & EK\times_K X \arrow[d] \\ & Y \end{tikzcd}$$

we see that the composition

$$IH^*_q(Y)\xrightarrow{f_K^*} IH^*_{p,K}(X)\xrightarrow{\imath_k^*} IH^*_p(EK_k\times_K X)$$

is f_k^* since $\imath_k^*\mathcal{IC}^{\cdot}_{p,X_K}\cong\mathcal{IC}^{\cdot}_{p,EK_k\times_K X}$ by construction. Hence $\{f_k^*\}$ determine f_K^* because of (2.3).

3.4. f^* **preserves the intersection pairing.** Our interest lies in the case where X is a smooth analytic manifold with an action by a compact connected Lie group K and $f:X\to Y$ is invariant. In particular we wish to relate the middle perversity intersection cohomology $IH^*_m(Y)$ with the equivariant cohomology $H^*_K(X)=H^*(EK\times_K X)$. From now on when using middle perversity, we will drop the subscript m for convenience.

Suppose X is smooth. Then since intersection cohomology is independent of stratification, $IH^*_p(X)\cong H^*(X)$ for any perversity p and hence for any (p,m)-placid subanalytic map f we have a homomorphism $f^*: IH^*_q(Y)\to H^*(X)$. Obviously (t,m)-placid condition is most general for us to get such a homomorphism.

If furthermore K acts on X preserving a subanalytic stratification $\mathfrak{X}$ for which f is (t,m)-placid stratified map with a stratification $\mathfrak{Y}$ of Y, we have a morphism

$$f_K^*:\mathcal{IC}^{\cdot}_Y\to Rf_{K*}\mathbb{C}_{X_K}$$

and the induced homomorphism

$$f_K^*: IH^*(Y) \to H_K^*(X).$$

Proposition 3.5. *Let $f: X \to Y$ be a (t,m)-placid map. Suppose X is smooth and Y is compact connected normal oriented. Let τ be the top degree class represented by a point in the smooth part. Then the induced map $f^*: IH^*(Y) \to H^*(X)$ preserves the intersection pairing in the sense that*

$$f^*(\alpha) \cup f^*(\beta) = \langle \alpha, \beta \rangle f^*(\tau) \tag{3.6}$$

for any $\alpha, \beta \in IH^*(Y)$ of complementary degrees. In the equivariant case, the same is true for $f_K^*: IH^*(Y) \to H_K^*(X)$.

Proof. Recall from §2.4 that α and β are represented by intersection cycles ξ and σ that intersect only at finitely many points in the smooth part. In this case, $\xi \cap \sigma \in IC^n(Y)$ represents $\langle \alpha, \beta \rangle \tau \in IH^n(Y)$. By McCrory's transversality result we can further assume that ξ and σ are dimensionally transverse to each stratum in X so that $f^*\alpha$ and $f^*\beta$ are represented by $f^{-1}(\xi)$ and $f^{-1}(\sigma)$. Because X is smooth, the complex $\mathcal{C}^\cdot$ of geometric chains is isomorphic to the constant sheaf $\mathbb{C}_X$ and the cup product is just the intersection of chains. Hence the cup product $f^*(\alpha) \cup f^*(\beta)$ is represented by $f^{-1}(\xi) \cap f^{-1}(\sigma) = f^{-1}(\xi \cap \sigma)$ which also represents $\langle \alpha, \beta \rangle f^*(\tau)$. So we proved (3.6).

For the equivariant case, observe that the statement is true for $f_k: EK_k \times_K X \to Y$ for any k since $EK_k \times_K X$ is a finite dimensional manifold. If we take a sufficiently large k, then $H_K^{\le l}(X) \cong H^{\le l}(EK_k \times_K X)$ and $f_k^* = f_K^*$ where $l = \dim Y$. So we are done. □

Corollary 3.6. *Suppose furthermore $f^*(\tau) \neq 0$ in $H^*(X)$. Then $f^*: IH^*(Y) \to H^*(X)$ is injective and the intersection pairing is given by the cup product structure of $H^*(X)$. A similar result is true for the equivariant case.*

Proof. The result follows from Proposition 3.5 since the intersection pairing is non-degenerate for $IH^*(Y)$. □

4. Symplectic reduction

Let (M, ω) be a connected Hamiltonian K-space with proper moment map $\mu: M \to \mathfrak{k}^*$ where $\mathfrak{k} = Lie(K)$. Then the symplectic reduction $X = \mu^{-1}(0)/K$, which we denote by $M/\!\!/K$, is in general a pseudomanifold, whose strata are symplectic manifolds. In this section, we describe the orbit type stratification and the infinitesimal orbit type decomposition of X from [SL91, MS99].

4.1. **Stratification of X.** Let $Z = \mu^{-1}(0)$ and $Z_{(H)} = \{x \in Z \,|\, \mathrm{Stab}(x) \in (H)\}$ for a subgroup H of K, where (H) denotes the conjugacy class of H. Also, let $Z_H = \{x \in Z \,|\, \mathrm{Stab}(x) = H\}$. Then

$$Z = \bigsqcup_{(H)} Z_{(H)} \tag{4.1}$$

and $Z_{(H)} = K \times_{N^H} Z_H$ where N^H is the normalizer of H in K. This decomposition induces a stratification

$$X = \bigsqcup_{(H)} X_{(H)} \tag{4.2}$$

where $X_{(H)} = Z_{(H)}/K = Z_H/N^H$.

For $x \in Z_H$, consider the *symplectic slice*

$$T_x(Kx)^{\perp_\omega}/T_x(Kx) \tag{4.3}$$

and let W be the symplectic complement of the H-fixed point set in the slice. We recall the following result from [SL91] Lemma 7.1.

Lemma 4.1. *There exists a neighborhood of the submanifold $Z_{(H)}$ of M that is symplectically and K-equivariantly diffeomorphic to a neighborhood of the zero section of a vector bundle $\mathcal{N} \to Z_{(H)}$. The space $\mathcal{N}$ is a symplectic fiber bundle over the stratum $X_{(H)}$*

$$F \to \mathcal{N} \to X_{(H)}$$

with fiber F given by

$$F = K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times W).$$

In particular, for any $x \in Z_H$, there is a neighborhood of the orbit Kx that is equivariantly diffeomorphic to $F \times \mathbb{R}^{\dim X_{(H)}}$.

Now the reduction of $\mathcal{N}$ is homeomorphic to a neighborhood of $X_{(H)}$ in X. The principle of reduction in stages gives us the following.

Proposition 4.2. *(*[SL91] *7.4) Given a stratum $X_{(H)}$ of X, there exists a fiber bundle over $X_{(H)}$ with typical fiber being the cone $W/\!\!/H$ such that a neighborhood of the vertex section of this bundle is symplectically diffeomorphic to a neighborhood of the stratum inside X.*

Consequently, $\{X_{(H)}\}$ gives us a topological stratification of X. This is called the *orbit type stratification.*

4.2. **Stratification of M^{ss}.** Let M^{ss} denote the open subset of elements in M whose gradient flow for $f = -|\mu|^2$ has a limit point in Z and put $r : M^{ss} \to Z$ denote the retraction by the flow.

Let ϕ be the composition

$$M^{ss} \to Z \to Z/K = X \tag{4.4}$$

of the retraction r and the quotient map. Let us call it the symplectic quotient map. The inverse image $\phi^{-1}(X_{(H)})$ of the stratum $X_{(H)}$ is diffeomorphic to a subfiber bundle of $\mathcal{N}$ in Lemma 4.1 with typical fiber $K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times \phi_W^{-1}(*))$ where $\phi_W : W \to W/\!\!/H$ is the symplectic quotient map for W and $* = \phi_W(0)$ is the vertex of the cone $W/\!\!/H$. If we assign a complex structure, compatible with the symplectic structure, it is well-known that $\phi_W^{-1}(*)$ is the affine cone over $\mathbb{P}W - \mathbb{P}W^{ss}$ where the superscript ss denotes the semistable set defined by Mumford [MFK94]. Hence, the affine cones over the unstable strata of $\mathbb{P}W$ minus 0, together with $\{0\}$, give us a stratification of $\phi^{-1}(X_{(H)})$ via the diffeomorphism in Lemma 4.1. Observe that M^{ss} is diffeomorphic to a neighborhood of Z by the gradient flow of $-|\mu|^2$. Since the diffeomorphism in Lemma 4.1 is K-equivariant and the stratification of $\phi^{-1}(X_{(H)})$ is completely determined by the group action, we get a K-invariant stratification of M^{ss} for which $\phi : M^{ss} \to X$ is stratified.

See [Kir94] for a description of the above stratification for GIT quotients and an application to the Atiyah-Jones conjecture.

4.3. **Infinitesimal orbit type decomposition.** There is another way to decompose X which is useful for partial desingularization. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{k}$, let $Z_{\mathfrak{h}} = \{x \in Z \,|\, \mathrm{LieStab}(x) = \mathfrak{h}\}$ and

$$Z_{(\mathfrak{h})} = \{x \in Z \,|\, \mathrm{LieStab}(x) \in (\mathfrak{h})\}$$

where $(\mathfrak{h})$ is the conjugacy class of $\mathfrak{h}$. Let $X_{(\mathfrak{h})} = Z_{(\mathfrak{h})}/K$. Then we have a decomposition

$$X = \bigsqcup_{(\mathfrak{h})} X_{(\mathfrak{h})}.$$

This is called the infinitesimal orbit type decomposition of X. From [MS99] §3, $X_{(\mathfrak{h})}$ are just orbifolds and thus homology manifolds. If we use the local normal form in [MS99] §3.1, it is elementary to show that Deligne's construction for this decomposition gives us a topologically constructible sheaf complex with respect to the orbit type stratification. This fact will not be used in this paper so we leave the details to the reader.

5. Almost-balanced action

We use the notations of the previous section. We assume that there is at least one point in $Z = \mu^{-1}(0)$ with finite stabilizer.

5.1. **Placid maps for symplectic quotients.** With the orbit type stratification, $\phi^{-1}(X_{(H)})$ is a fiber bundle over $X_{(H)}$ with fiber $K \times_H ((\mathfrak{k}/\mathfrak{h})^* \times \phi_W^{-1}(*))$ and hence

$$\mathrm{codim}_{M^{ss}}\, \phi^{-1}(X_{(H)}) = \mathrm{codim}_W\, \phi_W^{-1}(*).$$

On the other hand,

$$\mathrm{codim}_X\, X_{(H)} = \dim W /\!\!/ H$$

by Proposition 4.2. Recall that the map $\phi : M^{ss} \to X$ is (t, m)-placid if

$$t(\mathrm{codim}_{M^{ss}}\, \phi^{-1}(X_{(H)})) \geq m(\mathrm{codim}_X\, X_{(H)})$$

or equivalently

$$\mathrm{codim}_W\, \phi_W^{-1}(*) - 2 \geq \frac{1}{2} \dim W /\!\!/ H - 1.$$

Hence ϕ is (t, m)-placid if and only if

$$\mathrm{codim}_W\, \phi_W^{-1}(*) > \frac{1}{2} \dim W /\!\!/ H \tag{5.1}$$

for each (H).

The unstable strata of $\mathbb{P}W$ by the norm square of the moment map can be described using the weights of the maximal torus action, as follows: For any collection of weights of the maximal torus action, we consider the convex hull of them and get the closest point from the origin to the hull. Let $\mathcal{B}$ be the set of such closest points in the positive Weyl chamber. Then the unstable strata are in one-to-one correspondence with the set $\mathcal{B}$. (See [Kir84].)

For each $\beta \in \mathcal{B}$, let $n(\beta)$ denote the number of weights α such that $\langle \alpha, \beta \rangle < \langle \beta, \beta \rangle$. Then Kirwan proved in [Kir84] that the codimension of the stratum corresponding to $\beta \in \mathcal{B}$ is precisely

$$2n(\beta) - \dim H/\mathrm{Stab}\beta$$

where Stabβ is the stabilizer of β in H. Therefore, the (t,m)-placid condition is equivalent to

$$2n(\beta) - \dim H/\mathrm{Stab}\beta > \frac{1}{2}(\dim W - 2\dim H) \tag{5.2}$$

for each $\beta \in \mathcal{B}$. In particular, this condition is satisfied when

$$2n(\beta) \geq \frac{1}{2}\dim W$$

for all β. For example, if the set of weights is symmetric with respect to the origin, the above is satisfied . This is the case for the moduli spaces of vector bundles over a Riemann surface. (See Proposition 7.4.)

When $K = U(1)$ acts on $M = \mathbb{P}^n$ linearly and if n_+, n_0, n_- denote the number of positive, zero, negative weights respectively, then the condition (5.2) is satisfied if and only if $n_+ = n_-$. Hence the (t,m)-placid condition may be viewed as a condition on "balancedness of weights".

Definition 5.1. [KW] *The action on M is said to be* almost balanced *if the condition (5.2) is satisfied for all β and (H).*

Remark 5.2. By (5.1), if almost balanced, we have an isomorphism

$$H_H^{<a_H}(W) \cong H_H^{<a_H}\left(W - \phi_W^{-1}(*)\right)$$

where $a_H = \frac{1}{2}\dim W/\!\!/H = \frac{1}{2}\dim W - \dim H$. This is an easy consequence of the Gysin sequence (applied stratum by stratum) because $\phi_W^{-1}(*)$ is stratified.

5.2. **Almost-balanced action and GIT quotient.** We recall the following well-known facts from [Kir84]: The obvious action of $U(n+1)$ on $\mathbb{P}^n$ is Hamiltonian with moment map $\mu_{\mathbb{P}^n}$. When $M \subset \mathbb{P}^n$ is a smooth projective variety and K acts on M via a homomorphism $K \to U(n+1)$, the composition

$$\mu : M \hookrightarrow \mathbb{P}^n \xrightarrow{\mu_{\mathbb{P}^n}} \mathfrak{u}(n+1)^* \longrightarrow \mathfrak{k}^*$$

is the moment map for M and the set of semistable points in M is equal to the minimal Morse stratum M^{ss} for $-|\mu|^2$. The GIT quotient $M/\!\!/G$ of M is homeomorphic to the symplectic reduction $M/\!\!/K$ and the symplectic quotient map $\phi : M^{ss} \to M/\!\!/K$ is the GIT quotient map. Since we are interested in topology, we will not distinguish symplectic quotients from GIT quotients.

Theorem 5.3. *Let $M \subset \mathbb{P}^n$ be a smooth projective variety acted on by a compact connected Lie group K via a homomorphism $K \to U(n+1)$. Suppose the K action is almost balanced and there is at least one point in Z with finite stabilizer. Then we have a natural map*

$$\phi_K^* : IH^*(X) \to H_K^*(M^{ss}) \cong H_K^*(Z) \tag{5.3}$$

of the middle perversity intersection cohomology of $X = \mu^{-1}(0)/K$ into the K-equivariant cohomology of M^{ss}. Moreover, ϕ_K^* is injective and the intersection pairing of $IH^*(X)$ is given by the cup product of $H_K^*(Z)$ from the formula

$$\phi_K^*(\alpha) \cup \phi_K^*(\beta) = \langle \alpha, \beta \rangle \phi_K^*(\tau)$$

where τ is the class in $IH^{\dim X}(X)$ represented by a point.

Proof. Certainly the GIT quotient map is subanalytic and the stratification of M^{ss} in §4.2 is K-invariant. The almost balanced condition is equivalent to the (t,m)-placid condition and hence we have

$$\phi_K^* : IH^*(X) \to H_K^*(M^{ss}) \cong H_K^*(Z)$$

by Proposition 3.5. For injectivity and intersection pairing, it suffices to show that $\phi_K^*(\tau)$ is nonzero in $H_K^*(Z)$ by Corollary 3.6.

Let Σ denote the set of points in Z whose stabilizer is not finite. According to [MS99] §4, there is an equivariant proper map $\pi : \tilde{Z} \to Z$ such that $\pi|_{\pi^{-1}(Z-\Sigma)}$ is a homeomorphism and the stabilizer of every point in $\tilde{Z}$ is a finite group. Thus $H_K^*(\tilde{Z}) \cong H^*(\tilde{Z}/K)$.

Let $X^s = (Z-\Sigma)/K$ and consider the commutative diagram of natural maps

$$\begin{tikzcd} H_c^{\dim X}(X^s) \arrow[r, "\phi^*"] \arrow[d, "\cong"'] & H_K^{\dim X}(Z) \arrow[d] \\ H^{\dim X}(\tilde{Z}/K) \arrow[r, "\cong"] & H_K^{\dim X}(\tilde{Z}) \end{tikzcd}$$

where the subscript c denotes compact support. The class τ is the image of a nonzero class in $H_c^{\dim X}(X^s)$ and hence it follows from the above diagram that $\phi_K^*(\tau)$ is nonzero. □

Remark 5.4. Recall that (5.3) comes from a morphism (3.4) in the derived category $\mathbf{D}_c^+(X)$ of bounded below cohomologically constructible sheaves over X because $\phi : M^{ss} \to X$ is (t,m)-placid. In particular, for any open subset $U \subset X$ we get a map $\phi_K^* : IH^*(U) \to H_K^*(\phi^{-1}(U))$ and it is functorial with respect to restrictions.

6. The Kirwan map

In this section, we recall the definition of the Kirwan map from [Kir86b] and show that it is a left inverse of the pull-back homomorphism ϕ_K^*.

Let $M \subset \mathbb{P}^n$ be a connected nonsingular projective variety acted on linearly by a connected reductive group G via a homomorphism $G \to GL(n+1)$. We may assume that the maximal compact subgroup K of $G = K^{\mathbb{C}}$ acts unitarily possibly after conjugation. Let $\mu : \mathbb{P}^n \to u(n+1)^* \to \mathfrak{k}^*$ be the moment map for the action of K. Then $M^{ss} = M \cap (\mathbb{P}^n)^{ss}$ retracts onto $Z := \mu^{-1}(0) \cap M$ by the gradient flow of $-|\mu|^2$ and the GIT quotient $M/\!\!/G$ is homeomorphic to the symplectic quotient $M/\!\!/K = M \cap \mu^{-1}(0)/K =: X$.

6.1. **Definition of the Kirwan map.** In order to define the Kirwan map, we assume that there is at least one stable point in M, which amounts to saying that there is at least one point in Z whose stabilizer is finite.

We quote the following definitions from [Kir85].

Definition 6.1.
1. *Let $\mathcal{R}(M)$ be a set of representatives of the conjugacy classes of identity components of all subgroups of K which appear as the stabilizer of some point $x \in Z = \mu^{-1}(0)$.*
2. *Let M_H^{ss} denote the set of those $x \in M^{ss}$ fixed by $H \in \mathcal{R}(M)$.*
3. *Let $r(M) = \max\{\dim H |\, H \in \mathcal{R}(M)\}$.*

Of course, M_H^{ss} is a smooth complex manifold.

The definition of the Kirwan map is by induction on $r(M)$. When $r(M) = 0$, the action of K is locally free and X is an orbifold. Thus $\mathcal{IC}^{\cdot}_X \cong \mathbb{C}_X$. The pull-back morphism ϕ_K^* in this case is equal to the adjunction morphism $\mathbb{C}_X \to R\phi_{K*}\mathbb{C}$ which is an isomorphism by [BL94] Theorem 9.1 (ii). The Kirwan map is defined as the hypercohomology

$$\kappa_M^{ss} : H_K^*(M^{ss}) \to IH^*(X)$$

of the inverse $R\phi_{K*}\mathbb{C} \to \mathbb{C}_X \cong \mathcal{IC}^{\cdot}_X$ of ϕ_K^*.

Now suppose $r(M) > 0$. Let $\pi : \hat{M} \to M^{ss}$ be the blow-up of M^{ss} along the submanifold

$$\bigsqcup_{\dim H = r(M)} GM_H^{ss}.$$

If we choose a suitable linearization $\hat{M}$, the semistable points in the closure of $\hat{M}$ all lie in $\hat{M}$ and $r(\hat{M}) < r(M)$. Let $\hat{X} = \hat{M}/\!\!/G$ and $\hat{\phi} : \hat{M}^{ss} \to \hat{X}$ be the GIT quotient map. If we keep blowing up in this fashion, we get a quasi-projective variety $\widetilde{M}^{ss}$ birational to M whose quotient $\widetilde{X}$ has only finite quotient singularities. This is called the *partial desingularization* of X. See [Kir85] for details.

We have the following commutative diagram

$$\begin{tikzcd} \hat{M}^{ss} \arrow[r, hook, "\imath"] \arrow[d, "\hat{\phi}"'] & \hat{M} \arrow[r, "\pi"] & M^{ss} \arrow[d, "\phi"] \\ \hat{X} \arrow[rr, "\sigma"] & & X. \end{tikzcd} \tag{6.1}$$

Since the GIT quotient $\hat{X}$ is the categorical quotient of $\hat{M}^{ss}$, σ is defined uniquely by the universal property of the categorical quotient. (Recall that GIT quotients are categorical quotients [MFK94].)

Inductively, we may suppose that we have a morphism

$$\kappa_{\hat{M}}^{ss} : R(\hat{\phi}_K)_*\mathbb{C} \to \mathcal{IC}^{\cdot}_{\hat{X}}.$$

Then we have a morphism

$$R\phi_{K*}R\pi_{K*}\mathbb{C} \to R\phi_{K*}R\pi_{K*}R\imath_{K*}\mathbb{C} = R\sigma_*R(\hat{\phi}_K)_*\mathbb{C} \to R\sigma_*\mathcal{IC}^{\cdot}_{\hat{X}} \tag{6.2}$$

by composing the above with the adjunction morphism $\mathbb{C} \to R\imath_{K*}\imath_K^*\mathbb{C} = R\imath_{K*}\mathbb{C}$, where $\pi_K : EK \times_K \hat{M} \to EK \times_K M^{ss}$ is the induced map from π and $\imath_K$ is defined similarly. This induces a homomorphism

$$\kappa_{\hat{M}}^{ss} \circ \imath_K{}^* : H_K^*(\hat{M}) \to H_K^*(\hat{M}^{ss}) \to IH^*(\hat{M}/\!\!/G). \tag{6.3}$$

Next, compose (6.2) with the adjunction morphism

$$R\phi_{K*}\mathbb{C} \to R\phi_{K*}R\pi_{K*}\pi_K^*\mathbb{C} = R\phi_{K*}R\pi_{K*}\mathbb{C}$$

to get a morphism

$$R\phi_{K*}\mathbb{C} \to R\sigma_*\mathcal{IC}^{\cdot}_{\hat{X}}. \tag{6.4}$$

By [Kir85] §3, $\hat{X}$ is just the blow-up of X along

$$\bigsqcup_{\dim H = r(M)} GM_H^{ss}/\!\!/G.$$

In particular, σ is proper and hence we can apply the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber in [BBD82] which says

$$R\sigma_* \mathcal{IC}^\cdot_{\hat{X}} = \mathcal{IC}^\cdot_X \oplus \mathcal{F}^\cdot \tag{6.5}$$

where $\mathcal{F}^\cdot$ is a sheaf complex supported on the blow-up center. Therefore, we have a morphism

$$R\sigma_* \mathcal{IC}^\cdot_{\hat{X}} \to \mathcal{IC}^\cdot_X \tag{6.6}$$

whose kernel is $\mathcal{F}^\cdot$.

The composition of (6.4) with (6.6) is the desired morphism

$$\kappa^{ss}_M : R\phi_{K*}\mathbb{C} \to \mathcal{IC}^\cdot_X$$

which induces a homomorphism

$$H^*_K(\phi^{-1}(U)) \to IH^*(U)$$

for an open set U in X. This is denoted also by κ^{ss}_M by abuse of notations and called the *Kirwan map*.

6.2. **The pull-back is a right inverse.** From §5, when the K action on M is almost balanced we have a morphism

$$\phi^*_K : \mathcal{IC}^\cdot_X \to R\phi_{K*}\mathbb{C}.$$

This induces a homomorphism

$$IH^*(U) \to H^*_K(\phi^{-1}(U))$$

for any open set U in X which we also denote by ϕ^*_K by abuse of notations. The Kirwan map is a left inverse of ϕ^*_K.

Proposition 6.2. $\kappa^{ss}_M \circ \phi^*_K = 1$.

Proof. We know both ϕ^*_K and κ^{ss}_M come from morphisms in the derived category $\mathbf{D}^+_c(X)$. If we compose them, we get a morphism

$$\mathcal{IC}^\cdot_X \to R\phi_{K*}\mathbb{C} \to \mathcal{IC}^\cdot_X.$$

On the set of stable points M^s in M^{ss}, the action of K is locally free (i.e. the stabilizers are finite groups). Let $X^s = \phi(M^s)$ which is an orbifold. Then $\phi^*_K|_{X^s}$ is the adjunction morphism

$$\mathcal{IC}^\cdot_{X^s} \cong \mathbb{C}_{X^s} \to R\phi_{K*}\mathbb{C}|_{X^s}$$

which is an isomorphism by [BL94] Theorem 9.1 (ii) again, and $\kappa^{ss}_M|_{X^s}$ is its inverse by definition since X^s is untouched by the blow-ups in the partial desingularization process. Therefore, $\kappa^{ss}_M|_{X^s} \circ \phi^*_K|_{X^s}$ is the identity.

It is well-known ([B$^+$84], V §9) that a morphism $\mathcal{IC}^\cdot_X \to \mathcal{IC}^\cdot_X$ which restricts to the identity over the smooth part (that is obviously contained in X^s) is unique. Therefore, $\kappa^{ss}_M \circ \phi^*_K = 1$. □

In particular, ϕ^*_K is injective and κ^{ss}_M is surjective.

7. The image of ϕ^*_K

In this section, we identify the image of ϕ^*_K with a naturally defined subspace $V^*_M \subset H^*_K(M^{ss})$ under a slightly stronger assumption than almost balanced condition.

7.1. **Weakly-balanced action.** Let us make precise our assumption. We use the notations of §6.1.

Definition 7.1. *Let $M \subset \mathbb{P}^n$ be a projective variety with an action of a compact Lie group K via a homomorphism $K \to U(n+1)$. We say the K action on M is* weakly balanced *if it is almost balanced and so is the N^H/H action on the H-fixed submanifold M_H for each $H \in \mathcal{R}(M)$, where N^H is the normalizer of H in K.*

For practical application, the following 2-step equivalent definition is more useful. Recall that G is the complexification of K which acts on M via a homomorphism $G \to GL(n+1)$.

Definition 7.2. (1) *Suppose a nontrivial compact group H acts on a vector space W unitarily. Using the notations of §5, the action is said to be* weakly linearly balanced *if*

$$2n(\beta) - \dim H/\mathrm{Stab}\beta > \frac{1}{2}(\dim W - 2\dim H) \tag{7.1}$$

for every $\beta \in \mathcal{B}$.

(2) *The K-action on M is said to be* weakly balanced *if for each $H \in \mathcal{R}(M)$ and for a point $x \in \mu^{-1}(0)$ with $\mathrm{Lie}\,\mathrm{Stab}(x) = \mathrm{Lie}H$, the linear action of H on the normal space $\mathcal{N}_x$ to GM_H^{ss} is weakly linearly balanced and so is the action of $(H \cap N^L)/L$ on the L-fixed linear subspace $\mathcal{N}_{x,L}$ for each connected subgroup L of H whose conjugate appears in $\mathcal{R}(M)$.*

Lemma 7.3. *The two definitions 7.1 and 7.2 are equivalent.*

Proof. Let $x \in \mu^{-1}(0)$ and $P = \mathrm{Stab}(x)$. Let L be the identity component of P. By Lemma 4.1, a neighborhood of Kx is equivariantly diffeomorphic to

$$K \times_P ((\mathfrak{k}/\mathfrak{p})^* \times W) \times \mathbb{R}^{\dim X_{(P)}}$$

for some symplectic P-vector space W. Let us call W the *normal slice* at x.

Since P/L is discrete, $P \subset N^L$ and P acts on the L-fixed subspace W^L of W. By direct computation, one can check that M_L^{ss} in this neighborhood is

$$N^L \times_P \left((\mathfrak{n}^L/\mathfrak{p})^* \times W^L\right) \times \mathbb{R}^{\dim X_{(P)}}$$

and hence GM_L^{ss} is

$$K \times_P \left((\mathfrak{k}/\mathfrak{p})^* \times W^L\right) \times \mathbb{R}^{\dim X_{(P)}}$$

where $\mathfrak{n}^L$ (resp. $\mathfrak{p}$) is the Lie algebra of N^L (resp. P). Therefore the normal space $\mathcal{N}_x$ to GM_L^{ss} is the orthogonal complement of W^L in W.

If x is a generic point in $M_L^{ss} \cap \mu^{-1}(0)$ such that P is minimal among those containing L, then $\mathcal{N}_x = W$. Suppose the K-action on M is almost balanced, i.e. the action of $\mathrm{Stab}(x)$ on the normal slice W at x is weakly linearly balanced for all $x \in \mu^{-1}(0)$. Then by choosing a generic x for each $L \in \mathcal{R}(M)$, we deduce that the action of L on the normal space $\mathcal{N}_x = W$ to GM_L^{ss} is weakly linearly balanced.

In general, we only have $\mathcal{N}_x \subset W$. But L acts trivially on $\mathcal{N}_x^{\perp} \cap W$ and hence the weights of the maximal torus action on $\mathcal{N}_x^{\perp} \cap W$ are all zero. By examining the inequality (7.1) it is easy to see that if the L-action on $\mathcal{N}_x$ is weakly linearly balanced then so is the P action on W. Therefore, if for each $L \in \mathcal{R}(M)$ the action of L on the normal space to GM_L^{ss} at a generic point $x \in \mu^{-1}(0)$ with $\mathrm{LieStab}(x) = \mathrm{Lie}(L)$ is weakly linearly balanced, then the action of $\mathrm{Stab}(x)$ on the normal slice W at x is weakly linearly balanced for all $x \in \mu^{-1}(0)$, i.e. the K-action on M is almost balanced.

Now let $J \in \mathcal{R}(M)$ and suppose $x \in M_J^{ss} \cap \mu^{-1}(0)$. Then $L \supset J$. Using Lemma 8.2 which is purely a group theoretic result, it is direct to check that in the neighborhood of Kx, M_J^{ss} is

$$\begin{aligned} & N^J \times_{P\cap N^J} \left((\mathfrak{n}^J/\mathfrak{p}\cap\mathfrak{n}^J)^* \times W^J\right) \times \mathbb{R}^{\dim X_{(P)}} \\ \cong\ & N^J/J \times_{P\cap N^J/J} \left((\mathfrak{n}^J/\mathfrak{p}\cap\mathfrak{n}^J)^* \times W^J\right) \times \mathbb{R}^{\dim X_{(P)}} \end{aligned}$$

See the proof of Proposition 8.4 for a similar computation.

If $\mathrm{Stab}(x)$ is minimal with $\mathrm{LieStab}(x) = \mathrm{Lie}L$ so that $\mathcal{N}_x = W$, then the J-fixed sets $\mathcal{N}_{x,J}$ and W^J are isomorphic. In general, we only have $\mathcal{N}_{x,J} \subset W^J$. But by the arguments in the previous paragraphs, we deduce that the action of N^J/J on M_J is almost balanced if and only if the action of $L \cap N^J/J$ on $\mathcal{N}_{x,J}$ is weakly linearly balanced for $x \in \mu^{-1}(0)$ with $\mathrm{LieStab}(x) = \mathrm{Lie}L$ for all $L \supset J$. So we proved the lemma. $\square$

The weakly balanced condition is satisfied by many interesting spaces including the diagonal $SL(2)$ action on $(\mathbb{P}^1)^{2n}$. (See §9.) Also, it is satisfied by (the GIT construction of) the moduli spaces of holomorphic vector bundles over a Riemann surface of any rank and any degree. For the next proposition, let us use Definition 7.2.

Proposition 7.4. *Let $M(n,d)$ be the moduli space of rank n holomorphic vector bundles of degree $d > n(2g-1)$ over a Riemann surface Σ of genus g, which is a GIT quotient of a nonsingular quasiprojective variety $\mathfrak{R}(n,d)^{ss}$ by $G = SL(p)$ for $p = d - n(g-1)$. (See* [New78, Kir86a]*.) The action of $SL(p)$ on $\mathfrak{R}(n,d)^{ss}$ is weakly balanced.*

Proof. Let E be a semistable vector bundle such that $E \cong m_1E_1 \oplus \cdots \oplus m_sE_s$ where E_i's are non-isomorphic stable bundles with the same slope. Then the identity component of $\mathrm{Stab}\, E$ in G is $H^{\mathbb{C}} = S(\prod_{i=1}^s GL(m_i))$ where S denotes the subset of elements whose determinant is 1. The normal space to GM_H^{ss} at E is ([AB82, Kir86a])

$$\begin{aligned} H^1(\Sigma, End'_{\oplus}E) &= H^1(\Sigma, \oplus_{i,j}(m_im_j - \delta_{ij})Hom(E_i,E_j)) \\ &= \oplus_{i,j} H^1(\Sigma, (m_im_j-\delta_{ij})E_i^* \otimes E_j) \end{aligned}$$

More precisely,

$$\begin{aligned} H^1(\Sigma, End'_{\oplus}E) = \ & \oplus_{i<j}[H^1(\Sigma, E_i^*\otimes E_j) \otimes Hom(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \\ & \oplus H^1(\Sigma, E_i \otimes E_j^*) \otimes Hom(\mathbb{C}^{m_j}, \mathbb{C}^{m_i})] \\ & \oplus[\oplus_i H^1(\Sigma, End\, E_i) \otimes sl(m_i)] \end{aligned}$$

Because E_i is not isomorphic to E_j for $i \neq j$, $H^0(\Sigma, E_i^* \otimes E_j) = 0 = H^0(\Sigma, E_i \otimes E_j^*)$ and thus

$$\begin{aligned} \dim H^1(\Sigma, E_i^*\otimes E_j) &= -RR(\Sigma, E_i^* \otimes E_j) = (rank\, E_i)(rank\, E_j)(g-1) \\ &= -RR(\Sigma, E_i \otimes E_j^*) = \dim H^1(\Sigma, E_i \otimes E_j^*) \end{aligned}$$

where RR denotes the Riemann-Roch number. Therefore, the weights of the representation of $H^{\mathbb{C}}$ on $H^1(\Sigma, End'_{\oplus}E)$ are symmetric with respect to the origin. This implies that the action is weakly linearly balanced. As each subgroup $L^{\mathbb{C}}$ as in Definition 7.2 (2) is conjugate to $S(\prod_{i=1}^s GL(m_i'))$ for a "subdivision" $(m_1', m_2', ...)$

of $(m_1, m_2, ...)$, it is easy to check that such $H \cap N^L/L$ action on the L-fixed point set is also weakly linearly balanced. $\square$

7.2. **The image of the pull-back homomorphism.** For any $H \in \mathcal{R}(M)$, consider the natural map (sometimes called the "resolution")

$$K \times_{N^H} M_H^{ss} \to KM_H^{ss} \tag{7.2}$$

and the corresponding map on the cohomology ([Kir86a] Lemma 1.21)

$$H_K^*(KM_H^{ss}) \to H_K^*(K \times_{N^H} M_H^{ss}) \cong H_{N^H}^*(M_H^{ss}) \cong [H^*_{N_0^H/H}(M_H^{ss}) \otimes H_H^*]^{\pi_0 N^H} \tag{7.3}$$

where N_0^H is the identity component of N^H. For any $\zeta \in H_K^*(M^{ss})$ let $\zeta|_{K \times_{N^H} M_H^{ss}}$ denote the image of ζ by the composition of the above map with the restriction map $H_K^*(M^{ss}) \to H_K^*(KM_H^{ss})$. Now, we can describe the image of ϕ_K^*.

Definition 7.5. *Put $n_H = \frac{1}{2}\operatorname{codim} GM_H^{ss} - \dim H$ and $H_H^{<n_H} = \oplus_{i<n_H} H_H^i$. We define V_M^* as the set of $\zeta \in H_K^*(M^{ss})$ such that*

$$\zeta|_{K \times_{N^H} M_H^{ss}} \in H^*_{N_0^H/H}(M_H^{ss}) \otimes H_H^{<n_H} \tag{7.4}$$

for each $H \in \mathcal{R}(M)$.

Remark 7.6. The definition of V_M^* is independent of the choices of Hs in the conjugacy classes and the tensor product expressions: The former is easy to check by translating by g if H is replaced by gHg^{-1}. The latter can be immediately seen by considering the gradation of the degenerating spectral sequence for the cohomology of the fibration

$$\begin{array}{c} (EN_0^H \times EN_0^H/H) \times_{N_0^H} M_H^{ss} \\ \downarrow \\ EN_0^H/H \times_{N_0^H/H} M_H^{ss} \end{array} \tag{7.5}$$

The fiber is homotopically equivalent to BH. (See [Kir86a] Lemma1.21.) Though the last isomorphism in (7.3) is not canonical, the subspace in (7.4) is canonical.

From (7.4), we have

$$V_M^* = \operatorname{Ker}\left(H_K^*(M^{ss}) \to \bigoplus_{H \in \mathcal{R}(M)} H^*_{N_0^H/H}(M_H^{ss}) \otimes H_H^{\geq n_H} \right) \tag{7.6}$$

and thus V_M^* can be thought of as a subset of $H_K^*(M^{ss}) \cong H_K^*(Z)$, obtained by "truncating locally".

Now we can state the main theorem of the section which will be proved in the next section.

Theorem 7.7. *Let $M \subset \mathbb{P}^n$ be a projective smooth variety acted on unitarily by a compact connected group K with at least one stable point. Suppose that the weakly balanced condition is satisfied. Then we have $\phi_K^*(IH^*(X)) = V_M^*$. Moreover, for any open set U of X, if we define $V^*_{\phi^{-1}(U)} \subset H_K^*(\phi^{-1}(U))$ as in Definition 7.5, then we have $\phi_K^*(IH^*(U)) = V^*_{\phi^{-1}(U)}$.*

8. Proof of Theorem 7.7

This section is devoted to a proof of Theorem 7.7. Let us use the notations of §6.1 and §7. Recall that

$$r := r(M) = \max\{\dim H \mid H \in \mathcal{R}(M)\}. \tag{8.1}$$

Our proof is by induction on $r(M)$.

When $r = 0$, we have nothing to prove since

$$V_M^* = H_K^*(M^{ss}) \cong IH^*(M/\!\!/G).$$

So we consider the case $r > 0$. Suppose the theorem is true for all projective varieties Γ with $r(\Gamma) \le r-1$. Let $\hat{M}$ be the blowup of M^{ss} along the submanifold

$$\bigsqcup_{\dim H = r} GM_H^{ss}.$$

Then from [Kir85] §6, we have

$$\mathcal{R}(\hat{M}) = \{H \in \mathcal{R}(M) \mid \dim H \le r-1\}$$

and thus $r(\hat{M}) \le r-1$.

For simplicity, we assume from now on that there exists only one H such that $\dim H = r$. (The general case is no more difficult except for repetition. We can deal with each H one by one. See [Kir85], Cor.8.3.) We fix this H once and for all till the end of this section.

Remark 8.1. (1) To be precise, we have to take the closure of $\hat{M}$ with respect to a suitable linearization described in [Kir85] and then resolve the possible singularities. But as argued in [Kir85], this does not cause any trouble for us because all the semistable points are contained in $\hat{M}$ and we are only interested in the semistable points.

(2) By [Kir86b] 1.6, for $L \in \mathcal{R}(\hat{M})$, the L-fixed set $\hat{M}_L^{ss}$ in $\hat{M}^{ss}$ is the proper transform of the L-fixed set M_L^{ss} in M^{ss}. In particular, the normal space to $G\hat{M}_L^{ss}$ in $\hat{M}^{ss}$ at a generic point is isomorphic to the normal space to GM_L^{ss} in M^{ss} at a generic point. Notice that the weakly balanced condition in Definition 7.2 is purely about the actions of L on the normal spaces $\mathcal{N}_x$ to GM_L^{ss} for $L \in \mathcal{R}(M)$. (The fixed set by a subgroup of L is determined by the action of L.) Therefore, if the K-action on M^{ss} is weakly balanced, the action on $\hat{M}$ is also weakly balanced.

By our induction hypothesis, $\hat{\phi}_K^*(IH^*(\hat{M}/\!\!/G)) = V_{\hat{M}}^*$ and the same holds for any open subset of $\hat{X} = \hat{M}/\!\!/G$.

We start the proof with a few lemmas.

Lemma 8.2. *([Kir85] Proposition 8.10) Suppose $L \subset P$ are compact subgroups of K and L is connected. Then there exist finitely many elements $k_1, k_2, \cdots, k_m$ in K such that*

$$\{k \in K \mid k^{-1}Lk \subset P\} = \bigsqcup_{1 \le i \le m} N^L k_i P$$

where N^L is the normalizer of L in K.

Proof. See the proof of [Kir85] p77. □

Let E be the exceptional divisor in $\hat{M}$ of the blow-up.

Lemma 8.3. *By restriction, we have an isomorphism*

$$\operatorname{Ker}\left(H_K^*(\hat{M}) \to H_K^*(\hat{M}^{ss})\right) \cong \operatorname{Ker}\left(H_K^*(E) \to H_K^*(E^{ss})\right).$$

Proof. This follows from [Kir85] 7.5, 7.6 and 7.11. □

We first show that the image of ϕ_K^* is contained in V_M^*. Let U be an open subset of $X = M/\!\!/G$.

Proposition 8.4. $\phi_K^*(IH^*(U)) \subset V^*_{\phi^{-1}(U)}$.

Proof. Recall that $E\mathcal{G}$ denotes a contractible free $\mathcal{G}$-space for a Lie group $\mathcal{G}$ and $B\mathcal{G} = E\mathcal{G}/\mathcal{G}$. Let $L \in \mathcal{R}(M)$. From the obvious commutative diagram

$$\begin{tikzcd}
\big(EK \times E(N^L/L)\big) \times_{N^L} M_L^{ss} \arrow[r, "g"] \arrow[d, "\phi'_L"'] & EK \times_K M^{ss} \arrow[d, "\phi_K"] \\
E(N^L/L) \times_{N^L/L} M_L^{ss} \arrow[r, "f"] & X
\end{tikzcd}$$

we get a morphism

$$f^* R\phi_{K*}\mathbb{C} \to R\phi'_{L*} g^*\mathbb{C} \to \tau_{\geq n_L} R\phi'_{L*} g^*\mathbb{C} = \tau_{\geq n_L} R\phi'_{L*}\mathbb{C}.$$

This induces a morphism by adjunction

$$R\phi_{K*}\mathbb{C} \to Rf_* f^* R\phi_{K*}\mathbb{C} \to Rf_*\tau_{\geq n_L} R\phi'_{L*}\mathbb{C} =: \mathcal{A}^{\cdot}_L. \tag{8.2}$$

The fiber of ϕ'_L is homotopically equivalent to BL and thus this morphism induces the truncation homomorphism

$$H_K^*(M^{ss}) \to [H^*_{N_0^L/L}(M_L^{ss}) \otimes H_L^{\geq n_L}]^{\pi_0 N^L}$$

where N_0^L is the identity component of N^L. By composing (8.2) with ϕ_K^*, we get a morphism

$$\rho : \mathcal{IC}^{\cdot}_X \xrightarrow{\phi_K^*} R\phi_{K*}\mathbb{C} \longrightarrow \mathcal{A}^{\cdot}_L$$

whose hypercohomology gives us

$$IH^*(U) \to H_K^*(\phi^{-1}(U)) \to [H^*_{N_0^L/L}(M_L^{ss} \cap \phi^{-1}(U)) \otimes H_L^{\geq n_L}]^{\pi_0 N^L}.$$

Therefore it suffices to show that ρ is equal to zero in view of (7.6).

The sheaf complex $\mathcal{A}^{\cdot}_L$ is trivial on the complement of the closed subset $GM_L^{ss}/\!\!/G$. Hence ρ is zero on this open dense subset. Hence by adding stratum by stratum in the order of increasing codimension, it suffices to show the following: Let P be a subgroup of K and consider the stratum $X_{(P)}$ defined in §4. Suppose U is an open subset of X containing $X_{(P)}$ such that $U - X_{(P)}$ is open and $\rho|_{U-X_{(P)}}$ is equal to zero. Then $\rho|_U$ is also zero.

Let $\imath : U - X_{(P)} \hookrightarrow U$ and put $n_P = \frac{1}{2}\operatorname{codim} X_{(P)}$. We claim that

$$\tau_{<n_P}\mathcal{A}^{\cdot}_L|_U \cong \tau_{<n_P} R\imath_*\mathcal{A}^{\cdot}_L|_{U-X_{(P)}}. \tag{8.3}$$

This claim enables us to deduce that $\rho|_U$ is zero from $\rho|_{U-X_{(P)}}$ being zero because $\rho|_U$ is the composition

$$\mathcal{IC}^{\cdot}_X|_U \xrightarrow{\cong} \tau_{<n_P} R\imath_*\mathcal{IC}^{\cdot}_X|_{U-X_{(P)}} \xrightarrow{0} \tau_{<n_P} R\imath_*\mathcal{A}^{\cdot}_L|_{U-X_{(P)}} \xrightarrow{\cong} \tau_{<n_P}\mathcal{A}^{\cdot}_L|_U \longrightarrow \mathcal{A}^{\cdot}_L|_U$$

which is zero.

Let us now prove (8.3). If L is not conjugate to a subgroup of P, then $X_{(P)}$ does not intersect with $GM_L^{ss}/\!\!/G$ and thus we have nothing to prove. So we may assume $L \subset P$ after conjugation if necessary.

Consider the commutative diagram

$$\begin{tikzcd} M_L^{ss} \arrow[r, hook, "\jmath"] \arrow[d, "q"] & M^{ss} \arrow[d, "\phi"] \\ M_L^{ss}/\!\!/N^L \arrow[r, "h"] & M/\!\!/K = X \end{tikzcd}$$

where h is the unique map defined by the universal property of the categorical quotient $M_L^{ss}/\!\!/N^L$ of M_L^{ss}.

We compute the stalk cohomology of both sides of (8.3). By Lemma 4.1, the preimage of a contractible neighborhood Δ of a point in $X_{(P)}$ by ϕ is equivariantly homeomorphic to

$$K \times_P ((\mathfrak{k}/\mathfrak{p})^* \times W) \times \mathbb{R}^{\dim X_{(P)}} \tag{8.4}$$

for some symplectic P-vector space W. By Lemma 8.2, it is direct to check that M_L^{ss} in this neighborhood is

$$\bigsqcup_{1\le i\le m} k_i N^{L_i} \times_{P\cap N^{L_i}} \left((\mathfrak{n}^{L_i}/\mathfrak{p}\cap\mathfrak{n}^{L_i})^* \times W^{L_i}\right) \times \mathbb{R}^{\dim X_{(P)}} \tag{8.5}$$

where $L_i = k_i^{-1} L k_i$ and W^{L_i} is the L_i-fixed subspace of W. Also $\mathfrak{n}^L$ denotes the Lie algebra of N^L.

If we delete $X_{(P)}$ from the neighborhood Δ, then the preimage by ϕ is

$$K \times_P \left((\mathfrak{k}/\mathfrak{p})^* \times (W - \phi_W^{-1}(*))\right) \times \mathbb{R}^{\dim X_{(P)}}$$

where $\phi_W : W \to W/\!\!/P$ is the GIT quotient map and $*$ is the vertex of the cone $W/\!\!/P$. The intersection of this with M_L^{ss} is homeomorphic to

$$\bigsqcup_{1\le i\le m} k_i N^{L_i} \times_{P\cap N^{L_i}} \left((\mathfrak{n}^{L_i}/\mathfrak{p}\cap\mathfrak{n}^{L_i})^* \times (W^{L_i} - \phi_W^{-1}(*))\right) \times \mathbb{R}^{\dim X_{(P)}}.$$

Hence the stalk cohomology of the left hand side of (8.3) is

$$\bigoplus_i H^{<n_P-n_L}_{P\cap N^{L_i}/L_i}(W^{L_i}) \otimes H^{\ge n_L}_{L_i}$$

while the right hand side has

$$\bigoplus_i H^{<n_P-n_L}_{P\cap N^{L_i}/L_i}(W^{L_i} - \phi_W^{-1}(*)) \otimes H^{\ge n_L}_{L_i}.$$

Thus it suffices to show that

$$H^{<n_P-n_L}_{P\cap N^{L_i}/L_i}(W^{L_i}) \cong H^{<n_P-n_L}_{P\cap N^{L_i}/L_i}(W^{L_i} - \phi_W^{-1}(*)). \tag{8.6}$$

Without loss of generality, we may assume $L_i = L$.

By definition, we have

$$n_P = \frac{1}{2}\dim W/\!\!/P = \frac{1}{2}\dim W - \dim P. \tag{8.7}$$

From (8.5), it is easy to deduce that GM_L^{ss} in the preimage of Δ is

$$K \times_P \left((\mathfrak{k}/\mathfrak{p})^* \times P_{\mathbb{C}} W^L\right) \times \mathbb{R}^{\dim X_{(P)}} \tag{8.8}$$

where $P_{\mathbb{C}}$ is the complexification of P in G and W is assigned a complex structure compatible with the symplectic structure. Hence, we have

$$n_L = \frac{1}{2}\operatorname{codim} GM_L^{ss} - \dim L = \frac{1}{2}\left(\dim W - \dim P_{\mathbb{C}}W^L\right) - \dim L. \tag{8.9}$$

From the surjectivity of the morphism

$$P_{\mathbb{C}} \times_{P_{\mathbb{C}}\cap N_{\mathbb{C}}^L} W^L \to P_{\mathbb{C}}W^L$$

we see that

$$\dim P_{\mathbb{C}}W^L \leq \dim P_{\mathbb{C}} + \dim W^L - \dim(P_{\mathbb{C}} \cap N_{\mathbb{C}}^L). \tag{8.10}$$

Comparing (8.7), (8.9) and (8.10), we get

$$n_P - n_L \leq \frac{1}{2}\dim W^L - \dim(P \cap N^L/L). \tag{8.11}$$

Since the action of N^L/L on M_L^{ss} is almost balanced (Remark 5.2), by (8.11) we have

$$H^{<n_P-n_L}_{P\cap N^L/L}(W^L) \cong H^{<n_P-n_L}_{P\cap N^L/L}(W^L - \phi_{W^L}^{-1}(*)) \tag{8.12}$$

where $\phi_{W^L} : W^L \to W^L/\!\!/P \cap N^L$ is the GIT quotient map.

Finally, we observe that

$$\phi_{W^L}^{-1}(*) = \phi_W^{-1}(*) \cap W^L. \tag{8.13}$$

This is because we know the following from [Kir84]:

(1) For $x \in W^L$, $x \in \phi_{W^L}^{-1}(*) \Leftrightarrow \lim_{t\to\infty} x_t = 0$ where x_t is the gradient flow for $-|\mu_{W^L}|^2$ with $x_0 = x$ (μ_{W^L} is the moment map for W^L).
(2) For $x \in W$, $x \in \phi_W^{-1}(*) \Leftrightarrow \lim_{t\to\infty} x_t = 0$ where x_t is the gradient flow for $-|\mu_W|^2$ with $x_0 = x$ (μ_W is the moment map for W).
(3) For a moment map μ on a symplectic manifold, the gradient vector at x for $-|\mu|^2$ is $-2i\mu(x)_x$ if $\mathfrak{k}$ is identified with $\mathfrak{k}^*$ by the Killing form.
(4) $\mu_W(x) \in \mathfrak{n}^L$ if $x \in W^L$ and hence $\mu_W(x) = \mu_{W^L}(x)$.

(8.6) follows from (8.12) and (8.13). □

We need a few more lemmas.

Lemma 8.5. *Consider the diagram (6.1) in §6.1. The restriction to V_M^* of*

$$H_K^*(M^{ss}) \xhookrightarrow{\pi_K^*} H_K^*(\hat{M}) \xrightarrow{\imath_K^*} H_K^*(\hat{M}^{ss})$$

factors through $V_{\hat{M}}^*$ and is injective. A similar statement is true for $\phi^{-1}(U)$ where U is any open set in X.

Proof. Let ζ be a nonzero element in V_M^*. Then

$$\zeta|_{K\times_{N^L} M_L^{ss}} \in [H^*_{N_0^L/L}(M_L^{ss}) \otimes H_L^{<n_L}]^{\pi_0 N^L}$$

for each $L \in \mathcal{R}(M)$. Its image in $H_K^*(\hat{M}^{ss})$ satisfies

$$\zeta|_{K\times_{N^L} \hat{M}_L^{ss}} \in [H^*_{N_0^L/L}(\hat{M}_L^{ss}) \otimes H_L^{<n_L}]^{\pi_0 N^L}$$

for each $L \in \mathcal{R}(\hat{M}) = \{L \in \mathcal{R}(M) | \dim L < r(M)\}$. This follows from the commutative diagram

$$\begin{array}{ccccc} \hat{M}^{ss} & \longleftarrow & K\hat{M}^{ss}_L & \longleftarrow & K \times_{N^L} \hat{M}^{ss}_L \\ \downarrow & & \downarrow & & \downarrow \\ M^{ss} & \longleftarrow & KM^{ss}_L & \longleftarrow & K \times_{N^L} M^{ss}_L. \end{array}$$

Therefore, ζ is mapped to an element in $V^*_{\hat{M}}$.

Recall that H is the identity component of a stabilizer which has the maximal dimension $r(M)$ and the blowup center is the submanifold GM^{ss}_H. Since π^*_K is an injection by the well-known argument in [GH94] p605, we may think of ζ as an element of $H^*_K(\hat{M})$. By Lemma 8.3, if $\zeta|_{GM^{ss}_H} = 0$ i.e. $\zeta|_E = 0$, then $\zeta|_{\hat{M}^{ss}} \neq 0$.

Let us now consider the case when $\zeta|_{GM^{ss}_H} \neq 0$. Since H is maximal, we have an isomorphism

$$H^*_{N^H_0/H}(M^{ss}_H) \cong H^*(M^{ss}_H /\!\!/ N^H_0).$$

By the definition of V^*_M, $\zeta|_{GM^{ss}_H}$ lies in

$$[H^*(M^{ss}_H /\!\!/ N^H_0) \otimes H^{<n_H}_H]^{\pi_0 N^H} \tag{8.14}$$

and we have

$$H^*_K(E^{ss}) = [H^*(M^{ss}_H /\!\!/ N^H_0) \otimes H^*_H(\mathbb{P}\mathcal{N}^{ss}_x)]^{\pi_0 N^H} \tag{8.15}$$

from [Kir86b] Lemma 1.16 where $\mathcal{N}_x$ is the normal space to GM^{ss}_H. Because the K-action is almost balanced, the codimensions of the unstable strata in $\mathbb{P}\mathcal{N}_x$ are greater than n_H by (5.2). Therefore, by the equivariant Morse theory [Kir84], we deduce that the restriction homomorphism

$$H^{<n_H}_H(\mathbb{P}\mathcal{N}_x) \to H^{<n_H}_H(\mathbb{P}\mathcal{N}^{ss}_x)$$

is an isomorphism. In particular, $H^{<n_H}_H$ injects into $H^{<n_H}_H(\mathbb{P}\mathcal{N}^{ss}_x)$. By (8.15) and (8.14), the image of ζ in $H^*_K(E^{ss})$ is not zero and thus ζ injects into $H^*_K(\hat{M}^{ss})$.

It is obvious from our proof that the statement is true for any open set U in X. □

Let $\mathcal{N}$ be the normal bundle to GM^{ss}_H in M^{ss} and $\mathcal{N}_U$ be the normal bundle to $GM^{ss}_H \cap \phi^{-1}(U)$ for any open subset U of X. It is proved in [Kir86b] Lemma 2.9 that $\mathcal{N} /\!\!/ K$ is homeomorphic to a neighborhood of $X_{(\mathfrak{h})} = GM^{ss}_H /\!\!/ G$ and hence $\mathcal{N}_U /\!\!/ K$ is homeomorphic to a neighborhood of $U \cap X_{(\mathfrak{h})}$. We identify $\mathcal{N}_U$ with a tubular neighborhood of $GM^{ss}_H \cap \phi^{-1}(U)$ and identify $\mathcal{N}_U /\!\!/ K$ with a neighborhood U_1 of $U \cap X_{(\mathfrak{h})}$ in U.

By the gradient flow of $-|\mu|^2$, M^{ss} can be equivariantly retracted into $\mu^{-1}(D_\varepsilon)$ where D_ε is the disk of radius ε around 0 in $\mathfrak{k}^*$. By shrinking U_1 if necessary and taking ε sufficiently small, $\phi^{-1}(U_1)$ is retracted into $\mathcal{N}_U$. Conversely, if we decrease the radius of a tubular neighborhood of $GM^{ss}_H \cap \phi^{-1}(U)$, it is included in $\phi^{-1}(U_1)$. These two inclusions are clearly inverse to each other *homotopically* and K-equivariantly since μ is equivariant. In particular, $\mathcal{N}_U$ and $\phi^{-1}(U_1)$ are homotopically equivalent open neighborhoods of $GM^{ss}_H \cap \phi^{-1}(U)$. Therefore, $H^*_K(\phi^{-1}(U_1))$ is canonically isomorphic to $H^*_K(\mathcal{N}_U)$ and $V^*_{\phi^{-1}(U_1)} \cong V^*_{\mathcal{N}_U}$. Hence for cohomological purpose, we can think of $\mathcal{N}_U$ as the preimage of a neighborhood of $U \cap X_{(\mathfrak{h})}$.

Proposition 8.6. *The restriction of the Kirwan map gives us an isomorphism $V^*_{\mathcal{N}_U} \cong IH^*(\mathcal{N}_U /\!\!/ G)$.*

We postpone the proof of this proposition and prove Theorem 7.7.

Proof of Theorem 7.7. For an open subset U of X, let $B^*(U)$ be the kernel of the Kirwan map restricted to $V^*_{\phi^{-1}(U)}$. Then by Proposition 8.4, we can write

$$V^*_{\phi^{-1}(U)} = IH^*(U) \oplus B^*(U). \tag{8.16}$$

We have to show that $B^*(U)$ is zero.

Let $\hat{U}$ be the preimage of U by the blow-up map $\sigma : \hat{X} \to X$. By our induction hypothesis, the pull-back $\hat{\phi}^*_K$ is an isomorphism of $IH^*(\hat{U})$ onto $V^*_{\hat{\phi}^{-1}(\hat{U})}$ and the Kirwan map is its inverse.

Recall that we have the decomposition (6.5) which induces an isomorphism

$$IH^*(\hat{U}) \cong IH^*(U) \oplus F^*(U) \tag{8.17}$$

where $F^*(U)$ is the hypercohomology of $\mathcal{F}^\cdot$ over U. Since $B^*(U)$ is mapped to zero by

$$V^*_{\phi^{-1}(U)} \hookrightarrow V^*_{\hat{\phi}^{-1}(\hat{U})} \cong IH^*(\hat{U}) \cong IH^*(U) \oplus F^*(U) \to IH^*(U)$$

we see that $B^*(U)$ injects into $F^*(U)$.

As ϕ^*_K, κ^{ss}_M and (8.17) all came from sheaf complexes, we have the following commutative diagram by restriction

$$\begin{tikzcd}
V^*_{\phi^{-1}(U)} \arrow[r, hook] \arrow[d] & V^*_{\hat{\phi}^{-1}(\hat{U})} \arrow[r, "\cong"] \arrow[d] & IH^*(\hat{U}) \arrow[r, "\cong"] \arrow[d] & IH^*(U) \oplus F^*(U) \arrow[r] \arrow[d] & IH^*(U) \arrow[d] \\
V^*_{\mathcal{N}_U} \arrow[r, hook] & V^*_{\hat{\mathcal{N}}_U} \arrow[r] & IH^*(\hat{\mathcal{N}}_U /\!\!/ K) \arrow[r, "\cong"] & IH^*(\mathcal{N}_U /\!\!/ K) \oplus F^*(U) \arrow[r] & IH^*(\mathcal{N}_U /\!\!/ G)
\end{tikzcd}$$

where $\hat{\mathcal{N}}_U$ is the preimage of $\mathcal{N}_U$ in $\hat{M}^{ss}$. Since $\mathcal{F}^\cdot$ is supported over $X_{(\mathfrak{h})} = GM^{ss}_H /\!\!/ G$, the vertical map for $F^*(U)$ is the identity map.

Now let ζ be a nonzero element in $B^*(U)$. We know ζ is mapped to a nonzero element, say η in $F^*(U)$. In the above diagram, η is mapped to zero in $IH^*(\mathcal{N}_U /\!\!/ G)$. Then by Proposition 8.6, $\zeta|_{\mathcal{N}_U} = 0$ and thus $\eta = 0$. This is a contradiction! So we proved that $B^*(U) = 0$. □

It remains to prove Proposition 8.6. This is a consequence of the next three lemmas.

Let x and $\mathcal{N}_x$ be as in Definition 7.2. By [Kir85] Corollary 5.6 and [Kir86a] Lemma 1.21, we have an isomorphism

$$H^*_K(\mathcal{N}_U) \cong \left[H^*\left(M^{ss}_H \cap \phi^{-1}(U) /\!\!/ N^H_0\right) \otimes H^*_H(\mathcal{N}_x)\right]^{\pi_0 N^H} \tag{8.18}$$

from a degenerating spectral sequence.

Lemma 8.7. *Via the isomorphism (8.18), we have*

$$V^*_{\mathcal{N}_U} \cong \left[H^*\left(M^{ss}_H \cap \phi^{-1}(U) /\!\!/ N^H_0\right) \otimes V^*_{\mathcal{N}_x}\right]^{\pi_0 N^H}. \tag{8.19}$$

Proof. Let $L \in \mathcal{R}(M)$. If L is not conjugate to a subgroup of H, there is no L-fixed point in $\mathcal{N}$. Hence, after conjugation if necessary, we may assume that $L \subset P$. Let $\mathcal{N}_{U,L}$ be the L-fixed subset of $\mathcal{N}_U$. For $V^*_{\mathcal{N}_U}$ we have to consider the map

(8.20)

$$H^*_K(\mathcal{N}_U) \to H^*_K(K \times_{N^L} \mathcal{N}_{U,L}) \cong H^*_{N^L}(\mathcal{N}_{U,L}) \to [H^*_{N^L_0/L}(\mathcal{N}_{U,L}) \otimes H^{\geq n_L}_L]^{\pi_0 N^L}.$$

It is obvious that $\mathcal{N}_{U,L}$ is a vector bundle over $M_L^{ss} \cap GM_H^{ss} \cap \phi^{-1}(U)$. Using Lemma 8.2, it is easy to check that there are $k_1, \cdots, k_s$ in K so that

$$M_L^{ss} \cap GM_H^{ss} \cap \phi^{-1}(U) = \bigsqcup_i N_0^L M_{H_i}^{ss} \cap \phi^{-1}(U)$$

where $H_i = k_i H k_i^{-1}$. Since H is maximal in $\mathcal{R}(M)$ we have isomorphisms

$$N_0^L M_{H_i}^{ss} \cap \phi^{-1}(U) \cong N_0^L \times_{N_0^L \cap N^{H_i}} M_{H_i}^{ss} \cap \phi^{-1}(U) \cong N_0^L/L \times_{N_0^L \cap N^{H_i}/L} M_{H_i}^{ss} \cap \phi^{-1}(U)$$

by [Kir85] 5.6 and hence we have

$$H^*_{N_0^L/L}(\mathcal{N}_{U,L}) \cong \bigoplus_i H^*_{N_0^L \cap N^{H_i}/L}\left(M_{H_i}^{ss} \cap \phi^{-1}(U)\right). \tag{8.21}$$

If we apply Lemma 8.2 with $L \subset H$ as subgroups of N^H, we deduce that there exist $g_1, \cdots, g_t$ in N^H such that

$$\{k \in N^H \mid k^{-1} L k \subset H\} = \bigsqcup_j (N^H \cap N_0^L) g_j H.$$

But for $k \in N^H$, $k^{-1} L k \subset k^{-1} H k = H$ and hence we have

$$N^H = \bigsqcup_j (N^H \cap N_0^L) g_j H.$$

This implies that the natural embedding

$$N^H \cap N_0^L / H \cap N_0^L \hookrightarrow N^H/H$$

is of finite index. In particular, the identity component of $N^H \cap N_0^L / H \cap N_0^L$ is naturally isomorphic to the identity component N_0^H/H of N^H/H.

Let N_i be the identity component of $N^{H_i} \cap N_0^L$ and put $S_i = N_i \cap H_i$. Then $N_i/S_i \cong N_0^{H_i}/H_i$. Therefore,

$$H^*_{N_0^L \cap N^{H_i}/L}\left(M_{H_i}^{ss} \cap \phi^{-1}(U)\right)$$

is the $\pi_0(N^{H_i} \cap N_0^L)$-invariant part of

$$\begin{aligned} H^*_{N_i/L}\left(M_{H_i}^{ss} \cap \phi^{-1}(U)\right) &\cong H^*_{N_i/S_i}\left(M_{H_i}^{ss} \cap \phi^{-1}(U)\right) \otimes H^*_{S_i/L} \\ &\cong H^*_{N_0^{H_i}/H_i}\left(M_{H_i}^{ss} \cap \phi^{-1}(U)\right) \otimes H^*_{S_i/L} \\ &\cong H^*\left(M_{H_i}^{ss} \cap \phi^{-1}(U) /\!\!/ N_0^{H_i}\right) \otimes H^*_{S_i/L} \end{aligned} \tag{8.22}$$

The first isomorphism in (8.22) came from [Kir86a] Lemma 1.21. Our interest lies in finding the kernel of (8.20). Combining (8.20), (8.21) and (8.22), we see that $V^*_{\mathcal{N}_U}$ is the intersection of the kernels of

$$H^*_K(\mathcal{N}_U) \to \bigoplus_i H^*\left(M_{H_i}^{ss} \cap \phi^{-1}(U) /\!\!/ N_0^{H_i}\right) \otimes H^*_{S_i/L} \otimes H_L^{\geq n_L} \tag{8.23}$$

for all $L \in \mathcal{R}(M)$.

Now observe that the spaces that appear in (8.20) lie over $GM_H^{ss} /\!\!/ G \cong M_H^{ss} /\!\!/ N^H$. Applying spectral sequence, we get a homomorphism of spectral sequences whose E_2-terms give us

(8.24)

$$\left[H^* \quad M_H^{ss} \cap \phi^{-1}(U) /\!\!/ N_0^H \quad \otimes H^*_H(\mathcal{N}_x)\right]^{\pi_0 N^H} \to \bigoplus_i H^* \quad M_{H_i}^{ss} \cap \phi^{-1}(U) /\!\!/ N_0^{H_i} \quad \otimes H^*_{S_i/L}(\mathcal{N}_{x,L}) \otimes H_L^{\geq n_L}.$$

The right side of (8.24) is isomorphic to

$$\bigoplus_i H^*\left(M_H^{ss}\cap\phi^{-1}(U)/\!\!/N_0^H\right)\otimes H^*_{k_i^{-1}S_ik_i/L_i}(\mathcal{N}_{x,L_i})\otimes H^{\geq n_L}_{L_i} \tag{8.25}$$

by conjugation, where $L_i = k_i^{-1}Lk_i$. Note that $k_i^{-1}S_ik_i/L_i$ and $H\cap N_0^{L_i}/L_i$ share the same identity component say S_i'. Thus $H^*_{k_i^{-1}S_ik_i/L_i}(\mathcal{N}_{x,L_i})$ is the invariant subspace of $H^*_{S_i'}(\mathcal{N}_{x,L_i})$ with respect to a finite group action. With the isomorphism (8.25), the fiber direction in (8.24) is exactly the truncation map

$$H^*_H(\mathcal{N}_x)\to H^*_{S_i'}(\mathcal{N}_{x,L_i})\otimes H^{\geq n_L}_{L_i}$$

for $V^*_{\mathcal{N}_x}$. Therefore taking the kernel of (8.20) for all $L\in\mathcal{R}(M)$ gives us

$$\left[H^*\left(M_H^{ss}\cap\phi^{-1}(U)/\!\!/N_0^H\right)\otimes V^*_{\mathcal{N}_x}\right]^{\pi_0N^H}.$$

So we are done. □

Let $Y=\mathbb{P}(\mathcal{N}_x\oplus\mathbb{C})\supset\mathcal{N}_x$ and $\hat{Y}$ be the blow-up of Y at $0\in\mathcal{N}_x$. Consider the action of H on Y and $\hat{Y}$ where H acts trivially on the summand $\mathbb{C}$.

Lemma 8.8. *The actions of H on Y and $\hat{Y}$ are weakly balanced.*

Proof. We identify the hyperplane at infinity $Y-\mathcal{N}_x$ with $\mathbb{P}\mathcal{N}_x$. The equation for the hyperplane at infinity is H-invariant and hence Y^{ss} contains $\mathcal{N}_x$. Therefore $Y^{ss}=\mathcal{N}_x\cup\mathbb{P}\mathcal{N}_x^{ss}$. Let $0\neq y\in\mathcal{N}_x$ and suppose be the corresponding point $\overline{y}$ in $\mathbb{P}\mathcal{N}_x$ is semistable. By [Kir85] Lemma 4.3, y is fixed by L if and only if $\overline{y}$ is fixed by L. Hence we may consider only points in $\mathcal{N}_x$.

Let $\mathrm{Stab}(x)=P$. Since we could use any $x\in\mu^{-1}(0)$ as long as the infinitesimal stabilizer is $\mathrm{Lie}\,H$, we assume that P is minimal among the stabilizers whose Lie algebra is $\mathrm{Lie}\,H$ so that $\mathcal{N}_x=W$ in the notation of Lemma 4.1.

First since the action of K on M is weakly balanced, the action of H on $\mathcal{N}_x$ is weakly linearly balanced and so is the action of $H\cap N^L/L$ on the L-fixed subspace $\mathcal{N}_{x,L}$. Hence we checked the weakly balanced condition for H.

Now let $0\neq y\in\mu^{-1}_{\mathcal{N}_x}(0)$ and let L be the identity component of the stabilizer of y. Then from (8.8), we see that the normal space to GM_L^{ss} in M^{ss} is the same as the normal space to $P_{\mathbb{C}}W^L$ in W at a generic point. But H is the identity component of P and $\mathcal{N}_x=W$. Hence the normal space to GM_L^{ss} in M^{ss} at a generic point is isomorphic to the normal space to $H_{\mathbb{C}}\mathcal{N}_{x,L}$ in $\mathcal{N}_x$ at a generic point. Moreover since the diffeomorphism in Lemma 4.1 is equivariant, the actions of L on the normal spaces are identical. According to Definition 7.2, the weakly balanced condition is purely about the action of L on the normal spaces to $H_{\mathbb{C}}\mathcal{N}_{x,L}$ for all L. Because the action of K on M is weakly balanced, we deduce that the action of H on $\mathcal{N}_x$ is weakly balanced. So we proved that the action of H on Y is weakly balanced.

Note that $\hat{Y}$ is the first blow-up in the partial desingularization for Y and hence $r(\hat{Y})<r=r(M)$ by [Kir85] 6.1. By Remark 8.1 (2), the action on $\hat{Y}$ is also weakly balanced. □

If we let $\mathcal{N}_x=\mathrm{Spec}\,A$ for some polynomial ring A, then by definition [MFK94] there are homeomorphisms

$$\mathcal{N}_x/\!\!/H\cong\mathrm{Spec}\,A^H\qquad\mathbb{P}\mathcal{N}_x/\!\!/H\cong\mathrm{Proj}\,A^H$$

i.e. $\mathcal{N}_x/\!\!/H$ is the affine cone of $\mathbb{P}\mathcal{N}_x/\!\!/H$ and $Y/\!\!/H$ is the projective cone of $\mathbb{P}\mathcal{N}_x/\!\!/H$.

Since the equation for the hyperplane at infinity in Y is invariant, by construction of $Y/\!\!/G$ ([MFK94] Chapter 1 §4), it is obvious that the preimage of $\mathcal{N}_x/\!\!/H$ by the GIT quotient map $\phi_Y : Y^{ss} \to Y/\!\!/H$ is $\mathcal{N}_x$. Because the action of H on Y is weakly balanced by Lemma 8.8, the Kirwan map for Y restricted to $\mathcal{N}_x$

$$\kappa^{ss}_{\mathcal{N}_x} : H^*_H(\mathcal{N}_x) \to IH^*(\mathcal{N}_x/\!\!/H)$$

is surjective by Proposition 6.2.

Now the (sheaf-theoretic) Kirwan map gives us a homomorphism of the spectral sequence for $H^*_K(\mathcal{N}_U)$ to the spectral sequence for $IH^*(\mathcal{N}_U/\!\!/K)$. At E_2-level, we have a homomorphism

$$\Big[H^* \quad M^{ss}_H \cap \phi^{-1}(U)/\!\!/N^H_0 \quad \otimes H^*_H(\mathcal{N}_x)\Big]^{\pi_0 N^H} \to \Big[H^* \quad M^{ss}_H \cap \phi^{-1}(U)/\!\!/N^H_0 \quad \otimes IH^*(\mathcal{N}_x/\!\!/H)\Big]^{\pi_0 N^H}.$$

But we know the left side spectral sequence degenerates (8.18) and $H^*_H \cong H^*_H(\mathcal{N}_x)$ surjects onto $IH^*(\mathcal{N}_x/\!\!/H)$. Therefore the right side spectral sequence also degenerates and we have an isomorphism ([Kir86b] p495)

$$IH^*(\mathcal{N}_U/\!\!/K) \cong \left[H^*\left(M^{ss}_H \cap \phi^{-1}(U)/\!\!/N^H_0\right) \otimes IH^*(\mathcal{N}_x/\!\!/H)\right]^{\pi_0 N^H} \tag{8.26}$$

In view of (8.19) and (8.26), the proof of Proposition 8.6 is complete if we show the following lemma.

Lemma 8.9. *$V^*_{\mathcal{N}_x} \cong IH^*(\mathcal{N}_x/\!\!/H)$.*

Proof. As in the proof of Lemma 8.8, we may assume that $\mathrm{Stab}(x)$ is minimal among the stabilizers of points in $\mu^{-1}(0)$ whose Lie algebra is $\mathrm{Lie}\, H$. So we may use Lemma 4.1 with $\mathcal{N}_x = W$.

As $\mathcal{N}_x/\!\!/H$ is a cone with vertex point $*$, it is well-known that

$$IH^i(\mathcal{N}_x/\!\!/H) = 0$$

for $i \geq n_H$ and if $i < n_H$, we have

$$IH^i(\mathcal{N}_x/\!\!/H) \cong IH^i(\mathcal{N}_x/\!\!/H - *) \cong IH^i(\mathcal{N}_x - \phi_x^{-1}(*)/\!\!/H)$$

where $\phi_x : \mathcal{N}_x \to \mathcal{N}_x/\!\!/H$ is the GIT quotient map and the following diagram commutes:

$$\begin{tikzcd} IH^i(\mathcal{N}_x/\!\!/H) \arrow[r] \arrow[d, "\phi^*_{x,H}"'] & IH^i(\mathcal{N}_x - \phi_x^{-1}(*)/\!\!/H) \arrow[d, "\phi^*_{x,H}"'] \\ H^i_H(\mathcal{N}_x) \arrow[r] & H^i_H(\mathcal{N}_x - \phi_x^{-1}(*)) \end{tikzcd} \tag{8.27}$$

As $\phi_x^{-1}(*)$ is the union of the complex cones over the unstable strata of $\mathbb{P}\mathcal{N}_x$ and the real codimension of each unstable stratum is greater than n_H by the weakly balanced condition, the real codimension of $\phi_x^{-1}(*)$ is greater than n_H. Hence, the bottom horizontal map is an isomorphism.

We claim that the the right vertical in (8.27) is injective and the image is $V^i_{\mathcal{N}_x - \phi_x^{-1}(*)}$. Consider the following commutative diagram

$$\begin{tikzcd} \hat{Y}^{ss} \arrow[r, "\delta"] \arrow[d, "\tilde{\gamma}"'] & Y^{ss} \arrow[d, "\gamma"] \\ \hat{Y}/\!\!/H \arrow[r, "\epsilon"] & Y/\!\!/H \end{tikzcd}$$

we see that

$$\tilde{\gamma}^{-1}\epsilon^{-1}(\mathcal{N}_x/\!\!/H - *) = \delta^{-1}\gamma^{-1}(\mathcal{N}_x/\!\!/H - *) = \delta^{-1}\left(\mathcal{N}_x - \phi_x^{-1}(*)\right) \cong \mathcal{N}_x - \phi_x^{-1}(*).$$

For the last isomorphism, observe that $\mathcal{N}_x - \phi_x^{-1}(*)$ does not intersect with the blow-up center. Moreover, if $y \in \mathcal{N}_x - \phi_x^{-1}(*)$, the closure of $H_{\mathbb{C}} \cdot y$ does not meet 0 (if it does, the point belongs to $\phi_x^{-1}(*)$) and thus y is semistable as a point in $\hat{Y}$ by [Kir85] Remark 7.7.

Because the action of H on $\hat{Y}$ is weakly balanced and $r(\hat{Y}) < r = r(M)$, Theorem 7.7 is true for $\hat{Y}$. In particular, if we apply the theorem for the preimage $\mathcal{N}_x - \phi_x^{-1}(*)$ of the open set $\epsilon^{-1}(\mathcal{N}_x/\!\!/H - *)$ by $\tilde{\gamma}$ we obtain the isomorphism

$$IH^i(\mathcal{N}_x - \phi_x^{-1}(*)/\!\!/H) \cong V^i_{\mathcal{N}_x - \phi_x^{-1}(*)}.$$

Therefore, from (8.27), it suffices to show that

$$V^i_{\mathcal{N}_x} \cong V^i_{\mathcal{N}_x - \phi_x^{-1}(*)}$$

for $i < n_H$ by restriction. To see this, we note once again that for $i < n_H$,

$$H^i_H(\mathcal{N}_x) \to H^i_H(\mathcal{N}_x - \phi_x^{-1}(*))$$

is an isomorphism and the same is true for

$$H^j_{H\cap N_0^L/L}(\mathcal{N}_{x,L}) \otimes H^k_L \to H^j_{H\cap N_0^L/L}(\mathcal{N}_{x,L} - \phi_x^{-1}(*)) \otimes H^k_L$$

if $k \geq n_L$, $j < n_H - k \leq n_H - n_L$, because the action of $H \cap N_0^L/L$ on $\mathcal{N}_{x,L}$ is weakly linearly balanced. $\square$

9. Examples

Let

$$P_t^K(W) = \sum_{i\geq 0} t^i \dim H^i_K(W)$$
$$IP_t(W) = \sum_{i\geq 0} t^i \dim IH^i(W)$$

be the Poincaré series.

9.1. **$\mathbb{C}^*$-action on projective space.** Consider a $\mathbb{C}^*$-action on $M = \mathbb{P}^n$ via a representation $\mathbb{C}^* \to GL(n+1)$. Let n_+, n_0, n_- be the number of positive, zero, negative weights. Suppose the action is weakly balanced, i.e. $n_+ = n_-$. In this case, we can easily compute the intersection Betti nubmers by Theorem 7.7.

From the equivariant Morse theory [Kir84],

(9.1)
$$\begin{aligned} P_t^{S^1}((\mathbb{P}^n)^{ss}) &= \frac{1+t^2+\cdots+t^{2n}}{1-t^2} - \frac{t^{2n_0+2n_+}+\cdots+t^{2n}}{1-t^2} - \frac{t^{2n_0+2n_-}+\cdots+t^{2n}}{1-t^2} \\ &= \frac{1+t^2+\cdots+t^{2n_0+2n_--2} - t^{2n_0+2n_+} - \cdots - t^{2n}}{1-t^2}. \end{aligned}$$

In this case, $\mathcal{R} = \{S^1\}$ and $M^{ss}_{S^1} = \mathbb{P}^{n_0-1}$, $n_H = n_+ + n_- - 1 = 2n_- - 1$.

As $H^*_{S^1}(\mathbb{P}^n) \to H^*_{S^1}((\mathbb{P}^n)^{ss}) \to H^*_{S^1}(\mathbb{P}^{n_0-1})$ is surjective, we have only to subtract out the Poincaré series of $\oplus_{i\geq 2n_-} H^*(\mathbb{P}^{n_0-1}) \otimes H^i_{S^1}$, which is precisely

$$\frac{t^{2n_-}(1+t^2+\cdots+t^{2n_0-2})}{1-t^2}.$$

Therefore,

$$IP_t(\mathbb{P}^n/\!\!/\mathbb{C}^*) = \frac{1+t^2+\cdots+t^{2n_--2}-t^{2n_0+2n_+}-t^{2n_0+2n_++2}\cdots-t^{2n}}{1-t^2}$$

which is a palindromic polynomial of degree $2n-2$.

9.2. **Ordered $2n$-tuples of points of $\mathbb{P}^1$.** Let us consider $G = SL(2)$ action on the set $M = (\mathbb{P}^1)^{2n}$ of ordered $2n$-tuples of points in $\mathbb{P}^1$ as Möbius transformations. Then the semistable $2n$-tuples are those containing no point of $\mathbb{P}^1$ strictly more than n times and the stable points are those containing no point at least n times. Let H be the maximal torus of G. Then

$$\mathcal{R}(M) = \{H\}$$

$$M_H^{ss} = \{q_I | I \subset \{1,2,3,\cdots,2n\}, |I| = n\}$$

where $q_I \in M$, the j-th component of which is ∞ if $j \in I$, 0 otherwise. Therefore, the action is weakly balanced.

The normalizer N^H satisfies $N^H/H = \mathbb{Z}/2$ and $N_0^H = H$. Hence,

$$H^*_{N^H}(M_H^{ss}) = [\oplus_{|I|=n} H_H^*]^{\mathbb{Z}/2}.$$

The $\mathbb{Z}/2$ action interchanges ∞ and 0, i.e. q_I and q_{I^c}, and therefore, from now on, we only think of those I's that contain 1 so that we can forget the $\mathbb{Z}/2$ action.

$H_K^*(M) = H_K^*((\mathbb{P}^1)^{2n})$ has generators $\xi_1, \xi_2, \cdots, \xi_{2n}$ of degree 2 and ρ^2 of degree 4, subject to the relations $\xi_j^2 = \rho^2$ for $1 \le j \le 2n$. The I-th component of the restriction map

$$H_K^*(M) \to H^*_{N^H}(M_H^{ss}) = \oplus_I H_H^*$$

maps ρ^2 to ρ^2 and ξ_j to ρ if $j \in I$, $-\rho$ otherwise. From [Kir84],

$$P_t^K(M^{ss}) = \frac{(1+t^2)^{2n}}{1-t^4} - \sum_{n<r\le 2n} \binom{2n}{r} \frac{t^{2(r-1)}}{1-t^2}.$$

Proposition 9.1.

$$IP_t(M/\!\!/G) = P_t^K(M^{ss}) - \frac{1}{2}\binom{2n}{n}\frac{t^{2n-2}}{1-t^2}$$

Proof. By Theorem 7.7, we have only to subtract out the Poincaré series of

$$Im\{H_K^*(M^{ss}) \to \oplus_I H_H^*\} \ \cap \ \{\oplus_I \oplus_{i \ge n_H} H_H^i\}$$

where n_H is in this case $2n-3$. By the lemma below, which is essentially combinatorial, the image contains $\oplus_I \oplus_{i \ge 2n-3} H_H^i$ and thus the intersection is $\oplus_I \oplus_{i > 2n-3} H_H^i$, whose Poincaré series is precisely

$$\frac{1}{2}\binom{2n}{n}\frac{t^{2n-2}}{1-t^2}.$$

So we are done. □ □

Lemma 9.2. *The restriction map $H_K^{2k}(M^{ss}) \to \oplus_I H_H^{2k}$ is surjective for $k \ge n-1$.*

Proof. It is equivalent to show that $H_K^{2k}(M) \to \oplus_I H_H^{2k}$ is surjective. Let $\xi = \xi_2 + \cdots + \xi_{2n}$ and consider, for each $I = (1, i_2, \cdots, i_n)$,

$$\eta_I = (\xi - \xi_{i_2})^{k_2} \cdots (\xi - \xi_{i_n})^{k_n}.$$

Then since $\xi|_{q_J} = -\rho$ for all J, $\eta_I|_{q_J} = (-2\rho)^k$ if $J = I$ and 0 otherwise, where $k = k_2 + \cdots + k_n$, $k_i \geq 1$. Therefore, the images of those η_I span $\oplus_I H_H^{2k}$ for any $k \geq n-1$ and thus the restriction is surjective. □ □

9.3. **Intersection pairing.** Consider the $\mathbb{C}^*$ action on $\mathbb{P}^7$ by a representation with weights $+1, 0, -1$ of multiplicity $3, 2, 3$ respectively. Hence, $n_+ = n_- = 3$, $n_0 = 2$. Then

$$H^*_{S^1}(\mathbb{P}^7) = \mathbb{C}[\xi, \rho]/\langle \xi^2(\xi-\rho)^3(\xi+\rho)^3\rangle$$

where ξ is a generator in $H^2(\mathbb{P}^7)$ and ρ is a generator in $H^2_{S^1}$.

The equivariant Euler classes for the two unstable strata are $\xi^2(\xi-\rho)^3$, $\xi^2(\xi+\rho)^3$ respectively. Therefore,

$$H^*_{S^1}((\mathbb{P}^7)^{ss}) = \mathbb{C}[\xi, \rho]/\langle \xi^2(\xi-\rho)^3,\, \xi^2(\xi+\rho)^3\rangle$$

A Gröbner basis for the relation ideal is

$$\{\xi^5 + 3\xi^3\rho^2, \xi^4\rho + \frac{1}{3}\xi^2\rho^3, \xi^3\rho^3, \xi^2\rho^5\}$$

where $\xi > \rho$. Hence as a vector space,

$$H^*_{S^1}((\mathbb{P}^7)^{ss}) = \mathbb{C}\{\xi^i\rho^j | i = 0, 1,\, j \geq 0\} \oplus \mathbb{C}\{\xi^i\rho^j | 2i + j < 9, i \geq 2, j \geq 0\}$$

By definition, as $n_{S^1} = 5$, we remove $\mathbb{C}\{\xi^i\rho^j | i = 0, 1,\, j \geq 3\}$ to get

$$V = \oplus_{0 \leq i \leq 6} V^{2i}$$

$$V^0 = \mathbb{C}, \quad V^2 = \mathbb{C}\{\rho, \xi\}, \quad V^4 = \mathbb{C}\{\rho^2, \xi\rho, \xi^2\},$$

$$V^6 = \mathbb{C}\{\xi\rho^2, \xi^2\rho, \xi^3\}, \quad V^8 = \mathbb{C}\{\xi^2\rho^2, \xi^3\rho, \xi^4\},$$

$$V^{10} = \mathbb{C}\{\xi^2\rho^3, \xi^3\rho^2\}, \quad V^{12} = \mathbb{C}\{\xi^2\rho^4\}.$$

First, consider the pairing $V^2 \otimes V^{10} \to V^{12}$. As $\rho(\xi^2\rho^3) = \xi^2\rho^4$, $\rho(\xi^3\rho^2) = \xi^3\rho^3 = 0$, $\xi(\xi^2\rho^3) = \xi^3\rho^3 = 0$, $\xi(\xi^3\rho^2) = \xi^4\rho^2 = -\frac{1}{3}\xi^2\rho^4$, the pairing matrix is up to a constant

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

The determinant is $-\frac{1}{3} \neq 0$ and the signature is 0.

Next, consider the pairing $V^4 \otimes V^8 \to V^{12}$. One can similarly use the Gröbner basis to compute the pairing as above. The pairing matrix is up to a constant

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

The determinant is $-\frac{8}{27} \neq 0$ and the signature is 1.

Similarly, the intersection pairing matrix for $V^6 \otimes V^6 \to V^{12}$ is up to a constant

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

The determinant is $-\frac{8}{27} \neq 0$ and the signature is 1.

In this way, one can compute the intersection pairing for any $n_0, n_- = n_+$.

In a subsequent paper, we will compute the intersection pairing of the moduli spaces of holomorphic vector bundles over a Riemann surface of any rank and degree, using the nonabelian localization theorem of Jeffrey and Kirwan.

References

[AB82] M.F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. *Phil. Trans. Roy. Soc. Lond.*, A308:532–615, 1982.

[B+84] A. Borel et al. *Intersection cohomology.* Number 50 in Progress in mathematics. Birkhäuser, 1984.

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Astérisque*, 100, 1982. Proc. C.I.R.M. conférence: Analyse et topologie sur les espaces singuliers.

[BL94] J. Bernstein and V. Lunts. *Equivariant sheaves and functors.* Lecture Notes in Math. 1578, Springer, 1994.

[GeMa] S. Gelfand and Y. Manin. *Methods of homological algebra.* Springer-Verlag, 1996.

[GH94] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley, 1994.

[GM80] M. Goresky and R. MacPherson. Intersection homology theory I. *Topology*, 19:135–162, 1980.

[GM83] M. Goresky and R. MacPherson. Intersection homology theory II. *Inventiones Mathematicae*, 71:77–129, 1983.

[GM85] M. Goresky and R. MacPherson. Lefschetz fixed point theorem for intersection homology. *Comment. Math. Helvetici*, 60:366–391, 1985.

[Jef94] L.C. Jeffrey. Extended moduli spaces of flat connections on Riemann surfaces. *Math. Annalen*, 298:667–692, 1994.

[JK98] L.C. Jeffrey and F.C. Kirwan. Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface. *Ann. Math.*, 148:109–196, 1998.

[Kie] Y.-H. Kiem. Intersection cohomology of representation spaces of surface groups. Preprint.

[Kir84] F. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry.* Number 34 in Mathematical Notes. Princeton University Press, 1984.

[Kir85] F. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Annals of Mathematics*, 122:41–85, 1985.

[Kir86a] F. Kirwan. On the homology of compactifications of moduli spaces of vector bundles over a riemann surface. *Proc. Lon. Math. Soc.*, 53:237–266, 1986.

[Kir86b] F. Kirwan. Rational intersection homology of quotient varieties. *Inventiones Mathematicae*, 86:471–505, 1986.

[Kir92] F. Kirwan. The cohomology rings of moduli spaces of bundles over riemann surfaces. *Jour. A.M.S.*, 5:853–906, 1992.

[Kir94] F. Kirwan. Geometric invariant theory and the Atiyah-Jones conjecture. *Proc. S. Lie Mem. Conf.*, pages 161–188, 1994.

[KW] Y.-H. Kiem and J. Woolf. The cosupport axiom, equivariant cohomology and the intersection cohomology of certain symplectic quotients. Preprint.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* Springer–Verlag, third edition, 1994.

[MS99] E. Meinrenken and R. Sjamaar. Singular reduction and quantization. *Topology*, 38(4):699–762, 1999.

[New78] P. Newstead. *Introduction to moduli problems and orbit spaces.* Tata institute of fundamental research, Bombay, 1978.

[Sja95] R. Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. Math.*, 141:87–129, 1995.

[SL91] R. Sjamaar and E. Lerman. Stratified symplectic spaces and reduction. *Annals of Maths*, 134:375–422, 1991.

kiem@math.snu.ac.kr; Department of Mathematics, Seoul National University, Seoul, 151-747, Korea