ON THE EXISTENCE OF A SYMPLECTIC DESINGULARIZATION OF SOME MODULI SPACES OF SHEAVES ON A K3 SURFACE

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1. Introduction

Let X be a projective K3 surface with generic polarization $\mathcal{O}_X(1)$ and let $M_c = M(2,0,c)$ be the moduli space of semistable torsion-free sheaves on X of rank 2, with Chern classes $c_1 = 0$ and $c_2 = c$. When $c = 2n \geq 4$ is even, M_c is a singular projective variety. Recently O'Grady raised the following question ([OGr99] 0.1).

Question 1.1. *Does there exist a symplectic desingularization of M_{2n}?*

In [OGr99], he analyzes Kirwan's desingularization $\widehat{M}_c$ of M_c and proves that $\widehat{M}_c$ can be blown down twice and that as a result he gets a symplectic desingularization $\widetilde{M}_c$ of M_c in the case when $c = 4$. This turns out to be a new irreducible symplectic variety.

When $c \geq 6$, O'Grady conjectures that there is no smooth symplectic model of M_c ([OGr99] page 50). The purpose of this paper is to provide a partial answer to Question 1.1.

Theorem 1.2. *There is no symplectic desingularization of M_{2n} if $\frac{n\,a_n}{2n-3}$ is not an integer where a_n is the Euler number of the Hilbert scheme $X^{[n]}$ of n points in X.*

It is well-known that a_n is given by the equation

$$\sum_{n=0}^{\infty} a_n q^n = \prod_{n=1}^{\infty} 1/(1-q^n)^{24}.$$

By direct computation, one can check that $\frac{n\,a_n}{2n-3}$ is not an integer for $n =$ 5, 6, 8, 11, 12, 13, 15, 16, 17, 18, 19, 20,$\cdots$.

The idea of the proof is to use properties of the stringy Euler numbers. If there is an irreducible symplectic desingularization $\widetilde{M}_c$ of M_c, then the stringy Euler number of M_c is equal to the ordinary Euler number of $\widetilde{M}_c$ because the canonical divisors of both $\widetilde{M}_c$ and M_c are trivial (Theorem 2.2). In particular, we deduce that the stringy Euler number $e_{st}(M_c)$ must be an integer. Therefore, Theorem 1.2 is a consequence of the following.

Proposition 1.3. *The stringy Euler number $e_{st}(M_{2n})$ is of the form*

$$\frac{n\,a_n}{2n-3} + \textit{ integer }.$$

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We prove this proposition in section 3 after a brief review of preliminaries.

One motivation for Question 1.1 is to find a mathematical interpretation of Vafa-Witten's formula ([VW94] 4.17) which says that the "Euler characteristic" of M_{2n} is

$$e^{VW}(M_{2n}) = a_{4n-3} + \frac{1}{4}a_n.$$

Because $k/4 \neq l/(2n-3)$ for $1 \leq k \leq 3$, $1 \leq l < 2n-3$, we deduce the following from Proposition 1.3.

Corollary 1.4. *The stringy Euler number $e_{st}(M_{2n})$ is not Vafa-Witten's Euler characteristic $e^{VW}(M_{2n})$ in general.*

Independently, Kaledin and Lehn in [KaL04] prove that there is no symplectic desingularization of M_{2n} for any $n \geq 3$ by a very different method.

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2. Preliminaries

In this section, we recall the definition and basic facts about stringy Euler numbers. The references are [Bat98, DL99].

Let W be a variety with at worst *log-terminal* singularities, i.e.

- W is $\mathbb{Q}$-Gorenstein
- for a resolution of singularities $\rho : V \to W$ such that the exceptional locus of ρ is a divisor D whose irreducible components $D_1, \cdots, D_r$ are smooth divisors with only normal crossings, we have
$$K_V = \rho^* K_W + \sum_{i=1}^r a_i D_i$$
with $a_i > -1$ for all i, where D_i runs over all irreducible components of D. The divisor $\sum_{i=1}^r a_i D_i$ is called the *discrepancy divisor.*

For each subset $J \subset I = \{1, 2, \cdots, r\}$, define $D_J = \cap_{j \in J} D_j$, $D_\emptyset = Y$ and $D_J^0 = D_J - \cup_{j \in I-J} D_j$. Then the stringy E-function of W is defined by

$$E_{st}(W; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \tag{2.1}$$

where

$$E(Z; u, v) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(Z; \mathbb{C})) u^p v^q$$

is the Hodge-Deligne polynomial for a variety Z. Note that the Hodge-Deligne polynomials have

- the additive property: $E(Z; u, v) = E(U; u, v) + E(Z - U; u, v)$ if U is a smooth open subvariety of Z
- the multiplicative property: $E(Z; u, v) = E(B; u, v)\, E(F; u, v)$ if Z is a locally trivial F-bundle over B.

Definition 2.1. The stringy Euler number is defined as

$$e_{st}(W) = \lim_{u,v \to 1} E_{st}(W; u, v) = \sum_{J \subset I} e(D_J^0) \prod_{j \in J} \frac{1}{a_j + 1} \tag{2.2}$$

where $e(D_J^0) = E(D_J^0; 1, 1)$.

The "change of variable formula" (Theorem 6.27 in [Bat98], Lemma 3.3 in [DL99]) implies that the function E_{st} is independent of the choice of a resolution and the following holds.

Theorem 2.2. ([Bat98] Theorem 3.12) *Suppose W is a $\mathbb{Q}$-Gorenstein algebraic variety with at worst log-terminal singularities. If $\rho : V \to W$ is a crepant desingularization (i.e. $\rho^* K_W = K_V$) then $E_{st}(W; u, v) = E(V; u, v)$. In particular, $e_{st}(W) = e(V)$ is an integer.*

3. Proof of Proposition 1.3

We fix a generic polarization of X as in [OGr99] page 50. The moduli space M_{2n} has a stratification

$$M_{2n} = M_{2n}^s \sqcup (\Sigma - \Omega) \sqcup \Omega$$

where M_{2n}^s is the locus of stable sheaves and

$$\Sigma \cong (X^{[n]} \times X^{[n]})/\text{involution}$$

is the locus of sheaves of the form $I_Z \oplus I_{Z'}$ ($[Z], [Z'] \in X^{[n]}$) while

$$\Omega \cong X^{[n]}$$

is the locus of sheaves $I_Z \oplus I_Z$. Kirwan's desingularization $\rho : \widehat{M}_{2n} \to M_{2n}$ is obtained by blowing up M_c first along the deepest stratum Ω, next along the proper transform of the middle stratum Σ and finally along the proper transform of a subvariety Δ in the exceptional divisor of the first blow-up which is the locus of $\mathbb{Z}_2$ quotient singularities [Kir85]. This is indeed a desingularization by [OGr99] Proposition 1.8.3.

Let $D_1 = \widehat{\Omega}$, $D_2 = \widehat{\Sigma}$, $D_3 = \widehat{\Delta}$ be the (proper transforms of the) exceptional divisors of the three blow-ups. Then they are smooth divisors with only normal crossings and the discrepancy divisor of $\rho : \widehat{M}_{2n} \to M_{2n}$ is ([OGr99] 6.1)

$$(6n-7)D_1 + (2n-4)D_2 + (4n-6)D_3.$$

Therefore the singularities are terminal for $n \geq 2$ and from (2.2) the stringy Euler number of M_{2n} is given by

$$\begin{array}{l} e(M_{2n}^s) + e(D_1^0)\frac{1}{6n-6} + e(D_2^0)\frac{1}{2n-3} + e(D_3^0)\frac{1}{4n-5} \\ +e(D_{12}^0)\frac{1}{6n-6}\frac{1}{2n-3} + e(D_{23}^0)\frac{1}{2n-3}\frac{1}{4n-5} \\ +e(D_{13}^0)\frac{1}{6n-6}\frac{1}{4n-5} + e(D_{123}^0)\frac{1}{6n-6}\frac{1}{2n-3}\frac{1}{4n-5} \quad . \end{array} \tag{3.1}$$

We need to compute the (virtual) Euler numbers of D_J^0 for $J \subset \{1, 2, 3\}$. Let (E, ω) be a symplectic vector space of dimension $c = 2n$. Let $\mathrm{Gr}^\omega(k, c)$ be the Grassmannian of k dimensional subspaces of E isotropic with respect to the symplectic form ω (i.e. the restriction of ω to the subspace is zero).

Lemma 3.1. *For $k \leq n$, the Euler number of $\mathrm{Gr}^\omega(k, 2n)$ is $2^k \binom{n}{k}$.*

Proof. Consider the incidence variety

$$\{(a, b) \in \mathrm{Gr}^\omega(k-1, 2n) \times \mathrm{Gr}^\omega(k, 2n) \mid a \subset b\}.$$

This is a $\mathbb{P}^{2n-2k+1}$-bundle over $\mathrm{Gr}^\omega(k-1, 2n)$ and a $\mathbb{P}^{k-1}$-bundle over $\mathrm{Gr}^\omega(k, 2n)$. The formula follows from an induction on k. ◻

Let $\hat{\mathbb{P}}^5$ be the blow-up of $\mathbb{P}^5$ (projectivization of the space of 3×3 symmetric matrices) along $\mathbb{P}^2$ (the locus of rank 1 matrices). We have the following from [OGr99] §6 and [OGr97] §3.

Proposition 3.2. (1) *D_1 is a $\hat{\mathbb{P}}^5$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*
(2) *D_2^0 is a $\mathbb{P}^{2n-4}$-bundle over a $\mathbb{P}^{2n-3}$-bundle over $(X^{[n]}\times X^{[n]}-X^{[n]})$/involution.*
(3) *D_3 is a $\mathbb{P}^{2n-4}\times\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(4) *$D_1\cap D_2$ is a $\mathbb{P}^2\times\mathbb{P}^2$-bundle over $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*
(5) *$D_2\cap D_3$ is a $\mathbb{P}^{2n-4}\times\mathbb{P}^1$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(6) *$D_1\cap D_3$ is a $\mathbb{P}^2\times\mathbb{P}^{2n-5}$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(7) *$D_1\cap D_2\cap D_3$ is a $\mathbb{P}^1\times\mathbb{P}^{2n-5}$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*

For instance, (1) is just Proposition 6.2 of [OGr99] and (2) is Proposition 3.3.2 of [OGr97] while (3) is Lemma 3.5.4 in [OGr97].

From Proposition 3.2 and Lemma 3.1, we have the following by the additive and multiplicative properties of the (virtual) Euler numbers:

$$\begin{array}{ll} e(D_1^0)=0, & e(D_2^0)=(2n-3)(2n-2)\frac{1}{2}(a_n^2-a_n),\\ e(D_3^0)=2^2\binom{n}{2}a_n, & e(D_{12}^0)=3\cdot 2^3\binom{n}{3}a_n\\ e(D_{23}^0)=2\cdot 2^2\binom{n}{2}a_n, & e(D_{13}^0)=(2n-4)2^2\binom{n}{2}a_n\\ e(D_{123}^0)=2(2n-4)2^2\binom{n}{2}a_n & . \end{array}$$

Hence from the formula (3.1), the stringy Euler number of M_{2n} is given by

$$e_{st}(M_{2n})=e(M_{2n}^s)+(n-1)(a_n^2-a_n)+n\frac{2n-2}{2n-3}a_n=\frac{n\,a_n}{2n-3}+\text{ integer}$$

since $e(M_{2n}^s)$ is an integer. So we proved Proposition 1.3.

Remark 3.3. For the moduli space of rank 2 bundles over a smooth projective curve, the stringy E-function and the stringy Euler number are computed in [Kie03] and [KL04].

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