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Low degree GW invariants of surfaces II

To Fabrizio Catanese on the Occasion of his 60th Birthday

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Abstract We prove a conjectural formula of Maulik-Pandharipande on the degree one and two GW invariants of a surface with a smooth canonical divisor. We use the method of degeneration and the localized GW invariants introduced by the authors.

Keywords Gromov-Witten invariant, cosection, degeneration formula, localized virtual cycle

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1 Introduction

In this paper, we continue our study of the GW-invariants of algebraic surfaces with positive p_g. We will prove a deformation invariance of localized GW-invariants of surfaces, prove a degeneration formula of localized GW-invariants of spin surfaces, and in the end prove the formulas of low degree GW-invariants of surfaces with positive p_g conjectured by Maulik-Pandharipande.

In [7], in our attempt to understand Lee-Parker's work [16] on GW-invariants of Kähler surfaces with positive p_g, the authors constructed the cosection localized virtual class for a DM stack with a perfect obstruction theory and a cosection of its obstruction sheaf; the algebro-geometric construction of such cycles was completed after constructing the algebraic localized Gysin map [8]. Applied to the moduli of stable morphisms to surfaces, this reproduces Lee-Parker's localization of GW-invariants of surfaces with positive p_g.

Given an algebraic surface X, a holomorphic two-form $\theta \in \Gamma(\Omega^2_X)$ induces a cosection (i.e., a homomorphism to the structure sheaf) of the obstruction sheaf

$$\sigma_\theta : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet}$$

of the moduli of stable morphisms to X of not necessarily connected domains and of fundamental class $\beta \in H_2(X,\mathbb{Z})$ (cf. Definition 2.1). Let $Z(\sigma_\theta)$ be the non-surjective locus of σ_θ. On the one hand, the cosection localized virtual class (cf. (2.3))

$$[\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* Z(\sigma_\theta)$$

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coincides with the ordinary virtual class $[\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet]^{\mathrm{vir}}$, after applying the push-forward $A_*Z(\sigma_\theta)\to A_*\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$. On the other hand, since $Z(\sigma_\theta)$ consists of $[u,C]\in\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ such that $u(C)\subset\theta^{-1}(0)$ (cf. Lemma 2.3), it points to that localized virtual cycle only depends on the infinitesimal structure of X near $D=\theta^{-1}(0)$.

In this paper, we prove a deformation invariance result. It applies to cases when $\mathcal{X}\to T=\mathbb{A}^1$ is the deformation of X to the normal bundle $N_{D/X}$ of a smooth canonical divisor $D\subset X$, or when $\mathcal{X}\to T$ is a family of spin surfaces (they are total spaces of theta characteristics over smooth curves (cf. Example 2.5)). In each case we have a tautological holomorphic two-form $\Theta\in\Gamma(\Omega^2_{\mathcal{X}})$.

Let $\mathcal{X}\to T$ be a family as indicated. For $c\in T$ a closed point and a class $\beta\neq 0\in H_2(\mathcal{X}_c,\mathbb{Z})$, we let $\Theta_c\in\Gamma(\Omega^2_{\mathcal{X}_c})$ be the pull-back of Θ to $\mathcal{X}_c$; we form the moduli spaces of stable morphisms $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X},\beta)^\bullet$ and $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_c,\beta)^\bullet$. Using the two-form Θ, and applying [8, Sect. 6], we obtain cosections σ_Θ and σ_{Θ_c} of the respective obstruction sheaves of $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X},\beta)^\bullet$ and $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_c,\beta)^\bullet$, and then their respective cosection localized virtual classes.

Theorem 1.1. *Let the notation be as stated, let $Z(\sigma_{\Theta_c})$ be the non-surjective locus of σ_{Θ_c}, and let $\iota_c: c\to T$ be the inclusion. Then we have*

$$[\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_c,\beta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}=\iota_c^![\overline{\mathcal{M}}_{\chi,n}(\mathcal{X},\beta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*Z(\sigma_{\Theta_c}).$$

The corresponding invariance in GW-invariants was proved in [16].

Given a smooth curve D and a theta characteristic L over D, we form the surface S, called a spin surface, that is the total space of L; S has a tautological holomorphic two-form $\theta\in\Gamma(\Omega^2_S)$. Using its cosection localized virtual cycle $[\overline{\mathcal{M}}_{\chi,n}(S,d[D])^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$, $d>0$, we can form the localized GW-invariants with descendants

$$\langle\tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_n}(\gamma_n)\rangle^{S,\bullet}_{\chi,d,\mathrm{loc}},\quad \gamma_i\in H^*(S,\mathbb{Z}).$$

(We use d to mean the class $d[D]$.) The deformation invariance implies that the GW-invariants of an algebraic surface X having a smooth canonical divisor D are equal to the GW-invariants of S with the choice $L=N_{D/X}$.

Conjecture 1.2. *Let X be a smooth minimal general type surface with positive $p_g>0$. Its GW-invariants $\langle\cdots\rangle^{X,\bullet}_{\beta,g}$ vanish unless β is a non-negative integral multiple of $c_1(K_X)$. In case $\beta=dc_1(K_X)$ for an integer $d>0$, we let (D,L) be a pair of a smooth projective curve of genus K_X^2+1 and its theta characteristic with parity $\chi(\mathcal{O}_X)$, and let S be the total space of L. Then there is a canonical homomorphism $\rho: H^*(X,\mathbb{Z})\to H^*(S,\mathbb{Z})$ so that for any classes $\gamma_i\in H^*(X,\mathbb{Z})$ and integers $\alpha_i\geqslant 0$, $i=1,\dots,n$,*

$$\langle\tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_n}(\gamma_n)\rangle^{X,\bullet}_{\chi,\beta}=\langle\tau_{\alpha_1}(\rho(\gamma_1))\cdots\tau_{\alpha_n}(\rho(\gamma_n))\rangle^{S,\bullet}_{\chi,d,\mathrm{loc}}.$$

This conjecture was proved by Lee-Parker [16] when X has a smooth canonical divisor.

The main technical part of our paper is to prove a degeneration formula for spin surfaces. Let S be a spin surface that is the total space of a theta characteristic L over a smooth curve D. We pick a point $q\in D$ and let $E=S\times_D q$ (the fiber of S over q); we then form $\mathcal{X}\to\mathbb{A}^1$ the blowing up of $S\times\mathbb{A}^1$ along $E\times 0\subset S\times\mathbb{A}^1$. The family $\mathcal{X}\to\mathbb{A}^1$ has general fibers S and special fiber (over $0\in\mathbb{A}^1$) the union of two surfaces: one is S, which we denote by Y_1, and the other is $E\times\mathbb{P}^1$, which we denote by Y_2. Note that Y_1 and Y_2 intersect transversally along $E=Y_1\cap Y_2$.

We fix integers $\alpha_1,\dots,\alpha_n\in\mathbb{Z}^{\geqslant 0}$. Let $\mu=(\mu_1,\dots,\mu_{\ell(\mu)})$ be a partition of d. Using the holomorphic two-form θ on $S=Y_1$, we define in Section 5 a localized relative GW-invariants of the pair (Y_1,E):

$$\left\langle\prod_{j=1}^n\tau_{\alpha_j}(\gamma_j)\right\rangle^{Y_1/E,\bullet}_{\chi,\mu,\mathrm{loc}}\in A_*(e^{\ell(\mu)}),\quad \gamma_j\in H^*(S,\mathbb{Z}),$$

where e denotes the intersection of E with the zero section of $Y_1\to D$.

We fix a splitting $n_1+n_2=n$, and adopt the intersection pairing

$$\star: A_*E^l\times A_*e^l\longrightarrow\mathbb{Z},\quad [E^l]\cdot[\mathrm{pt}]=1.$$

Theorem 1.3. *Let the situation be as stated, and let $\gamma_i \in H^{\geqslant 1}(\mathcal{X})$ be classes such that for $\iota_i : Y_i \to \mathcal{X}$ the inclusion, $\iota_1^*(\gamma_i) = 0$ for $i > n_1$ and $\iota_2^*(\gamma_i) = 0$ for $i \leqslant n_1$. Then we have the degeneration formula*

$$\left\langle \prod_{j=1}^{n} \tau_{\alpha_j}(\gamma_j) \right\rangle_{\chi,d,\mathrm{loc}}^{S,\bullet} = \sum \frac{\mu!}{|\operatorname{Aut}(\mu)|} \cdot \left\langle \prod_{j=1}^{n_1} \tau_{\alpha_j}(\gamma_j) \right\rangle_{\chi_1,\mu,\mathrm{loc}}^{Y_1/E,\bullet} \star \left\langle \prod_{j=n_1+1}^{n} \tau_{\alpha_j}(\gamma_j) \right\rangle_{\chi_2,\mu}^{Y_2/E,\bullet},$$

where $\langle \prod_{j=n_1+1}^{n} \tau_{\alpha_j}(\gamma_j)\rangle_{\chi_2,\mu}^{Y_2/E,\bullet}$ is the ordinary relative GW-invariant of the pair (Y_2, E), and the summation is over all possible partitions $\mu \vdash d$ and $\chi = \chi_1 + \chi_2 - \ell(\mu)$.

In the end, by applying this degeneration formula, and a calculation of low degree GW invariants of surfaces [9], we prove the following low degree formulas originally conjectured by Maulik and Pandharipande [21].

Theorem 1.4. *Let $X \to D$ be a theta characteristic over a smooth curve of genus h. Let $\gamma \in H^2(D, \mathbb{Z})$ be the Poincaré dual of a point in D. Then the degree one and two GW-invariants with descendants are*

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{[D],\mathrm{loc}}^{X,\bullet} = (-1)^{h^0(L)} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{-\alpha_i}; \tag{1.1}$$

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{2[D],\mathrm{loc}}^{X,\bullet} = (-1)^{h^0(L)}\, 2^{h+n-1} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}. \tag{1.2}$$

The paper is organized as follows. In Section 2, the localized GW-invariants of not necessarily connected domains are introduced. Section 3 is devoted to prove the deformation invariance. In Section 4 and Section 5, we introduce localized relative GW-invariants of spin surfaces; in Section 6 we prove the degeneration formula of GW-invariants of spin curves. Finally, in Section 7 we prove the formulas (1.1) and (1.2).

Remarks on preprint [7]**.** In [16], Lee-Parker proved that a holomorphic two-form θ on a Kähler surface localized its GW-invariants to a neighborhood of the locus of $(\theta = 0)$. They conjectured a general form of the GW-invariants of surfaces with non-trivial holomorphic two-forms θ, and proved that a universal formula in case $(\theta = 0)$ is smooth.

In searching for an algebro-geometric analogue of Lee-Parker's work, the authors constructed the *cosection localized virtual cycles*. This construction was presented in the preprint [7], which contains a reduction of virtual normal cone at the presence of a cosection of obstruction sheaf, and defines the cosection localized virtual cycle in Borel-Moore homology using an analytic version of localized Gysin map. In the same preprint, they phrased a more general version of conjecture on the structure of GW-invariants of surfaces with holomorphic two-forms (cf. Conjecture 1.2), and proved the Maulik-Pandharipande formulas (Theorem 1.4) of low-degree GW-invariants of surfaces, *assuming* the degeneration formula (Theorem 1.3).

Shortly after, in [14] Lee proved the Maulik-Pandharipande's formulas of low degree GW-invariants by developing a degeneration formula of localized GW-invariants of surfaces in symplectic geometry.

Later, the authors constructed the algebraic version of the localized Gysin map in [8]. Because the construction of cosection localized virtual cycles has shown larger potential in applications in a wider range of problems, we have grouped the basic construction of cosection localized virtual cycles, the algebraic construction of localized Gysin map, and the application to GW-invariants of surfaces with holomorphic two-forms in the preprint [8].

The part of [7] on explicit calculation of low degree GW-invariants of surfaces, and the formula relating localized GW-invariants of spin surfaces with the twisted GW-invariants of surfaces form the preprint [9].

In this paper, we prove the deformation invariance, and the degeneration formula put forward in [7]. The proof is made possible after the algebraic construction of localized Gysin map.

2 Localized GW-invariants

Let X be a smooth complex quasi-projective variety and $\beta \in H_2(X,\mathbb{Z})$ be a curve class. In this paper, we focus on the moduli space of stable morphisms to X of not necessarily connected domains.

Definition 2.1. [23] *We define $\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ as the moduli stack of stable morphisms $u: C \to X$ of not necessarily connected n-pointed nodal curves C of Euler characteristic $\chi(\mathcal{O}_C)=\chi$ and of fundamental class $u_*([C])=\beta$, such that the restriction of u to each connected component of C is non-constant. We call u stable when the automorphism group of u is finite.*

Let θ be a non-trivial holomorphic two-form on X. The construction of localized GW-invariants in [7] can be applied to this moduli space. We divide $\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ into a disjoint union

$$\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet = \coprod_{r\geqslant 1} \mathcal{M}_r,$$

where $\mathcal{M}_r$ consists of stable morphisms $[u,C]\in \overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ whose domains C have r connected components. Let $(\pi_r,f_r):\mathcal{C}_r \to \mathcal{M}_r\times X$ be the universal family on $\mathcal{M}_r$. Then the cosection of the obstruction sheaf of $\mathcal{M}_r$ was constructed as a lift of (cf. [8, Sect. 6])

$$R^1\pi_{r*}f_r^*T_X \xrightarrow{\wedge f_r^*\theta} R^1\pi_{r*}f_r^*\Omega_X \xrightarrow{f_r^*} R^1\pi_{r*}\omega_{\mathcal{C}_r/\mathcal{M}_r} \xrightarrow{\psi_r} \mathcal{O}_{\mathcal{M}_r}. \tag{2.1}$$

When $r=1$, the last homomorphism ψ_r is the Serre duality $R^1\pi_{r*}\omega_{\mathcal{C}_r/\mathcal{M}_r}\cong \mathcal{O}_{\mathcal{M}_r}$. For $r\geqslant 2$, at each closed $\xi=[u,C]\in\mathcal{M}_r$, we have

$$R^1\pi_{r*}\omega_{\mathcal{C}_r/\mathcal{M}_r}\otimes_{\mathcal{O}_{\mathcal{M}_r}}\mathbf{k}(\xi)\cong \mathbf{k}^{\oplus r}, \tag{2.2}$$

where the summands are indexed by the connected components of C. In this case, we define ψ_r so that $\psi|_\xi:\mathbf{k}^{\oplus r}\to\mathbf{k}$ is the summation map. Since the summation is independent of the indexing of the summands of $\mathbf{k}^{\oplus r}$, it extends to a global homomomorphism ψ_r as in the sequence (2.1).

Proposition 2.2. *The composite* (2.1) *lifts to a cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$*

$$\sigma_\theta : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet} \to \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet}.$$

Proof. Since $\mathcal{M}_r\subset\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ is both open and closed, we only need to show that σ_θ exists over each $\mathcal{M}_r$.

We let $\mathfrak{M}_{\chi,n}$ be the Artin stack of n-pointed not necessarily connected nodal curves. By forgetting the maps, we obtain $q_r:\mathcal{M}_r\longrightarrow\mathfrak{M}_{\chi,n}$. The relative obstruction theory of $\mathcal{M}_r\to\mathfrak{M}_{\chi,n}$ is given by

$$\big(R^\bullet\pi_{r*}f_r^*T_X\big)^\vee \longrightarrow L_{\mathcal{M}_r},$$

where $L_{\mathcal{M}_r}$ is the cotangent complex of $\mathcal{M}_r$. Its relative obstruction sheaf is $R^1\pi_{r*}f_r^*T_X$, and its (absolute) obstruction sheaf fits into the exact sequence

$$q_r^*\Omega^\vee_{\mathfrak{M}_{\chi,n}} \longrightarrow R^1\pi_{r*}f_r^*T_X \longrightarrow \mathcal{O}b_{\mathcal{M}_r}\longrightarrow 0.$$

See [8, Sect. 6] for details.

Repeating the argument in [8, Sect. 6], we see that the arrow $R^1\pi_{r*}f_r^*T_X\to R^1\omega_{\mathcal{C}_r/\mathcal{M}_r}$ in (2.1) factors through

$$\mathcal{O}b_{\mathcal{M}_r}\longrightarrow R^1\pi_{r*}\omega_{\mathcal{C}_r/\mathcal{M}_r}.$$

Composed with the ψ_r constructed using the rule stated after (2.2), we obtain the desired cosection. ☐

We now describe the locus $Z(\sigma_\theta)\subset\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$ where σ fails to be surjective. We call a stable map $u: C\to X$ *θ-null* if the composite

$$u^*(\theta)\circ du : T_{C_{\mathrm{reg}}}\longrightarrow u^*T_X|_{C_{\mathrm{reg}}}\longrightarrow u^*\Omega_X|_{C_{\mathrm{reg}}}$$

is trivial over the regular locus C_{reg} of C.

Lemma 2.3. [8] *The locus $Z(\sigma_\theta)$ (i.e., the non-surjective locus of σ_θ) consists of all θ-null stable morphisms in $\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$.*

Proof. This is proved in [8, Prop. 6.4] for $\mathcal{M}_1 \subset \overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$. The case $\mathcal{M}_{r\geqslant 2}$ is similar, using that for every $[u,C] \in \mathcal{M}_r$, u restricted to each connected component of C is non-constant. We omit the details for it is a special case of Proposition 5.6. □

Using the cosection σ_θ and applying [8], we obtain a cosection localized virtual cycle

$$[\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* Z(\sigma_\theta). \tag{2.3}$$

In case $Z(\sigma_\theta)$ is proper, one defines the localized GW-invariants of the pair

$$(\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet, \sigma_\theta)$$

by pairing $[\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$ with the cohomology classes of $\overline{\mathcal{M}}_{\chi,n}(X,\beta)^\bullet$.

Application to surfaces is particularly interesting.

Example 2.4. [16] When X is a smooth algebraic surface and $\theta \in H^0(\Omega_X^2)$ is a non-zero holomorphic two form. Let $D = (\theta = 0) \subset X$. Then for $\beta \neq 0$, $Z(\sigma_\theta)$ is the union of $\overline{\mathcal{M}}_{\chi,n}(D,\beta')^\bullet$, where $\beta' \in H_2(D,\mathbb{Z})$ run through all classes such that $\iota_*(\beta') = \beta$ with $\iota : D \subset X$.

Note that in case D is smooth, $Z(\sigma) \neq \emptyset$ only if β is a positive integral multiple of $[D]$. Also, when X is proper, the localized GW-invariants coincide with the ordinary GW-invariants of X [16] (see also [8, Lemma 6.5]).

Example 2.5. (Spin surfaces) Let S be a spin surface, which is the total space of a theta characteristic of a smooth curve D, i.e. $L^{\otimes 2} \cong K_D$. We let $p: S \to D$ be the projection. Since $K_S \cong p^*K_D \otimes p^*L^\vee \cong p^*L$, the identity section of p^*L defines a section $\theta \in H^0(\Omega_S^2)$, which we call the standard two-form on S. Since $\theta^{-1}(0) = D$ is the zero-section of L and is proper, the localized GW-invariants of S are well-defined.

For a spin surface S associated with a theta characteristic L on D, and for $\gamma_i \in H^*(X)$, $\alpha_i \in \mathbb{Z}^{\geqslant 0}$, and ψ_i the first Chern class of the relative cotangent line bundle of the domain curves at the i-th marked point, using the evaluation morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{\chi,n}(S, d[D])^\bullet \longrightarrow S^n,$$

we define the localized GW-invariant of S with descendants to be

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{S,\bullet}_{\chi,d,\mathrm{loc}} = \int_{[\overline{\mathcal{M}}_{\chi,n}(S,d[D])]^{\mathrm{vir}}_{\mathrm{loc}}} \mathrm{ev}^*(\gamma_1 \times \cdots \times \gamma_n) \cdot \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}.$$

In [16], Lee-Parker conjectured the structure of GW-invariants of Kähler surfaces of positive p_g in terms of the structure of their canonical divisors. They proved the conjecture when the surfaces have smooth canonical divisors.

We make the following conjecture.

Conjecture 2.6. [7] *Let X be a smooth minimal general type surface with positive p_g. Its GW-invariants $\langle \cdots \rangle^X_{\beta,g}$ vanish unless β is a non-negative integral multiple of $c_1(K_X)$. In the case $\beta = dc_1(K_X)$ for an integer $d > 0$, we let (D, L) be a pair of a smooth curve of genus $K_X^2 + 1$ and its theta characteristic with parity $\chi(\mathcal{O}_X)$, and let S be the total space of L. Then there is a canonical homomorphism $\rho: H^*(X,\mathbb{Z}) \to H^*(S,\mathbb{Z})$ so that for classes $\gamma_i \in H^*(X,\mathbb{Z})$ and integers $\alpha_i \geqslant 0$,*

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{X,\bullet}_{\chi,\beta} = \langle \tau_{\alpha_1}(\rho(\gamma_1)) \cdots \tau_{\alpha_n}(\rho(\gamma_n)) \rangle^{S,\bullet}_{\chi,d,\mathrm{loc}}.$$

Using a deformation argument, one can verify the conjecture in several cases where the singularity of a canonical divisor of X is relatively simple.

3 Surfaces with smooth canonical divisors

Like the ordinary GW-invariants, the localized GW-invariants are expected to remain constant under deformation of complex structures. In the following, we shall prove this for the circumstances relevant to our study.

We consider a smooth family $\mathcal{X}/T$ of quasi-projective varieties over a connected smooth affine curve T; we assume that this family admits a regular (relative) homomorphic two-form $\Theta \in \Gamma(\mathcal{X}, \Omega^2_{\mathcal{X}/T})$. We let $\beta \in H_2(\mathcal{X}, \mathbb{Z})$ be a (fiber) curve class and denote by

$$M_T = \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}/T, \beta)^{\bullet}$$

the moduli space of stable morphisms (of not necessarily connected domains) to fibers of $\mathcal{X}/T$ of fundamental class β. For closed $t \in T$, we write

$$M_t = M_T \times_T t = \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_t, \beta), \quad \mathcal{X}_t = \mathcal{X} \times_T t.$$

Let $f : \mathcal{C} \to \mathcal{X}$ and $\pi : \mathcal{C} \to M_T$ be the universal family of this moduli stack; let $\kappa \in H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}/T})$ be the Kodaira-Spencer class of the first order deformation of $\mathcal{X}/T$— it is the extension class of the exact sequence of sheaves of tangent bundles

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/T} \longrightarrow \mathcal{T}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow 0.$$

As shown in [3, 20], the obstruction sheaf $\mathcal{O}b_{M_T}$ and its relative obstruction sheaf $\mathcal{O}b_{M_T/T}$, which is the sheaf whose restriction to each fiber M_t is the obstruction sheaf $\mathcal{O}b_{M_t}$ of M_t, fit into the exact diagram:

$$\begin{array}{ccccc}
\pi_* f^* \mathcal{O}_{\mathcal{X}} & \xrightarrow{f^*\kappa} & R^1\pi_* f^* \mathcal{T}_{\mathcal{X}/T} & \xrightarrow{\text{surj}} & \ker\big(R^1\pi_* f^*\mathcal{T}_{\mathcal{X}} \to R^1\pi_* f^* \mathcal{O}_{\mathcal{X}}\big) \\
\Big\| & & \text{surj}\big\downarrow & & \text{surj}\big\downarrow \\
\pi_* f^* \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}b_{M_T/T} & \xrightarrow{\text{surj}} & \mathcal{O}b_{M_T}.
\end{array} \tag{3.1}$$

Applying the previous construction, we check that the form Θ induces a cosection of $R^1\pi_* f^*\mathcal{T}_{\mathcal{X}/T}$ that descends to a cosection

$$\sigma_\Theta : \mathcal{O}b_{M_T/T} \longrightarrow \mathcal{O}_{M_T}. \tag{3.2}$$

The restriction of σ to each fiber M_t is the previously constructed cosection σ_{Θ_t} of $\mathcal{O}b_{M_t}$. We let $Z(\sigma)$ be the union of $Z(\sigma_{\Theta_t}) \subset M_t$ for all $t \in T$.

Suppose that $Z(\sigma)$ is proper over T, for each $t \in T$ we can define the localized GW-invariants of $\mathcal{X}_t$:

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle^{\mathcal{X}_t, \bullet}_{g,\beta,\mathrm{loc}}, \quad \gamma_i \in H^*(\mathcal{X}, \mathbb{Z}).$$

The deformation invariance principle states that the above is independent of t. In this section, we shall prove this principle for the localized GW-invariants for the circumstances relevant to our study.

According to [8, Thm. 5.2], the constancy of the localized GW-invariants follows from the lifting of the homomorphism σ_Θ to a homomorphism

$$\bar\sigma_\Theta : \mathcal{O}b_{M_T} \longrightarrow \mathcal{O}_{M_T}, \tag{3.3}$$

which by the lower exact sequence in (3.1) amounts to the vanishing of the composite

$$\pi_* f^* \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}b_{M_T/T} \xrightarrow{\sigma_\Theta} \mathcal{O}_{M_T}. \tag{3.4}$$

Lemma 3.1. *Suppose that the relative holomorphic two-form $\Theta \in \Gamma(\Omega^2_{\mathcal{X}/T})$ is the image of a $\tilde\Theta \in \Gamma(\Omega^2_{\mathcal{X}})$ via $\Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/T}$. Then the lifting $\bar\sigma_\Theta$ in (3.3) exists.*

Proof. This is because the form $\tilde{\Theta}$ defines a homomorphism $R^1\pi_* f^*\mathcal{T}_{\mathcal{X}} \to \mathcal{O}_{M_T}$ like in (2.1) with X replaced by $\mathcal{X}$, which is compatible with the cosection of $R^1\pi_* f^*\mathcal{T}_{\mathcal{X}/T}$. Because of the sequence (3.1), we see that the composite (3.4) is trivial. □

We now prove the vanishing of the composite (3.4) in some special situation without assuming the existence of $\tilde{\Theta}$. The first case is the deformation to the normal bundle of a smooth canonical divisor in a surface X. Let (X,θ) be a pair of a smooth surface and a holomorphic two-form with smooth $D=\theta^{-1}(0)$. By blowing up $X\times\mathbb{A}^1$ along $D\times 0$, and removing the proper transform of $X\times 0$ in the blown-up, we obtain a family of surfaces $\pi_{\mathbb{A}^1}: Z\to\mathbb{A}^1$ whose fiber over $t\neq 0$ is X and whose fiber over $0\in\mathbb{A}^1$ is a spin surface that is the total space of the normal bundle $N_{D/X}$.

Using the projection $\pi_X: Z\to X$, we obtain the pull back $\pi_X^*\theta$. We claim that the $t^{-2}\pi_X^*\theta$ induces a family of relative holomorphic two-form whose restriction to Z_0 is proportional to the standard holomorphic two-form on Z_0.

Let $U\subset X$ be an analytic open neighborhood of D with analytic coordinate functions (z_1,z_2) so that $\theta|_U = z_2 dz_1\wedge dz_2$. Then $\pi_X^{-1}(U)$ has analytic coordinate (z_1,t,ξ) with $z_2=t\cdot\xi$. The pull-back

$$\pi_X^*\theta|_{\pi_X^{-1}(U)} = t^2\xi\, dz_1\wedge d\xi + t\xi^2 dz_1\wedge dt \in \Gamma(\Omega^2_U)$$

has its image in $\Gamma(\Omega^2_{U/\mathbb{A}^1})$ of the form

$$\big(\pi_X^*\theta|_{\pi_X^{-1}(U)}\big)_{U/\mathbb{A}^1} = t^2\xi\, dz_1\wedge d\xi \in \Gamma(\Omega^2_{U/\mathbb{A}^1}).$$

This proves that the image of $t^{-2}\pi_X^*\theta$ in $\Omega^2_{\mathcal{X}/\mathbb{A}^1}$ is a regular relative holomorphic two-form whose restriction to $Z_t=X$ $(t\neq 0)$ and Z_0 are proportional to the form θ and the standard two-form on Z_0, respectively. We let

$$\Theta = \big(t^{-2}\pi_X^*\theta\big)_{\mathcal{X}/\mathbb{A}^1} \in \Gamma(\Omega^2_{\mathcal{X}/\mathbb{A}^1})$$

be this family of relative holomorphic forms.

Proposition 3.2. *Let $\mathcal{X}\to\mathbb{A}^1$ be the deformation of X to the normal bundle $N_{D/X}$, and let Θ be the relative holomorphic two-form specified above. Then the associated cosection σ_Θ in* (3.2) *lifts to a cosection $\bar{\sigma}_\Theta$ as in* (3.3).

Proof. We let $\beta = d[D]$, and let $M_T = \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}/T,\beta)^\bullet$. Following the diagram (3.1), we need to show that the composite

$$\pi_* f^*\mathcal{O}_{\mathcal{X}} \xrightarrow{f^*\kappa} R^1\pi_* f^*\mathcal{T}_{\mathcal{X}/T} \longrightarrow \mathcal{O}b_{M_T/T} \xrightarrow{\sigma_\Theta} \mathcal{O}_{M_T} \tag{3.5}$$

is zero, where κ is the Kodaira-Spencer class. In this case, since $Z\to\mathbb{A}^1$ over $\mathbb{A}^1-0$ is the constant family, κ is trivial over $Z\times_{\mathbb{A}^1}(\mathbb{A}^1-0)$.

We now consider $\mathcal{X}_0\subset\mathcal{X}$. Let $D_0\subset\mathcal{X}_0$ be the intersection of $\mathcal{X}_0$ with the proper transform of $D\times\mathbb{A}^1\subset X\times\mathbb{A}^1$. Then $T_{\mathcal{X}}|_{D_0}\cong T_{\mathcal{X}_0}|_{D_0}\oplus\mathcal{O}_{D_0}$. Since $\mathcal{X}_0$ is a line bundle over D_0, we have $T_{\mathcal{X}}|_{\mathcal{X}_0}\cong T_{\mathcal{X}_0}|_{\mathcal{X}_0}\oplus\mathcal{O}_{\mathcal{X}_0}$. This shows that κ restricted to $\mathcal{X}_0$ is a trivial cohomology class in $H^1(\mathcal{X}_0,T_{\mathcal{X}_0})$. In particular, $f^*\kappa=0$ in (3.5). This proves the proposition. □

We next consider the case where $\mathcal{X}$ is a smooth family of spin surfaces. We let $\mathcal{D}\to T$ be a smooth family of curves and $\mathcal{L}$ be a family of theta characteristics of $\mathcal{D}/T$; i.e. $\mathcal{L}^{\otimes 2}\cong\Omega_{\mathcal{D}/T}$. The total spaces $\mathcal{X}$ of $\mathcal{L}$ form a family of spin surfaces. We let $\Theta\in\Gamma(\mathcal{X},\Omega^2_{\mathcal{X}/T})$ be the standard relative holomorphic two-form. We let $\beta\in H_2(\mathcal{X},\mathbb{Z})$ be the class generated by a d-multiple of the zero section of one of $\mathcal{X}_t$. As before, we denote $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}/T,\beta)^\bullet$ by M_T.

Proposition 3.3. *The conclusion of Proposition* 3.2 *holds for the family of spin surfaces $\mathcal{X}$ and its associated two-form Θ.*

Proof. Let $0\in T$ be a closed point. We first check that restricting to $\mathcal{X}_0$ the Kodaira-Spencer class $\kappa_0\in H^1(\mathcal{X}_0,T_{\mathcal{X}_0})$ of the family $\mathcal{X}$ has the following property: Let $(f_0,\mathcal{C}_0)$ be the universal family of $M_0=\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_0,\beta)^\bullet$. Then the composite

$$\pi_* f^*\mathcal{O}_{\mathcal{X}_0} \xrightarrow{f_0^*\kappa_0} R^1\pi_* f^*\mathcal{T}_{\mathcal{X}_0} \longrightarrow \mathcal{O}b_{M_0} \xrightarrow{\sigma_{\Theta_0}} \mathcal{O}_{M_0} \tag{3.6}$$

is locally constant.

We first describe the Kodaira-Spencer class κ_0, which depends on the family $\mathcal{D}$. For simplicity, we shall work with the analytic charts of $\mathcal{D}_0 = \mathcal{D} \times_{\mathbb{A}^1} 0$. We pick an analytic open $U \subset \mathcal{D}_0$ so that U is isomorphic to the unit disk $\Delta \subset \mathbb{C}$. We then let $V = \mathcal{D}_0 - A$ with $A \subset U$ a compact subset so that $U - A$ is isomorphic to an annulus. The two open sets U and V form an open covering of $\mathcal{D}_0$. Since $H^1(T_U) = H^1(T_V) = 0$, $H^0(T_{U\cap V}) \to H^1(T_{\mathcal{D}_0})$ is surjective. Hence, for small t the family D_t can be realized by an analytic deformation of the gluing map

$$U \supset U \cap V \xrightarrow{\cong} U \cap V \subset V.$$

In concrete terms, if we let z and w be the analytic coordinates of U and V near $U \cap V$, and let $z = h(w,0)$ be the identity map of $U \cap V$ in coordinate variables z and w, then $\mathcal{D}_t$ can be realized by gluing U and V via $z = h(w,t)$ with $h(w,t)$ an analytic deformation of $h(w,0)$.

To proceed, we need the transition function of $\mathcal{X}_t$. Because $\mathcal{D}_t = U \cup V$, the surface $\mathcal{X}_t$ is the union of the total space of $K_U^{\frac{1}{2}}$ and $K_V^{\frac{1}{2}}$. To build a transition function of $\mathcal{X}_t$, we let $\xi = (dz)^{\frac{1}{2}}$ and $\eta = (dw)^{\frac{1}{2}}$ be bases of $K_U^{\frac{1}{2}}$ and $K_V^{\frac{1}{2}}$ over $U \cap V$. Then by adopting the convention that $h_w = \frac{\partial h}{\partial w}$ and $\dot{h} = \frac{dh}{dt}$, the two pairs of local charts (z,ξ) and (w,η) are related by

$$z = h(w,t) \quad \text{and} \quad \xi = (h_w)^{\frac{1}{2}}\eta.$$

Accordingly, the Kodaira-Spencer class of the first order deformation of $\mathcal{X}_t$ at $t=0$ can be represented by Čech 1-cocycle

$$\kappa_0(U \cap V) = \left(\frac{dh}{dt} \cdot \frac{\partial}{\partial z} + \frac{d}{dt}((h_w)^{\frac{1}{2}}\eta) \cdot \frac{\partial}{\partial \xi}\right)\bigg|_{t=0} = \left(\dot{h} \cdot \frac{\partial}{\partial z} + \frac{\xi\,\dot{h}_w}{2h_w} \cdot \frac{\partial}{\partial \xi}\right)\bigg|_{t=0}.$$

Because over $K_U^{\frac{1}{2}} \cap K_V^{\frac{1}{2}}$, the standard holomorphic two-form is $\Theta_0 = \xi\, d\xi \wedge dz$, the contraction is

$$\Theta_0(\kappa_0) = -\xi\dot{h}\, d\xi + \frac{1}{2}\xi^2 \dot{h}_w h_w^{-1} dz.$$

Therefore, using $\dot{h}_z = \dot{h}_w \frac{\partial w}{\partial z} = -\dot{h}_w h_w^{-1}$,

$$\partial(\Theta_0(\kappa_0)) = -\xi\dot{h}_z dz \wedge d\xi + \xi\dot{h}_w h_w^{-1} d\xi \wedge dz = 0.$$

Combined with the fact that $\bar{\partial}\Theta_0(\kappa_0) = 0$, we see that the form $\Theta_0(\kappa_0)$ is d-closed.

The lemma now follows easily. We let $p: \mathcal{X}_0 \to \mathcal{D}_0$ be the projection and let $W \subset M_0$ be the (analytic) open subset consisting of those $u: C \to \mathcal{X}_0$ so that $p \circ u: C \to \mathcal{D}_0$ are unramified over $U \cap V$. We then pick an oriented embedded circle $S^1 \subset U \cap V$ that separates the two boundary components of $U \cap V$. An easy argument shows that the homomorphism (3.6) is the function, up to sign,

$$[u, C] \in W \longmapsto \int_{u^{-1}(S^1)} \Theta_0(\kappa_0) \in \mathbb{C}.$$

Because $\Theta_0(\kappa_0)$ is d-closed, this integral only depends on the topological class of $u^{-1}(S^1)$, hence it must be locally constant over W. But then this constant must be zero since it vanishes on those u so that $u(C) \subset D_0 \subset \mathcal{X}_0$, and since by dilation along fibers of $\mathcal{L}_0$ each $u: C \to \mathcal{X}_0$ can be deformed to a stable map from C to $\mathcal{D}_0 \subset \mathcal{X}_0$ within W. This shows that (3.6) is zero over W.

Finally, we observe that for each stable map in M_0, we can choose $U \subset \mathcal{D}_0$ so that this stable map lies in the W associated with U. Therefore (3.6) must be zero on all M_0, completing the proof of the lemma. $\square$

These two deformation invariance properties provide an algebro-geometric proof of Conjecture 2.6 in case X has a smooth canonical divisor, which was originally proved in [16].

4 A degeneration formula

One of the major tools in studying GW-invariants of varieties is the degeneration formula. The symplectic version of this theory is completed in [6,19]; the algebraic version of this theory is developed by the second author in [17, 18]. In this section, we will work out a parallel theory of localized relative GW-invariants and prove a degeneration formula for spin surfaces. The degeneration will be used to prove a conjetured formula of low degree GW-invariants of surfaces [21].

We continue to denote by S a spin surface that is the total space of a theta-chracteristic L on a smooth curve D. We pick a point $q \in D$ and denote by $E \subset S$ the fiber of S over q; we blow up $S \times \mathbb{A}^1$ along $E \times 0$ to obtain a family $\mathcal{X}$ over $\mathbb{A}^1$ whose fiber over $t \neq 0 \in \mathbb{A}^1$ is the original S, and its central fiber $\mathcal{X}_0 := \mathcal{X} \times_{\mathbb{A}^1} 0$ is the union of S with $E \times \mathbb{P}^1$, intersecting transversally along $E \subset S$ and $E \times 0 \subset E \times \mathbb{P}^1$. To distinguish the $S \subset \mathcal{X}_0$ from $\mathcal{X}_t \cong S$, we denote by $Y_1 \subset \mathcal{X}_0$ the component $S \subset \mathcal{X}_0$ and denote $E \times \mathbb{P}^1 \subset \mathcal{X}_0$ by Y_2. We denote $E = Y_1 \cap Y_2$, viewed as a divisor in both Y_1 and Y_2. Let $e \in E$ be the intersection point of E with the zero section of $Y_1 \to D$.

The family $\mathcal{X}$ is the total space of a line bundle $\mathcal{L}$ on the blow-up of $D \times \mathbb{A}^1$ along $(q, 0)$. We let $\mathcal{D}$ be this blow-up. The line bundle $\mathcal{L}$ is the pull-back of L via the composite of the projections $\mathcal{D} \to D \times \mathbb{A}^1 \to D$.

The total space $\mathcal{X}$ has a holomorphic two-form by pulling back the standard two-form θ on S:

$$\Theta := \pi_S^* \theta \in \Gamma(\Omega^2_{\mathcal{X}}), \quad \pi_S : \mathcal{X} \to S \times \mathbb{A}^1 \to S. \tag{4.1}$$

We will show that this holomorphic two-form defines a cosection of the obstruction sheaf of the moduli of stable morphisms to the family $\mathcal{X}/\mathbb{A}^1$ in the sense of [18]. The cosection localized virtual cycle of this moduli space is the bridge to prove the degeneration formula for the localized GW-invariants.

The moduli of stable morphisms to $\mathcal{X}/\mathbb{A}^1$ constructed in [17] is the moduli of stable morphisms to the stack of expanded degenerations $\mathfrak{X}$ of $\mathcal{X}/\mathbb{A}^1$, whose construction we now recall. To each integer $m \geqslant 0$, we let $\mathbb{A}^{m+1} \to \mathbb{A}^1$ be $(z_1, \dots, z_{m+1}) \mapsto z_1 \cdots z_{m+1}$, and let the expanded degeneration

$$\mathcal{X}[m] \longrightarrow \mathbb{A}^{m+1}$$

be the small resolution of $\mathcal{X}_m := \mathcal{X} \times_{\mathbb{A}^1} \mathbb{A}^{m+1}$ characterized by the properties:

1. for each $\mathbf{t} \in \mathbb{A}^{m+1}$ the fiber $\mathcal{X}[m]_{\mathbf{t}} := \mathcal{X}[m] \times_{\mathbb{A}^{m+1}} \mathbf{t}$ is a surface with normal crossing singularities;

2. in case $\mathbf{t} = (t_i)$ has exactly k vanishing t_i's, the irreducible components of $\mathcal{X}[m]_{\mathbf{t}}$ are $Y_1 = S$, $(k-1)$ copies of $\Delta := \mathbb{P}^1 \times \mathbb{A}^1$, and $Y_2 = E \times \mathbb{P}^1$ (in case $k > 0$), with a chain like intersection patten;

3. the union of the fibers of $\mathcal{X}[m]$ over the i-th coordinate line in $\mathbb{A}^{m+1}$ is a smoothing of the i-th singular divisor of $\mathcal{X}[m]_0$, following the convention that $Y_1 \subset \mathcal{X}[m]_0$ is the 0-th component; Y_2 is the $(m+1)$-th component, and the i-th singular divisor is the intersection of the i-th and the $(i+1)$-th irreducible components of $\mathcal{X}[m]_0$.

The $(\mathbb{C}^*)^m$ action on $\mathbb{A}^{m+1}$ via

$$(z)^\tau = (\tau_1 z_1, \tau_1^{-1}\tau_2 z_2, \dots, \tau_m^{-1} z_{m+1})$$

lifts to a unique action on $\mathcal{X}[m] \to \mathbb{A}^{m+1}$. This group action defines a class of equivalences of $\mathcal{X}[m]$.

The space $\mathcal{X}[m]$ has another class of partial equivalences. For $1 \leqslant i \leqslant m+1$, we denote by

$$\mathbb{A}^{m+1}_{z_i=1} := \mathbb{A}^{i-1} \times 1 \times \mathbb{A}^{m-i+1} \subset \mathbb{A}^{m+1}$$

the hypersurface parallel to the coordinate hyperplane $(z_i = 0)$ and passing through $(1, \dots, 1)$. Following the construction, we have a canonical isomorphism

$$\imath_i : \mathcal{X}[m-1] \xrightarrow{\cong} \mathcal{X}[m] \times_{\mathbb{A}^{m+1}} \mathbb{A}^{m+1}_{z_i=1}. \tag{4.2}$$

By a direct inspection, via the inclusion

$$(\mathbb{C}^*)^{m-1} = (\mathbb{C}^*)^{i-1} \times \{1\} \times (\mathbb{C}^*)^{m-i} \subset (\mathbb{C}^*)^m, \tag{4.3}$$

the inclusion $\imath_i$ is $(\mathbb{C}^*)^{m-1}$ equivariant.

Using the standard isomorphism $\mathbb{C}^* \cong \mathbb{A}^1 - 0$ and $\mathbb{C}^*$-action induced by the i-th factor of $\mathbb{C}^*$ in $(\mathbb{C}^*)^m$, we obtain an isomorphism

$$\tilde{\imath}_i : \mathcal{X}[m-1] \times (\mathbb{A}^1 - 0) \xrightarrow{\cong} \mathcal{X}[m] \times_{\mathbb{A}^{m+1}} \big(\mathbb{A}^{m+1} - \mathbb{A}^{m+1}_{z_i=0}\big), \tag{4.4}$$

where $\mathbb{A}^{m+1}_{z_i=0}$ is the coordinate hyperplane $z_i = 0$ in $\mathbb{A}^{m+1}$.

We define the stack $[\mathcal{X}[m]/\sim]$ be $\mathcal{X}[m]$ quotient by the equivalence relations generated by the $(\mathbb{C}^*)^m$-action and the equivalences $\tilde{\imath}_i \circ \tilde{\imath}_j^{-1}$ (cf. (4.4)) for all possible $i \leqslant j$. Using (4.2), we have inclusion $[\mathcal{X}[m]/\sim] \subset [\mathcal{X}[m+1]/\sim]$; we define

$$\mathfrak{X} = \lim_{m} \,[\mathcal{X}[m]/\sim].$$

It was shown in [17] that $\mathfrak{X}$ is an Artin stack.

We next recall the construction of expanded relative pairs. We let $E \subset Y$ be either $E \subset Y_1$ or $E \subset Y_2$. Inductively,

1. we let $Y[0] = Y$ and $E[0] = E$;

2. after $E[i] \subset Y[i]$ and $Y[i] \to \mathbb{A}^i$ are constructed, we let $Y[i+1]$ be the blow-up of $Y[i] \times \mathbb{A}^1$ along $E[i] \times 0$, let $E[i+1]$ be the proper transform of $E[i] \times \mathbb{A}^1$, and let $Y[i+1] \to \mathbb{A}^{i+1}$ be induced by $Y[i] \times \mathbb{A}^1 \to \mathbb{A}^i \times \mathbb{A}^1 = \mathbb{A}^{i+1}$.

Note that the central fiber $Y[i] \times_{\mathbb{A}^i} 0$ is the union of Y with i copies of Δ's.

We denote by $\mathcal{Y}[m]$ the pair of a variety and a divisor plus the projection:

$$\mathcal{Y}[m] = (Y[m], E[m]), \quad Y[m] \longrightarrow \mathbb{A}^m. \tag{4.5}$$

The standard $(\mathbb{C}^*)^m$ action $(z)^\tau = (\tau_1 z_1, \dots, \tau_m z_m)$ on $\mathbb{A}^m$ lifts to a $(\mathbb{C}^*)^m$ action on $\mathcal{Y}[m]/\mathbb{A}^m$. Also, for each $1 \leqslant i \leqslant m$, we have isomorphisms and inclusions

$$\jmath_i : \mathcal{Y}[m-1] \xrightarrow{\cong} \mathcal{Y}[m] \times_{\mathbb{A}^m} \mathbb{A}^m_{z_i=1} \subset \mathcal{Y}[m], \tag{4.6}$$

which is $(\mathbb{C}^*)^{m-1}$-equivariant via the group homomorphism (4.3), and the isomorphisms

$$\tilde{\jmath}_i : \mathcal{Y}[m-1] \times (\mathbb{A}^1 - 0) \xrightarrow{\cong} \mathcal{Y}[m] \times_{\mathbb{A}^m} (\mathbb{A}^m - \mathbb{A}^m_{z_i=0}). \tag{4.7}$$

We define $[\mathcal{Y}[m]/\sim]$ to be $\mathcal{Y}[m]$ quotiented by the equivalences generated by the $(\mathbb{C}^*)^m$ action and the isomorphisms $\tilde{\jmath}_i \circ (\tilde{\jmath}_j)^{-1}$ for all $i \neq j$. Using (4.6), we have inclusions

$$[\mathcal{Y}[m]/\sim] \longrightarrow [\mathcal{Y}[m+1]/\sim];$$

we define

$$\mathfrak{Y} = \lim_{m} \,[\mathcal{Y}[m]/\sim].$$

It is an Artin stack.

The families $\mathcal{X}[m]$, $Y_1[k]$ and $Y_2[k]$[1)] are related by the following decomposition:

$$\mathcal{X}[m] \times_{\mathbb{A}^{m+1}} \mathbb{A}^m_{z_i=0} = Y_1[i] \times \mathbb{A}^{m-i} \cup Y_2[m-i+1] \times \mathbb{A}^i, \quad 1 \leqslant i \leqslant m+1,$$

which is a union of two smooth varieties (as shown) intersecting transversally along a smooth divisor. Passing to limits, we obtain a co-fiber product

$$\begin{array}{ccc} E & \longrightarrow & \mathfrak{Y}_1 \\ \big\downarrow & & \big\downarrow \\ \mathfrak{Y}_2 & \longrightarrow & \mathfrak{X} \times_{\mathbb{A}^1} 0. \end{array} \tag{4.8}$$

1) $Y_1[m]$ is $Y[m]$ with Y replaced by Y_1; same for $\mathcal{Y}_1$ and $\mathfrak{Y}_1$.

We introduce three more stacks. Using that S is (the total space of) a line bundle over D, the zero section $D \subset S$ provides us with closed substacks $\mathfrak{D} \subset \mathfrak{X}$ and $\mathfrak{B}_k \subset \mathfrak{Y}_k$ via fiber products

$$\mathfrak{D} = \mathfrak{X} \times_S D \quad \text{and} \quad \mathfrak{B}_k = \mathfrak{Y}_k \times_S D. \tag{4.9}$$

Here since the projection $\pi_S : \mathcal{X} \to S$ (cf. (4.1)) induces projections $\mathcal{X}[m] \to S$ that commute with the equivalence relations defining $\mathfrak{X}$, we have the induced $\mathfrak{X} \to S$; similarly, for projections $\mathfrak{Y}_k \to S$ for $k = 1, 2$. The fiber products above use these projections.

Like $\mathfrak{X}$, we can construct $\mathfrak{D}$ from quotients $[\mathcal{D}[m]/\sim]$, where $\mathcal{D}[m] = \mathcal{X}[m] \times_S D$ are small resolutions of $\mathcal{D} \times_{\mathbb{A}^1} \mathbb{A}^{m+1}$, parallel to the construction of $\mathcal{X}[m]$. By the same reason, if we let $B_k \subset Y_k$ be $B_k = Y_k \times_S D$ with relative divisor $e = E \cap B_k$, $\mathfrak{B}_k$ is the stack of expanded relative pair (B_k, e). By construction, both $\mathfrak{D}/\mathbb{A}^1$ and $\mathfrak{B}_k$ are proper.

We next recall the moduli stack $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$ of stable morphisms to $\mathfrak{X}$ of the given topological type and its key ingredients relevant to this paper. We form the moduli stack $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}[m], d)^\bullet$ of stable morphisms to $\mathcal{X}[m]$ of domain curves with indicated topological type and of fundamental classes d-multiple of the zero section of $\mathcal{X}[m]_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{A}^{m+1}$. (The zero section is $\mathcal{X}[m]_{\mathbf{t}} \times_S D$.)

Let $\iota : \mathcal{X}[m] \to \mathfrak{X}$ be the tautological morphism from the definition of $\mathfrak{X}$. For $\xi = [u, C] \in \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}[m], d)^\bullet$, we let $\iota(\xi)$ be the induced morphism $\iota \circ u : C \to \mathfrak{X}$. Following the definition of $\mathfrak{X}$, given $\xi = [u, C]$ and $\xi' = [u', C'] \in \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}[m], d)^\bullet$, $\iota(\xi) = \iota(u')$ if there is a pair (φ, τ) of an isomorphism $\varphi : C \to C'$ of pointed curves and $\tau \in (\mathbb{C}^*)^m$ so that $\tau \cdot u = u \circ \varphi$.

Definition 4.1. *We define an equivalence $\xi \sim_{\mathfrak{X}} \xi'$ to be a pair (φ, τ) that makes $\iota(\xi) = \iota(u')$. We define $\mathrm{Aut}_{\mathfrak{X}}(\xi)$ to be the set of self-equivalences of ξ; it is a group.*

We call $\xi = [u, C]$ a stable morphism to $\mathfrak{X}$ if $u : C \to \mathcal{X}[m]$ is pre-deformable 2) *and $|\mathrm{Aut}_{\mathfrak{X}}(\xi)| < \infty$.*

We let

$$\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{X}[m], d)^\bullet \subset \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}[m], d)^\bullet$$

be the locally closed substack of stable morphisms (to $\mathfrak{X}$). Because of the finite stabilizer assumption, $\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{X}[m], d)^\bullet/(\mathbb{C}^*)^m$ is a DM-stack.

Proposition 4.2. [17, Thm. 3.10] *The moduli stack $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$ of stable morphisms to $\mathfrak{X}$ of the given topological type is a separated DM-stack over $\mathbb{A}^1$. The set of its closed points is*

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet(\mathbb{C}) = \coprod_{m \geqslant 0} (\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{X}[m], d)^\bullet(\mathbb{C}))/\sim_{\mathfrak{X}} .$$

For each m, the tautological

$$\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{X}[m], d)^\bullet/(\mathbb{C}^*)^m \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \tag{4.10}$$

is finite and étale. For large m, (4.10) is surjective. For $c \neq 0 \in \mathbb{A}^1$, canonically

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c, d)^\bullet := \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \times_{\mathbb{A}^1} c \cong \overline{\mathcal{M}}_{\chi,n}(S, d)^\bullet.$$

We define the moduli of relative stable morphisms of not necessarily connected domains in $\mathcal{Y} = (Y, E)$, where $Y = Y_1$ or Y_2. We pick a partition $\mu = (\mu_1 \leqslant \cdots \leqslant \mu_\ell)$ of d, (we write $\ell = \ell(\mu)$,) and for (χ, n) and an $m \geqslant 0$, we call a stable map

$$u : (C, p_1, \ldots, p_n, q_1, \ldots, q_\ell) \longrightarrow Y[m] \tag{4.11}$$

in $\overline{\mathcal{M}}_{\chi,n+\ell}(Y[m], d)^\bullet$ of the contact type μ if as divisors

$$u^{-1}(E[m]) = \sum \mu_i q_i.$$

(Here, the stable maps having fundamental classes $u_*[C]$ are d-multiples of the zero sections of $Y[m]_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{A}^m$.)

2) We refer the definition of pre-deformability to [17]; it is repeated in this paper in Remark 5.2. It was proved that pre-deformability is a locally closed condition.

Definition 4.3. [17] *For two stable maps u and u' to $Y[m]$, we define an equivalence $u \sim_{\mathfrak{Y}} u'$ to be a pair (φ, τ) of an isomorphism $\varphi : C \to C'$ as pointed curves and $\tau \in (\mathbb{C}^*)^m$ so that $\tau \cdot u = u' \circ \varphi$. We define $\mathrm{Aut}_{\mathfrak{Y}}(u)$ to be the set of self-equivalences of u; it is a group.*

We call u a stable relative morphism of type μ to $\mathfrak{Y}$ if u has the contact type μ, it is pre-deformable and the group $\mathrm{Aut}_{\mathfrak{Y}}(u)$ is finite.

We let

$$\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{Y}[m],\mu)^\bullet \subset \overline{\mathcal{M}}_{\chi,n+\ell}(Y[m],d)^\bullet$$

be the locally closed substack of stable relative morphisms in $\overline{\mathcal{M}}_{\chi,n+\ell}(Y[m],d)^\bullet$ (cf. [17]). Because of the finiteness of $\mathrm{Aut}_{\mathfrak{Y}}$ assumption, $\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{Y}[m],\mu)^\bullet/(\mathbb{C}^*)^m$ is a DM-stack.

We list the relevant property of the stack $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet$ of the stable relative morphisms of type μ to $\mathfrak{Y}$ proved in [17].

Proposition 4.4. [17, Thm. 4.10] *The moduli stack $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet$ is a separated DM-stack. As sets,*

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet(\mathbb{C}) = \coprod_{m\geqslant 0} (\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{Y}[m],\mu)^\bullet(\mathbb{C}))/\sim_{\mathfrak{Y}}.$$

For each m, the tautological

$$\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{Y}[m],\mu)^\bullet/(\mathbb{C}^*)^m \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet \tag{4.12}$$

is finite and étale; for large m, it is surjective.

The moduli space $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet$ has two evaluation morphisms: one ordinary and one special: for $u : C \to Y[m]$ in $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y},\mu)^\bullet$, letting $\tilde{u}$ be the composite $C \xrightarrow{u} Y[m] \xrightarrow{\mathrm{pr}} Y$, we define

$$\mathrm{ev}(u) = \big(\tilde{u}(p_1),\dots,\tilde{u}(p_n)\big) \in Y^n, \quad \tilde{\mathrm{ev}}(u) = (\tilde{u}(q_1),\dots,\tilde{u}(q_\ell)) \in E^\ell, \quad \ell = \ell(\mu).$$

Using the co-fiber product (4.8), we have the gluing construction which we recall now. For an integer n, we denote $[n] = \{1,\dots,n\}$. We let

$$\gamma = \big(\mu \vdash d; \chi = \chi_1 + \chi_2 - \ell(\mu);\ \lambda : [n_1] \to [n] \text{ order preserving}\big). \tag{4.13}$$

(The role of λ will be clear shortly.) Using the special evaluation morphism, we define

$$\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)^\bullet := \overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{Y}_1,\mu)^\bullet \times_{E^{\ell(\mu)}} \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet. \tag{4.14}$$

We recall the construction of the gluing morphism

$$\Psi_\gamma : \overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)^\bullet \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,d)^\bullet := \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet \times_{\mathbb{A}^1} 0. \tag{4.15}$$

Given $(u_1,u_2) \in \overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)$, and suppose that u_i are of the forms $u_i : C_i \to Y_i[m_i]$ for some m_i, with ordinary marked points $p^i_1,\dots,p^i_{n_i}$ and special marked points $q^i_1,\dots,q^i_\ell$,

1. we let $C = C_1 \cup C_2/\sim$, where $\sim$ is identifying $q^1_j \in C_1$ with $q^2_j \in C_2$ for all j;
2. using (4.8), u_1 and u_2 patch to form a morphism $u : C \to \mathcal{X}[m]_0$, where $m = m_1 + m_2$;
3. we let $\lambda' : [n_2] \to [n]$ be the complement of λ[3)], and define the marked points $p_1,\dots,p_{n_1+n_2}$ of C to be $p_{\lambda(j)} = p^1_j$ and $p_{\lambda'(j)} = p^2_j$.

By construction, we see that $\chi(\mathcal{O}_C) = \chi_1 + \chi_2 - \ell$, and thus $[u,C] \in \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$. Working out the family version of this construction, we obtain the gluing morphism (4.15).

Let Γ be the collection of all possible γ in (4.13). It is clear that the union of the images of Ψ_γ for all $\gamma \in \Gamma$ surjects onto $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,d)^\bullet$.

We now state the degeneration of localized virtual classes. Recall that since $\mathfrak{D} \subset \mathfrak{X}$ is a substack, we have canonical inclusion $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet \subset \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$. For $c \in \mathbb{A}^1$, we let

$$\iota_c : \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_c,d)^\bullet := \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet \times_{\mathbb{A}^1} c \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet$$

3) Here, $[n] = \{1,\dots,n\}$; λ' is order preserving so that $\lambda([n_1]) \cup \lambda'([n_2]) = [n_1 + n_2]$.

be the fiber over c and its tautological embedding; we let

$$\iota_c^! : A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet \longrightarrow A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_c,d)^\bullet$$

be the associated Gysin map.

We also need the following commutative square. Given $\gamma \in \Gamma$ (cf. (4.13)), we let

$$\overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2,\gamma)^\bullet = \overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{B}_1,\mu)^\bullet \times_{E^{\ell(\mu)}} \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{B}_2,\mu)^\bullet.$$

We claim that the following commutative square is a Cartesian square:

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2,\gamma)^\bullet & \xrightarrow{\delta} & \overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{B}_1,\mu)^\bullet \times \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet \\ \Big\downarrow{\scriptstyle \tilde{\mathrm{ev}}} & & \Big\downarrow{\scriptstyle (\tilde{\mathrm{ev}},\tilde{\mathrm{ev}})} \\ E^{\ell(\mu)} & \xrightarrow{\mathrm{diag}} & E^{\ell(\mu)} \times E^{\ell(\mu)}. \end{array} \tag{4.16}$$

Indeed, because $Y_2 \to B_2$ is the trivial line bundle, the induced morphism (by composing $u : C \to Y_2[m]$ with $Y_2[m] \to B_2[m]$)

$$\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet \longrightarrow \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{B}_2,\mu)^\bullet$$

is a fiber bundle with fibers $(\mathbb{A}^1)^r$, where r is the number of connected components of the domain curves, which varies over different connected components of $\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet$. Because a relative stable map $[u,C] \in \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet$ restricted to every connected component of its domain is non-constant,

$$\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet \times_{E^{\ell(\mu)}} e^{\ell(\nu)} = \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{B}_2,\mu)^\bullet.$$

(Recall $e = E \cap B_2$, and $\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet \to E^{\ell(\mu)}$ is via special evaluation.) This proves that (4.16) is a Cartesian square. We let

$$\delta^! : A_*\big(\overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{B}_1,\mu)^\bullet \times \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet\big) \longrightarrow A_*\overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2,\gamma)^\bullet$$

be the associated Gysin map. We let

$$\psi_\gamma : \overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2,\gamma)^\bullet \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0,d)^\bullet \tag{4.17}$$

be induced by the gluing morphism Ψ_γ.

Let

$$[\overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(D,d)^\bullet$$

be the localized virtual class constructed in [8] (and Section 2) that defines the localized GW-invariants of S.

Theorem 4.5. *The holomorphic two-form Θ on $\mathcal{X}$ and its restriction to Y_1 define cosection localized virtual classes*

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet, \quad [\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\mu)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{B}_1,\mu)^\bullet;$$

these classes fit into the identities

$$\iota_c^![\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\big(\overline{\mathcal{M}}_{\chi,n}(D,d)^\bullet\big), \quad c \neq 0 \in \mathbb{A}^1; \tag{4.18}$$

$$\begin{aligned} \iota_0^![\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma\in\Gamma} \frac{\mu\,!}{|\operatorname{Aut}(\mu)|}(\psi_\gamma)_*\delta^!\big([\overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{Y}_1,\mu)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \\ \times [\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet]^{\mathrm{vir}}\big) \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0,d)^\bullet. \end{aligned} \tag{4.19}$$

Here $\mu! = \mu_1 \cdots \mu_{\ell(\mu)}$ and $\operatorname{Aut}(\mu)$ consists of all permutations $\alpha \in S_{\ell(\mu)}$ so that $\mu_i = \mu_{\alpha(i)}$ for all i.

The proof of this theorem will occupy the next two sections.

This theorem implies the degeneration formula of the localized GW-invariants of $\mathcal{X}$. We pick classes $\gamma_i \in H^*(Y_1)$ and non-negative integers α_i; we define the reduced relative GW-invariants

$$\langle \tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_n}(\gamma_n)\rangle^{Y_1/E,\bullet}_{\chi,\mu,\mathrm{loc}} \in A_* e^{\ell(\mu)}$$

to be the direct image

$$\tilde{\mathrm{ev}}_*(\mathrm{ev}^*(\gamma_i\times\cdots\times\gamma_n)\cdot\psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\mu)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}) \in A_* e^{\ell(\mu)}. \tag{4.20}$$

Here ψ_i is the ψ class of the universal curve of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\mu)^\bullet$ associated with the i-th marked points. Since the localized virtual class lies in $A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{B}_1,\mu)^\bullet$, the class (4.20) lies in $e^{\ell(\mu)}\subset E^{\ell(\mu)}$.

For $\mathcal{Y}_2=(Y_2,E)$, we take the virtual cycle of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,\mu)^\bullet$ using its perfect obstruction theory constructed in [18]:

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,\mu)^\bullet]^{\mathrm{vir}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,\mu)^\bullet.$$

Since $Y_2=B_2\times\mathbb{A}^1$ and E is one of the $\mathbb{A}^1$ in the product, the special evaluation morphism

$$\tilde{\mathrm{ev}}:\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,\mu)^\bullet \longrightarrow E^{\ell(\mu)}$$

is proper. Using the proper push-forward, we define

$$\langle \tau_{\alpha_1}(\gamma_1)\cdots\tau_{\alpha_n}(\gamma_n)\rangle^{Y_2/E,\bullet}_{\chi,\mu} \in A_* E^{\ell(\mu)} \tag{4.21}$$

to be (4.20) with $[\cdot]^{\mathrm{vir}}_{\mathrm{loc}}$ replaced by $[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,\mu)^\bullet]^{\mathrm{vir}}$.

We fix integers $\alpha_1,\ldots,\alpha_n\in\mathbb{Z}^{\geqslant 0}$; a splitting $n_1+n_2=n$ and classes $\gamma_i\in H^{\geqslant 1}(\mathcal{X})$ such that for $\iota_i: Y_i\to\mathcal{X}$ the inclusion, $\iota_1^*(\gamma_i)=0$ for $i>n_1$ and $\iota_2^*(\gamma_i)=0$ for $i\leqslant n_1$. We also adopt the intersection pairing

$$\star: A_*E^l\times A_*e^l\longrightarrow\mathbb{Z},\quad [E^l]\cdot[\mathrm{pt}]=1.$$

Theorem 4.6. *Let the situation be as stated. We have the degeneration formula*

$$\left\langle\prod_{j=1}^n\tau_{\alpha_j}(\gamma_j)\right\rangle^{S,\bullet}_{\chi,d,\mathrm{loc}} = \sum\frac{\mu!}{|\operatorname{Aut}(\mu)|}\cdot\left\langle\prod_{j=1}^{n_1}\tau_{\alpha_j}(\gamma_j)\right\rangle^{Y_1/E,\bullet}_{\chi_1,\mu,\mathrm{loc}}\star\left\langle\prod_{j=n_1+1}^{n}\tau_{\alpha_j}(\gamma_j)\right\rangle^{Y_2/E,\bullet}_{\chi_2,\mu}.$$

Here the summation is over all possible partitions $\mu\vdash d$ and $\chi=\chi_1+\chi_2-\ell(\mu)$.

Proof. Applying the class version of the degeneration formulas in Theorem 4.5, we obtain the formula in the statement of the theorem with summation over all possible partitions $\mu\vdash d$, $\chi=\chi_1+\chi_2-\ell(\mu)$ and $\lambda:[n_1]\to[n]$. Using the assumption $\iota_1^*(\gamma_i)=0$ for $i>n_1$ and $\iota_2^*(\gamma_i)=0$ for $i\leqslant n_1$, we see that the terms in the summation may possibly be non-vanishing only if $\lambda([n_1])=\{1,\ldots,n_1\}\subset[n]$. This proves the theorem. $\square$

5 Obstruction sheaves and their cosections

We prove Theorem 4.6 in this section and the next one. Our first step is to construct the cosection localized virtual class of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$. As constructed in [18], its perfect obstruction theory is the descent of that of $\overline{\mathcal{M}}^{st}_{\chi,n}(\mathcal{X}[m],d)^\bullet$. For simplicity, we denote $M_m=\overline{\mathcal{M}}^{st}_{g,n}(\mathcal{X}[m],d)$. Following [18], the obstruction sheaf of M_m is the cohomology sheaf $h^2(\mathfrak{E}^\bullet)$ of a Cěch complex $\mathfrak{E}^\bullet$ whose construction we now recall.

We begin with preferred charts of the universal family of M_m:

$$(f_m,\pi_m):\mathcal{C}_m\longrightarrow\mathcal{X}[m]\times M_m,\quad \mathcal{P}_m\subset\mathcal{C}_m\quad\text{marked points.}$$

We call $\mathcal{U}/\mathcal{V}$ an étale chart of $\mathcal{C}_m/M_m$ if $\mathcal{U}$ is an affine scheme over $\mathcal{V}$, and they fit into a commutative square with étale horizontal arrows

$$\begin{array}{ccc}\mathcal{U} & \longrightarrow & \mathcal{C}_m\\ \downarrow & & \downarrow\\ \mathcal{V} & \longrightarrow & M_m.\end{array} \tag{5.1}$$

Given $\mathcal{U}/\mathcal{V}$ an étale chart of $\mathcal{C}_m/M_m$, we let $\mathcal{C}_\mathcal{U} = \mathcal{C}_m \times_{M_m} \mathcal{U}$, and let $f_\mathcal{U} : \mathcal{C}_\mathcal{U} \to \mathcal{X}[m]$ be the morphisms induced by $f_m : \mathcal{C}_m \to \mathcal{X}[m]$. We let $z_i \in \Gamma(\mathcal{O}_{\mathbb{A}^{m+1}})$ be the i-th coordinate function of $\mathbb{A}^{m+1}$.

Definition 5.1. *We call $(\mathcal{U}/\mathcal{V}, f_\mathcal{U}, (w_{i,1}, w_{i,2})_{1\leqslant i\leqslant m+1})$ a preferred chart of f_m if*

(1) *$\mathcal{U}/\mathcal{V}$ is an étale chart of $\mathcal{C}_m/M_m$;*

(2) *there is a Zariski open $\mathcal{W} \subset \mathcal{X}[m]$ such that $f_\mathcal{U}(\mathcal{U}) \subset \mathcal{W}$, and*

(3) *$w_{i,1}, w_{i,2} \in \Gamma(\mathcal{O}_\mathcal{W})$ such that $w_{i,1} \cdot w_{i,2} = \rho^*(z_i)$, and both $w_{i,1} = 0$ and $w_{i,2} = 0$ are smooth for all i, where $\rho : \mathcal{W} \to \mathbb{A}^{m+1}$ is the composite $\mathcal{W} \subset \mathcal{X}[m] \to \mathbb{A}^{m+1}$.*

Recall that $\mathcal{W} \cap (\rho^*(z_i) = 0) \subset \mathcal{W}$ is a divisor of normal crossing singularity; the choice of $(w_{i,1}, w_{i,2})$ ensures that $(w_{i,1} = w_{i,2} = 0) \subset \mathcal{W}$ is the singular locus of $\mathcal{W} \cap (\rho^*(z_i) = 0)$.

Remark 5.2. Using preferred charts, we can describe the notion of pre-deformable as follows. Let $\mathcal{D}_i = (w_{i,1} = w_{i,2} = 0) \subset \mathcal{W}$. Suppose $f_\mathcal{U}^{-1}(\mathcal{D}_i) \neq \emptyset$. We let $\hat{\mathcal{U}}_i$ (resp. $\hat{\mathcal{V}}_i$) be the formal completion of $\mathcal{U}$ (resp. $\mathcal{V}$) along $f_\mathcal{U}^{-1}(\mathcal{D}_i)$ (resp. $\xi^{-1}(z_i = 0)$). (The morphisms ξ and others are shown in the squares below.)

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f_\mathcal{U}} & \mathcal{W} \\ \big\downarrow{\scriptstyle \pi_\mathcal{V}} & & \big\downarrow{\scriptstyle \rho} \\ \mathcal{V} & \xrightarrow{\xi} & \mathbb{A}^{m+1}, \end{array} \qquad \begin{array}{ccccc} (\hat{u}_{i,j} = 0) & \xrightarrow{\subset} & \hat{\mathcal{U}} & \xrightarrow{f_{\hat{\mathcal{U}}}} & \mathcal{W} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{\hat{\mathcal{V}}}} & & \big\downarrow{\scriptstyle \rho} \\ (\hat{v}_i = 0) & \xrightarrow{\subset} & \hat{\mathcal{V}} & \xrightarrow{\hat{\xi}} & \mathbb{A}^{m+1}. \end{array}$$

Then after shrinking $\mathcal{U}$, if necessary, that f is pre-deformable along $f_\mathcal{U}^{-1}(\mathcal{D}_i)$ is equivalent to that there are $\hat{u}_{i,1}, \hat{u}_{i,2} \in \Gamma(\mathcal{O}_{\hat{\mathcal{U}}})$, $\hat{v} \in \Gamma(\mathcal{O}_{\hat{\mathcal{V}}})$ and an integer n_i so that $\hat{u}_{i,1} = 0$ and $\hat{u}_{i,2} = 0$ are families of smooth curves over $\hat{v}_i = 0$,

$$\hat{z}_{i,1} = f_{\hat{\mathcal{U}}}^*(w_{i,1}^{n_i}), \quad \hat{z}_{i,2} = f_{\hat{\mathcal{U}}}^*(w_{i,2}^{n_i}) \quad \text{and} \quad \hat{u}_{i,1} \cdot \hat{u}_{i,2} = \pi_{\hat{\mathcal{V}}}^*(\hat{v}_i).$$

Let $(\mathcal{U}/\mathcal{V}, w_{i,1}, w_{i,2})$ be a preferred chart. We define $\Gamma(\mathcal{U}, f_m^*\Omega_{\mathcal{X}[m]}^\vee)^\dagger$ to be the set of data (cf. [18, Sect. 1.2])

$$\big(\varphi, (\eta_{i,1}, \eta_{i,2})_{1\leqslant i\leqslant m+1}\big) \in \Gamma(\mathcal{U}, f_m^*\Omega_{\mathcal{X}[m]}^\vee) \oplus \Gamma(\mathcal{O}_\mathcal{U})^{\oplus(2m+2)}$$

such that for all $1 \leqslant i \leqslant m+1$ and $j = 1, 2$,

$$\varphi(f_m^* dw_{i,j}) = f_m^*(w_{i,j}) \cdot \eta_{i,j}, \quad \varphi(f_m^* dz_i) \in \Gamma(\mathcal{O}_\mathcal{V}), \quad \eta_{i,1} + \eta_{i,2} \in \Gamma(\mathcal{O}_\mathcal{V}). \tag{5.2}$$

Here by $\varphi(f_m^* dz_i) \in \Gamma(\mathcal{O}_\mathcal{V})$ we mean that it lies in the image of the pull-back homomorphism $\Gamma(\mathcal{O}_\mathcal{V}) \to \Gamma(\mathcal{O}_\mathcal{U})$. Note that $\Gamma(\mathcal{U}, f_m^*\Omega_{\mathcal{X}[m]}^\vee)^\dagger$ is a $\Gamma(\mathcal{O}_\mathcal{V})$-module.

We cover f by finitely many preferred charts

$$\{(\mathcal{U}_\alpha/\mathcal{V}_\alpha, f_{\mathcal{U}_\alpha}, (w_{i,1}^\alpha, w_{i,2}^\alpha)_{1\leqslant i\leqslant m+1})\}_{\alpha\in\Lambda}.$$

For $A = (\alpha_0, \ldots, \alpha_k) \in \Lambda^{k+1}$, $\mathcal{U}_A = \mathcal{U}_{\alpha_0} \times_\mathcal{C} \cdots \times_\mathcal{C} \mathcal{U}_{\alpha_k}$ coupled with similarly defined $\mathcal{V}_A$ and $(w_{i,1}^A, w_{i,2}^A)$, we define $\Gamma(\mathcal{U}_A, f_m^*\Omega_{\mathcal{X}[m]}^\vee)^\dagger$ similarly.

We form

$$\mathfrak{D}_m^k = \bigoplus_{A\in\Lambda^{k+1}} \Gamma(\mathcal{U}_A, f_m^*\Omega_{\mathcal{X}[m]}^\vee)^\dagger. \tag{5.3}$$

We denote by $\mathbf{C}(M_m)$ (resp. $\mathbf{D}(M_m)$) the triangulated category (resp. derived category) of complexes of coherent sheaves of M_m.

In [18, Sect. 1.2], the second author constructed homomorphisms

$$\partial^k : \mathfrak{D}_m^k \to \mathfrak{D}_m^{k+1}$$

that make $(\mathfrak{D}_m^\bullet, \partial^\bullet)$ a complex in $\mathbf{C}(M_m)$; constructed a complex $\mathfrak{F}_m^\bullet$ in $\mathbf{C}(M_m)$ that is isomorphic to $\mathrm{RHom}_{\pi_m}(\Omega_{\mathcal{C}_m/M_m}(\mathcal{P}_m), \mathcal{O}_{\mathcal{C}_m})$ in $\mathbf{D}(M_m)$, and a homomorphism $\delta^\bullet : \mathfrak{F}_m^\bullet \to \mathfrak{D}_m^\bullet$ in $\mathbf{C}(M_m)$ that has the following properties.

Proposition 5.3. [18, Sect. 1] *Let $\mathfrak{E}_m^\bullet = c(\delta^\bullet)$ be the mapping cone of $\delta^\bullet$; it has $h^i(\mathfrak{E}_m^\bullet) = 0$ for $i \neq 1, 2$. The complex $\mathfrak{E}_m^\bullet$ is part of the perfect obstruction theory of M_m introduced in* [18]. *The obstruction sheaf $\mathcal{O}b_{M_m}$ is the cohomology sheaf $h^2(\mathfrak{E}_m^\bullet)$.*

The perfect obstruction theory of M_m constructed is $(\mathbb{C}^*)^m$-equivariant, and descends to a perfect obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$; the obstruction sheaf $\mathcal{O}b_{M_m}$ descends to the obstruction sheaf of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$.

Note that the vanishing $H^i(\mathfrak{F}_m^\bullet) = H^i(\mathfrak{D}_m^\bullet) = 0$ for $i \geqslant 2$ gives the vanishing $H^i(\mathfrak{E}_m^\bullet) = 0$ for $i \geqslant 3$.

Remark 5.4. Recently, Abramovich and Fantechi [1] have constructed the perfect obstruction theories of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$ and $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_k, \eta)^\bullet$ in the framework of [3]. The construction in this section should give the same cosections, which together with the technique developed in [8] will give the degeneration formula stated in Theorem 4.6.

We now construct the desired cosection of the obstruction sheaf $\mathcal{O}b_{M_m}$, which is built upon a homomorphism

$$\bar{\sigma}_m : \mathcal{O}b_{M_m} \longrightarrow R^1\pi_{m*}\omega_{\mathcal{C}_m/M_m}. \tag{5.4}$$

We let $\Theta_m \in \Gamma(\Omega^2_{\mathcal{X}[m]})$ be the pull-back of the two-form $\Theta \in \Gamma(\Omega^2_{\mathcal{X}[m]})$ (cf. (4.1)) via the projection $\mathcal{X}[m] \to \mathcal{X}$. Viewing it as a homomorphism $\Omega^\vee_{\mathcal{X}[m]} \to \Omega_{\mathcal{X}[m]}$, its pull-back $f_m^*\Theta_m$ defines a homomorphism

$$\Gamma(\mathcal{U}_A, f_m^*\Omega^\vee_{\mathcal{X}[m]})^\dagger \xrightarrow{\mathrm{pr}} \Gamma(\mathcal{U}_A, f_m^*\Omega^\vee_{\mathcal{X}[m]}) \xrightarrow{f_m^*\Theta_m} \Gamma(\mathcal{U}_A, f_m^*\Omega_{\mathcal{X}[m]}), \tag{5.5}$$

where "pr" sends $(\varphi, (\eta_{i,j}))$ to φ. Composed with $f_m^*\Omega_{\mathcal{X}[m]} \to \omega_{\mathcal{C}_m/M_m}$, it defines a homomorphism

$$\Gamma(\mathcal{U}_A, f_m^*\Omega^\vee_{\mathcal{X}[m]})^\dagger \longrightarrow \Gamma(\mathcal{U}_A, \omega_{\mathcal{C}_m/M_m}). \tag{5.6}$$

Let $\mathfrak{C}_m^\bullet = C^\bullet(\Lambda, \omega_{\mathcal{C}_m/M_m})$ be the Cěch complex of the sheaf $\omega_{\mathcal{C}_m/M_m}$ associated with the covering $\{\mathcal{U}_\alpha\}_\Lambda$. Then (5.6) defines a homomorphism of complexes $\mathfrak{D}_m^\bullet \to \mathfrak{C}_m^\bullet$. Taking cohomologies, we obtain

$$\tilde{\sigma}_m : H^1(\mathfrak{D}_m^\bullet) \longrightarrow H^1(\mathfrak{C}_m^\bullet) = R^1\pi_{m*}\omega_{\mathcal{C}_m/M_m}. \tag{5.7}$$

Let

$$(\pi, f) : \mathcal{C} \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \times \mathfrak{X}$$

be the universal family of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$.

Proposition 5.5. *The homomorphism $\tilde{\sigma}_m$ lifts to a homomorphism*

$$\bar{\sigma}_m : \mathcal{O}b_{M_m} = H^2(\mathfrak{E}_m^\bullet) \longrightarrow R^1\pi_{m*}\omega_{\mathcal{C}_m/M_m}.$$

For m large, $\bar{\sigma}_m$ descends to a homomorphism, independent of m,

$$\bar{\sigma} : \mathcal{O}b_{\overline{\mathcal{M}}_{g,n}(\mathfrak{X},d)^\bullet} \longrightarrow R^1\pi_*\omega_{\mathcal{C}/\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet}.$$

Proof. By definition, $H^2(\mathfrak{E}^\bullet)$ is the cokernel of $H^1(\delta^\bullet) : H^1(\mathfrak{F}^\bullet) \to H^1(\mathfrak{D}^\bullet)$; since $H^1(\mathfrak{F}^\bullet) \to H^1(\mathfrak{D}^\bullet)$ is

$$\mathcal{E}xt^1_{\pi_m}(\Omega_{\mathcal{C}_m/M_m}(\mathcal{P}_m), \mathcal{O}_{\mathcal{C}_m}) \longrightarrow H^1(\mathfrak{D}^\bullet),$$

to prove this proposition, it suffices to check that the composite

$$\mathcal{E}xt^1_{\pi_m}(\Omega_{\mathcal{C}_m/M_m}(\mathcal{P}_m), \mathcal{O}_{\mathcal{C}_m}) \longrightarrow H^1(\mathfrak{D}^\bullet) \longrightarrow R^1\pi_{m*}\omega_{\mathcal{C}_m/M_m}$$

is trivial.

According to the construction (5.5), we see that this composition is induced by the sequence

$$\omega^\vee_{\mathcal{C}_m/\mathcal{M}_m} \longrightarrow f^*\Omega^\vee_{\mathcal{X}[m]} \xrightarrow{f_m^*\Theta_m} f^*\Omega_{\mathcal{X}[m]} \longrightarrow \Omega_{\mathcal{C}_m/\mathcal{M}_m}, \tag{5.8}$$

where the first arrow is the dual of the composite $f^*\Omega_{\mathcal{X}[m]} \to \Omega_{\mathcal{C}_m/\mathcal{M}_m} \to \omega_{\mathcal{C}_m/\mathcal{M}_m}$.

Applying [8, Prop. 3.4], we conclude that the composite (5.8) vanishes. This proves that $\bar\sigma_m$ exists. That $\bar\sigma_m$ descends to $\bar\sigma$ follows from Theorem 4.2. Because $\bar\sigma_m$ is constructed canonically, using (4.2), we see that $\bar\sigma$ is independent of m. $\square$

The desired cosection

$$\sigma_m : \mathcal{O}b_{M_m} \longrightarrow \mathcal{O}_{M_m} \tag{5.9}$$

is defined to be the composite of $\bar\sigma_m$ with the sum homomorphism

$$\mathrm{sum} : R^1\pi_{m*}\omega_{\mathcal{C}_m/M_m} \longrightarrow \mathcal{O}_{M_m}$$

specified in (2.2) and after. Since the "sum" homomorphism is canonical, σ_m is well defined, and descends to a homomorphism

$$\sigma_{\mathfrak{X}} : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet}. \tag{5.10}$$

The degeneracy (non-surjective) locus of $\sigma_{\mathfrak{X}}$ is easy to describe. Let

$$Z(\sigma_{\mathfrak{X}}) = \{\xi \in \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet \mid \sigma(\xi) = 0 : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet}|_\xi \to \mathbf{k}(\xi)\}.$$

Proposition 5.6. *For $d > 0$, the set $Z(\sigma_{\mathfrak{X}})$ coincides with the set $\overline{\mathcal{M}}_{g,n}(\mathfrak{D},d)^\bullet \subset \overline{\mathcal{M}}_{g,n}(\mathfrak{X},d)^\bullet$. In particular, it is proper over $\mathbb{A}^1$.*

Proof. Let $\xi \in Z(\sigma_{\mathfrak{X}})$ be a closed point, represented by a stable morphism $u : C \to \mathcal{X}[m]_{\mathbf{t}}$, where $\mathbf{t} \in \mathbb{A}^{m+1}$, and $P \subset C$ its marked points. Let $\mathbb{A}^{1\dagger}$ be the standard log-structure of the pair $0 \in \mathbb{A}^1$; let $\mathbb{A}^{m+1\dagger}$ (resp. $\mathcal{X}[m]^\dagger$) be $\mathbb{A}^{m+1}$ (resp. $\mathcal{X}[m]$) endowed with the pull-back log-structure via $\mathbb{A}^{m+1} \to \mathbb{A}^1$ (resp. $\mathcal{X}[m] \to \mathbb{A}^1$).

Let $\Omega_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}}$ be the sheaf of relative log-differentials of $\mathcal{X}[m]^\dagger \to \mathbb{A}^{m+1\dagger}$. By [18, Prop. 5.1], we have the exact sequences

$$\mathrm{Ext}^1_C(\Omega_C(P), \mathcal{O}_C) \longrightarrow H^1(\mathfrak{D}^\bullet|_\xi) \longrightarrow \mathcal{O}b_{M_m}|_\xi \longrightarrow 0$$

and

$$H^1(C, u^*\Omega^\vee_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}}) \longrightarrow H^1(\mathfrak{D}^\bullet|_\xi) \longrightarrow \bigoplus_{l=1}^{k+1} H^1_{\mathrm{et}}(\mathfrak{R}^\bullet_l|_\xi).$$

Here the exact meaning of $\mathfrak{R}^\bullet_l$ is irrelevant to our discussion. What is crucial is that the explicit form of the homomorphism (5.5) shows that the dual of the composite

$$H^1(C, u^*\Omega^\vee_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}}) \longrightarrow H^1(\mathfrak{D}^\bullet|_\xi) \longrightarrow \mathcal{O}b_{M_m}|_\xi \xrightarrow{\bar\sigma_m|_\xi} H^1(C, \omega_C)$$

is the arrow

$$H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, u^*\Omega_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}} \otimes \omega_C) = H^1(C, u^*\Omega^\vee_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}})^\vee \tag{5.11}$$

induced by the holomorphic two-form $u^*\Theta_m$.

We now suppose that $\mathbf{t}$ lies in the coordinate hyperplane of $\mathbb{A}^{m+1}$. In this case, $\mathcal{X}[m]_{\mathbf{t}} = Y_1 \sqcup \Delta_1 \sqcup \cdots \sqcup \Delta_k \sqcup Y_2$, where $k \geqslant 0$ depends on the number of vanishing coordinates of $\mathbf{t} \in \mathbb{A}^{m+1}$. We let $Y_1^c = \Delta_1 \sqcup \cdots \sqcup \Delta_k \sqcup Y_2$, and denote $E' = Y_1 \cap Y_1^c$.

We write $C = \bigcup_{i=1}^r C_i$ for the connected component decomposition of C. Let

$$\phi_i : H^0(C_i, \mathcal{O}_{C_i}) \longrightarrow H^0(C_i, u^*\Omega_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}} \otimes_{\mathcal{O}_C} \omega_{C_i})$$

be the summands in (5.11), which are individually induced by $u^*\Theta_m|_{C_i}$. Note that restricting to $Y_1 - E' \subset \mathcal{X}[m]_{\mathbf{t}}$, $\Theta_m|_{Y_1-E'}$ induces an isomorphism

$$\Theta_m|_{Y_1-E'} : \Omega^\vee_{Y_1-E'} \longrightarrow \Omega_{Y_1-E'}. \tag{5.12}$$

We now suppose $\phi_i = 0$. Because $d > 0$, by the pre-deformability requirement of stable morphisms in $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$, we have $u(C_i) \cap (Y_1 - E') \neq \emptyset$; because of the isomorphism (5.12), we have $u(C_i) \cap (Y_1 - E') \subset$

B_1 ($B_1 = Y_1 \times_S D$). In particular, $u(C_i) \cap E' \subset B_1 \cap E'$. Further more, because $Y_1^c \to Y_1^c \times_S D := B_1^c$ is the trivial line bundle over B_1^c, knowing C_i is connected and

$$\emptyset \neq u(C_i) \cap E' \subset B_1 \cap E',$$

we must have $u(C_i) \subset B_1 \cup B_1^c$.

Finally, we let $\epsilon : \mathbf{k} \to \mathbf{k}^{\oplus r}$ be the diagonal homomorphism. Then

$$\mathbf{k} \xrightarrow{\epsilon} \mathbf{k}^{\oplus r} = \bigoplus_{i=1}^{r} H^0(C_i, \mathcal{O}_{C_i}) \longrightarrow \bigoplus_{i=1}^{r} H^0(C_i, u^*\Omega_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}} \otimes \omega_{C_i}) \tag{5.13}$$

is zero if and only if all $\phi_i = 0$, which is true only if $u(C_i) \subset \mathcal{X}[m] \times_{\mathcal{X}} \mathcal{D}$ for all i; namely, $u(C) \subset \mathcal{X}[m] \times_{\mathcal{X}} \mathcal{D}$.

Since (5.13) is (5.11) composed with $\mathbf{k} \to H^1(C, \mathcal{O}_C)$, and is dual to the composite

$$H^1(C, u^*\Omega^\vee_{\mathcal{X}[m]^\dagger/\mathbb{A}^{m+1\dagger}}) \longrightarrow H^1(\mathfrak{D}^\bullet|_\xi) \longrightarrow \mathcal{O}b_{M_m}|_\xi \xrightarrow{\sigma_m|_\xi} \mathbf{k},$$

we conclude that $\sigma_m|_\xi = 0$ only if $\xi \in \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}, d)^\bullet$.

The remainder case is when $\mathbf{t}$ does not lie in the coordinate hyperplanes of $\mathbb{A}^{m+1}$. In this case $Y_1 = \mathcal{X}[m]_\mathbf{t}$ is smooth and Y_1^c defined previously is the empty set. Then the same argument shows that $\xi \in Z(\sigma_\mathfrak{X})$ only if $\xi \in \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}, d)^\bullet$. This proves $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}, d)^\bullet \supset Z(\sigma_\mathfrak{X})$.

Finally, since $\Theta|_\mathcal{D} \equiv 0$, $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}, d)^\bullet \subset Z(\sigma_\mathfrak{X})$. This proves the proposition. □

We have a parallel construction for the moduli of relative stable morphisms to the pair (Y_1, E_1). Fixing a partition μ of d, we form the moduli space $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1, \mu)^\bullet$ of relative stable morphisms to $\mathfrak{Y}_1$, as stated in Proposition 4.4. By definition, it is étale covered by $\overline{\mathcal{M}}^{\mathrm{st}}_{\chi,n}(\mathcal{Y}_1[m], \mu)^\bullet/(\mathbb{C}^*)^m$. For simplicity, we abbreviate $M'_m = \overline{\mathcal{M}}^{\mathrm{st}}_{\chi,n}(\mathcal{Y}_1[m], \mu)^\bullet$; we denote by $\mathcal{O}b_{M'_m}$ its obstruction sheaf.

We give a description of the obstruction sheaf of M'_m (cf. [18, Sect. 1.3]). Let

$$(\pi'_m, f'_m) : \mathcal{C}'_m \longrightarrow M'_m \times Y_1[m]$$

(with the marked points implicitly understood) be the universal family of M'_m. We cover $\mathcal{C}'_m \to M'_m$ by preferred charts $\mathcal{U}/\mathcal{V}$ as in (5.1) so that in addition to (5.1) and the validity of Definition 5.1 (with $\mathcal{C}_m$, f_m, $f_\mathcal{U}$, etc. replaced by $\mathcal{C}'_m$, f'_m, $f'_\mathcal{U}$, etc.), we require

(4) there is a $w \in \Gamma(\mathcal{O}_\mathcal{W})$ so that $(w = 0) = \mathcal{W} \cap E[m]$.

We then define $\Gamma(\mathcal{U}, f'^*_m\Omega^\vee_{Y_1[m]})^\dagger$ to be the set of data

$$\big(\varphi, (\eta_{i,1}, \eta_{i,2})_{1\leqslant i\leqslant m+1}, \eta\big) \in \Gamma(\mathcal{U}, f^*_m\Omega^\vee_{\mathcal{X}[m]}, \mathcal{O}_{\mathcal{C}_m}) \oplus \Gamma(\mathcal{O}_\mathcal{U})^{\oplus(2m+2)} \oplus \Gamma(\mathcal{O}_\mathcal{U})$$

such that in addition to (5.2), we have

$$\varphi(f'^*_m dw) = f'^*_m(w) \cdot \eta.$$

Like the case for M_m, using $\Gamma(\mathcal{U}, f'^*_m\Omega^\vee_{Y_1[m]})^\dagger$, we form the complex $\mathfrak{D}'^\bullet$, and then the complex $\mathfrak{E}'^\bullet$. The sheaf cohomology

$$H^2(\mathfrak{E}'^\bullet) = \mathcal{O}b_{M'_m}.$$

Using $Y_1 = S$, we have the holomorphic two-form $\theta \in \Gamma(\Omega^2_{Y_1})$. Like the case for M_m, the form θ induces a homomorphism

$$\sigma'_m : \mathcal{O}b_{M'_m} = H^2(\mathfrak{E}'^\bullet) \longrightarrow \mathcal{O}_{M'_m}.$$

Because the construction of σ'_m is canonical, it is $(\mathbb{C}^*)^m$-equivariant and descends to a homomorphism (independent of m)

$$\sigma_{\mathfrak{Y}_1} : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\eta)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\eta)^\bullet}. \tag{5.14}$$

As before, we denote $\mathfrak{B}_1 = \mathfrak{Y}_1 \times_{Y_1} B_1$, where $B_1 = D \subset S = Y_1$.

Proposition 5.7. *The locus $Z(\sigma_{\mathfrak{Y}_1})$ of non-surjectivity of $\sigma_{\mathfrak{Y}_1}$ is*

$$Z(\sigma_{\mathfrak{Y}_1}) = \overline{\mathcal{M}}_{\chi,n}(\mathfrak{B}_1,\eta)^\bullet \subset \overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\eta)^\bullet.$$

It is proper.

Proof. The proof is similar to that of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$, and will be omitted. □

6 Proof of Theorem 4.5

Applying cosection localized virtual class construction [8], we obtain

Definition-Proposition 6.1. *The cosection $\sigma_{\mathfrak{X}}$ in* (5.10) *and $\sigma_{\mathfrak{Y}_1}$ in* (5.14) *define cosection localized virtual classes:*

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D},d)^\bullet \quad \text{and} \quad [\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_1,\eta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{B}_1,\eta)^\bullet.$$

To prove Theorem 4.5, the first step is to use $[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$ to produce a numerical equivalence of $[\overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$ in (4.18) with the cosection localized virtual class of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,d)^\bullet$.

For $c \in \mathbb{A}^1$, we endow $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet$ with the perfect obstruction theory induced by the Cartesian product above and the obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$. (Recall that for $c \in \mathbb{A}^1$, $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet = \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet \times_{\mathbb{A}^1} c$; for $c \neq 0$, using $\mathfrak{X}_c \cong S$, we have $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet = \overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet$.)

Lemma 6.2. *Let $c \in \mathbb{A}^1$ be any closed point. The obstruction sheaves of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$ and $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet$ fit into the following exact sequence, and the cosection $\sigma_{\mathfrak{X}}$ of* (5.10) *restricting to $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet$ lifts to a cosection $\sigma_{\mathfrak{X}_c}$ of $\mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet}$:*

$$\begin{array}{ccccc} \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet}|_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet} & \longrightarrow & \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet} & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \sigma_{\mathfrak{X}}|_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet}} & & \Big\downarrow{\scriptstyle \sigma_{\mathfrak{X}_c}} & & \\ \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet} & = & \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet}. & & \end{array} \tag{6.1}$$

Proof. Recall that Θ is the pull-back of $\theta \in \Gamma(\Omega^2_S)$ via the projection $\mathcal{X} \to S$; for $c \neq 0$, using $\mathcal{X}_c \cong S$, we obtain the following commutative square

$$\begin{array}{ccc} \Omega^\vee_{\mathcal{X}}|_{\mathcal{X}_c} & \xrightarrow{\Theta} & \Omega_{\mathcal{X}}|_{\mathcal{X}_c} \\ \Big\uparrow & & \Big\downarrow \\ \Omega^\vee_{\mathcal{X}_c} & \xrightarrow{\theta} & \Omega_{\mathcal{X}_c}. \end{array}$$

Since the cosection $\sigma_{\mathfrak{X}}$ is constructed using Θ, this commutative square guarantees that the cosection $\sigma_{\mathfrak{X}_c}$ shown in (6.1) constructed using θ on $\mathcal{X}_c \cong S$ commutes with $\sigma_{\mathfrak{X}}$ as shown in (6.1). This proves the lemma for $c \neq 0$.

For $c = 0$, the proof is similar. In this case, the square (6.1) is commutative with $\Omega_{\mathcal{X}_c}$ replaced by $\Omega_{\mathcal{X}_0}(\log E)$. Since maps $[u,C] \in \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,d)^\bullet$ are pre-deformable, the pull-back u^* is a homomophism $u^* : u^*\Omega_{\mathcal{X}_0}(\log E) \to \omega_C$. Using these, we see that $\sigma_{\mathcal{X}_0}$ exists and fits into the commutative diagram (6.1). □

Remark 6.3. For $c \neq 0$, using $\mathcal{X}_c \cong S$, the obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet$ induced from $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$ coincides with that of $\overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet$ (without referring to the family $\mathcal{X} \to \mathbb{A}^1$); the cosection $\sigma_{\mathfrak{X}_c}$ coincides with the cosection of $\mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(S,d)^\bullet}$ constructed in Section 2.

Applying Proposition 5.6, we see that the non-surjective locus $Z(\sigma_{\mathfrak{X}_c})$ of $\sigma_{\mathfrak{X}_c}$ is $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_c,d)^\bullet$. We let

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_c,d)^\bullet$$

be the cosection localized virtual class of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c,d)^\bullet$, using the cosection $\sigma_{\mathfrak{X}_c}$.

Proposition 6.4. *Let $c \in \mathbb{A}^1$ be any closed point, and let $\iota_c : c \to \mathbb{A}^1$ be the inclusion. Then*

$$\iota_c^! [\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_c, d)^\bullet.$$

Proof. This follows from [8, Thm 5.2]. □

Corollary 6.5. *For $c \neq 0 \in \mathbb{A}^1$, using the canonical isomorphism $\mathcal{X}_c \cong S$, we have*

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{\chi,n}(S, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \overline{\mathcal{M}}_{\chi,n}(D, d)^\bullet.$$

Proof. For $c \neq 0 \in \mathbb{A}^1$, by Remark 6.3, the obstruction theory and the cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c, d)^\bullet$ coincide with that of $\overline{\mathcal{M}}_{\chi,n}(S, d)^\bullet$. Thus $[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_c, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{\chi,n}(S, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$. This proves the corollary, which implies (4.18) in Theorem 4.5. □

We quote the following existence result. Let $\pi_{\mathbb{A}^1} : \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \to \mathbb{A}^1$ be the projection; let $t \in \Gamma(\mathcal{O}_{\mathbb{A}^1})$ be the standard coordinate function. We continue to denote by Γ the set of possible decompositions defined in and after (4.13).

Lemma 6.6. [18, Sect. 3.1] *There are pairs (L_γ, s_γ), indexed by $\gamma \in \Gamma$, of line bundles and sections $s_\gamma \in \Gamma(L_\gamma)$ on $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$ such that*

$$\bigotimes_{\gamma\in\Gamma} L_\gamma \cong \pi_{\mathbb{A}^1}^* \mathcal{O}_{\mathbb{A}^1}, \quad \text{and} \quad \prod_{\gamma\in\Gamma} s_\gamma = \pi_{\mathbb{A}^1}^* t.$$

Let $L_0 = \pi_{\mathbb{A}^1}^* \mathcal{O}_{\mathbb{A}^1}$ and $s_0 = \pi_{\mathbb{A}^1}^* t \in \Gamma(L_0)$. Let

$$c_1(L_0, s_0) : A_* \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}, d)^\bullet \longrightarrow A_{*-1} \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0, \gamma)^\bullet$$

be the localized first Chern class of the pair (L_0, s_0).

Proposition 6.7. *We have the identity*

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = c_1(L_0, s_0)[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\gamma\in\Gamma} c_1(L_\gamma, s_\gamma)[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Proof. The first identity follows from that $c_1(L_0, s_0) = \iota_0^!$ and Proposition 6.4; the second identity follows from Lemma 6.6. □

The summands in the last summation have their own virtual cycle interpretations. We fix a $\gamma \in \Gamma$. We define

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet := (s_\gamma = 0) \subset \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet. \tag{6.2}$$

Because it is defined by the vanishing of a section of a line bundle on $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$, it has an induced perfect obstruction theory from that of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$ [18, Prop. 3.8].

Proposition 6.8. *The cosection $\sigma_{\mathfrak{X}}$ restricted to $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet$ lifts to a cosection $\sigma_{\mathfrak{X}_0,\gamma}$:*

$$\begin{array}{ccccc} \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet}|_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet} & \longrightarrow & \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet} & \longrightarrow & 0 \\ \big\downarrow {\scriptstyle \sigma_{\mathfrak{X}}|_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet}} & & \big\downarrow {\scriptstyle \sigma_{\mathfrak{X}_0,\gamma}} & & \\ \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet} & = & \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet}. & & \end{array} \tag{6.3}$$

Let $[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$ be the cosection localized virtual class of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet$ using the cosection $\sigma_{\mathfrak{X}_0,\gamma}$. Then

$$[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = c_1(L_\gamma, s_\gamma)[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0, \gamma)^\bullet.$$

Proof. We comment that the first line in (6.3) is an exact sequence, which follows from the fact that the obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet$ is induced from that of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$. That $\sigma_{\mathfrak{X}_0,\gamma}$ exists follows the same reasoning as in the proof of Lemma 6.2. Finally, the proof of the identity of cycles follows from the proof of [8, Thm. 5.2]. □

We next give $\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet$, defined via the Cartesian product (4.14), a perfect obstruction theory. Using the Cartesian product, and that each factor $\overline{\mathcal{M}}_{\chi_i,n_i}(\mathfrak{Y}_i, \mu)^\bullet$ has perfect obstruction theory, we obtain a perfect obstruction theory of $\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet$. We call this the perfect obstruction theory induced by the Cartesian product (4.14).

It has a second induced perfect obstruction theory. We let $\mathfrak{M}$ be the Artin stack of not necessarily connected nodal curves; for μ the partition appearing in the decomposition index γ (cf. (4.13)) and $\ell = \ell(\mu)$, we denote by $\mathfrak{M}_\ell$ be the Artin stack of (ordered) ℓ-pointed not necessarily connected nodal curves. By identifying the q_i^1 with q_i^2 for all i of all pairs $\big((C_1, q_i^1), (C_2, q_i^2)\big)$ in $\mathfrak{M}_\ell \times \mathfrak{M}_\ell$, we obtain a gluing morphism

$$\mathfrak{M}_\ell \times \mathfrak{M}_\ell \longrightarrow \mathfrak{M}. \tag{6.4}$$

By the Cartesian product above and the pre-deformable assumption, one sees that the tautological inclusion

$$\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet \xrightarrow{\subset} \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \times_{\mathfrak{M}} \big(\mathfrak{M}_\ell \times \mathfrak{M}_\ell\big) \tag{6.5}$$

is both open and closed.

Since (6.4) is a regular local immersion of smooth stacks, together with the perfect obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$, it induces a perfect obstruction theory of the fiber product (on the right hand side) in (6.5). As the arrow in (6.5) is both open and closed, it defines a perfect obstruction theory of $\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet$. We call this perfect obstruction theory the one induced by (6.5).

Lemma 6.9. [18] *The two perfect obstruction theories of $\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet$ induced by the Cartesian product* (4.14) *and by the open and closed immersion* (6.5) *are identical.*

We now look at the product

$$\overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{Y}_1, \mu)^\bullet \times \overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2, \mu)^\bullet.$$

Since its obstruction sheaf is the direct sum of the pull-back of the obstruction sheaves of the two individual factors, it has a cosection that is the pull-back of the cosection $\sigma_{\mathfrak{Y}_1}$ in (5.14) and the zero cosection $\sigma_{\mathfrak{Y}_2} = 0$ of $\mathcal{O}b_{\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2,\mu)^\bullet}$. We denote this cosection by $\sigma_{\mathfrak{Y}_1 \times \mathfrak{Y}_2}$.

Proposition 6.10. *The cosections $\sigma_{\mathfrak{Y}_1 \times \mathfrak{Y}_2}$ and $\sigma_{\mathfrak{X}}$ restricted to $\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet$ lift to cosections of $\mathcal{O}b_{\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)^\bullet}$ via* (4.14) *and* (6.5), *respectively. The two* (*lifted*) *cosections are identical, which we denote by*

$$\sigma_{\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2} : \mathcal{O}b_{\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2,\gamma)^\bullet}. \tag{6.6}$$

The locus of its non-surjectivity is $\overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2, \gamma)^\bullet$. Let $[\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}$ be the cosection localized virtual class, then

$$[\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = \delta^!\big([\overline{\mathcal{M}}_{\chi_1,n_1}(\mathfrak{Y}_1, \eta)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \times [\overline{\mathcal{M}}_{\chi_2,n_2}(\mathfrak{Y}_2, \eta)^\bullet]^{\mathrm{vir}}\big),$$

as classes in $A_*\overline{\mathcal{M}}(\mathfrak{B}_1 \sqcup \mathfrak{B}_2, \gamma)^\bullet$.

Proof. The proof of the existence of the lifting is parallel to the proof of Lemma 6.2; the two lifted cosections coincide because both are induced by the same two-form $\theta \in \Gamma(\Omega_S^2)$. Finally, the identity on cycles follows from the proof of [8, Thm. 5.2]. $\square$

Theorem 4.5 will follow after we prove

Proposition 6.11. *Let μ be the partition appearing in the data $\gamma \in \Gamma$, and let ψ_γ be the morphism defined in* (4.17). *Then*

$$\frac{\mu!}{|\mathrm{Aut}\,\mu|} \cdot (\psi_\gamma)_*[\overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, \gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0, d)^\bullet.$$

We prove the proposition. Consider the morphism (cf. (4.15))

$$\Psi_\gamma : \overline{\mathcal{M}}(\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2, \gamma)^\bullet \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet.$$

We let $\hat{\phi}_\gamma : \hat{\varXi}_\gamma \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$ (resp. $\phi_\ell : \hat{\mathfrak{M}}_{\ell\times\ell} \to \mathfrak{M}$) be the formal completion of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$ (resp. $\mathfrak{M}$) along the image $\operatorname{Im}\Psi_\gamma$ (resp. the image of (6.4)). We have the following commutative diagram

$$\begin{array}{ccccc}
\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet & \xrightarrow{\Phi_\gamma} & \hat{\varXi}_\gamma & \xrightarrow{\hat{\phi}_\gamma} & \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}_\ell\times\mathfrak{M}_\ell & \xrightarrow{\zeta_\ell} & \hat{\mathfrak{M}}_{\ell,\ell} & \xrightarrow{\phi_\gamma} & \mathfrak{M}.
\end{array}$$

Here Φ_γ is the lift of Ψ_γ. Note that the left square is a Cartesian square. Since the lower left horizontal arrow is finite and étale (of degree $\ell!$) to its image, by the topological invariance of étale morhisms, we can find an Artin stack $\tilde{\mathfrak{M}}_{\ell,\ell}$ that contains $\mathfrak{M}_\ell\times\mathfrak{M}_\ell$ as its closed substack and a pure degree $\ell!$ finite and étale morphism $\tilde{\mathfrak{M}}_{\ell,\ell}\to\hat{\mathfrak{M}}_{\ell,\ell}$ that extends $\mathfrak{M}_\ell\times\mathfrak{M}_\ell\to\hat{\mathfrak{M}}_{\ell,\ell}$.

We let $\tilde{\varXi}_\gamma=\hat{\varXi}_\gamma\times_{\hat{\mathfrak{M}}_{\ell,\ell}}\tilde{\mathfrak{M}}_{\ell,\ell}$; by definition, it contains $\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet$ as its closed substack, and Φ_γ lifts to a finite and étale $\vartheta_\gamma:\tilde{\varXi}_\gamma\to\hat{\varXi}_\gamma$. A direct inspection shows that ϑ_γ has pure degree $|\operatorname{Aut}\gamma|$.

We can insert $\tilde{\varXi}_\gamma$ and $\tilde{\mathfrak{M}}_{\ell,\ell}$ in the above diagram to get a commutative

$$\begin{array}{ccccccc}
\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet & \xrightarrow{\subset} & \tilde{\varXi} & \xrightarrow{\vartheta_\gamma} & \hat{\varXi}_\gamma & \xrightarrow{\hat{\phi}_\gamma} & \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet \\
\downarrow & & \downarrow{\scriptstyle \tilde{h}_\gamma} & & \downarrow & & \downarrow \\
\mathfrak{M}_\ell\times\mathfrak{M}_\ell & \xrightarrow{\zeta_\ell} & \tilde{\mathfrak{M}}_{\ell,\ell} & \longrightarrow & \hat{\mathfrak{M}}_{\ell\times\ell} & \xrightarrow{\phi_\ell} & \mathfrak{M}.
\end{array} \tag{6.7}$$

This time, the two left horizontal arrows are closed immersions, and the two middle horizontal arrows are finite and étale. (We comment that $\tilde{\mathfrak{M}}_{\ell,\ell}$ is the $\mathbf{Q}$ mentioned in [18, Lemma 4.10].)

The importance of the $\tilde{\mathfrak{M}}_{\ell,\ell}$ is the following existence result [18, Lemma 4.12]. There are pairs of line bundles and sections $(\tilde{L}_i,\tilde{s}_i)$, where $\tilde{L}_i$ are line bundles on $\tilde{\mathfrak{M}}_{\ell,\ell}$, $\tilde{s}_i\in\Gamma(\tilde{L}_i)$ and $1\leqslant i\leqslant\ell$, such that

(i) $(\vartheta_\gamma\circ\hat{\phi}_\gamma)^*L_\gamma=\tilde{h}_\gamma^*(\bigotimes_{i=1}^\ell L_i^{\otimes\mu_i})$, and $(\vartheta_\gamma\circ\hat{\phi}_\gamma)^*s_\gamma=\tilde{h}_\gamma^*(\prod_{i=1}^\ell s_i^{\mu_i})$;

(ii) $(s_i=0)\subset\tilde{\mathfrak{M}}_{\ell,\ell}$ is the divisor of the locus where the i-th gluing-nodes of curves in $\operatorname{Im}(\zeta_\ell)$ are not smoothed.

Because of (ii), Φ_γ factors and effects an isomorphism

$$\Phi_\gamma:\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,d)^\bullet\xrightarrow{\cong}\bigcap_{i=1}^{\ell}(\tilde{h}_\gamma^*s_i=0)\subset\tilde{\varXi}_\gamma.$$

We let

$$\mathbf{N}_0=\bigcap_{i=1}^{\ell}(s_i=0)\ \subset\ \mathbf{N}=\bigcap_{i=1}^{\ell}(s_i^{\mu_i}=0)\ \subset\ \tilde{\mathfrak{M}}_{\ell,\ell};$$

We define

$$\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet:=\bigcap_{i=1}^{\ell}(\tilde{h}_\gamma^*s_i^{\mu_i}=0)=\tilde{\varXi}_\gamma\times_{\tilde{\mathfrak{M}}_{\ell,\ell}}\mathbf{N}\ \subset\ \tilde{\varXi}_\gamma. \tag{6.8}$$

Then we have the Cartesian square

$$\begin{array}{ccccc}
\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet & \longrightarrow & \overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet & & \\
\downarrow & & \downarrow & & \\
\mathbf{N}_0 & \longrightarrow & \mathbf{N} & \xrightarrow{\subset} & \tilde{\mathfrak{M}}.
\end{array} \tag{6.9}$$

Using the top line of (6.7), we also have an induced morphism

$$\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet\longrightarrow\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet. \tag{6.10}$$

Lemma 6.12. *The stack $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ has a perfect obstruction theory induced by the arrow* (6.10).

Proof. Using the definition of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ (in (6.2)), and using the item (i) above, the morphism (6.10) factors through a finite, étale morphism

$$\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet \longrightarrow \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet \tag{6.11}$$

of pure degree $|\mathrm{Aut}\,\gamma|$. Thus, the perfect obstruction theory of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ (defined after (6.2), using the defining equation $s_\gamma=0$ and that of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},d)^\bullet$) induces a perfect obstruction theory of $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$. □

Like in [18, p. 278], one works out the perfect relative obstruction theory of the pair $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet\to\mathbf{N}$; using that $\mathbf{N}_0\subset\mathbf{N}$ is a closed substack, and the Cartesian square (6.9), we obtain an induced perfect relative obstruction theory of $\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet\to\mathbf{N}_0$.

Lemma 6.13. [18, Lemma 4.13] *The relative obstruction theory of $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet\to\mathbf{N}$ is compatible with the perfect obstruction theory of $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ given in Lemma* 6.12*; the relative obstruction theory of $\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet\to\mathbf{N}_0$ is compatible with the perfect obstruction theory of $\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet$.*

Repeating the previous argument, using that both cosections $\sigma_{\mathfrak{X}}$ and $\sigma_{\mathfrak{Y}_1\sqcup\mathfrak{Y}_2}$ are induced by the two-form $\theta\in\Gamma(\Omega^2_S)$, one checks that the cosection $\sigma_{\mathfrak{X}_0,\gamma}$ (cf. Proposition 6.8) lifts to a cosection $\sigma_{\mathfrak{X}_0^{\sim},\gamma}$ of the obstruction sheaf of $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$, and that the later fits into the following commuting square

$$\begin{array}{ccc}
\mathcal{O}b_{\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)}|_{\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet} & \xrightarrow{\text{surj}} & \mathcal{O}b_{\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet} \\
\Big\downarrow{\scriptstyle\sigma_{\mathfrak{X}_0^{\sim},\gamma}} & & \Big\downarrow{\scriptstyle\sigma_{\mathfrak{Y}_1\sqcup\mathfrak{Y}_2}} \\
\mathcal{O}_{\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet} & = & \mathcal{O}_{\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet}.
\end{array} \tag{6.12}$$

Recall that using the relative obstruction theories, we can construct the virtual cycles $[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet/\mathbf{N}]^{\mathrm{vir}}$ and $[\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet/\mathbf{N}_0]^{\mathrm{vir}}$ (cf. [18, Sect. 4], see also [3]). Applying the cosection localized virtual cycle construction, using $\sigma_{\mathfrak{X}_0^{\sim},\gamma}$ and $\sigma_{\mathfrak{Y}_1\sqcup\mathfrak{Y}_2}$, we obtain cosection localized virtual cycles

$$[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet/\mathbf{N}]^{\mathrm{vir}}_{\mathrm{loc}} \quad\text{and}\quad [\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet/\mathbf{N}_0]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}(\mathfrak{B}_1\sqcup\mathfrak{B}_2,\gamma)^\bullet.$$

Here the inclusion $\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet\subset\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ is a bijection.

Since $\mathbf{N}_0$ is smooth, using the second part of Lemma 6.13, we have identity

$$[\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet/\mathbf{N}_0]^{\mathrm{vir}}_{\mathrm{loc}}=[\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}(\mathfrak{B}_1\sqcup\mathfrak{B}_2,\gamma)^\bullet, \tag{6.13}$$

without relying on any rational equivalence.

Because $\tilde{\mathfrak{M}}_{\ell,\ell}$ is smooth, $\mathbf{N}_0\subset\tilde{\mathfrak{M}}_{\ell,\ell}$ is smooth of codimension ℓ and defined by $\cap(s_i=0)$, and because $\mathbf{N}\subset\tilde{\mathfrak{M}}_{\ell,\ell}$ is defined by $\cap(s_i^{\eta_i}=0)$, parallel to the proof of [18, Lemma 4.8], we conclude

$$\mu!\cdot[\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet/\mathbf{N}_0]^{\mathrm{vir}}_{\mathrm{loc}}=[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)/\mathbf{N}]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}(\mathfrak{B}_1\sqcup\mathfrak{B}_2,\gamma)^\bullet, \tag{6.14}$$

without relying on any rational equivalence.

At last, using the cosection $\sigma_{\mathfrak{X}_0^{\sim},\gamma}$, repeating the argument in [8, Thm. 5.2], we conclude that the canonical rational equivalence constructed by Vistoli [24] (see also [11] and [10]) provides a rational equivalence

$$[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet/\mathbf{N}]^{\mathrm{vir}}_{\mathrm{loc}}=[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}(\mathfrak{B}_1\sqcup\mathfrak{B}_2\gamma)^\bullet. \tag{6.15}$$

Since the argument is similar, we omit the details here.

Proof of Proposition 6.11. Combining identities (6.13), (6.14) and (6.15), we obtain

$$\mu!\cdot[\overline{\mathcal{M}}(\mathfrak{Y}_1\sqcup\mathfrak{Y}_2,\gamma)^\bullet/\mathbf{N}_0]^{\mathrm{vir}}_{\mathrm{loc}}=[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}(\mathfrak{B}_1\sqcup\mathfrak{B}_2\gamma)^\bullet.$$

Because (6.11) is finite, étale and of pure degree $|\mathrm{Aut}\,\mu|$, and that the obstruction theory of $\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$ is the pull-back of that of $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet$, for ψ_γ the morphism defined in (4.17), we have

$$(\psi_\gamma)_*[\overline{\mathcal{M}}^{\sim}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}=|\mathrm{Aut}\,\gamma|\cdot[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0,\gamma)^\bullet]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*\overline{\mathcal{M}}_{\chi,n}(\mathfrak{D}_0,d)^\bullet.$$

This proves the proposition. □

7 Low-degree GW-invariants of surfaces

In this section, we use degeneration formula to prove the formulas of low degree GW-invariants of surfaces.

Applying the deformation invariance results in Section 3, for smooth algebraic surfaces X with $\theta \in H^0(K_X)$ and $(\theta = 0)$ smooth, its GW-invariants are given by the local GW-invariants of any spin surface S that is the total space of a theta-characteristic L of a smooth curve D of genus

$$h = K_X^2 + 1$$

such that $h^0(L) \equiv \chi(\mathcal{O}_X) \mod 2$.

We denote the localized GW-invariants of S by

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{\chi, d[D], \mathrm{loc}}^{S, \bullet} = \langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle_{\chi, d[D], \mathrm{loc}}^{X, \bullet}, \quad \text{for } \gamma_i \in H^*(S). \tag{7.1}$$

In this section, we use $d[D]$ to signify that d is the degree of stable maps.

Following the convention, since (7.1) is possibly non-trivial only when

$$-\chi = d\, K_S^2 + \sum_{i=1}^{n} \alpha_i, \quad \alpha_i \in \mathbb{Z}_{\geqslant 0}, \tag{7.2}$$

we shall omit the reference to χ in the notation of (7.1) with the understanding that it is given by (7.2).

Let $\gamma \in H^2(D, \mathbb{Z})$ be the Poincaré dual of a point in D. The main result of this section is the following theorem, conjectured by Maulik and Pandharipande [21, (8)–(9)].

Theorem 7.1. *Let $S \to D$ and $h = g(D)$ be as before. Then the degree one and two localized GW-invariants with descendants are*

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{[D], \mathrm{loc}}^{S, \bullet} = (-1)^{h^0(L)} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i + 1)!} (-2)^{-\alpha_i}, \tag{7.3}$$

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{2[D], \mathrm{loc}}^{S, \bullet} = (-1)^{h^0(L)}\, 2^{h+n-1} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i + 1)!} (-2)^{\alpha_i}. \tag{7.4}$$

The first identity is proved in [9, Prop. 1.3] (see also [7]). Before we prove (7.4) using the degeneration formula proved above, we recall the following two identities proved in [9].

Let Y_0 be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Then we have (see [9, Prop. 3.3])

$$\left\langle \prod_{i=1}^{n} \tau_{\alpha_i}(\gamma) \right\rangle_{2[\mathbb{P}^1], \mathrm{loc}}^{Y_0, \bullet} = 2^{n-1} \prod_{i=1}^{n} \frac{\alpha_i!}{(2\alpha_i + 1)!} (-2)^{\alpha_i}; \tag{7.5}$$

we also have (see [9, Prop. 3.4])

$$\langle \tau_1(\gamma) \rangle_{2[D], \mathrm{loc}}^{S, \bullet} = (-1)^{h^0(L)} \binom{2^h}{-3}. \tag{7.6}$$

Let $\mu = (1, 1)$ be the obvious partition of 2.

Lemma 7.2. *Let (Y_1, E) and (Y_2, E) be the relative pairs resulting from the degeneration constructed in the previous section. Then*

$$\langle 1 \rangle_{(1,1), \mathrm{loc}}^{Y_1/E, \bullet} = (-1)^{h^0(L)}\, 2^h\, [pt^2] \quad \textit{and} \quad \langle \tau_1(\gamma) \rangle_{(1,1)}^{Y_2/E, \bullet} \star [pt^2] = -\frac{1}{6}.$$

Proof. We first look at the first identity. It is easy to see, from the construction of localized relative invariants (Definition-Proposition 6.1), that $\langle 1 \rangle_{(1,1), \mathrm{loc}}^{Y_1/E, \bullet}$ is a scalar multiple of $[pt^2]$. Then, an easy virtual dimension counting shows that the stable maps that contribute to this invariant must have $-\chi = 2(h-1)$.

By (7.2), the composites of these stable maps with $p: X \to D$ are étale covers of D. Hence the (relevant) moduli space of relative stable maps to (Y_1, E) is a disjoint union of 2^{2h} vector spaces, each consists of all liftings of an étale cover of D. By the proof of [9, Prop. 2.5], we obtain

$$\langle 1\rangle^{Y_1/E,\bullet}_{(1,1),\mathrm{loc}} = \sum_{u:2\text{-fold étale covers of } D} (-1)^{h^0(u^*L)}[pt^2].$$

It is known that étale double covers of D are parameterized by the set of order 2 line bundles on D, and exactly $2^{h-1}(2^h+1)$ of them satisfy $h^0(u^*L) \equiv h^0(L) \bmod 2$ (see [5]). Therefore we have

$$\langle 1\rangle^{Y_1/E,\bullet}_{(1,1),\mathrm{loc}} = (-1)^{h^0(L)}\left(2^{h-1}(2^h+1) - 2^{h-1}(2^h-1)\right)[pt^2] = (-1)^{h^0(L)}2^h[pt^2].$$

This proves the first equation.

Since Y_2 is the total space of the trivial line bundle over $\mathbb{P}^1$, any stable map in $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{Y}_2,(1,1))^\bullet$ with two distinct intersection points with E has two irreducible components, one with the marked point and the other without. Therefore, because

$$\langle \tau_1(\gamma)\rangle^{Y_2,E}_{(1)} \star [pt] = \langle \tau_1(\gamma)\rangle^{Y_0,\bullet}_{[\mathbb{P}^1],\mathrm{loc}} = -\frac{1}{12} \quad \text{and} \quad \langle 1\rangle^{Y_2/E}_{(1)} \star [pt] = 1,$$

we have

$$\begin{aligned}\langle \tau_1(\gamma)\rangle^{Y_2/E,\bullet}_{(1,1)} \star [pt^2] &= (\langle \tau_1(\gamma)\rangle^{Y_2/E}_{(1)} \star [pt])(\langle 1\rangle^{Y_2/E}_{(1)} \star [pt]) \\ &\quad + (\langle 1\rangle^{Y_2/E}_{(1)} \star [pt])(\langle \tau_1(\gamma)\rangle^{Y_2/E}_{(1)} \star [pt]) = -\frac{1}{6}.\end{aligned}$$

This proves the lemma. □

We prove (7.4). By the degeneration formula, we have

$$\left\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\right\rangle^{S,\bullet}_{2[D],\mathrm{loc}} = \frac{1}{2}\langle 1\rangle^{Y_1/E,\bullet}_{(1,1),\mathrm{loc}} \star \left\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\right\rangle^{Y_2/E,\bullet}_{(1,1)} + 2\langle 1\rangle^{Y_1/E,\bullet}_{(2),\mathrm{loc}} \star \left\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\right\rangle^{Y_2/E,\bullet}_{(2)}. \tag{7.7}$$

In particular, from (7.6) and Lemma 7.2, we have

$$(-1)^{h^0(L)}\left(\frac{2^h}{-3}\right) = \langle \tau_1(\gamma)\rangle^{S,\bullet}_{2[D],\mathrm{loc}} = \frac{1}{2}(-1)^{h^0(L)}\left(\frac{2^h}{-6}\right) + 2\langle 1\rangle^{Y_1/E,\bullet}_{(2),\mathrm{loc}} \star \langle \tau_1(\gamma)\rangle^{Y_2/E,\bullet}_{(2)}.$$

Comparing this with the case where $D = \mathbb{P}^1$ ($h = 0$), we see that the relative invariants $\langle 1\rangle^{Y_1/E,\bullet}_{(1,1),\mathrm{loc}}$ and $\langle 1\rangle^{Y_1/E,\bullet}_{(2),\mathrm{loc}}$ are exactly those for $D = \mathbb{P}^1$, multiplied by $(-1)^{h^0(L)}\,2^h$. Therefore by (7.7) and (7.5), we have

$$\left\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\right\rangle^{S,\bullet}_{2[D],\mathrm{loc}} = (-1)^{h^0(L)}\,2^h\left\langle \prod_{i=1}^n \tau_{\alpha_i}(\gamma)\right\rangle^{Y_0,\bullet}_{2[\mathbb{P}^1],\mathrm{loc}} = (-1)^{h^0(L)}\,2^{h+n-1}\prod_{i=1}^n \frac{\alpha_i!}{(2\alpha_i+1)!}(-2)^{\alpha_i}.$$

This proves Theorem 7.1.

We end this section by commenting on other possible degenerations of spin surfaces. Let S be a spin surface over a smooth curve D of genus g with the associated theta-characteristic L. Besides the situation studied in details in this paper, we can also generate S to a normal crossing surface having three irreducible components, which we describe now.

We first degenerate D to a (chain like) nodal curve D' of three smooth irreducible components $D'_1 \cup D'_2 \cup D'_3$ so that D'_1 and D'_3 have genus g_1 and g_3 with $g_1+g_3 = g$, and $D'_2 \cong \mathbb{P}^1$. The theta characteristic L on D can be specialized to a line bundle L' on D' so that $L'|_{D'_1}$ and $L'|_{D'_3}$ are theta-characteristics of D'_1 and D'_3, respectively, and $L'|_{D'_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. We let S' be the total space of L'; it is the union of smooth components $S'_i = S' \times_{D'} D'_i$. Like before, we denote by $\mathcal{X} \to V$ the total space of this degeneration, where $0 \in V$ is a smooth curve; we agree $\mathcal{X}_0 \cong S'$. Using this family $\mathcal{X}$, we can form the stack $\mathfrak{X}$ and the moduli of stable morphisms

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet$$

as in Proposition 4.2.

To construct the cosection of its obstruction sheaf, we notice that the tautological holomorphic two-form θ on S extends to a section $\Theta \in \Gamma(\mathcal{X}, \omega_{\mathcal{X}/V})$ that vanishes along $S_2' \subset \mathcal{X}_0$. Parallel to the discussion we had, $\theta' = \Theta|_{\mathcal{X}_0}$ defines a cosection σ' of the obstruction sheaf of the moduli space

$$\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}_0, d)^\bullet = \overline{\mathcal{M}}_{\chi,n}(\mathfrak{X}, d)^\bullet \times_V 0.$$

One checks that the non-surjective loci of σ' contains stable maps $u : C \to S'$ so that $u(C) \subset D_1' \cup S_2' \cup D_3'$. Suppose $d \geqslant 2$. One sees that the non-surjective loci of σ' is not proper, of which we can not apply the proof derived in this paper. Nevertheless, it is hoped that a detailed study of this non-properness will yield information on higher degree localized GW-invariants of spin surfaces.

8 Comment on reduced GW-invariants of K3 surfaces

Let X be a smooth projective K3 surface with a non-trivial algebraic class $\alpha \neq 0 \in H_2(X, \mathbb{Z})$. The GW-invariant satisfies

$$\langle\cdot\rangle^{X,\bullet}_{\chi,n,\alpha} := [\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet]^{\mathrm{vir}} = 0 \in A_*\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet. \tag{8.1}$$

(We use $\langle\cdot\rangle$ to represent the cycle class.) This can be seen as follows. Let $\theta \in \Gamma(\Omega^2_X)$ be a no-where vanishing holomorphic two-form; it induces a cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet$

$$\sigma : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet}$$

that is everywhere surjective (since $\alpha \neq 0$). Applying the cosection localized virtual cycles [8], we obtain the vanishing (8.1).

Toward enumerating curves in K3 surfaces, modified GW-invariants of K3 surfaces were introduced. In [4], using twisted family of K3 surfaces Bryan-Leung introduced the family GW-invariants of a K3 surface. Later, by deforming almost complex structures of surfaces Lee introduced the family GW-invariants of surfaces with $p_g > 0$ [12]. For the case of K3 surfaces, Lee showed that the two versions of family GW-invariants coincide.

Algebraic version of family GW-invariants of a K3 surface can be defined as follows. Given a pair (X, α) of an algebraic K3 surface X with an algebraic class $\alpha \neq 0 \in H_2(X, \mathbb{Z})$, we pick a family of K3 surfaces $\mathcal{X} \to T$ over a disk $0 \in T$ so that $\mathcal{X}_0 \cong X$ and that the class $\alpha \in H_2(X, \mathbb{Z}) = H_2(\mathcal{X}, \mathbb{Z})$ ceases to be algebraic at the first order deformation of $\mathcal{X}_0 \subset \mathcal{X}$. We then form the moduli space $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}, \alpha)^\bullet$ of stable morphisms to $\mathcal{X}$ of class α; it is a Deligne-Mumford stack proper over T. Since α does not deform in the first order as an algebraic class, the tautological embedding

$$\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet = \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}_0, \alpha)^\bullet \xrightarrow{\cong} \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}, \alpha)^\bullet \tag{8.2}$$

is an isomorphism.

Let $(\pi, f) : \mathcal{C} \to \overline{\mathcal{M}}_{\chi,n}(\mathcal{X}, \alpha)^\bullet \times \mathcal{X}$ be its universal family. To define the family GW-invariants, we form the perfect relative obstruction theory

$$\big(R\pi_* f^* T_{\mathcal{X}/T}\big)^\vee \longrightarrow \mathbb{L}^\bullet_{\overline{\mathcal{M}}_{\chi,n}(\mathcal{X},\alpha)^\bullet/\mathfrak{M}_{\chi,n}\times T}. \tag{8.3}$$

Let $\mathfrak{C}_T \subset h^1/h^0\big(R\pi_* f^* T_{\mathcal{X}/T}\big)$ be the intrinsic normal cone embedded via the obstruction theory (8.3). Because of (8.2), $\overline{\mathcal{M}}_{\chi,n}(\mathcal{X}, \alpha)^\bullet = \overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet$ is proper. We define the algebraic version of family GW-invariants of X to be

$$\langle\cdot\rangle^{X,\bullet,\mathrm{fl}}_{\chi,n,\alpha} := 0^![\mathfrak{C}_T] \in A_*\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet, \tag{8.4}$$

where $0^!$ is intersecting with the zero section of $h^1/h^0\big(R\pi_* f^* T_{\mathcal{X}/T}\big)$. (Here we use "fl" to stand for "family".)

In [22], Okounkov-Pandharipande introduced the reduced GW-invariants of an algebraic K3 surface. It can be phrased using cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{\chi,n}(X, \alpha)^\bullet$. Because of the identity (8.2),

$(\pi, f, \mathcal{C})$ is also the universal family of $\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet$. The holomorphic two form $\theta \in \Gamma(\Omega_X^2)$ defines a surjective cosection (homomorphism) [8]

$$\sigma' : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet}.$$

Since σ' is surjective, it induces a surjective bundle stack homomorphism

$$[\sigma'] : h^1/h^0\big(R\pi_* f^* T_X\big) \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet}.$$

We let $\mathbf{R}$ be its kernel bundle stack.

Let $\mathfrak{C}' \subset h^1/h^0\big(R\pi_* f^* T_X\big)$ be the intrinsic normal cone of $\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet$. The cosection localization of [8] shows that the cycle $[\mathfrak{C}']$ lifts to a cycle $[\mathfrak{C}'_{\mathrm{lift}}] \in Z_*\mathbf{R}$. The reduced GW-invariants of X (in [22]) is

$$\langle \cdot \rangle^{X,\bullet,\mathrm{red}}_{\chi,n,\alpha} = 0^!_{\mathbf{R}}[\mathfrak{C}'_{\mathrm{lift}}] \in A_*\overline{\mathcal{M}}_{\chi,n}(X,\alpha)^\bullet. \tag{8.5}$$

Lemma 8.1. *The algebraic version of family GW-invariants* (8.4) *and the reduced GW-invariants* (8.5) *of a* K3 *surface X are identical.*

Proof. By (8.2), $R\pi_* f^* T_{\mathcal{X}/T} = R\pi_* f^* T_X$. The Lemma then follows from the fact that $\mathfrak{C}_T$ intersects $\mathbf{R} \subset h^1/h^0\big(R\pi_* f^* T_X\big)$ transversally and the cycle of the (stack-theoretic) intersection $[\mathfrak{C}_T \cap \mathbf{R}]$ is the lift of $[\mathfrak{C}']$. This can be proved using the algebraic class $\alpha \in H_2(X,\mathbb{Z})$ when ceases to be algebraic in the first order deformation of X in $\mathcal{X}$. We leave the details to the readers. □

In [15], Lee-Leung proved the index two Yau-Zaslow conjecture using a degeneration formula of the family GW-invariants of an elliptic K3 surface. By the above equivalence result, this degeneration is also a degeneration of reduced GW-invariants of K3. We show that the degeneration formula can be derived parallel to the method developed in this paper.

We let $X \to \mathbb{P}^1$ be an elliptic K3 surface. Pick a $q \in \mathbb{P}^1$ so that $F = X \times_{\mathbb{P}^1} q$ is a smooth fiber. We form the family $\mathcal{X} \to \mathbb{A}^1$ that is the blow-up of $X \times \mathbb{A}^1$ along $F \times 0$. Note that the fiber of $\mathcal{X}$ over $t \neq 0 \in \mathbb{A}^1$ is X, and the special fiber $\mathcal{X}_0$ is the union of X with $F \times \mathbb{P}^1$, intersecting transversally along $F \subset X$ and $F \times 0 \subset F \times \mathbb{P}^1$. To avoid confusing the $X \subset \mathcal{X}_0$ with the general fiber of $\mathcal{X}$, we denote the two irreducible components of $\mathcal{X}_0$ by $Y_1 = X$ and $Y_2 = F \times \mathbb{P}^1$.

Let $p : \mathcal{X} \to X$ be the projection, and let $\Theta = p^*\theta \in \Gamma(\Omega^2_{\mathcal{X}})$. Let $\iota_i : Y_i \to \mathcal{X}$ be the tautological inclusion. Then

$$\iota_1^*\Theta = \theta \in \Gamma(\Omega^2_{Y_1}) \quad \text{and} \quad \iota_2^*\Theta = 0 \in \Gamma(\Omega^2_{Y_2}). \tag{8.6}$$

Parallel to the case studied in this paper, for the algebraic class $\alpha \in H_2(X,\mathbb{Z})$, we form the moduli $\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},\alpha)^\bullet$ of stable morphisms to the stack $\mathfrak{X}$ of expanded degenerations of the family $\mathcal{X}/\mathbb{A}^1$; the form Θ induces a cosection

$$\sigma : \mathcal{O}b_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},\alpha)^\bullet} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},\alpha)^\bullet}.$$

Suppose $\alpha \cdot [F] \neq 0$, one checks that this cosection is surjective, thus the cosection localization lemma in [8] implies that we can define a reduced virtual cycle $[\overline{\mathcal{M}}_{\chi,n}(\mathfrak{X},\alpha)^\bullet]^{\mathrm{vir}}_{\mathrm{red}}$. Then using (8.6), one checks that the resulting degeneration formula of the reduced GW-invariants of X is the usual degeneration formula after pairing the reduced relative GW-invariants of (Y_1, F) with the (ordinary) relative GW-invariants of $(Y_2, F \times 0)$. The degeneration formula used in [15] can be derived along this line.

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