LOCALIZING VIRTUAL CYCLES BY COSECTIONS

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ABSTRACT. We show that a cosection of the obstruction sheaf of a perfect obstruction theory localizes the virtual cycle to the non-surjective locus of the cosection. We construct a localized Gysin map and localized virtual cycles. Various applications of this construction are discussed.

1. INTRODUCTION

Invariants defined by virtual cycles of moduli spaces have played important roles in research in algebraic geometry. Invariants of this kind include the much studied Gromov-Witten (in short GW) invariants of varieties, and the recently introduced Donaldson-Thomas (in short DT) invariants of Calabi-Yau three-folds.

One of the main challenges in studying such invariants is to develop techniques for investigating virtual cycles. In this paper, we will present a new technique, which we call localization by a cosection of the obstruction sheaf (Theorem 1.1). This is achieved after constructing a localized Gysin map (Proposition 1.3).

Theorem 1.1 (Localization by a cosection)**.** *Let $\mathcal{M}$ be a Deligne-Mumford stack endowed with a perfect obstruction theory. Suppose the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ admits a surjective homomorphism $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_U$ over an open $U \subset \mathcal{M}$. Let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$. Then $(\mathcal{M}, \sigma)$ has a localized virtual cycle*

$$[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \mathcal{M}(\sigma).$$

This cycle satisfies the usual properties of virtual cycles; it relates to the usual virtual cycle $[\mathcal{M}]^{\mathrm{vir}}$ via $[\mathcal{M}]^{\mathrm{vir}} = \iota_* [\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \mathcal{M}$, where $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$ is the inclusion.

This work originated from our effort to understand Lee-Parker's discovery that GW-invariants of surfaces with holomorphic two-forms localize along the loci of stable maps to canonical divisors [21]. We show that a holomorphic two-form induces a cosection of the obstruction sheaf of the moduli space; applying localization by cosection we recover and generalize Lee-Parker's theorem as follows.

Let X be a smooth quasi-projective variety with a holomorphic two-form $\theta \in H^0(\Omega^2_X)$. Let $\overline{\mathcal{M}}_{g,n}(X, \beta)$ denote the moduli stack of n-pointed stable maps of genus g to X with homology class β. The two-form θ on X induces a cosection σ of the obstruction sheaf of $\overline{\mathcal{M}}_{g,n}(X, \beta)$; the degeneracy loci $\mathcal{M}(\sigma)$ consist of stable maps $[u : C \to X]$ satisfying $\theta(u_* TC) = 0$ (called θ-null stable maps) where θ is viewed as an antisymmetric homomorphism $T_X \to \Omega_X$. Applying the localization by cosection, we obtain

Theorem 1.2. *For a pair (X, θ) of a smooth quasi-projective variety and a holomorphic two-form, the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ vanishes unless β is represented by a θ-null stable map.*

The localization by cosection has other applications in the study of GW-invariants and of DT-invariants. One is the work on quantum cohomology of the Hilbert scheme of points by W.-P. Li and the second named author. Using meromorphic two-forms of algebraic surfaces, they determined the two-point extreme quantum cohomology of the Hilbert scheme of points of any algebraic surface [23].

A special but important case is when the cosection is regular and surjective everywhere. In this case, a reduced virtual cycle can be defined using localization by a cosection. The notion of reduced virtual fundamental class was first introduced by Okounkov-Pandharipande in [29, §3.4.3] for GW-invariants of holomorphic symplectic varieties. However, the idea was already around before, e.g. in the work of Donaldson [8], Lee [19], Bryan-Leung [3, 4] and J. Li [22]. In studying curve counting [28], Maulik-Pandharipande-Thomas use localization of the virtual cycle by a cosection to define the reduced Gromov-Witten and Pandharipande-Thomas invariants of a class of Calabi-Yau three-folds; in the appendix of the same paper, they provide an in-depth discussion on the relationships between the reduced class construction and usual deformation-obstruction theories.

Another application of reduced virtual cycles appears in a recent paper [15]: the authors use localization by a cosection to prove a $\mathbb{C}^*$-wall crossing formula of DT-invariants by producing a reduced virtual fundamental class of the master space.

Quite recently, H.-L. Chang and the second named author, in [6], use the cosection localization to define the Gromov-Witten invariants of stable maps to $\mathbb{P}^4$ *with p-fields* and relate them to those of stable maps to a quintic 3-fold.

The proof of localization by a cosection (Theorem 1.1) consists of two parts. In one part, we prove that the intrinsic normal cone of the perfect obstruction theory of $\mathcal{M}$ lies in the kernel cone stack of the cosection σ. In particular, if there is a surjective homomorphism $\sigma : \mathcal{O}b_{\mathcal{M}} \to \mathcal{L}$ to a locally free sheaf $\mathcal{L}$, the intrinsic normal cone lies in a smaller cone stack and we can define a *reduced virtual fundamental class*.

The other part of the proof is the construction of the following localized Gysin map.

Proposition 1.3 (Localized Gysin Map)**.** *Let E be a rank r vector bundle on a Deligne-Mumford stack $\mathcal{M}$, and let σ be a meromorphic surjective cosection of E, meaning that there is an open $U \subset \mathcal{M}$ so that σ is a surjective homomorphism $\sigma : E|_U \to \mathcal{O}_U$. Let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$ and let $E(\sigma) = E|_{\mathcal{M}(\sigma)} \cup \ker\{\sigma : E|_U \to \mathcal{O}_U\}$. The Basic Construction stated in Section 2 defines a homomorphism*

$$s^!_{E,\sigma} : A_* E(\sigma) \to A_{*-r} \mathcal{M}(\sigma),$$

which we call the localized Gysin map. Furthermore, if we denote the inclusions by $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$, $\tilde{\iota} : E(\sigma) \to E$, and let $s_E : A_* E \to A_{*-r} \mathcal{M}$ be the usual Gysin map of intersecting with the zero section of E, then we have

$$\iota_* \circ s^!_{E,\sigma} = s^!_E \circ \tilde{\iota}_* : A_* E(\sigma) \to A_{*-r} \mathcal{M}.$$

By applying the localized Gysin map to the intrinsic normal cone in $E(\sigma)$, we obtain the localized virtual cycle $[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}}$ in Theorem 1.1.

The paper is organized as follows. In Sections 2 and 3, we construct the localized Gysin map. In Section 4, we show that the intrinsic normal cone lies in the kernel cone stack. In Section 5, we define the localized virtual cycles and prove its deformation invariance. An application of this localization technique to GW-invariants of varieties with holomorphic two-forms is presented in Section 6. In Section 7, we discuss other possible applications. In the Appendix, we give an analytic definition of the localized Gysin map and prove its equivalence with the algebraic definition.

Addendum. The current version is the replacement of the first half of [12]. Our prior treatment of localization by a cosection used a topological definition of the localized Gysin map [12], which limits its application in algebraic geometry. In this paper, we provide an algebraic construction of the localized Gysin map, including the Chow groups of the total space of a cone-stack over a Deligne-Mumford stack. This makes it possible to directly apply other developed techniques on virtual cycles to localized virtual cycles.

In the sequel to this paper, we will prove a degeneration formula of localized GW-invariants, and include its application worked out in [12], in proving Maulik-Pandharipande's conjecture ([27]) on degree two GW-invariants of surfaces.

Notation: In this paper, all schemes and stacks are defined over the complex number field $\mathbb{C}$. We will use Z_*X (resp. A_*X; resp. W_*X) to denote the group of algebraic cycles (resp. group of algebraic cycles modulo rational equivalence; resp. group of rational equivalences) with $\mathbb{Q}$-coefficients.

Since we will be working with locally free sheaves and cycles in the total spaces of the vector bundles associated to the locally free sheaves, to streamline the notation, we will use the same symbol to denote a locally free sheaf as well as its associated vector bundle. Thus, given a vector bundle (locally free sheaf) E, by Z_*E we mean the group of cycles of the total space of E, and by $E \to \mathcal{O}b_{\mathcal{M}}$ we mean a sheaf homomorphism $\mathcal{O}_{\mathcal{M}}(E) \to \mathcal{O}b_{\mathcal{M}}$.

Given a subvariety $T \subset E$, we denote by $[T] \in Z_*E$ its associated cycle, and denote by $[T] \in A_*E$ its rational equivalence class in A_*E.

2. Localized Gysin maps

Let $\pi : E \to \mathcal{M}$ be a rank r vector bundle over a Deligne-Mumford (DM for short) stack $\mathcal{M}$. The usual Gysin map $s_E^! : A_d E \to A_{d-r}\mathcal{M}$ is defined by "intersecting" cycles in E with the zero section s_E of E.

In this section, we suppose that E has a surjective meromorphic cosection σ.

Definition 2.1. *A surjective meromorphic cosection is a surjective homomorphism* $\sigma : E|_U \to \mathcal{O}_U$ *for an open substack* $U \subset \mathcal{M}$.

Remark 2.2. *For the purpose of defining the localized Gysin map, the degeneracy locus of a meromorphic cosection $\sigma : E- \to \mathcal{O}_\mathcal{M}$ includes the loci where σ is undefined and where σ is not surjective. In the definition above, the open U is the locus where σ is defined and surjective.*

For such a σ, we let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$, let $G = \ker\{\sigma : E|_U \to \mathcal{O}_U\}$ and let $E(\sigma) = E|_{\mathcal{M}(\sigma)} \cup G$, which is closed in E. The goal of this section is to define a localized Gysin map

$$s^!_{E,\sigma} : A_d E(\sigma) \to A_{d-r}\mathcal{M}(\sigma)$$

that has the usual properties of the Gysin map and coincides with $s^!_E$ when composed with the tautological $A_{d-r}\mathcal{M}(\sigma) \to A_{d-r}\mathcal{M}$.

Definition 2.3. *Let $\rho : X \to \mathcal{M}$ be a morphism from a variety X to $\mathcal{M}$ such that $\rho(X) \cap U \neq \emptyset$. We call ρ a σ-regularizing morphism if ρ is proper, and $\rho^*(\sigma)$ extends to a surjective homomorphism*

$$\tilde{\sigma} : \tilde{E} := \rho^* E \to \mathcal{O}_X(D)$$

for a Cartier divisor $D \subset X$. We adapt the convention that $\tilde{\rho} : \tilde{E} \to E$ is the projection; $\tilde{G} := \ker\{\tilde{\sigma}\} \subset \tilde{E}$; $|D| \subset X$ is the support of D, and $\rho(\sigma) : |D| \to \mathcal{M}(\sigma)$ is the ρ restricted to $|D|$.

Basic Construction: *Let $[B] \in Z_d E(\sigma)$ be a cycle represented by a closed integral substack $B \subset E(\sigma)$. In case $B \subset E|_{\mathcal{M}(\sigma)}$, we define $s^!_{E,\sigma}([B]) = s^!_{E|_{\mathcal{M}(\sigma)}}([B]) \in A_{d-r}\mathcal{M}(\sigma)$. Otherwise, we pick a variety X and a σ-regularizing $\rho : X \to \mathcal{M}$ such that there is a closed integral $\tilde{B} \subset \tilde{G}$ so that $\tilde{\rho}_*([\tilde{B}]) = k \cdot [B] \in Z_d E$ for some $k \in \mathbb{Z}$. We define*

$$(2.1) \qquad s^!_{E,\sigma}([B])_{\rho,\tilde{B}} = k^{-1} \cdot \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}([\tilde{B}])) \in A_{d-r}\mathcal{M}(\sigma).$$

Here $[D]\cdot : A_* X \to A_{*-1}|D|$ is the intersection with the divisor D.

Here is a simple example.

Example 2.4. *Let M be an n-dimensional smooth scheme, E a vector bundle of rank n on M and $\sigma : E \to \mathcal{O}_M$ a cosection so that $\sigma^{-1}(0) = p$ is a simple point in M. Let $[M]$ be the cycle of the zero section of E in $Z_n E(\sigma)$. Then $s^!_{E,\sigma}([M]) = (-1)^n[p]$.*

The proof is straightforward. Let $\rho : \tilde{M} \to M$ be the blow-up at p and let $D \subset \tilde{M}$ be the exceptional divisor. We let $F = \ker\{\rho^* E \to \mathcal{O}_{\tilde{M}}(-D)\}$, and compute $s^!_F([\tilde{M}]) = c_{n-1}(F)[\tilde{M}] = [D]^{n-1}$. Since $s^!_{E,\sigma}([M]) = \rho_*([-D] \cdot s^!_F([\tilde{M}]))$,

$$s^!_{E,\sigma}([M]) = \rho_*([-D] \cdot c_{n-1}(F)) = \rho_* c_n(\rho^* E) = c_n(E) = (-1)^n[p].$$

See [5] for a direct application of Example 2.4.

Lemma 2.5. *Let the notation be as in the basic construction. Then for each closed integral $B \subset E(\sigma)$ not contained in $E|_{\mathcal{M}(\sigma)}$, we can find a pair $(\rho, \tilde{B})$ so that $s^!_{E,\sigma}([B])_{\rho,\tilde{B}}$ is defined. Furthermore the resulting cycle class $s^!_{E,\sigma}([B])_{\rho,\tilde{B}} \in A_{d-r}\mathcal{M}(\sigma)$ is independent of the choice of $(\rho, \tilde{B})$.*

Proof. We let $B \subset E$ be as in the statement of the Lemma; we let $B_0 = \pi(B) \subset \mathcal{M}$, where $\pi : E \to \mathcal{M}$ is the projection. To pick ρ, we then pick a normal variety X and a proper and generically finite morphism $\rho : X \to B_0$. Since $\mathcal{M}$ is a DM-stack, such ρ exists. Using ρ, we pull back $\tilde{E} = \rho^* E$, and $\rho^* \sigma : \rho^* E|_{\rho^{-1}U} \to \mathcal{O}_{\rho^{-1}U}$. Since $B \times_{\mathcal{M}} U \neq \emptyset$, $\rho^{-1}U \neq \emptyset$.

Next, possibly after replacing X by a blow-up of X, we can assume that $\rho^* \sigma$ extends to a surjective homomorphism $\tilde{\sigma} : \tilde{E} \to \mathcal{O}_X(D)$ for a Cartier divisor $D \subset X$. Thus ρ is a σ-regularizing morphism. We let $\tilde{\rho} : \tilde{E} \to E$ and $\tilde{G} \subset \tilde{E}$ be as defined in Definition 2.3.

Since $\rho : X \to B_0$ is generically finite, there is an open $O \subset B_0$ so that $\rho|_{\rho^{-1}O} : \rho^{-1}O \to O$ is flat and finite. We let $\tilde{B}$ be an irreducible component of $\tilde{\rho}^{-1}(B \cap \pi^{-1}(O))$. Since B is integral and $\tilde{\rho}$ is proper, $\tilde{\rho}(\tilde{B}) = B$, thus $\tilde{\rho}_*([\tilde{B}]) = k[B]$ for an integer k. This shows that $s^!_{E,\sigma}([B])_{\rho,\tilde{B}}$ is defined according to the Basic Construction.

We next check that $s^!_{E,\sigma}([B])_{\rho,\tilde{B}}$ is independent of the choice of $(\rho, \tilde{B})$. Let $\rho' : X' \to \mathcal{M}$, $\tilde{E}' := \rho'^* E$ and $\tilde{B}' \subset \tilde{E}'$ be another choice that fulfills the requirement of the Basic Construction, thus giving rise to the class $s^!_{E,\sigma}([B])_{\rho',\tilde{B}'} \in A_{d-r}\mathcal{M}(\sigma)$. We show that $s^!_{E,\sigma}([B])_{\rho,\tilde{B}} = s^!_{E,\sigma}([B])_{\rho',\tilde{B}'}$.

We form $Y = X \times_{\mathcal{M}} X'$; denote by $q : Y \to X$ and $q' : Y \to X'$ the projections, and by $p : Y \to \mathcal{M}$ the composite $\rho \circ q = \rho' \circ q'$. Since $\rho : X \to \mathcal{M}$ is generically finite and $\tilde{B} \to X$ is dominant, $\tilde{B} \times_E \tilde{B}'$ contains a pure dimension d irreducible component $\bar{B}$ that surjects onto $\tilde{B}$ and $\tilde{B}'$ via the tautological projections $\tilde{q} : \bar{E} := p^* E \to \tilde{E}$ and $\tilde{q}' : \bar{E} \to \tilde{E}'$, respectively.

We claim that there is an isomorphism $\mathcal{O}_Y(q^* D) \cong \mathcal{O}_Y(q'^* D')$ so that

$$(2.2) \qquad q^* \tilde{\sigma} = q'^* \tilde{\sigma}' : \bar{E} = q^* \tilde{E} = q'^* \tilde{E}' \longrightarrow \mathcal{O}_Y(q^* D) \cong \mathcal{O}_Y(q'^* D').$$

Indeed, since $q^* \tilde{\sigma}|_{p^{-1}U} = q'^* \tilde{\sigma}'|_{p^{-1}U} = p^* \sigma|_{p^{-1}U}$, the kernel sheaves $\ker\{q^* \tilde{\sigma}\}|_{p^{-1}U} = \ker\{q'^* \tilde{\sigma}'\}|_{p^{-1}U}$. Since both are subsheaves of the locally free sheaf $\bar{E}$, the two kernels are identical. Therefore, since both $q^* \tilde{\sigma}$ and $q'^* \tilde{\sigma}'$ are surjective, we obtain (2.2). Furthermore, since when restricted to $p^{-1}U$ the last isomorphism in (2.2) is the identity, the arrow in (2.2) sends $1 \in \Gamma(\mathcal{O}_Y(q^* D)|_{p^{-1}U})$ to $1 \in \Gamma(\mathcal{O}_Y(q'^* D')|_{p^{-1}U})$. We conclude that $q^* D = q'^* D'$ as divisors. We denote this divisor by $\bar{D}$.

We let $\bar{\sigma} = q^* \tilde{\sigma} = q'^* \tilde{\sigma}'$, and let $\bar{G} = \ker\{\bar{\sigma}\}$. By (2.2), $\bar{G}$ is the pull-back of $\tilde{G}$ via q^* and of $\tilde{G}' = \ker\{\tilde{\sigma}'\}$ via q'^*. Back to $\bar{B}$, since $B \not\subset E|_{\mathcal{M}(\sigma)}$, $B \subset E(\sigma)$ and $\tilde{p}(\bar{B}) = B$, we have $\bar{B} \subset \bar{G}$. We let k_i be the integers so that

$$\tilde{q}_*([\bar{B}]) = k_1[\tilde{B}], \quad \tilde{q}'_*([\bar{B}]) = k_2[\tilde{B}'], \quad \tilde{\rho}_*([\tilde{B}]) = k_3[B], \quad \tilde{\rho}'_*([\tilde{B}']) = k_4[B].$$

Then since $\tilde{\rho} \circ \tilde{q} = \tilde{\rho}' \circ \tilde{q}'$, we have $k_1 k_3 = k_2 k_4$.

We let $p(\sigma) : |\bar{D}| \to \mathcal{M}(\sigma)$, $q(\sigma) : |\bar{D}| \to |D|$ and $\rho(\sigma) : |D| \to \mathcal{M}(\sigma)$ be the restrictions of p, q and ρ to $|\bar{D}|$ and $|D|$, respectively. Since the projections $\tilde{q}$ and $\tilde{\rho}$ are proper and because $\tilde{q}^{-1}\tilde{G} = \bar{G}$, applying the projection formula (and using $\bar{D} = q^* D$), we obtain

$$s^!_{E,\sigma}([B])_{p,\bar{B}} = (k_1 k_3)^{-1} \cdot p(\sigma)_*([\bar{D}] \cdot s^!_{\bar{G}}([\bar{B}])) = (k_1 k_3)^{-1} \cdot \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}([\tilde{q}_* \bar{B}]))$$

$$= k_3^{-1} \cdot \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}([\tilde{B}])) = s^!_{E,\sigma}([B])_{\rho,\tilde{B}}.$$

Similarly, we have $s^!_{E,\sigma}([B])_{p,\bar{B}} = s^!_{E,\sigma}([B])_{\rho',\tilde{B}'}$. This proves the Lemma. $\qquad \square$

The Lemma shows that the Basic Construction defines a homomorphism

(2.3) $$s^!_{E,\sigma} : Z_d E(\sigma) \longrightarrow A_{d-r} \mathcal{M}(\sigma).$$

We next check that this homomorphism descends to a homomorphism $A_d E(\sigma) \to A_{d-r} \mathcal{M}(\sigma)$. We need a simple Lemma. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of vector bundles (locally free sheaves) on a variety Y. Let $r_i = \mathrm{rank}\, F_i$ and $p_i : F_i \to Y$ be the projections. Let $\iota : F_1 \to F_2$ be the inclusion.

Lemma 2.6. *For any cycle $\alpha \in A_d F_1$, we have*

$$s^!_{F_2}(\iota_* \alpha) = c_{r_3}(F_3) \cap s^!_{F_1}(\alpha) = s^!_{F_1}(c_{r_3}(F_3) \cap \alpha)$$

Proof. By [10, Thm 3.3], there is a cycle $B \in Z_{d-r_1} Y$ so that $\alpha = p_1^* B \in A_* F_1$. Since $p_1^* B$ is the total space of F_1 restricted to B, its Segre class is $s(p_1^* B) = c(F_1)^{-1} \cap B$, where $c(F_1)$ is the total Chern class of F_1. By the formula expressing Gysin map in terms of the Segre class [10, Prop. 6.1],

$$s^!_{F_2}(\iota_* \alpha) = s^!_{F_2}(p_1^* B) = [c(F_2) \cap (c(F_1)^{-1} \cap B)]_{d-r_2} = [c(F_3) \cap B]_{d-r_2} = c_{r_3}(F_3) \cap B.$$

On the other hand,

$$s^!_{F_1}(c_{r_3}(F_3) \cap \alpha) = s^!_{F_1}(c_{r_3}(F_3) \cap [p_1^* B]) = s^!_{F_1}(p_1^*(c_{r_3}(F_3) \cap [B])) = c_{r_3}(F_3) \cap [B].$$

This proves the Lemma.   □

The same method proves the following Lemma that will be useful later. Let $\rho' : X' \to \mathcal{M}$ be a σ-regularizing morphism; let $\tilde{\sigma}' : \tilde{E}' = \rho'^* E \to \mathcal{O}_{X'}(D')$, $\tilde{G}' = \ker\{\tilde{\sigma}'\}$, and $\rho'(\sigma) : |D'| \to \mathcal{M}(\sigma)$ be the tautological projection, as in the proof of Lemma 2.5.

Lemma 2.7. *Let the notation be as before. Suppose $[\tilde{B}'] \in Z_d \tilde{G}'$ such that $\tilde{\rho}'_*[\tilde{B}'] = 0 \in Z_d E$. Then $\rho'(\sigma)_*([D'] \cdot s^!_{\tilde{G}'}([\tilde{B}'])) = 0 \in A_{d-r} \mathcal{M}(\sigma)$.*

Proof. We let $B = \tilde{\rho}'(\tilde{B}')$. Since $\tilde{\rho}'$ is proper, B is closed and integral in E; since $\tilde{\rho}'_*([\tilde{B}']) = 0$ in $Z_d E$, $\dim B < d$. We let $B_0 = \pi(B) \subset \mathcal{M}$. We then pick $\rho : X \to \mathcal{M}$ a σ-regularizing morphism so that $\rho(X) = B_0$ and $\rho : X \to B_0$ is generically finite.

We form $Y = X \times_{\mathcal{M}} X'$ and let $p : Y \to \mathcal{M}$ be the induced morphism. As in the proof of Lemma 2.5 (and using the notations developed in the proof), we have $\bar{E} = p^* E \to Y$, $\tilde{q}' : \bar{E} \to \tilde{E}'$ and an integral $\bar{B} \subset \bar{E}$ so that $\tilde{q}'(\bar{B}) = \tilde{B}'$. Let $\tilde{q} : \bar{E} \to \tilde{E}$ and let $\tilde{B} = \tilde{q}(\bar{B})$. Since $\tilde{B}$ is integral, since $\tilde{\rho}(\tilde{B}) = B$, since $B \to B_0$ is dominant, and since $\rho : X \to B_0$ is dominant and generically finite, $\dim \tilde{B} = \dim \tilde{\rho}(\tilde{B}) = \dim B_0 < d$. Thus $\tilde{q}_*([\bar{B}]) = 0 \in Z_d \tilde{E}$. Therefore,

$$k \cdot \rho'(\sigma)_*([D'] \cdot s^!_{\tilde{G}'}([\tilde{B}'])) = p(\sigma)_*([\bar{D}] \cdot s^!_{\bar{G}}([\bar{B}])) = \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}([\tilde{q}_* \bar{B}])) = 0.$$

This proves the Lemma.   □

We next show that the map $s^!_{E,\sigma}$ preserves rational equivalence. In this paper, we adopt the the convention that a rational equivalence $R \in W_d E$ is a sum $R = \sum_{\alpha \in \Lambda} n_\alpha [S_\alpha, \phi_\alpha]$, where $S_\alpha \subset E$ are $(d+1)$-dimensional closed integral and $\phi_\alpha \in \Bbbk(S_\alpha)^*$. We also use $\partial_0[S, \phi] = (\phi = 0) \cdot [S]$ and $\partial_\infty[S, \phi] = (\phi = \infty) \cdot [S]$.

Lemma 2.8. *For $R \in W_d E(\sigma)$, $s^!_{E,\sigma}(\partial_0 R) = s^!_{E,\sigma}(\partial_\infty R)$.*

Proof. We only need to prove the Lemma for $R = [S, \phi] \in W_d E(\sigma)$, where $S \subset E(\sigma)$ is closed and integral. In case $S \subset E|_{\mathcal{M}(\sigma)}$, then both $\partial_0 R$ and $\partial_\infty R \in Z_d E|_{\mathcal{M}(\sigma)}$, and the required identity holds because the usual Gysin map $s^!_{E|_{\mathcal{M}(\sigma)}} : Z_d(E|_{\mathcal{M}(\sigma)}) \to A_{d-r}\mathcal{M}(\sigma)$ preserves the rational equivalence.

We now suppose $S \not\subset E|_{\mathcal{M}(\sigma)}$. We let $S_0 = \pi(S) \subset \mathcal{M}$. We pick a proper and generically finite σ-regularizing $\rho : X \to \mathcal{M}$ so that $\rho(X) = S_0$. We let $\tilde{\sigma} : \tilde{E} := \rho^* E \to \mathcal{O}_X(D)$ be the surjective homomorphism extending $\rho^*\sigma$. By Lemma 2.5, such ρ exists. Because ρ is generically finite, we can find a closed integral $d+1$ dimensional $\tilde{S} \subset \tilde{E}$ so that with $\tilde{\rho} : \tilde{E} \to E$ the projection, $\tilde{\rho}(\tilde{S}) = S$.

We let $\tilde{G} = \ker\{\tilde{\sigma} : \tilde{E} \to \mathcal{O}_X(D)\}$. Because $S \subset E(\sigma)$ and $S \not\subset E|_{\mathcal{M}(\sigma)}$, $\tilde{S} \subset \tilde{G}$. We let $\tilde{\phi}$ be the pull-back $\tilde{\rho}^*\phi$, and let $\tilde{R} = [\tilde{S}, \tilde{\phi}]$. Thus, $\tilde{\rho}_* \tilde{R} \in W_d E(\sigma)$ and $\tilde{\rho}_* \tilde{R} = k \cdot R$, for an integer k; since $\tilde{\rho}$ is proper,

$$(2.4) \qquad \tilde{\rho}_*(\partial_0 \tilde{R}) = k \cdot \partial_0 R \quad \text{and} \quad \tilde{\rho}_*(\partial_\infty \tilde{R}) = k \cdot \partial_\infty R.$$

We now decompose the cycle $\partial_0 R = B_1 + B_2$ so that each summand $[T] \in B_1$ (resp. $[T] \in B_2$) satisfies $T \subset E|_{\mathcal{M}(\sigma)}$ (resp. $T \not\subset E|_{\mathcal{M}(\sigma)}$). We also decompose $\partial_\infty R = C_1 + C_2$ according to the same rule with B_i replaced by C_i.

For $\partial_0 \tilde{R}$, we decompose it into the sum of three parts $\partial_0 \tilde{R} = \tilde{B}_0 + \tilde{B}_1 + \tilde{B}_2$ so that $[T] \in \tilde{B}_0$ (resp. $[T] \in \tilde{B}_1$; resp. $[T] \in \tilde{B}_2$) satisfying $\tilde{\rho}_*([T]) = 0$ (resp. $T \subset \tilde{E}|_{|D|}$; resp. $T \not\subset \tilde{E}|_{|D|}$). By moving all summands $[T]$ with $\tilde{\rho}_*[T] = 0$ to $\tilde{B}_0$, no $[T]$ appears simultaneously in two of the three factors $\tilde{B}_0$, $\tilde{B}_1$ and $\tilde{B}_2$. We decompose $\partial_\infty \tilde{R} = \tilde{C}_0 + \tilde{C}_1 + \tilde{C}_2$ according to the same rule with $\tilde{B}_i$ replaced by $\tilde{C}_i$. By (2.4), and using the property of the decompositions, we have $\tilde{\rho}_*(\tilde{B}_i) = k \cdot B_i$ and $\tilde{\rho}_*(\tilde{C}_i) = k \cdot C_i$ for $i = 1, 2$.

Applying the definition of $s^!_{E,\sigma}$, to $\partial_0 R$ (resp. $\partial_\infty R$):

$$s^!_{E,\sigma}(\partial_0 R) = s^!_{E|_{\mathcal{M}(\sigma)}}(B_1) + k^{-1}\rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\tilde{B}_2)),$$

(resp. same formula with $\partial_0 R$ replaced by $\partial_\infty R$), we obtain identities in $A_{d-r}\mathcal{M}(\sigma)$:

$$s^!_{E,\sigma}(\partial_0 R) - s^!_{E,\sigma}(\partial_\infty R) = s^!_{E|_{\mathcal{M}(\sigma)}}(B_1 - C_1) + k^{-1} \cdot \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\tilde{B}_2 - \tilde{C}_2)).$$

We claim that

$$(2.5) \qquad \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\partial_0 \tilde{R} - \partial_\infty \tilde{R})) = 0 \in A_{d-r}\mathcal{M}(\sigma).$$

This is true because $s^!_{\tilde{G}}(\partial_0 \tilde{R} - \partial_\infty \tilde{R}) = 0 \in A_{d-r+1}X$, and $\rho(\sigma)_*$ and $[D]\cdot$ preserve rational equivalence.

Applying Lemma 2.7, we also have

$$(2.6) \qquad \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\tilde{B}_0)) = \rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\tilde{C}_0)) = 0 \in A_{d-r}\mathcal{M}(\sigma).$$

Furthermore, because of Lemma 2.6 and the fact that $\tilde{B}_1$ and $\tilde{C}_1$ lie over $|D|$,

$$\rho(\sigma)_*([D] \cdot s^!_{\tilde{G}}(\tilde{B}_1 - \tilde{C}_1)) = \rho(\sigma)_* s^!_{\tilde{E}|_{|D|}}(\tilde{B}_1 - \tilde{C}_1) = s^!_{E|_{\mathcal{M}(\sigma)}}(\tilde{\rho}_* \tilde{B}_1 - \tilde{\rho}_* \tilde{C}_1).$$

Therefore, using $\tilde{B}_2 - \tilde{C}_2 = (\partial_0 \tilde{R} - \partial_\infty \tilde{R}) - (\tilde{B}_0 - \tilde{C}_0) - (\tilde{B}_1 - \tilde{C}_1)$, (2.5) and (2.6), and $\tilde{\rho}_*(\tilde{B}_1) = k \cdot B_1$ and same for C_1, we obtain $s^!_{E,\sigma}(\partial_0 R) - s^!_{E,\sigma}(\partial_\infty R) = 0$. This proves the Lemma. $\qquad \square$

Corollary 2.9. *The Basic Construction defines a homomorphism*

$$s^!_{E,\sigma} : A_* E(\sigma) \to A_{*-r}\mathcal{M}(\sigma),$$

which we call the localized Gysin map. Furthermore, if we let $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$ and $\tilde{\iota} : E(\sigma) \to E$ be the inclusions, then $\iota_* \circ s^!_{E,\sigma} = s^!_E \circ \tilde{\iota}_* : A_* E(\sigma) \to A_{*-r}\mathcal{M}$.

Proof. The first part is the combination of Lemma 2.8 and 2.5. The second part is the consequence of Lemma 2.6. $\qquad\square$

3. Localized Gysin maps for bundle stacks

To construct localized virtual cycles, we need to generalize the localized Gysin map to bundle stacks over a DM stack $\mathcal{M}$.

Let $E^\bullet \in D(\mathcal{M})$ be a derived (category) object that is locally quasi-isomorphic to a two-term complex of locally free sheaves concentrated at $[0, 1]$. We let $\mathbf{E} = h^1/h^0(E^\bullet)$, which is a bundle stack isomorphic to $[E^1/E^0]$ in case $E^\bullet \cong_{qis} [E^0 \to E^1]$ (cf. [2]).

In case there is a vector bundle V that surjects onto $\mathbf{E}$, one defines $s^!_{\mathbf{E}} : A_*\mathbf{E} \to A_*\mathcal{M}$ as the composite of the flat pull-back $A_*\mathbf{E} \to A_*V$ and the Gysin map $s^!_V : A_*V \to A_*\mathcal{M}$.

Without such a vector bundle, one can either use the intersection theory developed by Kresch [16] to define $s^!_{\mathbf{E}}$, or follow the recipe developed in [25] by the second named author. We will follow the latter approach in this paper.

We suppose there is a surjective homomorphism of sheaves on an open substack $U \subset \mathcal{M}$

$$(3.1) \qquad\qquad \sigma : h^1(E^\bullet)|_U \longrightarrow \mathcal{O}_U.$$

It induces a morphism from the bundle stack $\mathbf{E}|_U$ to the line bundle $\mathbb{C}_U$. As before, we let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$. We let $\mathbf{E}(\sigma)$ be the kernel cone stack in $\mathbf{E}$,

$$(3.2) \qquad\qquad \mathbf{E}(\sigma) = \mathbf{E}|_{\mathcal{M}(\sigma)} \cup \ker\{\mathbf{E}|_U \to \mathbb{C}_U\} \subset \mathbf{E}$$

endowed with the reduced structure. (Since σ is surjective on U, $\mathbf{E}|_U \to \mathbb{C}_U$ is surjective; thus the kernel is well-defined and is a closed substack of $\mathbf{E}|_U$.)

Our goal is to construct a localized Gysin map

$$(3.3) \qquad\qquad s^!_{\mathbf{E},\sigma} : A_*\mathbf{E}(\sigma) \longrightarrow A_*\mathcal{M}(\sigma).$$

It is constructed by finding for each irreducible cycle $[\mathbf{c}] \in Z_*\mathbf{E}(\sigma)$ a proper representative $m_X^{-1}[C] \in Z_*F$ of $[\mathbf{c}]$ in a vector bundle F over a variety X with a surjective meromorphic cosection (F, σ_X) and a proper $\rho : X \to \mathcal{M}$, and then defining

$$s^!_{\mathbf{E},\sigma}([\mathbf{c}]) = m_X^{-1}\rho(\sigma)_*(s^!_{F,\sigma_X}([C])) \in A_*\mathcal{M}(\sigma).$$

We remark here that any homomorphism $V \to h^1(E^\bullet)$ from a locally free sheaf V to $h^1(E^\bullet)$ induces canonically a bundle-stack morphism $V \to \mathbf{E}$. Indeed, let $\eta : M \to \mathcal{M}$ be an étale open so that $\eta^*E^\bullet \cong_{qis} [F^0 \to F^1]$ with both F^i locally free on M. We lift the pull-back $\eta^*V \to \eta^*h^1(E^\bullet)$ to a homomorphism $\eta^*V \to F^1$, which defines a homomorphism of complexes $\eta^*V[-1] \to [F^0 \to F^1]$. By the functorial construction of h^1/h^0, we obtain a morphism

$$\eta^*V = h^1/h^0(\eta^*V[-1]) \to h^1/h^0(\eta^*E^\bullet) = \mathbf{E} \times_{\mathcal{M}} M.$$

One checks that any two liftings $\eta^*V \to F^1$ define homotopic $\eta^*V[-1] \to [F^0 \to F^1]$, and thus induce canonically isomorphic maps $\eta^*V \to \mathbf{E} \times_{\mathcal{M}} M$. Since the so constructed morphism is canonical, it descends to a morphism

$$(3.4) \qquad\qquad V \longrightarrow \mathbf{E}.$$

We now construct representatives of irreducible $[\mathbf{c}] \in Z_*\mathbf{E}$. Let $\pi : \mathbf{E} \to \mathcal{M}$ be the projection; let $M_{\mathbf{c}}$ be the closure of the image $\pi(\mathbf{c}) \subset \mathcal{M}$. Applying Chow's Lemma [18, Cor. 16.6.1], we can find a quasi-projective variety X, a proper and surjective morphism $\rho : X \to M_{\mathbf{c}}$ such that for a Zariski dense open $X_0 \subset X$, $\rho|_{X_0} : X_0 \to M_{\mathbf{c}}$ is étale. Since X is quasi-projective, we can find a complex of locally free sheaves $F^\bullet = [F_0 \to F_1]$ of $\mathcal{O}_X$-modules that is quasi-isomorphic to $\rho^* E^\bullet$. By abuse of notation, we view F_1 as a vector bundle over X, and obtain a flat morphism

$$\gamma : F_1 \longrightarrow h^1/h^0(F^\bullet) = h^1/h^0(\rho^* E^\bullet) = \rho^* \mathbf{E}.$$

Since $\rho|_{X_0} : X_0 \to M_{\mathbf{c}}$ is étale, the induced

$$\gamma|_{X_0} : F_1 \times_X X_0 \longrightarrow \mathbf{E}|_{M_{\mathbf{c}}} = h^1/h^0(E^\bullet) \times_{\mathcal{M}} M_{\mathbf{c}}$$

is flat.

Definition 3.1. *A proper representative of an irreducible $[\mathbf{c}] \in Z_*\mathbf{E}$ consists of (ρ, F_1) just constructed and a cycle $C_X = \sum_i m_i[C_i] \in Z_*F_1$, where C_i are closed and integral in F_1 and $m_i \in \mathbb{Z}$, such that $\sum_i m_i[C_i \times_X X_0] = (\gamma|_{X_0})^*(\mathbf{c})$, and $C_i = \overline{C_i \times_X X_0}$ for all i.*

Using proper representatives of $[\mathbf{c}] \in Z_*\mathbf{E}$, we can define $s^!_{\mathbf{E},\sigma}$ at the cycle level. Let $[\mathbf{c}] \in Z_*\mathbf{E}(\sigma)$ be an irreducible cycle. Let (ρ, X, F_1) with $\sum_i m_i C_i$ be a proper representative of $[\mathbf{c}]$. Let $\sigma_X : F_1|_{\rho^{-1}U} \to \mathcal{O}_{\rho^{-1}U}$ be the composite of $F_1|_{\rho^{-1}U} \to \rho^* h^1(E^\bullet)|_{\rho^{-1}U}$ with $\rho^*\sigma$. Automatically $[C_i] \in Z_*F_1(\sigma_X)$. We let $\rho(\sigma) : X(\sigma_X) \to \mathcal{M}(\sigma)$ be the restricton of ρ to $X(\sigma_X) = X \setminus \rho^{-1}U$.

Definition 3.2. *We define $s^!_{\mathbf{E},\sigma}([\mathbf{c}]) = m_X^{-1}\rho(\sigma)_*(s^!_{F_1,\sigma_X}([C_X]))$, where m_X is the degree of $\rho : X \to M_{\mathbf{c}}$. We extend it to $s^!_{\mathbf{E},\sigma} : Z_*\mathbf{E} \to A_*\mathcal{M}(\sigma)$ by linearity.*

Proposition 3.3. *The map $s^!_{\mathbf{E},\sigma}$ in Definition 3.2 is well-defined. The map $s^!_{\mathbf{E},\sigma}$ preserves the rational equivalence, and thus defines a (localized Gysin) homomoprhism*

$$(3.5) \qquad\qquad s^!_{\mathbf{E},\sigma} : A_*\mathbf{E}(\sigma) \longrightarrow A_*\mathcal{M}(\sigma).$$

Proof. Since the existence of proper representatives of $[\mathbf{c}]$ is guaranteed by the Chow Lemma, to show that the localized Gysin map is well-defined at the cycle level we need to check that it is independent of the choice of proper representatives.

To check that it is well-defined, we need to show that if $m_{X'}^{-1}[C_{X'}]$ with (ρ', X', F'_1) is another proper representative of $[\mathbf{c}]$, then

$$m_X^{-1}\rho(\sigma)_*(s^!_{F_1,\sigma_X}([C_X])) = m_{X'}^{-1}\rho'(\sigma)_*(s^!_{F'_1,\sigma_{X'}}([C_{X'}])).$$

This can be proved by choosing a third proper representative using $Y \subset X \times_{\mathcal{M}} X' \to \mathcal{M}$ an irreducible component, choosing a complex $\bar{F}^\bullet$ quasi-isomorphic to the pull-backs of $F^\bullet$ and $(F')^\bullet$ and whose projections commute with the projections to the pull-backs of $h^1(E^\bullet)$. Since $X \to M_{\mathbf{c}}$ and $X' \to M_{\mathbf{c}}$ are étale over dense open subsets, we can choose Y so that $Y \to M_{\mathbf{c}}$ is étale over a dense open subset as well. The remaining proof is parallel to the proof of Lemma 2.5, and will be omitted. This defines the homomorphism $s^!_{\mathbf{E},\sigma} : Z_*\mathbf{E}(\sigma) \longrightarrow A_*\mathcal{M}(\sigma)$.

Finally, we check that it preserves the rational equivalence. This time the proof is a line by line repetition of the proof of Lemma 2.8, incorporating the need to use proper representatives to make sense of degrees of maps. Since the modification is routine, we will omit the details here. $\qquad\square$

4. Reducing intrinsic normal cones by cosections

In this section, we show that a Deligne-Mumford stack equipped with a perfect obstruction theory and a meromorphic cosection of its obstruction sheaf has "restricted" intrinsic normal cones. Applying the localized Gysin maps, we obtain the localized virtual cycles.

We let $\pi : \mathcal{M} \to \mathcal{S}$ be a DM stack $\mathcal{M}$ over a smooth Artin stack $\mathcal{S}$; we assume π is representable. We assume $\mathcal{M}/\mathcal{S}$ admits a relative perfect obstruction theory $E^\bullet \to L^\bullet_{\mathcal{M}/\mathcal{S}}$ formulated in [2] using cotangent complexes.[1] As part of the definition, locally $E^\bullet$ is quasi-isomorphic to two-term complexes of locally free sheaves concentrated at $[-1, 0]$.

We introduce the obstruction sheaf of a relative perfect obstruction theory. We recall that $\mathcal{O}b_{\mathcal{M}/\mathcal{S}} = h^1((E^\bullet)^\vee)$ is the relative obstruction sheaf of $E^\bullet \to L^\bullet_{\mathcal{M}/\mathcal{S}}$. To introduce its (absolute) obstruction sheaf, we pick a smooth chart M/S of $\mathcal{M}/\mathcal{S}$ by affine schemes M and S such that $S \to \mathcal{S}$ is smooth, $M \subset \mathcal{M} \times_\mathcal{S} S$ is open and $M \to \mathcal{M}$ is étale. (This is possible since $\mathcal{M} \to \mathcal{S}$ is representable.) We pick an S-embedding $M \to V$ into an affine V, smooth over S. Since M is affine, we pick a presentation $E^\bullet|_M = [E^{-1} \to E^0]$ as a complex of locally free sheaves so that the perfect obstruction theory of $\mathcal{M}/\mathcal{S}$ lifted to M/S is given by a homomorphism of complexes of sheaves

$$(4.1) \qquad (\phi^{-1}, \phi^0) : [E^{-1} \to E^0] \to \tau^{\geq -1} L^\bullet_{M/S} = [I_M/I_M^2 \to \Omega_{V/S}|_M],$$

where I_M is the ideal sheaf of $M \subset V$. We let $\pi_S : M \to S$ be the projection.

We denote

$$\mathcal{O}b_{M/S} := \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{O}_M = h^1((E^\bullet|_M)^\vee).$$

From the distinguished triangle

$$\pi_S^* L^\bullet_S \longrightarrow L^\bullet_M \longrightarrow L^\bullet_{M/S} \longrightarrow \pi_S^* L^\bullet_S[1] = [\pi_S^* \Omega_S \to 0],$$

we have the composition

$$E^\bullet \longrightarrow \tau^{\geq -1} L^\bullet_{M/S} \longrightarrow \pi_S^* \Omega_S[1]$$

and a distinguished triangle

$$\hat{E}^\bullet \longrightarrow E^\bullet \longrightarrow \pi_S^* \Omega_S[1],$$

which fits into a commutative diagram of distinguished triangles

$$(4.2) \qquad
\begin{array}{ccccc}
\hat{E}^\bullet & \longrightarrow & E^\bullet & \longrightarrow & \pi_S^* \Omega_S[1] \xrightarrow{+1} \\
\downarrow & & \downarrow & & \| \\
L^\bullet_M & \longrightarrow & L^\bullet_{M/S} & \longrightarrow & \pi_S^* \Omega_S[1] \xrightarrow{+1} .
\end{array}$$

---

[1] We recall that there are two versions of perfect obstruction theories. One formulated by Behrend-Fantechi [2] using an arrow from a derived object to the relative cotangent complex $E^\bullet \to L^\bullet_{\mathcal{M}/\mathcal{S}}$; the other by Li-Tian [24] using obstruction to deformation assignment in the cohomology of a derived object. By [2, Theorem 4.5], the BF's version of perfect obstruction theories induces LT's version of perfect obstruction theories.

Conversely, it will be shown in [7] that LT's version of perfect obstruction theory is affine locally equivalent to BF's version. Furthermore, all available technical tools concerning cycles constructed from BF's version of perfect obstruction theory work for LT's version as well.

By the standard 5-lemma, we find that $\hat{E}^\bullet \to L_M^\bullet$ is a perfect obstruction theory of M and that the obstruction sheaf $\mathcal{O}b_M$ is the quotient

$$(4.3) \qquad \mathcal{O}b_{M/S} \longrightarrow \mathcal{O}b_M = \operatorname{coker}\{\pi_S^* \Omega_S^\vee \longrightarrow \mathcal{O}b_{M/S}\}$$

of $\mathcal{O}b_{M/S}$.

Since this construction is canonical, the object $\hat{E}^\bullet$ is unique up to quasi-isomorphism; the arrow $\pi_S^* \Omega_S \to \hat{E}^\bullet$ (in (4.2)) is unique up to homotopy. Thus $\mathcal{O}b_M = h^1((\hat{E}^\bullet)^\vee)$ in (4.3) is canonically defined, and the first arrow in (4.3) is unique. This proves that (4.3) descends to a quotient homomorphism of sheaves on $\mathcal{M}$

$$(4.4) \qquad \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \longrightarrow \mathcal{O}b_{\mathcal{M}}.$$

Definition 4.1. *We call $\mathcal{O}b_{\mathcal{M}}$ the (absolute) obstruction sheaf of the obstruction theory $E^\bullet \to L_{\mathcal{M}/\mathcal{S}}^\bullet$.*

We denote

$$\mathbf{E} := h^1/h^0((E^\bullet)^\vee)$$

and let $\mathbf{c}_{\mathcal{M}} \subset \mathbf{E}$ be the relative intrinsic normal cone introduced in [2]. A meromorphic cosection of $\mathcal{O}b_{\mathcal{M}}$ will reduce the intrinsic normal cone $[\mathbf{c}_{\mathcal{M}}]$ to a subcone-stack of $\mathbf{E}$. Let $U \subset \mathcal{M}$ be an open substack and let

$$(4.5) \qquad \sigma : \mathcal{O}b_{\mathcal{M}}|_U \longrightarrow \mathcal{O}_U$$

be a surjective homomorphism. As before, we call σ a meromorphic cosection surjective on U; we call $\mathcal{M}(\sigma) = \mathcal{M} - U$ the degeneracy locus of σ.

The homomorphism σ induces a homomorphism $(E^\bullet)^\vee|_U \to [0 \to \mathcal{O}_U]$, and thus a surjective cone-stack morphism $\bar{\sigma} : \mathbf{E}|_U \longrightarrow h^1/h^0([0 \to \mathcal{O}_U]) = \mathbb{C}_U$. (Here we use $\mathbb{C}_{\mathcal{M}}$ to denote the trivial line bundle $\mathbb{A}^1 \times \mathcal{M} \to \mathcal{M}$.)

Definition 4.2. *We define $\mathbf{E}(\sigma)$ to be the union of $\mathbf{E} \times_{\mathcal{M}} \mathcal{M}(\sigma)$ with $\ker\{\bar{\sigma} : \mathbf{E}|_U \to \mathbb{C}_U\}$, endowed with the reduced structure. The closed substack $\mathbf{E}(\sigma) \subset \mathbf{E}$ is called the kernel cone-stack of σ.*

Proposition 4.3. *Let the notation be as stated, and let $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_U$ be surjective. Then the cycle $[\mathbf{c}_{\mathcal{M}}] \in Z_* \mathbf{E}$ lies in $Z_* \mathbf{E}(\sigma)$.*

We consider a simple case. Let $M \subset V$ be a closed subscheme of a smooth scheme V defined by the vanishing $s = 0$ of a section s of a vector bundle E on V; let $C_{M/V}$ be the normal cone to M in V, embedded in $E|_M$ via the section s. We suppose $\bar{\sigma} : E|_M \to \mathcal{O}_M$ is a surjective homomorphism. Let I_M be the ideal sheaf of $M \subset V$.

Lemma 4.4 (Cone reduction criterion)**.** *Suppose the defining equation s satisfies the following criterion: for any germ $\varphi : \operatorname{Spec} \Bbbk[\![\xi]\!] \to V$, $\varphi(0) \in M$, the section $\bar{\sigma} \circ s \circ \varphi \in \Bbbk[\![\xi]\!]$ satisfies $\bar{\sigma} \circ s \circ \varphi \in \xi \cdot \varphi^* I_M$. Then the support of the cone $C_{M/V}$ lies entirely in the kernel $F = \ker\{\bar{\sigma} : E|_M \to \mathcal{O}_M\} \subset E$.*

Proof. The cone $C_{M/V} \subset E$ is the specialization of the section $t^{-1}s \subset E$ as $t \to 0$. More precisely, we consider the subscheme

$$\Gamma = \{(t^{-1}s(w), t) \in E \times (\mathbb{A}^1 \setminus 0) \mid w \in V,\ t \in \mathbb{A}^1 \setminus 0\}.$$

For $t \in \mathbb{A}^1 \setminus 0$, the fiber Γ_t of Γ over $t \in \mathbb{A}^1$ is the section $t^{-1}s$ of E. We let $\bar{\Gamma}$ be the closure of Γ in $E \times \mathbb{A}^1$. The central fiber $\bar{\Gamma} \times_{\mathbb{A}^1} 0 \subset E$ is the normal cone $C_{M/V}$ embedded in $E|_M$. Clearly, $C_{M/V}$ is of pure dimension $\dim V$.

Now let $N \subset C_{M/V}$ be any irreducible component. Let $v \in N$ be a general closed point of N. Then we can find a smooth affine curve $0 \in S$ and a morphism $\rho : (0, S) \to (v, \bar{\Gamma})$ such that $\rho(S \setminus 0) \subset \Gamma$. We let $\rho_V : S \to V$ and $\rho_{\mathbb{A}^1} : S \to \mathbb{A}^1$ be the composites of ρ with the projections from $E \times \mathbb{A}^1$ to V and to $\mathbb{A}^1$. Since $\rho(S \setminus 0) \subset \Gamma$, $\rho_{\mathbb{A}^1}$ dominates $\mathbb{A}^1$.

We then choose a uniformizing parameter ξ of S at 0 so that $(\rho_{\mathbb{A}^1})^*(t) = \xi^n$ for some n. Because $\rho(0) = v$, $\xi^{-n} \cdot s \circ \rho_V : S \setminus 0 \to E$ specializes to v; hence $s \circ \rho_V$ has the expression

$$s \circ \rho_V = v\xi^n + O(\xi^{n+1}).$$

Applying $\bar{\sigma}$, we obtain $\bar{\sigma} \circ s \circ \rho_V = \bar{\sigma}(v)\xi^n + O(\xi^{n+1})$.

Now suppose $N \not\subset E(\sigma)$; in particular v does not lie in the zero section of E. Then $\rho_V^* I_M = (\xi^n)$. By assumption, $\bar{\sigma} \circ s \circ \rho_V \in (\xi^{n+1}) = \xi \cdot \rho_V^* I_M$, we must have $\bar{\sigma}(v) = 0$. This proves that $v \in F = \ker\{\bar{\sigma}\}$. Since v is general in $C_{M/V}$, $v \in F$ implies that the support of $C_{M/V}$ lies in F. This proves the Lemma. $\qquad\square$

Let $\mathcal{O}b_M = \operatorname{coker}\{ds : T_V|_M \to E|_M\}$; let $\operatorname{pr} : E|_M \to \mathcal{O}b_M$ be the projection.

Corollary 4.5. *Let the notation be as in Lemma 4.4. Suppose we have a surjective homomorphism $\sigma : \mathcal{O}b_M \to \mathcal{O}_M$. Then the support of the cone $C_{M/V}$ lies entirely in the kernel $F = \ker\{\sigma \circ \operatorname{pr} : E|_M \to \mathcal{O}_M\} \subset E$.*

Proof. We verify the cone reduction criterion. Let $\varphi : \operatorname{Spec} \Bbbk[\![\xi]\!] \to V$, $\varphi(0) \in M$, be any morphism. Suppose $\varphi^* I_M = (\xi^n)$. Pulling back the exact sequence

$$\mathcal{O}_M(T_V) \xrightarrow{ds} \mathcal{O}_M(E) \longrightarrow \mathcal{O}b_M \longrightarrow 0$$

via the induced morphism $\bar{\varphi} = \varphi|_{\varphi^{-1}(M)} : \varphi^{-1}(M) = (\xi^n = 0) \to M$, we obtain

$$(4.6) \qquad\qquad \bar{\varphi}^*(T_V) \xrightarrow{\bar{\varphi}^*(ds)} \bar{\varphi}^*(E) \xrightarrow{\bar{\varphi}^*(\operatorname{pr})} \bar{\varphi}^*(\mathcal{O}b_M) \longrightarrow 0,$$

where $\bar{\varphi}^*(\operatorname{pr})$ is the pullback of the projection $\operatorname{pr} : E|_M \to \mathcal{O}b_M$.

Let $v \in E|_{\varphi(0)}$ be the element so that $s \circ \varphi = v\xi^n + O(\xi^{n+1})$. Thus $\bar{\varphi}^*(ds) = d(v\xi^n) = nv\xi^{n-1}d\xi$.

Let $\bar{\sigma} = \operatorname{pr} \circ \sigma$. Because (4.6) is exact, we have the vanishing $\bar{\varphi}^*(\bar{\sigma})(\bar{\varphi}^*(ds)) = 0$, which proves that $\bar{\sigma}(v) = 0$; hence $\bar{\sigma} \circ s \circ \varphi = O(\xi^{n+1})$. Thus the section s satisfies the cone reduction criterion (Lemma 4.4). This proves the corollary. $\qquad\square$

Assume there is a vector bundle (locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules) F that surjects onto $\mathcal{O}b_{\mathcal{M}/\mathcal{S}}$. This homomorphism induces a flat morphism $F \to h^1/h^0((E^\bullet)^\vee)$ (cf. (3.4)), which pulls back $[\mathbf{c}_{\mathcal{M}}]$ to a cycle $[C_{\mathcal{M}}] \in Z_* F$. We let $\tilde{\sigma}$ be the composition

$$(4.7) \qquad\qquad \tilde{\sigma} : F|_U \longrightarrow \mathcal{O}b_{\mathcal{M}/\mathcal{S}}|_U \longrightarrow \mathcal{O}b_{\mathcal{M}}|_U \longrightarrow \mathcal{O}_U,$$

which is surjective. Let $F(\tilde{\sigma}) \subset F$ be $F|_{\mathcal{M}(\sigma)} \cup \ker\{\tilde{\sigma}\}$, endowed with the reduced structure.

Corollary 4.6. *Let the notation be as stated and let σ be a surjective homomorphism $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_U$ over an open U. Then the cycle $[C_{\mathcal{M}}] \in Z_* F$ lies in $Z_* F(\tilde{\sigma})$.*

Proof. Because of the way $F(\tilde{\sigma})$ is defined, we only need to show that $[C_{\mathcal{M}} \times_{\mathcal{M}} U] \in Z_*(F(\tilde{\sigma}) \times_{\mathcal{M}} U)$. By replacing $\mathcal{M}$ with the open substack U, we can assume that σ is regular and surjective on $\mathcal{M}$.

Since the statement is local, we only need to consider the case where $M/S \to \mathcal{M}/\mathcal{S}$ is as introduced before (4.1). Since M is affine, we can pick $E^\bullet|_M = [E^{-1} \to E^0]$ so that $E^\bullet|_M \to \tau^{\geq -1} L^\bullet_{M/S}$ is given by (4.1) and that in addition satisfies $\phi^0 : E^0 \to \Omega_{V/S}|_M$ is an isomorphism. Because of the exact sequence $(E^0)^\vee \to (E^{-1})^\vee \to \mathcal{O}b_{M/S} \to 0$, possibly by replacing $E^{-1} \to E^0$ with a quasi-isomorphic complex we can assume $F|_M \to \mathcal{O}b_{M/S}$ lifts to a surjective $F|_M \to (E^{-1})^\vee$, thus defining a homomorphism of complexes

$$\gamma : [0 \to F] \to [(E^0)^\vee \to (E^{-1})^\vee].$$

Two γ's coming from two liftings $F \to (E^{-1})^\vee$ are homotopy equivalent, and hence the induced morphism of bundle stacks

$$(4.8) \qquad\qquad \tilde{\gamma} : F \longrightarrow \mathbf{E} = h^1/h^0((E^\bullet)^\vee)$$

is canonically defined.

Let $\tilde{E}^{-1}$ be a locally free sheaf on V such that $\tilde{E}^{-1}|_M \cong E^{-1}$. This is possible since V is affine. By the same reason, we can lift $\phi^{-1} \in I_M/I_M^2 \otimes_{\mathcal{O}_M} (E^{-1})^\vee$ to an $f \in I_M \otimes_{\mathcal{O}_V} (\tilde{E}^{-1})^\vee$. Then since $\phi^{-1} : E^{-1} \to I_M/I_M^2$ is surjective [2, Thm. 4.5], $M = (f = 0) \subset V$. We let $C_{M/V}$ be the normal cone to $M = (f = 0)$ in V. The cone $C_{M/V}$ canonically embeds in $(E^{-1})^\vee = (\tilde{E}^{-1})^\vee|_M$ via the defining equation f.

On the other hand, because the arrow in $[I_M/I_M^2 \to \Omega_{V/S}|_M]$ is via sending $u \in I_M/I_M^2$ to its relative differential $d_{/S} u \in \Omega_{V/S}|_M$, and because ϕ^0 is an isomorphism, after identifying E^0 with $\Omega_{V/S}|_M$ using ϕ^0, the arrow in $[E^{-1} \to E^0]$ is the relative differential $d_{/S} f \in \Omega_{V/S}|_M \otimes_{\mathcal{O}_M} (\tilde{E}^{-1})^\vee$. Thus $\mathcal{O}b_{M/S} = \mathrm{coker}\{d_{/S} f : T_{V/S}|_M \to (E^{-1})^\vee\}$. Following the definition of the obstruction sheaf $\mathcal{O}b_M$,

$$\mathcal{O}b_M = \mathrm{coker}\{df : T_V|_M \longrightarrow (E^{-1})^\vee\}.$$

Finally, we apply Corollary 4.5 to conclude that the support of the cone $C_{M/V}$ lies in the kernel of the composite $(E^{-1})^\vee \to \mathcal{O}b_M \to \mathcal{O}_M$. Since the pull-back of $\mathbf{c}_\mathcal{M}$ to F under (4.8) is the pull-back of $C_{M/V}$ under the surjective $F|_M \to (E^{-1})^\vee$, and since the support of $C_{M/V}$ lies in the kernel of $(E^{-1})^\vee \to \mathcal{O}_M$, the support of $C_\mathcal{M}|_M$ lies in the kernel $F(\tilde{\sigma})$. This proves the Corollary. $\qquad \square$

Proof of Proposition 4.3. The proof is a direct application of Corollary 4.6. We pick an $M/S \to \mathcal{M}/\mathcal{S}$ as in the proof of Corollary 4.6; we only need to consider the case where $M \to \mathcal{M}$ factors through $U \subset \mathcal{M}$. We pick a vector bundle (locally free sheaf) F on M so that F surjects onto $\mathcal{O}b_{M/S}$. This is possible since M is affine. Following the proof of Corollary 4.6, $\gamma : F \to \mathcal{O}b_{M/S}$ induces a bundle stack homomorphism $\tilde{\gamma} : F \to \mathbf{E} \times_\mathcal{M} M$. Let $\tilde{\sigma}$ be as in (4.7). (Note that $M \to \mathcal{M}$ factors through $U \subset \mathcal{M}$.) Since for $x \in M$ and $(E^\bullet)^\vee = [E_0 \to E_1]$ near x, $\mathrm{im}(E_0|_x) \subset \ker(E_1|_x \to \mathcal{O}b_{M/S}|_x \overset{\sigma}{\to} \mathbb{C}) \subset \tilde{\gamma}^{-1}([E_1/E_0](\sigma)|_x)$, we have $\tilde{\gamma}^{-1}(\mathbf{E}(\sigma)) = F(\tilde{\sigma})$. Hence, the statement of the Corollary is equivalent to $\tilde{\gamma}^*[\mathbf{c}_\mathcal{M}] \in Z^* F(\tilde{\sigma})$. But this is what is proved in Corollary 4.6. This proves the Proposition. $\qquad \square$

We have an equivariant version of the Corollary. We suppose $\mathcal{M}/\mathcal{S}$ as before has a $\mathbb{C}^*$-structure; we suppose its relative obstruction theory is $\mathbb{C}^*$-equivariant. Let $\mathcal{M}^c$ be the $\mathbb{C}^*$-fixed part of $\mathcal{M}$. We suppose there is a surjective sheaf homomorphism

$$(4.9) \qquad\qquad \sigma_c : (\mathcal{O}b_\mathcal{M}|_{\mathcal{M}^c})^{\mathbb{C}^*} \longrightarrow \mathcal{O}_{\mathcal{M}^c},$$

from the $\mathbb{C}^*$-fixed part to $\mathcal{O}_{\mathcal{M}^c}$, and let F be a $\mathbb{C}^*$-locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules and $F \to \mathcal{O}b_{\mathcal{M}}$ be a $\mathbb{C}^*$-equivariant surjective homomorphism. We let σ be the composite

$$\sigma : F|_{\mathcal{M}^c} \longrightarrow \mathcal{O}b_{\mathcal{M}/\mathcal{S}}|_{\mathcal{M}^c} \longrightarrow \mathcal{O}b_{\mathcal{M}}|_{\mathcal{M}^c} \longrightarrow (\mathcal{O}b_{\mathcal{M}}|_{\mathcal{M}^c})^{\mathbb{C}^*} \xrightarrow{\sigma_c} \mathcal{O}_{\mathcal{M}^c},$$

where the second arrow is the projection onto the invariant part.

Lemma 4.7. *Let the notation be as before and let $C_{\mathcal{M}} \subset F$ be the cone-cycle that is the pull-back of the intrinsic normal cone $\mathbf{c}_{\mathcal{M}}$ (cf. Corollary 4.6). Then the support of the restriction $C_{\mathcal{M}}|_{\mathcal{M}^c} \subset F|_{\mathcal{M}^c}$ lies in the kernel vector bundle $F|_{\mathcal{M}^c}(\sigma) = \ker\{F|_{\mathcal{M}^c} \to \mathcal{O}_{\mathcal{M}^c}\}$.*

Proof. We let $M/S \to \mathcal{M}/\mathcal{S}$ be as in the proof of Corollary 4.6. Without loss of generality, we can assume both S and M are $\mathbb{C}^*$-schemes and $M/S \to \mathcal{M}/\mathcal{S}$ is a $\mathbb{C}^*$-morphism. (We can avoid this assumption by working with the formal completion of M at a closed $p \in M \times_{\mathcal{M}} \mathcal{M}^c$; the remaining arguments go through.) We take a $\mathbb{C}^*$-equivariant S-embedding $M \to V$ as before.

We repeat the proof of Corollary 4.6. Since the obstruction theory is $\mathbb{C}^*$-equivariant, we can choose a $\mathbb{C}^*$-complex $E^\bullet|_M = [E^{-1} \to E^0]$ so that the obstruction theory $E^\bullet|_M \to \tau^{\geq -1}L_{M/S}^\bullet$ is given by a $\mathbb{C}^*$-equivariant (4.1). We extend E^{-1} to a $\mathbb{C}^*$-equivariant $\tilde{E}^{-1}$ on V. Since (4.1) is $\mathbb{C}^*$-equivariant, we can choose a $\mathbb{C}^*$-equivariant lift $f \in I_M \otimes_{\mathcal{O}_V} (\tilde{E}^{-1})^\vee$ of ϕ^{-1}.

We let V^c (resp. M^c; resp E_c^i) be the $\mathbb{C}^*$ fixed part of V (resp. M; resp. E^i); let I_{M^c} be the ideal sheaf of $M^c \subset V^c$. Then the $\mathbb{C}^*$-fixed part $f^c := (f)^{\mathbb{C}^*} \in I_{M^c} \otimes_{\mathcal{O}_{V^c}} (\tilde{E}_c^{-1})^\vee$ defines $M^c = (f^c = 0) \subset V^c$.

On the other hand, since the $\mathbb{C}^*$-invariant part of (4.1) is a perfect obstruction theory of M^c, the cokernel of df^c, which is a quotient of $(E_c^{-1})^\vee$, is the obstruction sheaf $\mathcal{O}b_{M^c}$ of M^c. One checks that it is identical to the invariant part $(\mathcal{O}b_M|_{M^c})^{\mathbb{C}^*}$ (cf. [11]).

We now look at the normal cone C_{M^c/V^c} (resp. $C_{M/V}$) to M^c (resp. M) in V^c (resp. V); it is a subcone of $(E_c^{-1})^\vee$ (resp. $(E^{-1})^\vee$). By the previous Corollary, the cycle $[C_{M^c/V^c}]$ is a cycle in $\ker\{(E_c^{-1})^\vee \to \mathcal{O}_{M^c}\}$, where the arrow is the composite $(E_c^{-1})^\vee \to \mathcal{O}b_{M^c} \xrightarrow{\sigma_c} \mathcal{O}_{M^c}$ (as in the statement of the Lemma).

We claim that

$$(4.10) \qquad\qquad (C_{M/V})^{\mathbb{C}^*} = C_{M^c/V^c} \subset (E_c^{-1})^\vee.$$

To prove this, we consider the graph Γ of $t^{-1}f$ ($t \in \mathbb{A}^1$), considered as a subscheme in $(\mathbb{A}^1 \setminus 0) \times (\tilde{E}^{-1})^\vee$. By viewing it as a family over $\mathbb{A}^1 \setminus 0$, we can take its $\mathbb{A}^1$-flat closure $\overline{\Gamma} \subset \mathbb{A}^1 \times (\tilde{E}^{-1})^\vee$. By the definition of normal cone, the central fiber $\overline{\Gamma}_0 := \overline{\Gamma}|_{t=0}$ is the cone $C_{M/V}$ canonically embedded in $(E^{-1})^\vee$. Thus $\overline{\Gamma}_0 = C_{M/V}$ and $(\overline{\Gamma}_0)^{\mathbb{C}^*} = (C_{M/V})^{\mathbb{C}^*}$.

On the other hand, notice that Γ is $\mathbb{C}^*$-invariant with $\mathbb{C}^*$ acting on $\mathbb{A}^1$ trivially; the fixed part $\Gamma^c := \Gamma^{\mathbb{C}^*}$ of Γ is

$$(4.11) \qquad\qquad \Gamma^c = \{\text{the graph of } t^{-1}f^c\} \subset (\mathbb{A}^1 \setminus 0) \times (\tilde{E}_c^{-1})^\vee.$$

We let $\overline{\Gamma^c}$ be the flat closure of Γ^c (over $\mathbb{A}^1$). Thus, $(\overline{\Gamma^c})|_{t=0} = C_{M^c/V^c} \subset (E_c^{-1})^\vee$.

Therefore, to conclude the Lemma we only need to check that $(\overline{\Gamma^c})|_{t=0} = (\overline{\Gamma}_0)^c$. Because $\mathbb{C}^*$ is reductive, the flatness of $\overline{\Gamma}$ over $\mathbb{A}^1$ implies that the fixed part $(\overline{\Gamma})^{\mathbb{C}^*}$

is also flat over $\mathbb{A}^1$. Then since $(\overline{\Gamma})^{\mathbb{C}^*} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) = \Gamma^c$, and $\overline{\Gamma^c}$ is $\mathbb{A}^1$-flat, we obtain $\overline{\Gamma^c} = (\overline{\Gamma})^{\mathbb{C}^*}$. This proves $\overline{\Gamma^c}|_{t=0} = (\overline{\Gamma}_0)^{\mathbb{C}^*}$.

Finally, like in the proof of Corollary 4.6, possibly after replacing $[E^{-1} \to E^0]$ by a quasi-isomorphic complex, we can assume that $F|_M \to \mathcal{O}b_{\mathcal{M}}|_M$ factors through a $\mathbb{C}^*$-equivariant $F|_M \to (E^{-1})^\vee \to \mathcal{O}b_M$. This proves the Lemma. $\qquad\square$

5. Localized virtual cycles

We state and prove the main theorem of this paper.

Theorem 5.1. *Let $\mathcal{M}/\mathcal{S}$ be a Deligne-Mumford stack as before endowed with a relative perfect obstruction theory. Suppose there is a surjective homomorphism $\sigma : \mathcal{O}b_{\mathcal{M}}|_U \to \mathcal{O}_U$ on an open $U \subset \mathcal{M}$. Let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$ be the degeneracy locus of σ. Then $\mathcal{M}$ admits a localized virtual cycle*

$$[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} \in A_*\mathcal{M}(\sigma).$$

It relates to its ordinary virtual cycle by $\iota_*[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} = [\mathcal{M}]^{\mathrm{vir}}$, where $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$ is the inclusion.

Proof. Let $E^\bullet \to L_{\mathcal{M}/\mathcal{S}}^\bullet$ be the obstruction theory of $\mathcal{M}/\mathcal{S}$. Let $\mathbf{E} = h^1/h^0((E^\bullet)^\vee)$ and let $[\mathbf{c}_{\mathcal{M}}] \in Z_*\mathbf{E}$ be the corresponding intrinsic virtual cycle [2]. The cosection σ defines a surjective bundle stack morphism $\tilde{\sigma} : \mathbf{E}|_U \to \mathbb{C}_U$. As before, we let $\mathbf{E}(\sigma) = \mathbf{E}|_{\mathcal{M}(\sigma)} \cup \ker\{\tilde{\sigma}\}$, endowed with the reduced structure; $\mathbf{E}(\sigma)$ is a closed substack of $\mathbf{E}$.

By Proposition 4.3, $[\mathbf{c}_{\mathcal{M}}]$ is a cycle in $Z_*\mathbf{E}(\sigma)$. We apply the localized Gysin map constructed in Proposition 3.3 to define

$$[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} = s_{\mathbf{E},\sigma}^!([\mathbf{c}_{\mathcal{M}}]) \in A_*\mathcal{M}(\sigma).$$

By the property of localized Gysin map, we have $\iota_*[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} = [\mathcal{M}]^{\mathrm{vir}}$. This proves the theorem. $\qquad\square$

Like the ordinary virtual cycle, the localized virtual cycle is expected to remain constant in some naturally arisen situations. We let $\mathcal{M}/\mathcal{S}$ as before be a DM stack over a smooth Artin stack $\mathcal{S}$, $\mathcal{M} \to \mathcal{S}$ representable, with a relative perfect obstruction theory $E^\bullet \to L_{\mathcal{M}/\mathcal{S}}^\bullet$. We let $0 \in T$ be a pointed smooth affine curve. We suppose $\mathcal{N}/\mathcal{S}$ is a DM stack over $\mathcal{S}$, $\mathcal{N} \to \mathcal{S}$ representable, together with a morphism $\pi : \mathcal{N} \to T$ such that

$$\mathcal{M} \cong \mathcal{N} \times_T 0.$$

We assume there is a perfect relative obstruction theory $F^\bullet \to L_{\mathcal{N}/\mathcal{S}}^\bullet$ that is compatible to that of $\mathcal{M}/\mathcal{S}$ in that we have a homomorphism of distinguished triangles in the derived category $D(\mathcal{M})$:

(5.1)
$$\begin{array}{ccccccc}
F^\bullet|_{\mathcal{M}} & \xrightarrow{\ g\ } & E^\bullet & \longrightarrow & \mathcal{O}_{\mathcal{M}}[1] & \xrightarrow{\ +1\ } & \\
\downarrow & & \downarrow & & \downarrow & & \\
L_{\mathcal{N}/\mathcal{S}}^\bullet|_{\mathcal{M}} & \xrightarrow{\ h\ } & L_{\mathcal{M}/\mathcal{S}}^\bullet & \longrightarrow & L_{\mathcal{M}/\mathcal{N}}^\bullet & \xrightarrow{\ +1\ } & .
\end{array}$$

This way, the (relative) obstruction sheaf $\mathcal{O}b_{\mathcal{N}/\mathcal{S}} = h^1((F^\bullet)^\vee)$ fits into the exact sequence $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \to \mathcal{O}b_{\mathcal{N}/\mathcal{S}}|_{\mathcal{M}} \to 0$. Applying the construction of the

(absolute) obstruction sheaves, we obtain the exact sequence

(5.2) $$\mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}b_{\mathcal{N}}|_{\mathcal{M}} \longrightarrow 0.$$

We suppose there is an open $\mathcal{U} \subset \mathcal{N}$ and a surjective homomorphism

$$\sigma_{\mathcal{U}} : \mathcal{O}b_{\mathcal{N}}|_{\mathcal{U}} \longrightarrow \mathcal{O}_{\mathcal{U}}.$$

We let $U = \mathcal{U} \times_{\mathcal{N}} \mathcal{M}$; let $\sigma : \mathcal{O}b_{\mathcal{M}}|_{U} \to \mathcal{O}_U$ be the composition of $\mathcal{O}b_{\mathcal{M}} \to \mathcal{O}b_{\mathcal{N}}|_{\mathcal{M}}$ with $\sigma_{\mathcal{U}}|_U$. As before, we let $\mathcal{N}(\sigma_{\mathcal{U}}) = \mathcal{N} \backslash \mathcal{U}$ and $\mathcal{M}(\sigma) = \mathcal{M} \backslash U$ be the degeneracy loci of $\sigma_{\mathcal{U}}$ and σ. Note that $\mathcal{M}(\sigma) = \mathcal{N}(\sigma_{\mathcal{U}}) \times_T 0$.

We let $\tau : 0 \to T$ be the inclusion and let $\tau^! : A_*\mathcal{N}(\sigma_{\mathcal{U}}) \to A_*\mathcal{M}(\sigma)$ be the Gysin map that is "intersecting" with $0 \in T$.

Theorem 5.2. *Let the notation be as stated; let $[\mathcal{N}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\mathcal{N}(\sigma_{\mathcal{U}})$ and $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*\mathcal{M}(\sigma)$ be the localized virtual cycles. Then $[\mathcal{M}]^{\mathrm{vir}}_{\mathrm{loc}} = \tau^!([\mathcal{N}]^{\mathrm{vir}}_{\mathrm{loc}})$.*

We will prove the Theorem by applying the rational equivalence inside the deformations of ambient cone-stacks constructed by Kim-Kresch-Pantev [17].

We begin with recalling the convention used in [17]. For an object $G^\bullet$ in the derived category $D(\mathcal{M})$ of coherent sheaves on $\mathcal{M}$, we denote $p_{\mathcal{M}}^* G^\bullet \otimes p_{\mathbb{P}^1}^* \mathcal{O}(-1)$ by $G^\bullet(-1)$, where $p_{\mathcal{M}}, p_{\mathbb{P}^1}$ are the two projections of $\mathcal{M} \times \mathbb{P}^1$. Accordingly, whenever we see a $G^\bullet \in D(\mathcal{M})$ appearing in a sequence involving elements in $D(\mathcal{M} \times \mathbb{P}^1)$, we understand it as $p_{\mathcal{M}}^* G^\bullet$.

For the top line in (5.1), we mimick the construction of [17]. Let $[x, y]$ be the homogeneous coordinates of $\mathbb{P}^1$, let $\tilde{g} : F^\bullet|_{\mathcal{M}}(-1) \to F^\bullet|_{\mathcal{M}} \oplus E^\bullet$ be defined by $(x \cdot 1, y \cdot g)$. The mapping cone $c(\tilde{g})$ of $\tilde{g}$ is locally quasi-isomorphic to a two-term complex of locally free sheaves, and fits into the distinguished triangle

(5.3) $$F^\bullet|_{\mathcal{M}}(-1) \xrightarrow{\tilde{g}} F^\bullet|_{\mathcal{M}} \oplus E^\bullet \longrightarrow c(\tilde{g}) \xrightarrow{+1} .$$

Following [17], $h^1/h^0(c(\tilde{g})^\vee)$ is a vector bundle stack over $\mathcal{M} \times \mathbb{P}^1$, thus flat over $\mathbb{P}^1$; its fibers over $a = [1, 0]$ and $b = [0, 1] \in \mathbb{P}^1$ are

$$h^1/h^0(c(\tilde{g})^\vee) \times_{\mathbb{P}^1} a = h^1/h^0((E^\bullet)^\vee), \ h^1/h^0(c(\tilde{g})^\vee) \times_{\mathbb{P}^1} b = h^1/h^0((F^\bullet|_{\mathcal{M}})^\vee) \times \mathbb{A}^1.$$

Here the $\mathbb{A}^1$ in the product is the fiber of the vector bundle $h^1/h^0([0 \to \mathcal{O}_{\mathcal{M}}]) \cong \mathbb{C}_{\mathcal{M}}$.

We let $\mathbf{M}^0_{\mathcal{N}/\mathcal{S}}$ be the deformation of $\mathcal{N}$ to its normal cone $\mathbf{c}_{\mathcal{N}/\mathcal{S}}$; let $\mathbf{c}_{\mathcal{M} \times \mathbb{P}^1 / \mathbf{M}^0_{\mathcal{N}/\mathcal{S}}}$ be the normal cone to $\mathcal{M} \times \mathbb{P}^1$ in $\mathbf{M}^0_{\mathcal{N}/\mathcal{S}}$. We let $\mathbf{e} = h^1/h^0(c(\tilde{g})^\vee)$, which is a bundle stack over $\mathcal{M} \times \mathbb{P}^1$. According to [17], we have a canonical closed immersions

(5.4) $\quad \mathbf{c}_{\mathcal{N}/\mathcal{S}} \subset h^1/h^0((F^\bullet)^\vee) \quad$ and $\quad \mathcal{D} := \mathbf{c}_{\mathcal{M} \times \mathbb{P}^1 / \mathbf{M}^0_{\mathcal{N}/\mathcal{S}}} \subset \mathbf{e} = h^1/h^0(c(\tilde{g})^\vee).$

We will see that $\mathcal{D}$ is flat over $\mathbb{P}^1 - b$; for $\mathbf{E} := h^1/h^0((E^\bullet)^\vee)$, $\mathbf{e} \times_{\mathbb{P}^1} a \cong \mathbf{E}$, and

(5.5) $$a^![\mathcal{D}] = [\mathcal{D} \times_{\mathbb{P}^1} a] = [\mathbf{c}_{\mathcal{M}/\mathcal{S}}] \in Z_*\mathbf{E}.$$

(Following [10], for $c \in \mathbb{P}^1$ we define $c^![\mathcal{D}] = [\mathcal{D}^{\mathrm{fl}} \times_{\mathbb{A}^1} c]$, where $\mathcal{D}^{\mathrm{fl}}$ is the $\mathbb{P}^1$-flat part of $\mathcal{D}$.) Consequently, (the flat part of) $\mathcal{D}$ provides a rational equivalence between $[\mathbf{c}_{\mathcal{M}/\mathcal{S}}]$ and $b^![\mathcal{D}]$.

We let $\mathcal{D}_b = \mathcal{D} \times_{\mathbb{P}^1} b$. In [30] and [17], a canonical rational equivalence $[\mathcal{R}] \in W_*(\mathcal{D}_b)$ is constructed such that

(5.6) $\quad \partial_0[\mathcal{R}] = [b^!\mathcal{D}] \quad$ and $\quad \partial_\infty[\mathcal{R}] = [\mathbf{c}_{\mathcal{M}/\mathbf{c}_{\mathcal{N}/\mathcal{S}}}] \in Z_*(\mathbf{e} \times_{\mathbb{P}^1} b).$

Combining, we obtain a pair of rational equivalences giving the equivalence of

$$[\mathbf{c}_{\mathcal{M}/\mathcal{S}}] \sim [\mathbf{c}_{\mathcal{M}/\mathbf{c}_{\mathcal{N}/\mathcal{S}}}] \quad \text{via} \quad ([\mathcal{D}], [\mathcal{R}]).$$

This rational equivalence implies $\tau^![\mathcal{N}]^{\mathrm{vir}} = [\mathcal{M}]^{\mathrm{vir}}$.

To prove the constancy of the localized virtual cycles, we need to show that both $[\mathcal{D}]$ and $[\mathcal{R}]$ lie in the appropriate kernel bundle stack. First, $\sigma_{\mathcal{U}} : \mathcal{O}b_{\mathcal{N}}|_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}$ induces $(F^\bullet)^\vee|_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}[-1]$, which together with the $(E^\bullet)^\vee|_U \to \mathcal{O}_U[-1]$ induced by σ, defines the two right vertical arrows β_1 and β_2 shown below, and making the right square a commutative square of homomorphisms of complexes. Using β_1 and β_2, we construct a homomorphism of complexes σ' (shown), which together with the β_i defines an arrow between distinguished triangles (after restricting to $U \times \mathbb{P}^1$):
(5.7)

$$
\begin{array}{ccccc}
c(\tilde{g})^\vee|_{U \times \mathbb{P}^1} & \longrightarrow & (F^\bullet)^\vee|_U \oplus (E^\bullet)^\vee|_U & \longrightarrow & (F^\bullet)^\vee|_U(1) \xrightarrow{+1} \\
\downarrow{\sigma'} & & \downarrow{\beta_1} & & \downarrow{\beta_2} \\
c(\mathbf{x})[-1] & \xrightarrow{(x,y)} & \mathcal{O}_{U \times \mathbb{P}^1}^{\oplus 2}[-1] & \xrightarrow{\mathbf{x}=(x,y)[-1]} & p_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1)[-1] \xrightarrow{+1}
\end{array}
$$

Let L be the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Then, $c(\mathbf{x}) \cong_{qis.} p_{\mathbb{P}^1}^* L$; thus

$$
h^1/h^0(c(\mathbf{x})[-1]) \cong p_{\mathbb{P}^1}^* L,
$$

which is a line bundle on $\mathcal{M} \times \mathbb{P}^1$. On the other hand, since both β_1 and β_2 are defined and surjective on $U \times \mathbb{P}^1$, $\bar{\sigma}$ is surjective on $U \times \mathbb{P}^1$. Thus $\bar{\sigma}$ induces a surjective bundle-stack homomorphism (which we still denote by $\bar{\sigma}$)

(5.8) $$\bar{\sigma} : \mathbf{e}|_{U \times \mathbb{P}^1} \longrightarrow p_{\mathbb{P}^1}^* L|_{U \times \mathbb{P}^1}.$$

We let $\mathbf{e}(\bar{\sigma})$ be the union of $\mathbf{e}|_{\mathcal{M}(\sigma) \times \mathbb{P}^1}$ with the kernel $\ker\{\bar{\sigma}\}$, endowed with the reduced structure; $\mathbf{e}(\bar{\sigma}) \subset \mathbf{e}$ is closed. As mentioned, we let $W_* \mathbf{e}_b$ be the group of rational equivalences of $\mathbf{e}_b = \mathbf{e} \times_{\mathbb{P}^1} b$.

Lemma 5.3. *The cycle $[\mathcal{D}] \in Z_* \mathbf{e}$ and the rational equivalence $[\mathcal{R}] \in W_* \mathbf{e}_b$ lie in $Z_* \mathbf{e}(\bar{\sigma})$ and $W_* \mathbf{e}(\bar{\sigma})_b$, respectively.*

Proof. Since $\mathcal{R} \subset \mathcal{D} \times_{\mathbb{P}^1} b$, the support of $\mathcal{R}$ lies in $\mathcal{D}$. Thus we only need to show that the support of $\mathcal{D}$ lies in $\mathbf{e}(\bar{\sigma})$. Furthermore, since $\mathbf{e}|_{\mathcal{M}(\sigma) \times \mathbb{P}^1} = \mathbf{e}(\bar{\sigma})|_{\mathcal{M}(\sigma) \times \mathbb{P}^1}$, to prove the Lemma we only need to show that the support of $\mathcal{D} \times_{\mathcal{M} \times \mathbb{P}^1} (U \times \mathbb{P}^1)$ lies in $\mathbf{e}(\bar{\sigma})|_{U \times \mathbb{P}^1}$. Thus, it suffices to prove the Lemma in case $\mathcal{N} = \mathcal{U}$; namely, $\sigma_{\mathcal{U}}$ is regular and surjective everywhere.

In the following, we assume $\sigma_{\mathcal{N}} : \mathcal{O}b_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$ is regular and surjective on $\mathcal{N}$. Since the statement of the Lemma is local, it suffices to investigate the situation over $N/S \to \mathcal{N}/\mathcal{S}$ for S smooth over $\mathcal{S}$, $N \to \mathcal{N}$ étale, and $N \subset \mathcal{N} \times_{\mathcal{S}} S$ is an affine open subscheme.

We then pick a smooth affine scheme V over S and T, an embedding $N \to V$ that is both an S and T-embedding. Using $N \to V$, we have a representative

(5.9) $\tau^{\geq -1} L_{N/S}^\bullet = [I_N/I_N^2 \to \Omega_{V/S}|_N]$ and $\tau^{\geq -1} L_{M/S}^\bullet = [I_M/I_M^2 \to \Omega_{V/S}|_M]$,

where $M = N \times_T 0$ is the corresponding chart of $\mathcal{M}$, I_N and I_M are the ideal sheaves of $N \subset V$ and $M \subset V$, respectively.

Since N and V are affine, we can assume that there is a vector bundle (locally free sheaf) F on N (resp. E on M) so that

(5.10) $F^\bullet|_N = [F^\vee \to \Omega_{V/S}|_N]$ and $E^\bullet|_M = [E^\vee \to \Omega_{V/S}|_M]$,

and the diagram (5.1) restricted to M is represented by the following commuting homomorphisms of complexes of sheaves[2]

(5.11)
$$
\begin{array}{ccccc}
[F^\vee \to \Omega_{V/S}|_N]|_M & \xrightarrow{g_N} & [E^\vee \to \Omega_{V/S}|_M] & \longrightarrow & [\mathcal{O}_M \to 0] \\
\downarrow{[\phi^{-1},\phi^0]|_M} & & \downarrow{[\psi^{-1},\psi^0]} & & \downarrow \\
\tau^{\geq-1}L^\bullet_{N/S}|_M & \longrightarrow & \tau^{\geq-1}L^\bullet_{M/S} & \longrightarrow & [I_{M\subset N}/I^2_{M\subset N} \to 0]
\end{array}
$$

which in addition satisfy $\phi^0 = \mathrm{id} : \Omega_{V/S}|_N \to \Omega_{V/S}|_N$ and $\psi^0 = \mathrm{id}$, and the part of the top line at place $[-1]$ is an exact sequence

$$0 \longrightarrow F^\vee|_M \longrightarrow E^\vee \longrightarrow \mathcal{O}_M \longrightarrow 0.$$

Here for the terms in the second line we use representatives (5.9).

Since V is affine, we can split this exact sequence to get $E^\vee \cong F^\vee|_M \oplus \mathcal{O}_M$. We then extend F to a vector bundle (locally free sheaf) $\tilde{F}$ on V; because of the isomorphism $E^\vee \cong F^\vee|_M \oplus \mathcal{O}_M$, $\tilde{F} \oplus \mathcal{O}_V$ is an extension of E.

We now give an explicit description of $\mathbf{e}|_{M\times\mathbb{P}^1}$ $(:= \mathbf{e} \times_{\mathcal{M}\times\mathbb{P}^1} (M \times \mathbb{P}^1))$. By the canonical construction of $c(\tilde{g})$, we have a canonical isomorphism

(5.12)
$$\mathbf{e}|_{M\times\mathbb{P}^1} \cong h^1/h^0((c(\tilde{g}_N)^\vee),$$

where $\tilde{g}_N = (x \cdot 1, y \cdot g_N)$ is

$$\tilde{g}_N : [F^\vee \to \Omega_{V/S}|_N]|_M(-1) \longrightarrow [F^\vee \to \Omega_{V/S}|_N]|_M \oplus [E^\vee \to \Omega_{V/S}|_M]$$

(cf. $\tilde{g}$ in (5.3)). Using the isomorphism $E \cong F|_M \oplus \mathcal{O}_M$ and that the arrow $F^\vee|_M \to E^\vee$ is the inclusion under the splitting, we see that the mapping cone

(5.13)
$$c(\tilde{g}_N) \cong_{qis.} [F^\vee|_M \to \Omega_{V/S}|_M](1) \oplus [\mathcal{O}_M \to 0].$$

Thus, following the convention that $F|_M(-1) = p_M^* F|_M \otimes p_{\mathbb{P}^1}^* L$ is a vector bundle on $M\times\mathbb{P}^1$, where $L \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, we obtain a tautological flat bundle stack morphism

$$\Phi : F_M(-1) \times \mathbb{A}^1 \longrightarrow \mathbf{e}_N = h^1/h^0(c(\tilde{g}_N)^\vee).$$

We let $\mathcal{D}|_{M\times\mathbb{P}^1} \subset \mathbf{e}|_{M\times\mathbb{P}^1}$ be the pull-back of $\mathcal{D} \subset \mathbf{e}$ using the isomorphism (5.12), and let

$$D = \Phi^*(\mathcal{D}|_{M\times\mathbb{P}^1}) \subset F|_M(-1) \times \mathbb{A}^1.$$

We next describe the pull-back $\Phi^*(\mathbf{e}(\bar\sigma))$. ($\bar\sigma$ is defined in (5.8).) Like the construction of σ' in (5.7), the surjective homomorphism $\sigma_N : Ob_N \to \mathcal{O}_N$ (surjectivity is assumed at the beginning of the proof) induces a surjective homomorphism

(5.14)
$$\delta : F \longrightarrow h^1((F^\bullet)^\vee) \longrightarrow \bar{\mathbb{C}}_N.$$

Cf. (5.10). Pulling back δ to $N \times \mathbb{P}^1$ and twisting it by $\mathcal{O}_{\mathbb{P}^1}(-1)$, we obtain the second arrow shown below, which composed with the projection to the first factor of $F(-1) \oplus \mathbb{C}_{N\times\mathbb{P}^1}$ defines γ_N:

(5.15)
$$\gamma_N : F(-1) \oplus \mathbb{C}_{N\times\mathbb{P}^1} \xrightarrow{\mathrm{pr}} F(-1) \longrightarrow \bar{p}_{\mathbb{P}^1}^* L,$$

where $\bar{p}_{\mathbb{P}^1} : N \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection.

We let K be the kernel bundle of γ_N. It is direct to check that

(5.16)
$$K|_{M\times\mathbb{P}^1} = \Phi^*(\mathbf{e}(\bar\sigma)) \subset F|_M(-1) \times \mathbb{A}^1.$$

---

[2]The bottom line of (5.11) is a truncated version of the distinguished triangle of cotangent complexes for $M \to N \to S$. It is not necessarily exact

This way, $[\mathcal{D}] \in Z_*\mathbf{e}(\bar{\sigma})$ is equivalent to the reduced part D_{red} (of D) lying in $K|_{M\times\mathbb{P}^1}$.

To prove $D_{\mathrm{red}} \subset K|_{M\times\mathbb{P}^1}$, we give a graph construction of D. We pick a lifting $f_1 \in I_N \otimes_{\mathcal{O}_V} \tilde{F}$ of $\phi^{-1} \in I_N/I_N^2 \otimes_{\mathcal{O}_N} F$, and extend f_1 to a lifting $(f_1, f_2) \in I_M \otimes_{\mathcal{O}_V} (\tilde{F} \oplus \mathbb{C}_V)$ of ψ^{-1}. After shrinking V if necessary, $f_1 = 0$ defines N and $(f_1, f_2) = 0$ defines M. By shrinking T if necessary, we can pick a uniformizing parameter $t \in \Gamma(\mathcal{O}_T)$ so that $t = 0$ consists of a single point $0 \in T$.

We next view x^{-1} as a meromorphic section of $L = \mathcal{O}_{\mathbb{P}^1}(-1)$ with no zero and only pole at $b = [0, 1]$. Then $((tx)^{-1}f_1, t^{-1}f_2)$ is a section of $F(-1) \oplus \mathbb{C}_{N\times\mathbb{P}^1}$ over $(N \setminus M) \times (\mathbb{P}^1 - b)$; we denote by

(5.17)
$$\Gamma \subset (F(-1) \times \mathbb{A}^1)|_{(N\setminus M)\times(\mathbb{P}^1-b)}$$

this section (graph). We let $\bar{\Gamma}$ be the closure of Γ in $F(-1) \times \mathbb{A}^1$. According to the construction in [17],

$$D = \bar{\Gamma} \times_{N\times\mathbb{P}^1} (M \times \mathbb{P}^1) \subset F|_M(-1) \times \mathbb{A}^1.$$

We now prove $D_{\mathrm{red}} \subset K|_{M\times\mathbb{P}^1}$. First, let

$$\Gamma_1 \subset (F \oplus \mathbb{C}_N)|_{N\setminus M}$$

be the graph of $(t^{-1}f_1, t^{-1}f_2)$ (recall $t \in \Gamma(\mathcal{O}_T)$ and $M = N \cap (f_2 = 0)$); let $\bar{\Gamma_1}$ be its closure in $F \oplus \mathbb{C}_N$; then $\bar{\Gamma_1} \times_N M$ is the normal cone $C_{M/V}$ embedded in $F|_M \oplus \mathbb{C}_M = F|_M \times \mathbb{A}^1$ by the defining equation (f_1, f_2) of M.

We next form the commutative diagram

$$
\begin{array}{ccc}
(F(-1) \oplus \mathbb{C}_{N\times\mathbb{P}^1})|_{N\times(\mathbb{P}^1-b)} & \xrightarrow{(x\cdot 1, 1)} & \bar{p}_N^*(F \oplus \mathbb{C}_N)|_{N\times(\mathbb{P}^1-b)} \\
\downarrow{\scriptstyle \gamma_N} & & \downarrow{\scriptstyle (\bar{p}_N^*\delta, 0)} \\
\bar{p}_{\mathbb{P}^1}^* L|_{N\times(\mathbb{P}^1-b)} & \xrightarrow{\quad x \quad} & \mathbb{C}_{N\times\mathbb{P}^1}|_{N\times(\mathbb{P}^1-b)}.
\end{array}
$$

Note that the two horizontal arrows are isomorphisms of vector bundles. By our construction, Γ is the preimage of $\bar{p}_N^*\Gamma_1$ under the top horizontal arrow. Therefore, $\bar{\Gamma}|_{N\times(\mathbb{P}^1-b)}$ is the preimage of $\bar{p}_N^*\bar{\Gamma_1}|_{N\times(\mathbb{P}^1-b)}$, and thus

$$D|_{M\times(\mathbb{P}^1-b)} \subset (p_M^*F|_M \times \mathbb{A}^1)|_{M\times(\mathbb{P}^1-b)}$$

is the preimage of $p_M^{-1}C_{M/V} \subset p_M^*(F|_M \oplus \mathbb{C}_M)$. This proves that $\mathcal{D}$ is flat over $\mathbb{P}^1 - b$ and also the identity (5.5).

Furthermore, by Corollary 4.5, the support of $C_{M/V} \subset E|_M \cong F|_M \oplus \mathbb{C}_M$ lies in the kernel of $(\delta, 0) : F|_M \oplus \mathbb{C}_M \to \mathbb{C}_M$. Thus the support of $p_M^{-1}C_{M/V}$ lies in the kernel of $(\bar{p}_N^*\delta, 0)$, and the support of $D|_{M\times(\mathbb{P}^1-b)}$ lies in the kernel of γ_N.

It remains to show that every irreducible component of D that lies over $M \times_{\mathbb{P}^1} b$ lies in $K|_{M\times\mathbb{P}^1}$ (cf. (5.16)). Let $A \subset D$ be an irreducible component lying over $M \times_{\mathbb{P}^1} b$, and let $v \in A$ be a general closed point. Since $D \subset \bar{\Gamma} - \Gamma$, we can find a smooth curve $0 \in S$ and a morphism $\tilde{\rho} : (0, S) \to (v, \bar{\Gamma})$ so that $\tilde{\rho}(S \setminus 0) \in \Gamma$. We let $(\rho_1, \rho_2) : S \to M \times \mathbb{P}^1$ be $\tilde{\rho}$ composed with the projection $\bar{\Gamma} \to M \times \mathbb{P}^1$. By shrinking S if necessary, we can assume $\rho_2(S) \subset \mathbb{P}^1 - a$.

We let $x_S := x \circ \rho_2$ (resp. $t_S := t \circ \rho_1$) be the composition of $x \in \Gamma(\mathcal{O}_{\mathbb{P}^1-a})$ with ρ_2 (resp. $t \in \Gamma(\mathcal{O}_T)$ with the composite $S \xrightarrow{\rho_1} M \to T$). Since $v = \lim_{s\to 0} \tilde{\rho}(s)$, by the definition of Γ, we have

$$v = (v_1, v_2) = \left(\lim_{s\to 0} (x_S \cdot t_S)^{-1} \cdot f_1 \circ \rho, \ \lim_{s\to 0} (t_S)^{-1} \cdot f_2 \circ \rho \right).$$

Let $u = \rho_1(0) \in M$. Repeating the proof of Lemma 4.4 and Corollary 4.5, the surjective homomorphism $\sigma_{\mathcal{N}} : \mathcal{O}b_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$ forces $v_1 \in \ker\{\delta(u) : F|_u \to \mathbb{C}\}$ (cf. (5.14)). Finally, by the explicit form of γ_N in (5.15), $\ker\{\delta(u) : F|_u \to \mathbb{C}\} \times \mathbb{A}^1 = K|_{u \times b}$. Therefore, v_1 lies in this kernel, which proves $A \subset K|_{M \times \mathbb{P}^1}$. This proves the Lemma. $\qquad\square$

Proof of Theorem 5.2. This follows from that the localized Gysin map $s^!_{\mathbf{e},\bar{\sigma}}$ preserves rational equivalence. $\qquad\square$

6. Application: Localized GW-invariants

We let X be a smooth quasi-projective variety equipped with a holomorphic two-form $\theta \in \Gamma(\Omega_X^2)$. This form induces a cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, the moduli space of n-pointed genus g stable morphisms to X of class β. This cosection defines a localized virtual class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, thus localized GW-invariants of (X,θ).

We begin with the construction of the cosection of the obstruction sheaf of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. For simplicity, we will fix the data g, n, X and β for the moment and abbreviate $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to $\mathcal{M}$. We let $f : \mathcal{C} \to X$ and $\pi : \mathcal{C} \to \mathcal{M}$ be the universal family of $\mathcal{M}$. If we denote by $\mathcal{S}$ the Artin stack of genus g connected nodal curves with n marked points, $\mathcal{M}$ is a representable DM stack over $\mathcal{S}$; and the relative obstruction sheaf of the standard relative obstruction theory of $\mathcal{M}/\mathcal{S}$ (cf. [2]) is $\mathcal{O}b_{\mathcal{M}/\mathcal{S}} = R^1\pi_* f^* T_X$.

We now show that a holomorphic two-form θ defines a cosection of $\mathcal{O}b_{\mathcal{M}/\mathcal{S}}$. Indeed, by viewing it as an anti-symmetric homomorphism

$$(6.1) \qquad\qquad \hat{\theta} : T_X \longrightarrow \Omega_X, \quad (\hat{\theta}(v), v) = 0,$$

it defines the first arrow in the following sequence of homomorphisms

$$(6.2) \qquad R^1\pi_* f^* T_X \longrightarrow R^1\pi_* f^* \Omega_X \longrightarrow R^1\pi_* \Omega_{\mathcal{C}/\mathcal{M}} \longrightarrow R^1\pi_* \omega_{\mathcal{C}/\mathcal{M}},$$

where the second is induced by $f^*\Omega_X \to \Omega_{\mathcal{C}/\mathcal{M}}$, and the last by the tautological $\Omega_{\mathcal{C}/\mathcal{M}} \to \omega_{\mathcal{C}/\mathcal{M}}$. Because $R^1\pi_* \omega_{\mathcal{C}/\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}$, the composite of this sequence provides

$$(6.3) \qquad\qquad \sigma_\theta^{\mathrm{rel}} : R^1\pi_* f^* T_X = \mathcal{O}b_{\mathcal{M}/\mathcal{S}} \longrightarrow \mathcal{O}_{\mathcal{M}}.$$

The obstruction sheaf of $\mathcal{M}$ is the cokernel of $p^* T_{\mathcal{S}} \to \mathcal{O}b_{\mathcal{M}/\mathcal{S}}$, where $p : \mathcal{M} \to \mathcal{S}$ is the projection. Using the universal family f and that $R^1\pi_* f^* T_X = \mathcal{E}xt_\pi^1(f^*\Omega_X, \mathcal{O}_{\mathcal{C}})$, we have the exact sequence

$$(6.4) \qquad \mathcal{E}xt_\pi^1(\Omega_{\mathcal{C}/\mathcal{M}}(x), \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt_\pi^1(f^*\Omega_X, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{O}b_{\mathcal{M}} \longrightarrow 0,$$

where the first arrow is induced by $f^*\Omega_X \to \Omega_{\mathcal{C}/\mathcal{M}}$. Here $x = x_1 + \cdots + x_n$ denotes the divisor of marked points.

Lemma 6.1. *The composition*

$$\mathcal{E}xt_\pi^1(\Omega_{\mathcal{C}/\mathcal{M}}(x), \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt_\pi^1(f^*\Omega_X, \mathcal{O}_{\mathcal{C}}) \xrightarrow{\sigma_\theta^{\mathrm{rel}}} \mathcal{O}_{\mathcal{M}}$$

is trivial.

Proof. Applying the definition of $\sigma_\theta^{\mathrm{rel}}$, one sees that the stated composition is the composition of the following sequence

$$\mathcal{E}xt_\pi^1(\Omega_{\mathcal{C}/\mathcal{M}}(x), \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt_\pi^1(f^*\Omega_X, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt_\pi^1(f^* T_X, \mathcal{O}_{\mathcal{C}}) \longrightarrow \mathcal{E}xt_\pi^1(\omega_{\mathcal{C}/\mathcal{M}}^\vee, \mathcal{O}_{\mathcal{C}})$$

that is induced by the sequence

$$\Theta : \omega_{\mathcal{C}/\mathcal{M}}^{\vee} \longrightarrow f^*T_X \xrightarrow{f^*\hat{\theta}} f^*\Omega_X \longrightarrow \Omega_{\mathcal{C}/\mathcal{M}}(x),$$

where the first arrow is the dual of the composite $f^*\Omega_X \to \Omega_{\mathcal{C}/\mathcal{M}} \to \omega_{\mathcal{C}/\mathcal{M}}$.

We now prove that the composite $\Theta = 0$. First, let $\mathcal{C}_{\text{reg}}$ be the smooth loci of the fibers of $\mathcal{C}/\mathcal{M}$. Since $\Omega_{\mathcal{C}/\mathcal{M}}|_{\mathcal{C}_{\text{reg}}} = \omega_{\mathcal{C}/\mathcal{M}}|_{\mathcal{C}_{\text{reg}}}$, and since $\hat{\theta}$ (in (6.1)) is anti-symmetric, $\Theta|_{\mathcal{C}_{\text{reg}}}$ is anti-symmetric. Therefore, because $\omega_{\mathcal{C}/\mathcal{M}}$ is invertible, $\Theta|_{\mathcal{C}_{\text{reg}}} = 0$.

Now let $q \in \mathcal{C} \setminus \mathcal{C}_{\text{reg}}$; let $\xi = \pi(q) \in \mathcal{M}$. We pick an affine scheme M and an étale morphism $M \to \mathcal{M}$ whose image contains ξ; let $\bar{\xi} \in M$ be a lift of ξ. We let C be an affine open $C \subset \mathcal{C} \times_{\mathcal{M}} M$ such that $(q, \xi) \in \mathcal{C} \times_{\mathcal{M}} M$ lifts to a $\bar{q} \in C$. We let $g : C \to X$ be the restriction of f to C, and let $\bar{\xi} \in M$ be the image of $\bar{q}$ under $C \to M$. Since both C and M are affine, we can find a closed immersion $M \subset \tilde{M}$ into a smooth scheme $\tilde{M}$ and extend C/M to a family of nodal curves $\tilde{C}/\tilde{M}$ so that the node $\bar{q} \in C_{\bar{\xi}}$ is smoothed in the family $\tilde{C}/\tilde{M}$, and that the morphism $g : C \to X$ extends to $\tilde{g} : \tilde{C} \to X$.

For the family $\tilde{g} : \tilde{C} \to X$, we form the similarly defined homomorphism

(6.5) $$\tilde{\Theta} : \omega_{\tilde{C}/\tilde{M}}^{\vee} \longrightarrow \Omega_{\tilde{C}/\tilde{M}}.$$

Like Θ, $\tilde{\Theta}$ vanishes away from the singularities of the fibers of $\tilde{C}/\tilde{M}$. On the other hand, since $\tilde{M}$ is smooth and the node $\bar{q} \in \tilde{C}_{\bar{\xi}}$ is smoothed in the family $\tilde{C}/\tilde{M}$, $\Omega_{\tilde{C}/\tilde{M}}$ has no torsion near $\bar{q}$. Therefore, the fact that $\tilde{\Theta}$ vanishes away from the singular points of the fibers of $\tilde{C}/\tilde{M}$ implies that $\tilde{\Theta}$ vanishes near $\tilde{q}$.

Finally, since $\Theta|_{C_{\bar{\xi}}} = \tilde{\Theta}|_{C_{\bar{\xi}}}$, we conclude that Θ vanishes at $q \in \mathcal{C}$. Since $q \in \mathcal{C}$ is an arbitrary node, this shows that $\Theta = 0$. This proves that the composite in the statement of the Lemma vanishes. $\qquad\square$

Corollary 6.2. *The homomorphism σ_θ^{rel} lifts to a homomorphism $\sigma_\theta : \mathcal{O}b_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$.*

Proof. This follows from Lemma 6.1 and the exact sequence (6.4). $\qquad\square$

Definition 6.3. *A stable map $u : C \to X$ is called θ-null if the composite*

$$u^*(\hat{\theta}) \circ du : T_{C_{reg}} \longrightarrow u^*T_X|_{C_{reg}} \longrightarrow u^*\Omega_X|_{C_{reg}}$$

is trivial over the regular locus C_{reg} of C.

Proposition 6.4. *Any holomorphic two-form $\theta \in H^0(\Omega_X^2)$ on a smooth quasi-projective variety X induces a homomorphism $\sigma_\theta : \mathcal{O}b_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}}$ of the obstruction sheaf $\mathcal{O}b_{\mathcal{M}}$ of the moduli of stable morphisms $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$. The homomorphism σ is surjective away from the set of θ-null stable maps in $\mathcal{M}$.*

Proof. Only the last part needs to be proved. Since σ_θ is the lift of $\sigma_\theta^{\text{rel}}$, it is surjective at $[u : C \to X] \in \mathcal{M}$ if and only if $\sigma_\theta^{\text{rel}}$ is surjective at $[u]$. Since the latter at $[u]$ is the composition

$$H^1(C, u^*T_X) \xrightarrow{u^*\hat{\theta}} H^1(C, u^*\Omega_X) \longrightarrow H^1(C, \Omega_C) \longrightarrow H^1(C, \omega_C) = \mathbb{C},$$

whose Serre dual is

$$\mathbb{C} = H^0(C, \mathcal{O}_C) \longrightarrow H^0(f^*T_X \otimes \omega_C) \longrightarrow H^0(f^*\Omega_X \otimes \omega_C).$$

Because $\mathcal{O}_C$ is generated by global sections, the composite of the above sequence is trivial if and only if the composite

$$T_C \otimes \omega_C|_{C_{\mathrm{reg}}} \longrightarrow f^* T_X \otimes \omega_C|_{C_{\mathrm{reg}}} \longrightarrow f^* \Omega_X \otimes \omega_C|_{C_{\mathrm{reg}}}$$

is trivial. But this is equivalent to u being θ-null. This proves the Proposition. $\square$

Using the cosection σ_θ, we can localize the virtual cycle of $\mathcal{M}$. Let $\mathcal{M}(\sigma_\theta) \subset \mathcal{M}$ be the collection of θ-null stable maps. Clearly, $\mathcal{M}(\sigma_\theta) \subset \mathcal{M}$ is closed. Because $\sigma_\theta : \mathcal{O}b_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ is surjective away from $\mathcal{M}(\sigma_\theta)$, applying Theorem 5.1, we obtain the localized virtual cycle

$$[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} \in A_* \mathcal{M}(\sigma_\theta).$$

In case $\mathcal{M}(\sigma_\theta)$ is proper, we define the localized GW-invariants as follows. We let $\mathrm{ev} : \mathcal{M} \to X^n$ be the evaluation morphism, let $\gamma_1, \cdots, \gamma_n \in H^*(X)$, let $\alpha_1, \cdots, \alpha_n \in \mathbb{Z}^{\geq 0}$, and let ψ_i be the first Chern class of the relative cotangent line bundle of the domain curves at the i-th marked point. The localized GW-invariant of X with descendants is defined to be

$$\langle \tau_{\alpha_1}(\gamma_1) \cdots \tau_{\alpha_n}(\gamma_n) \rangle_{g,\beta}^{X,\mathrm{loc}} = \int_{[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}}} \mathrm{ev}^*(\gamma_1 \times \cdots \times \gamma_n) \cdot \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$$

In case X is proper, then $\mathcal{M}(\sigma_\theta)$ is automatically proper.

Lemma 6.5. *If X is proper, the localized GW-invariant coincides with the ordinary GW-invariant of X.*

Proof. This follows from the last statement in Theorem 5.1. $\square$

A Corollary of this is the following generalization of the vanishing results of J. Lee and T. Parker [21, 20] for compact algebraic surfaces.

Corollary 6.6 (First vanishing). *Let X be a smooth projective variety endowed with a holomorphic two-form θ. The virtual cycle of the moduli of stable morphisms $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is trivial unless the class β can be represented by a θ-null stable morphism.*

Proof. Let $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$. Suppose β cannot be represented by a θ-null stable morphism, i.e. $\mathcal{M}(\sigma_\theta) = \emptyset$. Then obviously $[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}} = 0$. Since $[\mathcal{M}]^{\mathrm{vir}}$ is the image of $[\mathcal{M}]_{\mathrm{loc}}^{\mathrm{vir}}$ in $A_* \mathcal{M}$, $[\mathcal{M}]^{\mathrm{vir}} = 0$. $\square$

A generalization of the second vanishing of Lee-Parker is as follows. Since $\hat{\theta}$ is a homomorphism from $T_X \to \Omega_X$, $\det \hat{\theta}$ can be viewed as a section of the line bundle $K_X^{\otimes 2}$. Let $D = (\det \hat{\theta} = 0)$. Every $[u, C] \in \mathcal{M}(\sigma_\theta)$ has $u(C) \subset D$. Then an easy argument shows that

Corollary 6.7 (Second vanishing). *Let (X, θ) be as in Corollary 6.6. For $\beta \neq 0$ and $\gamma_i \in H^*(X)$, the GW-invariant $\langle \prod \tau_{\alpha_i}(\gamma_i) \rangle_{g,\beta}^X$ vanishes if one of the classes γ_i is Poincaré dual to a cycle disjoint from D.*

Further applications of this cycle localization technique will appear in the sequel of this paper.

7. Other Applications

We conclude our paper with comments on other possible applications; some have been worked out and some are under development.

One application is the study of extremal GW-invariants of the Hilbert schemes of points of surfaces. After Beauville, we know that every holomorphic two-form θ of an algebraic surface X induces a holomorphic two-form on the Hilbert scheme $X^{[k]}$ of k-points of X. Thus the second vanishing applied to this case gives us vanishing of a lot of the GW-invariants of $X^{[k]}$. For general X, we can pick a meromorphic two-form θ. If $\beta \in H_2(X^{[k]}, \mathbb{Z})$ is an extremal class (i.e. a class in the kernel of $H_2(X^{[k]}, \mathbb{Z}) \to H_2(X^{(k)}, \mathbb{Z})$), then the meromorphic θ induces a meromorphic two-form on $\overline{\mathcal{M}}_{g,n}(X^{[k]}, \beta)$. This form can be used to study the GW-invariants of $X^{[k]}$. This has been exploited by W-P. Li and the second named author in [23].

Another application is in the study of Donaldson-Thomas invariants of a Calabi-Yau threefold. In [15], the authors showed that the modified Kirwan blow-up of the moduli of semistable sheaves has an obstruction theory whose obstruction sheaf has a regular surjective cosection. Using this cosection, we can localize the virtual cycle and prove a wall crossing formula using the master space construction and $\mathbb{C}^*$-localization. In comparison with Joyce's proof of the wall crossing formula, our proof does not use the Chern-Simons functional, and thus can possibly apply to a wider classes of moduli of derived objects over Calabi-Yau three-folds.

Another potential application is the study of the GW-invariants of a three-fold X that is a $\mathbb{P}^1$-bundle over a surface S equipped with a holomorphic two-form θ. The two-form θ on S pulls back to a two-form of X. Since the geometry of this two-form is explicit, one hopes that one can essentially reduce the study of the GW-invariants of X to a ruled surface over the canonical divisor $D \subset S$. It will be interesting to see some part of this carried out in the near future.

Addendum. The cosection localized virtual cycle has been applied to construct the GW-invariants of stable morphisms with fields [6]. This is the all genus generalization of the $g = 0$ theory in Super-String theories. The cosection localized virtual cycle can also give an algebro-geometric approach to FJRW-invariants of spin curves with fields.

In [13, 14], the authors derive a degeneration formula for certain localized Gromov-Witten invariants and use it to prove a conjecture of Maulik-Pandharipande [27] on degree two Gromov-Witten invariants of surfaces.

In [5], the first named author and H.-L. Chang use cosection localization to prove a conjecture of Dürr, Kabanov and Okonek [9], which gives an algebraic geometric definition of Seiberg-Witten invariants of any smooth projective surface.

Appendix A. Analytic analogue of localized Gysin maps

In the case of schemes over $\mathbb{C}$, we can construct an analytic analogue of the localized Gysin map by picking a smooth (C^∞) section of a bundle and intersecting with the cycle representative. This was the original approach adopted in [12]. Due to its potential applications, we include it here. We will also prove the equivalence of the two constructions.

For simplicity, we will work out the case of schemes. The case of orbifolds (DM stacks) is similar using multiple sections. Let $\pi : E \to M$ be a rank r holomorphic vector bundle over a reduced scheme M, $U \subset M$ an open subset and $\sigma : E|_U \to \mathcal{O}_U$

be a surjective homomorphism. We denote $M(\sigma) = M \setminus U$ and denote $E(\sigma)$ the kernel cone of σ, as defined before. We assume $M(\sigma)$ is compact.

We first pick a splitting of σ over U. Because σ is surjective, possibly by picking a hermitian metric on E we can find a smooth section $\check{\sigma} \in C^\infty(E|_U)$ so that $\sigma \circ \check{\sigma} = 1$. Next, we pick an analytic neighborhood V of $M(\sigma) \subset M$ such that V has compact (analytic) closure in M and the homomorphism

$$\imath_* : H_*(M(\sigma), \mathbb{Z}) \xrightarrow{\cong} H_*(V, \mathbb{Z})$$

induced by the inclusion $\imath : M(\sigma) \to V$ is an isomorphism. Because M is algebraic and $M(\sigma)$ is compact, such a neighborhood V always exists ([26]). We then extend $\check{\sigma}|_{M-V}$ to a smooth section $\check{\sigma}_{\mathrm{ex}} \in C^\infty(E)$. and pick a smooth function $\rho : M \to \mathbb{R}^{>0}$ so that $\xi = \rho \cdot \check{\sigma}_{ex} \in C^\infty(E)$ is a small perturbation of the zero section of E.

Now let $B \subset E(\sigma)$ be a complex d-dimensional closed subvariety. By fixing a stratification of B and of M by complex subvarieties, we can choose the extension $\check{\sigma}_{ex}$ and the function ρ so that the section ξ intersects B transversely. As a consequence, the intersection $B \cap \xi$, which is of pure dimension, has no real codimension 1 strata. Hence, it defines a closed oriented Borel-Moore cycle in E.

But on the other hand, since $\sigma \circ \xi|_{M-V} = \rho \in C^\infty(M - V)$, ξ is disjoint from B over $M - V$. Thus $B \cap \xi \subset E|_V$. Adding that V has compact (analytic) closure in M, $\pi(B \cap \xi)$ defines a closed chain in V, thus defines a homology class $[\pi(B \cap \xi)] \in H_{2d-2r}(V, \mathbb{Z})$. Finally, applying the inverse of $\imath_*$, we define

$$s_{E,\sigma}^{!,\mathrm{an}}([B]) = \imath_*^{-1}([\pi(B \cap \xi)]) \in H_{2d-2r}(M(\sigma), \mathbb{Z}).$$

Applying the standard transversality argument, one easily shows that this class is independent of the choice of V and the section ξ; thus it only depends on the cycle B we begin with.

Definition-Proposition A.1. *The map* $s_{E,\sigma}^{!,an} : Z_d E(\sigma) \to H_{2d-2r}(M(\sigma), \mathbb{Z})$ *so defined descends to a homomorphism*

$$s_{E,\sigma}^{!,an} : A_d E(\sigma) \longrightarrow H_{2d-2r}(M(\sigma), \mathbb{Z}),$$

which we call the analytic localized Gysin map. Furthermore, via the cycle-to-homology homomorphism $cl : A_k M(\sigma) \to H_{2k}(M(\sigma), \mathbb{Z})$, *the two versions of localized Gysin maps coincide, i.e.* $cl \circ s_{E,\sigma}^! = s_{E,\sigma}^{!,an}$.

Proof. The proof that $s_{E,\sigma}^{!,\mathrm{an}}$ preserves rational equivalence is standard and will be omitted. We now prove the comparison by using the notation of Section 2. If $B \subset E|_{M(\sigma)}$, the result follows directly from [10, Lemma 19.2 (d)]. So we suppose $B \cap G \neq \emptyset$. We choose a σ-regularizing morphism $\rho : X \to M$ and a closed integral $\tilde{B} \subset \tilde{G}$ such that $\tilde{\rho}_*[\tilde{B}] = k \cdot [B]$ where $\tilde{G}$ is the kernel of $\tilde{\sigma} : \tilde{E} \to \mathcal{O}_X(D)$ with $\tilde{E} = \rho^* E$ and $\tilde{\rho} : \tilde{E} \to E$ is induced from ρ.

Observe that the intersections $\xi \cap B$ for defining $s_{E,\sigma}^{!,\mathrm{an}}[B]$ and $D \cdot s_{\tilde{G}}^![\tilde{B}]$ for defining $s_{E,\sigma}^![B]$ take place in the tubular neighborhood V of $M(\sigma)$ in the sense that if we replace M by V and X by the inverse image of V and so on, we get the same classes. So for the purpose of the comparison result, we may assume that $M = V$ is a tubular neighborhood of $M(\sigma)$.

Consider the fiber square of zero sections

$$
\begin{array}{ccc}
X & \xrightarrow{\;\tilde{\imath}\;} & \tilde{E} \\
\rho \downarrow & & \downarrow \tilde{\rho} \\
M & \xrightarrow{\;\imath\;} & E.
\end{array}
$$

By [10, Theorem 6.2],

$$\rho_*(s^!_{\tilde{E}}[\tilde{B}]) = s^!_E \tilde{\rho}_*[\tilde{B}] = k \cdot s^!_E[B].$$

By the excess intersection formula [10, Theorem 6.3],

$$\rho_*(D \cdot s^!_{\tilde{G}}[\tilde{B}]) = \rho_*(s^!_{\tilde{E}}[\tilde{B}]) = k \cdot s^!_E[B].$$

Applying the cycle-to-homology homomorphism cl, we obtain

$$cl(s^!_{E,\sigma}[B]) = k^{-1} \cdot cl(\rho_*(D \cdot s^!_{\tilde{G}}[\tilde{B}])) = cl(s^!_E[B]).$$

By [10, Lemma 19.2 (d)] again, this equals $u_\imath \cap cl([B])$, where $u_\imath \in H^{2r}(E, E - M)$ is the orientation class of E. By standard arguments, the cap product of the orientation class is the same as intersecting B with a transversal perturbation ξ of the zero section. This proves the Proposition. $\qquad\square$

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