BIRATIONAL GEOMETRY OF MODULI SPACES OF RATIONAL CURVES IN PROJECTIVE VARIETIES

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1. Introduction

Let $X \subset \mathbb{P}^n$ be a smooth projective variety over the complex number field $\mathbb{C}$. Let $\mathbf{R}^s_d(X)$ be the moduli space of all smooth rational curves of degree d in X. When X is convex in the sense that $H^1(\mathbb{P}^1, f^*T_X) = 0$ for any $f : \mathbb{P}^1 \to X$ of degree d, $\mathbf{R}^s_d(X)$ is smooth. In particular when X is a projective homogeneous variety, $\mathbf{R}^s_d(X)$ is a smooth quasi-projective variety. However, even for projective spaces, $\mathbf{R}^s_d(X)$ is not compact for $d \geq 2$. So from moduli theoretic point of view, the following questions are quite natural:

(1) Does $\mathbf{R}^s_d(X)$ admit a natural moduli theoretic compactification?
(2) If there are more than one such compactifications, how can we compare them?
(3) Can we calculate the Betti numbers and intersection numbers of these compactifications?

The purpose of this paper is to provide a survey of our recent results [4, 5, 17, 18] concerning these questions for a projective homogeneous variety

$$X \subset \mathbb{P}^n$$

with fixed projective embedding and $d \leq 3$.

Let us recall several important compactifications of $\mathbf{R}^s_d(X)$. Since X is a projective variety, Grothendieck's general construction gives us the Hilbert scheme $Hilb^{dm+1}(X)$ of closed subschemes of X with Hilbert polynomial $h(m) = dm + 1$ as a closed subscheme of $Hilb^{dm+1}(\mathbb{P}^n)$. The inclusion $R^s_d(X) \subset Hilb^{dm+1}(X)$ is an open immersion and thus the irreducible component(s) of $Hilb^{dm+1}(X)$ containing smooth rational curves is a compactification which we call the *Hilbert compactificaiton* and denote by $\mathbf{H}_d(X)$.

In 1994, Kontsevich and Manin proposed another way to compactify $\mathbf{R}^s_d(X)$ by using the notion of stable maps. A stable map is a morphism of a connected nodal curve $f : C \to X$ with finite automorphism group. Recall that two maps $f : C \to X$ and $f' : C' \to X$ are isomorphic if there exists an isomorphism $\eta : C \to C'$ satisfying $f' \circ \eta = f$. Let $\mathcal{M}_0(X, d)$ denote the (coarse) moduli space of isomorphism classes of stable maps $f : C \to X$ with arithmetic genus of C equal to 0 and $\deg(f^*\mathcal{O}_X(1)) = d$. The obvious inclusion $\mathbf{R}^s_d(X) \to \mathcal{M}_0(X, d)$ is an open immersion and hence the closure $\mathbf{M}_d(X)$ of $\mathbf{R}^s_d(X)$ in $\mathcal{M}_0(X, d)$ is a compactification, which we call the *Kontsevich compactification*.

Yet another natural compactification is obtained by using C. Simpson's general construction of moduli spaces of stable sheaves on a projective variety $X \subset \mathbb{P}^n$. A coherent sheaf E on X is *pure* if any nonzero subsheaf of E has the same dimensional

support as E. A pure sheaf E is called *semistable* if

$$\frac{\chi(E(m))}{r(E)} \leq \frac{\chi(E''(m))}{r(E'')} \qquad \text{for } m >> 0$$

for any nontrivial pure quotient sheaf E'' of the same dimension, where $r(E)$ denotes the leading coefficient of the Hilbert polynomial $\chi(E(m)) = \chi(E \otimes \mathcal{O}_X(m))$. We obtain *stability* if $\leq$ is replaced by $<$. If we replace the quotient sheaves E'' by subsheaves E' and reverse the inequality, we obtain an equivalent definition of (semi)stability.

Simpson proved that there is a projective moduli scheme $\mathcal{S}imp^P(X)$ of semistable sheaves of given Hilbert polynomial P. If C is a smooth rational curve in X, then the structure sheaf $\mathcal{O}_C$ is a stable sheaf on X. Hence we get an open immersion $\mathbf{R}^s_d(X) \hookrightarrow \mathcal{S}imp^{dm+1}(X)$. By taking the closure we obtain a compactifiction $\mathbf{S}_d(X)$, which we call the *Simpson compactification*.

We will often write $\mathbf{M}$ or $\mathbf{M}(X)$ (resp. $\mathbf{S}$ or $\mathbf{S}(X)$, resp. $\mathbf{H}$, or $\mathbf{H}(X)$) instead of $\mathbf{M}_d(X)$ (resp. $\mathbf{S}_d(X)$, resp. $\mathbf{H}_d(X)$) when the meaning is clear from the context.

The problem that we are interested in can be more precisely phrased as follows:

Problem 1.1. *Study the birational geometry of* $\mathbf{H}$, $\mathbf{M}$, $\mathbf{S}$ *and calculate their Betti numbers.*

In §2, we provide two motivations for this problem. In §3, we survey our comparison results for $X = \mathbb{P}^n$. The birational maps of $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{S}$ are factorized into a sequence of explicit blow-ups and -downs. In §4, we explain the comparison results for the case where X is a projective homogeneous variety and find the Betti numbers of $\mathbf{H}$ and $\mathbf{S}$ by using the blow-up formula of cohomology. In §5, we apply the same line of ideas to compare the moduli spaces of weighted pointed stable curves of genus 0 and explain how they appear as the log canonical models of the Knudsen-Mumford compactification $\overline{M}_{0,n}$.

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2. Motivations

There are two major motivations for Problem 1.1: (1)Enumerative geometry and (2)Log MMP of moduli spaces. In this section, we explain these motivations.

Classical algebraic geometers have already used various compactifications of the space of rational curves in projective space. For instance in mid 19th century, J. Steiner proposed the problem of finding the number of conics in $\mathbb{P}^2$, which are tangent to given general five conics. It turned out that the intersection theory on the Hilbert compactification $\mathbf{H} = \mathbb{P}^5$ does not give us the correct answer in this case. But if we use the Kontsevich compactification $\mathbf{M}$ which is the blow-up of $\mathbb{P}^5$ along the Veronese surface in this case and calculate the intersection numbers on $\mathbf{M}$, then we find the correct number of conics.

Since mid 1990s the focus of enumerative geometry has been laid on the (virtual) intersection theory on the moduli space of stable maps and various techniques to calculate the (virtual) intersection numbers, which are called the Gromov-Witten invariants (GW invariants for short), have been developed. The known techniques for calculating GW invariants so far include (1) WDVV equations, (2) localization

by torus action, (3) quantum Lefschetz and Grothendieck-Riemann-Roch, (4) localization by holomorphic 2-form and (5) degeneration method. However each of them has limited applications and complete calculation of GW invariants seems impossible at this stage. On the other hand, GW invariants are rational numbers and it has been believed that there should be more fundamental integer invariants that generate all the GW invariants and that are more directly related to enumerative geometry. Various approaches have been proposed for the integer invariants but there is one common feature in all of them: They all involve counting on moduli of sheaves (or sheaf complexes).

The best known moduli space of sheaves is perhaps the Hilbert schemes of ideal sheaves on projective schemes. A little less prestigious but still very important is Simpson's moduli of stable sheaves. In late 1990s, R. Thomas constructed virtual cycles on the Hilbert schemes and moduli spaces of stable sheaves on a Calabi-Yau or Fano 3-fold (i.e. the anticanonical bundle $-K_X$ is numerically effective). Thus obtained are new deformation invariant intersection numbers, which are called the Donaldson-Thomas invariants. Philosophically both GW and DT invariants count curves, and hence they should be related by some formulas which account for the differences of the boundaries. In fact precise formulas comparing DT and GW invariants were conjectured by Maulik, Nekrasov, Okounkov and Pandharipande. Here they used the Hilbert compactification. The MNOP conjectures were proved for the base case of degree 0 and for toric three-folds but no systematic program has been announced to attack this problem in general. Also, there is a conjectural formula of S. Katz which compares the GW invariants with the DT invariants given by the Simpson compactifications when the genus is 0.

In the absence of systematic programs for the comparison conjectures, it seems natural to restrict ourselves to the simple case of $g = 0$ and think about the most barbaric approach as follows:

(1) Compare the compactifications $\mathbf{M}, \mathbf{H}, \mathbf{S}$ as explicitly as possible, for instance factorize the birational maps $\mathbf{M} \dashrightarrow \mathbf{S} \dashrightarrow \mathbf{H}$ into a sequence of explicit blow-ups/-downs.
(2) Compare the (virtual) intersection numbers by using the blow-up formula of cohomology groups, the residue formula or other techniques.

This is our original motivation for Problem 1.1.

Quite recently, there has been a strong interest in the Mori theory of moduli spaces of curves. Since there are lots of compactifications of the space of smooth curves, it is certainly a good idea to give an order in the wild world of moduli spaces. To algebraic geometers of the 21st century, Mori theory is not so alien and in some sense it is quite natural to apply the Mori program to the moduli spaces to find relationships among them. The most prominent result in this direction in recent years is the following result of D. Chen.

Theorem 2.1. [2] *When $X = \mathbb{P}^3$ and $d = 3$, $\mathbf{H}$ is a log flip of $\mathbf{M}$ with respect to $K_{\mathbf{M}} + \alpha\Delta$ where Δ is the boundary divisor.*

We will see below that this flip is more precisely the composition of three blow-ups and three blow-downs. Furthermore, we will see that this result holds true for any $\mathbb{P}^r$ with $r \geq 3$ if we replace $\mathbf{H}$ by $\mathbf{S}$. Note that when $X = \mathbb{P}^3$, $\mathbf{H} = \mathbf{S}$. Another result in this line is due to D. Chen and I. Coskun as follows.

Theorem 2.2. [3] *When $X = Gr(2,4)$ and $d = 2$, $\mathbf{H}$ is obtained from $\mathbf{M}$ by a blow-up followed by a blow-down.*

We will see below that this theorem is true for any projective homogeneous variety X.

3. When the target is a projective space

In this section, we compare $\mathbf{M}, \mathbf{H}, \mathbf{S}$ when $X = \mathbb{P}^n$ and $d \leq 3$. Note that when $d = 1$, $\mathbf{R}^s_d(\mathbb{P}^n) = Gr(2, n+1)$ is already compact and hence $\mathbf{M} = \mathbf{S} = \mathbf{H} = \mathbf{R}^s_d(\mathbb{P}^n)$. So we consider the cases where $d = 2, 3$.

3.1. **From Hilbert to Simpson.** Suppose $d = 2$. If $C \in \mathbf{H} = \mathbf{H}_2(\mathbb{P}^n)$, then C is a smooth conic or the union of two distinct lines or the thickening of a line in a plane. In all these cases, it is straightforward to see that the structure sheaf $\mathcal{O}_C$ is stable and the induced morphism $\mathbf{H} \to \mathbf{S}$ is injective. Therefore when $d \leq 2$, we have $\mathbf{H} \cong \mathbf{S}$. So from now on we assume $d = 3$.

We fix $n \geq 3$. We will see that the Hilbert compactification $\mathbf{H}$ is the blow-up of the Simpson compactification $\mathbf{S}$ along a smooth subvariety. One way to prove this is as follows: The universal family $\mathcal{Z} \subset \mathbf{H} \times \mathbb{P}^n$ defines a family of sheaves $\mathcal{O}_{\mathcal{Z}}$ on $\mathbf{H} \times \mathbb{P}^n$ which is flat over $\mathbf{H}$. Therefore we have a birational map

$$\mathbf{H} \dashrightarrow \mathbf{S} \qquad C \mapsto \mathcal{O}_C.$$

The locus of unstable sheaves is a smooth divisor Δ_H and each point C in Δ_H has support C' entirely contained in a plane in $\mathbb{P}^n$. Moreover, for $C \in \Delta_H$, we have an exact sequence

$$0 \to \mathbb{C}_p \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0 \tag{3.1}$$

where p is an embedded point. These quotient sheaves $\mathcal{O}_{C'}$ form a flat family A over Δ_H. We then apply the elementary modification

$$\mathcal{F} := \ker(\mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}}|_{\Delta_H \times \mathbb{P}^n} \to A)$$

over $\mathbf{H} \times \mathbb{P}^n$ and check that $\mathcal{F}$ is now a flat family of stable sheaves. The effect of elementary modification is the interchange of the sub and quotient sheaves so that (3.1) becomes an exact sequence

$$0 \to \mathcal{O}_{C'} \to F \to \mathbb{C}_p \to 0.$$

For instance, if C' is a nodal cubic plane curve with exactly one node p, then the result of the elementary modification is $\nu_* \mathcal{O}_{\tilde{C}}$ where $\nu : \tilde{C} \to C$ is the normalization of the node. Hence we obtain a morphism $\mathbf{H} \to \mathbf{S}$. By further analyzing the fibers and applying the Fujiki-Nakano criterion for blow-ups ([10]), we can show that this is a smooth blow-up along Δ.

Theorem 3.1. [4] *For $d = 3$ and $X = \mathbb{P}^n$, there exists a morphism $\psi : \mathbf{H} \to \mathbf{S}$ which is the smooth blow-up along Δ_S which is the (smooth) locus of stable sheaves which are planar (i.e. the support is contained in a plane).*

Another way to prove Theorem 3.1 is to use the results of [26, 9] as follows. We write $\mathbf{H}(\mathbb{P}^n)$ for $\mathbf{H}$ when there is any need to emphasize the target.

Theorem 3.2. [26] *$\mathcal{H}ilb^{3m+1}(\mathbb{P}^3)$ has only two irreducible components. They are smooth and intersect transversely. One of them is $\mathbf{H}(\mathbb{P}^3)$ and the other is a 15 dimensional variety which parameterizes planar cubics in $\mathbb{P}^3$ together with a point. Their intersection is a divisor $\Delta(\mathbb{P}^3)$ of $\mathbf{H}(\mathbb{P}^3)$.*

Theorem 3.3. [9]

(1) *$\mathcal{S}imp^{3m+1}(\mathbb{P}^3)$ is the fine moduli space of stable sheaves, i.e. semistable sheaves are stable.*
(2) *$\mathcal{S}imp^{3m+1}(\mathbb{P}^3)$ has two irreducible components which intersect transversely along $\Delta(\mathbb{P}^3)$. One is $\mathbf{S}(\mathbb{P}^3)$ and the other is a 13 dimensional variety which parameterizes planar cubics in $\mathbb{P}^3$ together with a point on the curve.*
(3) *$\mathbf{S}(\mathbb{P}^3)$ is isomorphic to $\mathbf{H}(\mathbb{P}^3)$.*

Now Theorem 3.1 is a direct consequence of the following.

Proposition 3.4. [4]

(1) *$\mathbf{H}(\mathbb{P}^n)$ is isomorphic to $\mathbf{H}(\mathbb{P}\mathcal{U})$ which is a component of the relative Hilbert scheme for the bundle $\mathbb{P}\mathcal{U} \to Gr(4, n+1)$ of $\mathbb{P}^3$'s, where $\mathcal{U}$ is the universal rank 4 vector bundle on the Grassmannian $Gr(4, n+1)$.*
(2) *$\mathbf{S}(\mathbb{P}\mathcal{U})$ is the blow-up of $\mathbf{S}(\mathbb{P}^n)$ along the smooth locus of planar stable sheaves where $\mathbf{S}(\mathbb{P}\mathcal{U})$ is the relative moduli space of stable sheaves of $\mathbb{P}\mathcal{U} \to Gr(4, n+1)$.*

Proof. (1) Looking at the complete list of elements in $\mathbf{H}(\mathbb{P}^3)$ in [13], we see immediately that the obvious map

$$\phi : \mathbf{H}(\mathbb{P}\mathcal{U}) \to \mathbf{H}(\mathbb{P}^n)$$

is injective because the image $\phi(y)$ of every point $y \in \mathbf{H}(\mathbb{P}\mathcal{U})$ determines a unique $\mathbb{P}^3$. This certainly implies that ϕ is an isomorphism because $\mathbf{H}(\mathbb{P}\mathcal{U})$ is smooth.

(2) A stable sheaf F on a $\mathbb{P}^3$ in $\mathbb{P}^n$ gives us a stable sheaf $\imath_* F$ where $\imath$ is the inclusion $\mathbb{P}^3 \hookrightarrow \mathbb{P}^n$. Thus we have a morphism

$$\mathbf{S}(\mathbb{P}\mathcal{U}) \to \mathbf{S}(\mathbb{P}^n) \qquad F \mapsto \imath_* F.$$

The exceptional locus is the divisor $\tilde{\Delta}$ of planar sheaves in $\mathbf{S}(\mathbb{P}\mathcal{U})$ which is a $\mathbf{S}(\mathbb{P}^2)$-bundle on $\mathbb{P}\mathcal{U}^*$ over $Gr(4, n+1)$. By the isomorphism $\mathbb{P}\mathcal{U}^* \cong \mathbb{P}(\mathbb{C}^{n+1}/\mathcal{U}')$ where $\mathcal{U}'$ is the tautological bundle on $Gr(3, n+1)$,

$$\tilde{\Delta} \cong \mathbf{S}(\mathbb{P}\mathcal{U}') \times_{Gr(3,n+1)} \mathbb{P}(\mathbb{C}^{n+1}/\mathcal{U}').$$

It is obvious that $\mathbf{S}(\mathbb{P}\mathcal{U}) \to \mathbf{S}(\mathbb{P}^n)$ is constant on the fibers $\mathbb{P}^{n-3}$ of $\mathbb{P}(\mathbb{C}^{n+1}/\mathcal{U}') \to Gr(3, n+1)$. The normal bundle of $\tilde{\Delta}$ restricted to a fiber of $\mathbb{P}(\mathbb{C}^{n+1}/\mathcal{U}') \to Gr(3, n+1)$ is $\mathcal{O}_{\mathbb{P}^{n-3}}(-1)$ by a direct check. By the Fujiki-Nakano criterion ([10]), we conclude that $\mathbf{S}(\mathbb{P}\mathcal{U}) \to \mathbf{S}(\mathbb{P}^n)$ is a smooth blow-up along Δ_S. □

3.2. **From Kontsevich to Simpson.** In this subsection, we compare the Kontsevich compactification $\mathbf{M}$ and the Simpson compactification $\mathbf{S}$ for $X = \mathbb{P}^n$ and $d \leq 3$. For any family of stable maps

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathbf{f}} & \mathbb{P}^n \\ {\scriptstyle \pi}\downarrow & & \\ Z & & \end{array}$$

parameterized by a reduced scheme Z, we can associate a coherent sheaf $(\mathrm{id}_Z \times \mathbf{f})_*\mathcal{O}_{\mathcal{C}}$ on $Z \times \mathbb{P}^n$, flat over Z. Hence we have a sheaf $\mathcal{E}_0$ on $\mathbf{M} \times \mathbb{P}^n$ which is flat over $\mathbf{M}$. So we have a birational map

$$\mathbf{M} \dashrightarrow \mathbf{S} \qquad (f : C \to \mathbb{P}^n) \mapsto f_*\mathcal{O}_C.$$

If $f : \mathbb{P}^1 \to L \subset \mathbb{P}^n$ is a d-fold covering onto a line L, then the direct image sheaf $f_*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^{d-1}$ is unstable. Our strategy for decomposing the birational map $\mathbf{M} \dashrightarrow \mathbf{S}$ into blow-ups and -downs is as follows:

(1) Find the locus of unstable sheaves in $\mathbf{M}$.
(2) Blow up a component of the indeterminacy locus and apply elementary modification.
(3) Repeat (2) until all sheaves become stable so that we have a diagram

$$\mathbf{M} \leftarrow \widetilde{\mathbf{M}} \to \mathbf{S}.$$

(4) Factorize the morphism $\widetilde{\mathbf{M}} \to \mathbf{S}$ into a sequence of blow-ups.

For the last item, a very useful tool is the variation of GIT quotients [6, 29].

3.2.1. *Degree 2 case.* We let $d = 2$. The locus Γ of $f : C \to \mathbb{P}^n$ in $\mathbf{M}$ where $f_*\mathcal{O}_C$ is unstable is exactly the locus of 2:1 maps $f : C \to L$ to a line L in $\mathbb{P}^n$ where $C = \mathbb{P}^1$ or $\mathbb{P}^1 \cup \mathbb{P}^1$ glued at a point. The choice of L is parameterized by the Grassmannian $Gr(2, n+1)$ and the space of 2:1 maps to L is $\mathcal{M}_0(\mathbb{P}^1, 2) \cong \mathbb{P}^2$ which parameterizes a (unordered) pair of branch points. Therefore Γ is a $\mathbb{P}^2$-bundle over $Gr(2, n+1)$. More precisely,

$$\Gamma = \mathcal{M}_0(\mathbb{P}\mathcal{U}, 2) \to Gr(2, n+1)$$

is the relative moduli space of stable maps of the family $\mathbb{P}\mathcal{U} \to Gr(2, n+1)$ where $\mathcal{U}$ is the tautological rank 2 bundle over $Gr(2, n+1)$. For $f \in \Gamma$, the inclusion $\mathcal{O}_L \hookrightarrow f_*\mathcal{O}_C = \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ comes from adjunction and hence the quotient maps $f_*\mathcal{O}_C \to \mathcal{O}_L(-1)$ form a flat family. Another way is to see it is as follows: The destabilizing subsheaf $\mathcal{O}_L$ is the first term in the Harder-Narasimhan filtration and so they form a flat family ([15]). Therefore the factors $\mathcal{O}_L(-1)$ also form a flat family over Γ.

We blow up $\mathbf{M}$ along Γ and then apply the following elementary modification of the family $\mathcal{E}_0$ along the exceptional divisor $\tilde{\Gamma}$ which is a $\mathbb{P}^{n-2}$-bundle over Γ:

$$\mathcal{E} := \ker(\mathcal{E}_0 \to \mathcal{E}_0|_{\tilde{\Gamma} \times \mathbb{P}^n} \to \tilde{A})$$

where $\tilde{A}$ is the pull-back of A to $\tilde{\Gamma} \times \mathbb{P}^n$. As mentioned above, the effect of elementary modification is the interchange of sub and quotient sheaves. In fact we can analyze the Kodaira-Spencer map of the family $\mathcal{E}$ to find that all sheaves in $\mathcal{E}$ over $\tilde{\Gamma}$ are nontrivial extensions

$$0 \to \mathcal{O}_L(-1) \to E \to \mathcal{O}_L \to 0.$$

In particular, all sheaves in $\mathcal{E}$ are now stable and thus we obtain a diagram

$$\mathbf{M} \leftarrow \widetilde{\mathbf{M}} \to \mathbf{S}$$

whose left arrow is the blow-up along Γ. It is rather obvious that the morphism $\widetilde{\mathbf{M}} \to \mathbf{S}$ is constant along fibers $\mathbb{P}^2$ of $\Gamma \to Gr(2, n+1)$. In fact one can directly verify that this contraction is actually a blow-up. So we obtain the following.

Theorem 3.5. [16] *The birational map* $\mathbf{M} \dashrightarrow \mathbf{S}$ *is the blow-up of* $\mathbf{M}$ *along* Γ *followed by the contraction of* $\mathbb{P}^2$ *in* Γ.

Note that $\mathbf{S} = \mathbf{H}$ is isomorphic to $\mathbb{P}(\mathrm{Sym}^2\mathcal{U}^*)$, a $\mathbb{P}^5$-bundle over $Gr(3, n+1)$, where $\mathcal{U}$ is the tautological rank 3 bundle. Actually, the morphism $\widetilde{\mathbf{M}} \to \mathbf{S}$ is the blow-up along the relative Veronese surface over $Gr(3, n+1)$ and $\widetilde{\mathbf{M}}$ is isomorphic to the variety of complete conics $CC(\mathcal{U})$ over $Gr(3, n+1)$.

3.2.2. *Degree 3 case.* Let $d = 3$. As in the degree 2 case, we have a family of coherent sheaves $\mathcal{E}_0$ on $\mathbf{M} \times \mathbb{P}^n$, flat over $\mathbf{M}$, and a birational map $\mathbf{M} \dashrightarrow \mathbf{S}$ defined by $(f : C \to \mathbb{P}^n) \mapsto f_*\mathcal{O}_C$.

The locus of unstable sheaves has now two irreducible components

$$\Gamma^1 = \{f \in \mathbf{M} \,|\, \mathrm{im}(f) \text{ is a line}\}, \quad \Gamma^2 = \{f \in \mathbf{M} \,|\, \mathrm{im}(f) \text{ is a union of two lines}\}.$$

For a point $f \in \Gamma^1$, $f_*\mathcal{O}_C = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^2$ and the normal space of Γ^1 to $\mathbf{M}$ at f is canonically

$$Hom(\mathbb{C}^2, Ext^1_{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L(-1))).$$

Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}$ be the blow-up along Γ^1. The destabilizing quotients $f_*\mathcal{O}_C = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^2 \to \mathcal{O}_L(-1)^2$ form a flat family A over the exceptional divisor Γ^1_1 of π_1 and so we can apply the elementary modification

$$\mathcal{E}_1 = \ker((\pi_1 \times \mathrm{id}_{\mathbb{P}^n})^*\mathcal{E}_0 \to (\pi_1 \times \mathrm{id}_{\mathbb{P}^n})^*\mathcal{E}_0|_{\Gamma^1_1 \times \mathbb{P}^n} \to A).$$

By direct calculation, we find that the locus of unstable sheaves in $\mathcal{E}_1$ still has two irreducible components. One is the proper transform Γ^2_1 of Γ^2 and the other is a subvariety Γ^3_1 of the exceptional divisor Γ^1_1 which is the $\mathbb{P}Hom_1(\mathbb{C}^2, Ext^1_{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L(-1)))$-bundle over Γ^1 where

$$\mathbb{P}Hom_1(\mathbb{C}^2, Ext^1_{\mathbb{P}^n}(\mathcal{O}_L, \mathcal{O}_L(-1))) \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$$

is the locus of rank 1 homomorphisms.

Let $\pi_2 : \mathbf{M}_2 \to \mathbf{M}_1$ be the blow-up along Γ^2_1. Apply elementary modification with respect to the first term in the Harder-Narasimhan filtration along the exceptional divisor Γ^2_2 to obtain a family $\mathcal{E}_2$ of sheaves on $\mathbf{M}_2 \times \mathbb{P}^n$. Let Γ^j_2 be the proper transform of Γ^j_1 for $j = 1, 3$. It turns out that the locus of unstable sheaves in $\mathcal{E}_2$ is precisely Γ^3_2. We repeat the same. Let $\pi_3 : \mathbf{M}_3 \to \mathbf{M}_2$ be the blow-up along Γ^3_2 and apply elementary modification along the exceptional divisor Γ^3_3. After this, all sheaves become stable and so we obtain a morphism $\mathbf{M}_3 \to \mathbf{S}$.

To analyze the morphism $\mathbf{M}_3 \to \mathbf{S}$, we keep track of analytic neighborhoods of Γ^1 and Γ^2 through the sequence of blow-ups (and -downs). It turns out that the local geometry is completely determined by variation of GIT quotients. For instance, a neighborhood of Γ^1_1 is the geometric invariant theory quotient of $\mathcal{O}_{\mathbb{P}^7 \times \mathbb{P}^{2n-3}}(-1, -1)$ by $SL(2)$ with respect to the linearization $\mathcal{O}(1, \alpha)$ for $0 < \alpha << 1$. As we vary α from 0^+ to ∞, the GIT quotient goes through two flips, or two blow-ups followed by two blow-downs. The two blow-ups correspond to our two blow-ups $\mathbf{M}_3 \to \mathbf{M}_2 \to \mathbf{M}_1$ and we can blow down twice $\mathbf{M}_3 \to \mathbf{M}_4 \to \mathbf{M}_5$ in the neighborhoods of Γ^1_j. For $\alpha >> 1$, the GIT quotient of $\mathbb{P}^7 \times \mathbb{P}^{2n-3}$ by $SL(2)$ is a $\mathbb{P}^7$-bundle which can be contracted in the open neighborhood. A similar analysis for a neighborhood of Γ^2 tells us that we can blow down $\mathbf{M}_3$ three times

$$\mathbf{M}_3 \to \mathbf{M}_4 \to \mathbf{M}_5 \to \mathbf{M}_6$$

and the morphism $\mathbf{M}_3 \to \mathbf{S}$ is constant on the fibers of the blow-downs. Hence we obtain an induced morphism $\mathbf{M}_6 \to \mathbf{S}$ which turns out to be injective. So we conclude that $\mathbf{M}_6 \cong \mathbf{S}$.

We can summarize the above discussion as follows.

Theorem 3.6. [4] *For $X = \mathbb{P}^n$ and $d = 3$, $\mathbf{S}$ is obtained from $\mathbf{M}$ by blowing up along Γ^1, Γ^2_1, Γ^3_2 and then blowing down along Γ^2_3, Γ^3_4, Γ^1_5 where Γ^j_i is the proper transform of Γ^j_{i-1} if Γ^j_{i-1} is not the blow-up/-down center and the image/preimage of Γ^j_{i-1} otherwise.*

The following diagram summarizes the comparison results for $X = \mathbb{P}^n$ and $d = 3$:

$$\begin{tikzcd}
 & & \mathbf{M}_3 \arrow[dl, "\Gamma^3_2"'] \arrow[dr, "\Gamma^2_4"] & & & \\
 & \mathbf{M}_2 \arrow[dl, "\Gamma^2_1"'] & & \mathbf{M}_4 \arrow[dr, "\Gamma^3_5"] & & \\
\mathbf{M}_1 \arrow[dl, "\Gamma^1"'] & & & & \mathbf{M}_5 \arrow[dr, "\Gamma^1_6"] & \mathbf{H} \arrow[d] \\
\mathbf{M} & & & & \mathbf{M}_6 \arrow[r, "\cong"] & \mathbf{S}
\end{tikzcd}$$

All the arrows are blow-ups and the blow-up centers are indicated above the arrows.

See §4 for the Betti number calculation of $\mathbf{M}$, $\mathbf{S}$ and $\mathbf{H}$.

4. When the target is a homogeneous variety

The results of the previous section can be generalized to the case where $X \subset \mathbb{P}^n$ is a homogeneous variety. Actually all we need is the following condition.

Assumption 4.1. *(1) $H^1(\mathbb{P}^1, f^*T_X) = 0$ for any $f : \mathbb{P}^1 \to X$ of degree $\leq d$.*
(2) $\mathrm{ev} : \mathcal{M}_{0,1}(X, 1) \to X$ is smooth where

$$\mathcal{M}_{0,1}(X, 1) = \{(f : \mathbb{P}^1 \to X, p \in \mathbb{P}^1) \,|\, \deg f^*\mathcal{O}_X(1) = 1\}$$

is the moduli space of 1-pointed stable maps of degree 1 to X and ev is the evaluation map at the marked point.

If X is projective homogeneous, these two conditions are obviously satisfied. The second condition above guarantees that the irreducible component Γ^2 of the undefined locus of the birational map $\mathbf{M} \dashrightarrow \mathbf{S}$ has a fiber bundle description which is convenient for analyzing blow-ups/-downs.

By repeating the same constructions as in the previous section, we obtain the following.

Theorem 4.2. [5] *Theorem 3.5 for $d = 2$ and Theorem 3.6 for $d = 3$ hold if X satisfies Assumption 4.1.*

As an application, we can calculate the Betti numbers of $\mathbf{H}$ and $\mathbf{S}$ for all Grassmannians $X = Gr(k, n)$ with $k < n$ and $d \leq 3$. We use the Plücker embedding of $Gr(k, n)$ for our choice of $\mathcal{O}_X(1)$. For a variety Z, let

$$P(Z) = \sum_{i \geq 0} \dim_{\mathbb{Q}} H^{2i}(Z, \mathbb{Q}) q^i$$

be the Poincaré polynomial of Z. The degree 1 case is quite elementary.

Lemma 4.3. *When $d = 1$ and $X = Gr(k, n)$ with Plücker embedding,*

$$\mathbf{M} = \mathbf{H} = \mathbf{S} = \mathbf{R}_1^s(X) = Gr(k-1, \mathcal{U})$$

where $\mathcal{U}$ is the tautological vector bundle of rank $k+1$ over $Gr(k+1, n)$ and $Gr(k-1, \mathcal{U})$ is the relative Grassmannian. In particular, the Poincaré polynomial of $\mathcal{M}_0(X, 1) = \mathbf{M}$ is

$$P(Gr(k+1, n)) \times P(Gr(k-1, k+1)) = \prod_{i=1}^{k+1} \frac{1-q^{n-i+1}}{1-q^i} \cdot \prod_{i=1}^{k-1} \frac{1-q^{k-i+2}}{1-q^i}.$$

If we fix a $k-1$ dimensional subspace W in a $k+1$ dimensional subspace V in $\mathbb{C}^n$, the family of k-dimensional subspaces in V containing W gives us a line in $Gr(k, n)$ and it is easy to see that all lines in $Gr(k, n)$ are obtained in this fashion. Here we also used the well-known formula

$$P(Gr(k, n)) = \prod_{i=1}^{k} \frac{1-q^{n-i+1}}{1-q^i}.$$

For the Kontsevich compactification $\mathbf{M}$, the Betti numbers were calculated by A. Martín.

Theorem 4.4. [23] *Let $X = Gr(k, n)$ with Plücker embedding.*
(1) When $d = 2$, the Poincaré polynomial of $\mathbf{M}$ is

$$P(\mathbf{M}_2(X)) = \frac{((1+q^n)(1+q^3) - q(1+q)(q^k + q^{n-k})) \prod_{i=k}^{n}(1-q^i)}{(1-q)^2(1-q^2)^2 \prod_{i=1}^{n-k-1}(1-q^i)}.$$

(2) When $d = 3$, the Poincaré polynomial of $\mathbf{M}$ is

$$\frac{F_1(q)(1+q^{2n}) + (1+q)^2(F_2(q)q^n(1+q^2) - F_3(q)q(1+q^n)(q^k+q^{n-k})) + F_4(q)q^2(q^{2k}+q^{2n-2k})}{(1-q)(1-q^2)^2(1-q^3)^2}$$
$$\cdot P(Gr(k+1, n)) \cdot P(Gr(k-1, k+1))$$

where

$$F_1(q) = 1 + 2q^2 + 3q^3 + 3q^4 - q^5 + q^6 - 3q^7 - 3q^8 - 2q^9 - q^{11},$$
$$F_2(q) = 1 + 5q^2 + 2q^3 - 2q^4 - 5q^5 - q^7,$$
$$F_3(q) = 2 + 3q^2 + q^3 - q^4 - 3q^5 - 2q^7,$$
$$F_4(q) = 1 + 6q + 3q^2 + 2q^3 - 2q^4 - 3q^5 - 6q^6 - q^7.$$

The Poincaré polynomial of $\mathbf{H} = \mathbf{S}$ when $d = 2$ and $X = Gr(k, n)$ is calculated as follows.

Theorem 4.5. [5] *For $X = Gr(k, n)$ and $d = 2$,*

$$P(\mathbf{S}) = \frac{[(1+q^n)(1+q^3) - q(1+q)(q^k+q^{n-k}) + (1-q^2)(q^3-q^{n-2})] \prod_{i=n-k}^{n}(1-q^i)}{(1-q)^2(1-q^2)^2 \prod_{i=1}^{k-1}(1-q^i)}$$

where $\prod_{i=1}^{0}(1-q^i)$ is defined to be 1.

Proof. We know that $\mathbf{S}$ is obtained from $\mathbf{M}$ by a blow-up and a blow-down. By the blow-up formula in [12], the blow-up adds

$$P(Gr(k-1, k+1))P(Gr(k+1, n))P(\mathbb{P}^2)(P(\mathbb{P}^{n-3}) - 1)$$

to $P(\mathbf{M})$ and the blow-down subtracts

$$P(Gr(k-1, k+1))P(Gr(k+1, n))P(\mathbb{P}^{n-3})(P(\mathbb{P}^2) - 1).$$

By direct calculation we obtain the theorem. □

It is straightforward to check that the polynomial in Theorem 4.5 is palindromic with degree $(k+2)n-k^2-3$.

Next we turn to the degree 3 case. By applying the blow-up formula through the blow-ups and -downs from $\mathbf{M}$ to $\mathbf{S}$ (Theorem 3.6), we obtain the following.

Theorem 4.6. [5] *For $X = Gr(k,n)$ and $d=3$, $P(\mathbf{S})$ is*

$$P(Gr(k+1,n))\cdot P(Gr(k-1,k+1))$$

multiplied by

$$\frac{F_1(q)(1+q^{2n})+(1+q)^2(F_2(q)q^n(1+q^2)-F_3(q)q(1+q^n)(q^k+q^{n-k}))+F_4(q)q^2(q^{2k}+q^{2n-2k})}{(1-q)(1-q^2)^2(1-q^3)^2}$$

$$+(1+q+2q^2+q^3+q^4)(\frac{1-q^{2n-4}}{1-q}-1)$$

$$+\frac{1-q^2}{1-q}(\frac{(1-q^{n-k})(1-q^k)}{(1-q)^2}+\frac{1-q^{n-2}}{1-q}-1)(1+q+q^2)(\frac{1-q^{n-1}}{1-q}-1)$$

$$+\frac{1-q^{n-2}}{1-q}((1+q)(1+q+2q^2+q^3+q^4)+q(1+q)(1+q+q^2))(\frac{1-q^{n-2}}{1-q}-1)$$

$$-\frac{1-q^2}{1-q}[\frac{1-q^{n-1}}{1-q}(\frac{(1-q^{n-k})(1-q^k)}{(1-q)^2}+\frac{1-q^{n-2}}{1-q}-1)+\frac{1-q^2}{1-q}\frac{1-q^{n-2}}{1-q}(\frac{1-q^{n-2}}{1-q}-1)](\frac{1-q^3}{1-q}-1)$$

$$-\frac{1-q^2}{1-q}\frac{1-q^{n-2}}{1-q}\frac{1-q^{n-2}}{1-q}(\frac{1-q^5}{1-q}-1)$$

$$-\frac{1-q^{n-2}}{1-q}\frac{1-q^{n-3}}{1-q^2}(\frac{1-q^8}{1-q}-1).$$

Each line above shows the terms added or subtracted at each stage of blow-ups/-downs.

When $d=3$, $k=1$ and $n=4$, the Poincaré polynomial was calculated by G. Ellingsrud, R. Piene and S. Stromme [7]. It is direct to check that Theorem 4.6 specializes to their formula.

Remark 4.7. For $d=3$ and $X=Gr(k,n)$, we can also calculate the Poincaré polynomial of the Hilbert compactification $\mathbf{H}$ by applying the blow-up formula because $\mathbf{H}$ is the blow-up of $\mathbf{S}$ along the locus of planar sheaves. Because linear subspaces of projective homogeneous varieties are classified ([21]), we can calculate the Poincaré polynomial of the blow-up center and complete the calculation of the Poincaré polynomial of $\mathbf{H}$. The details are in [5].

5. Birational Geometry of $\overline{M}_{0,n}$

Let $M_{0,n}=\{(p_1,\cdots,p_n)\in(\mathbb{P}^1)^n \mid \text{all distinct}\}/\mathrm{Aut}(\mathbb{P}^1)$ be the moduli space of n distinct points of $\mathbb{P}^1$ modulo isomorphisms. This is not compact for $n\geq 4$ and there are many compactifications. The most famous compactification $\overline{M}_{0,n}$ is due to Knudsen and Mumford and is obtained by adding nodal curves $(C,p_1,\cdots,p_n)$ with p_i smooth distinct points of C with finite automorphism group $\mathrm{Aut}(C,p_1,\cdots,p_n)=\{f\in\mathrm{Aut}(C)\mid f(p_i)=p_i\}$. Finiteness of the automorphism group is equivalent to saying that each irreducible component has at least 3 nodal or marked points.

In 2003, Hassett generalized this construction by introducing the notion of weighted pointed stable curves. We let $w=(w_1,\cdots,w_n)\in\mathbb{Q}^n$ with $0<w_i\leq 1$ and assign weight w_i to each marked point p_i. Then an n-pointed nodal curve

$(C, p_1, \cdots, p_n)$ of arithmetic genus 0 is called w-stable if for each irreducible component C_1 of C, the sum of weights of the marked points in C_1 and the number of nodes is greater than 2. Hassett in [14] proves that there is a projective fine moduli space of w-stable curves $\overline{M}_{0,w}$ for each choice of weights w. In analogy with Problem 1.1 it seems reasonable to consider the following.

Problem 5.1. *Study the birational geometry of* $\overline{M}_{0,w}$.

For two weights $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n)$, we say $v \geq w$ if $v_i \geq w_i$ for all i. Hassett proves that if $v \geq w$, there is a morphism $\overline{M}_{0,v} \to \overline{M}_{0,w}$ but the structure of the morphisms is not understood completely. When $w = (\epsilon, \cdots, \epsilon)$ for $0 < \epsilon \leq 1$, we write $w = n \cdot \epsilon$.

Let $\Delta = \overline{M}_{0,n} - M_{0,n}$ be the boundary divisor of the Knudsen-Mumford space. From Mori theoretic point of view, it seems natural to consider the log canonical models

$$\overline{M}_{0,n}(\alpha) = \mathrm{Proj}\left(\oplus_{m\geq 0} H^0(\overline{M}_{0,n}, m(K_{\overline{M}_{0,n}} + \alpha\Delta))\right)$$

of $\overline{M}_{0,n}$. In 2008, M. Simpson proved the following beautiful theorem.

Theorem 5.2. [28]

(1) *If* $\frac{2}{m-k+2} < \alpha \leq \frac{2}{m-k+1}$ *for* $1 \leq k \leq m-2$, *then* $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,n\cdot\epsilon_k}$ *where* $m = \lfloor \frac{n}{2} \rfloor$ *and* $\frac{1}{m+1-k} < \epsilon_k \leq \frac{1}{m-k}$.

(2) *If* $\frac{2}{n-1} < \alpha \leq \frac{2}{m+1}$, *then* $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n//SL(2)$ *where the quotient is taken with respect to the symmetric linearization* $\mathcal{O}(1, \cdots, 1)$.

Actually M. Simpson proved this theorem under the assumption that Fulton's conjecture about extremal rays of $\overline{M}_{0,n}$ holds. Subsequently in 2009, two unconditional proofs of Theorem 5.2 were given by Fedorchuk-Smyth [8] and Alexeev-Swinarski [1] although the latter seems incomplete because they use an ampleness criterion which has not been proven in any article or preprint yet.

By using the line of ideas of §3 and §4 above, we proved the following.

Theorem 5.3. [18] *There is a sequence of blow-ups*

$$\overline{M}_{0,n} = \overline{M}_{0,n\cdot\epsilon_{m-2}} \to \overline{M}_{0,n\cdot\epsilon_{m-3}} \to \cdots \to \overline{M}_{0,n\cdot\epsilon_2} \to \overline{M}_{0,n\cdot\epsilon_1} \to (\mathbb{P}^1)^n//SL(2)$$

where $m = \lfloor \frac{n}{2} \rfloor$ *and* $\frac{1}{m+1-k} < \epsilon_k \leq \frac{1}{m-k}$. *Except for the last arrow when* n *is even, the center for each blow-up is a union of transversal smooth subvarieties of the same dimension. When* n *is even, the last arrow is the blow-up along the singular locus which consists of* $\frac{1}{2}\binom{n}{m}$ *points in* $(\mathbb{P}^1)^n//SL(2)$, *i.e.* $\overline{M}_{0,n\cdot\epsilon_1}$ *is Kirwan's partial desingularization (see* [19]*) of the GIT quotient* $(\mathbb{P}^1)^{2m}//SL(2)$.

If the center of a blow-up is the transversal union of smooth subvarieties in a nonsingular variety, the result of the blow-up is isomorphic to that of the sequence of smooth blow-ups along the irreducible components of the center in any order (see [22]). So each of the above arrows can be decomposed into the composition of smooth blow-ups along the irreducible components.

For the moduli spaces of *unordered* weighted pointed stable curves

$$\widetilde{M}_{0,n\cdot\epsilon_k} = \overline{M}_{0,n\cdot\epsilon_k}/S_n$$

we can simply take the S_n quotient of the sequence in Theorem 5.3 and thus $\widetilde{M}_{0,n\cdot\epsilon_k}$ can be constructed by a sequence of *weighted blow-ups* from $\mathbb{P}^n//SL(2) =$

$\left((\mathbb{P}^1)^n//SL(2)\right)/S_n$. In particular, $\widetilde{M}_{0,n\cdot\epsilon_1}$ is a weighted blow-up of $\mathbb{P}^n//SL(2)$ at its singular point when n is even.

The proof of Theorem 5.3 is obtained by taking the $SL(2)$ quotient of Mustata-Mustata's sequence of blow-ups in [25, §1] from the Fulton-MacPherson space $\mathbb{P}^1[n]$ of configurations of n points in $\mathbb{P}^1$ to the product $(\mathbb{P}^1)^n$.

Theorem 5.2 now follows straightforwardly from Theorem 5.3 together with some calculation of nef divisors in [1]. See [18] for further details.

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