REMARKS ON HYPERKÄHLER GEOMETRY

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Foreword

I intend to talk about some results and open problems on hyperkähler geometry. I will explain basic notions on hyperkähler geometry and discuss the known examples of hyperkähler manifolds which include Hilbert schemes of points on K3 or abelian surfaces, moduli of instantons, moduli of monopoles and quiver varieties. Also I will discuss hyperkähler quotient which is a method to construct new hyperkähler manifolds from the known examples. I don't (to be honest, can't) provide a systematic or complete survey of this fascinating subject. Rather I will just discuss several topics that make sense to me (or topics that please me). For instance, important recent progresses concerned with periods, mirror symmetry and 3-manifold invariants will not be considered in this article.

1. What is geometry?

(1.1) It is well-known that the word "geometry" is a combination of "geo" and "metry". Of course, "geo" means *earth* and "metry" means *measure*. To mathematicians, earth is a *manifold*[1] and measure is a *metric*.[2] Hence the word "geometry" literally stands for the study of

manifolds equipped with a metric.

(1.2) Suppose we have a Riemannian manifold (M, g), i.e. a manifold M equipped with a metric g. In order to say anything geometric (like "straight line") we need the notion of differentiation. Let us consider an example: $M = \mathbb{R}^n$, $g =$ standard Euclidean metric. Let F be a vector field on M. Its derivative DF is defined by the equation

$$\lim_{h\to 0} \frac{F(x+h) - F(x) - DF(x)h}{|h|} = 0.$$

[1]A manifold is a paracompact Hausdorff locally Euclidean topological space whose transition maps are differentiable. In this talk, we consider only *orientable* manifolds which means that the determinant of the Jacobian of a transition map is positive everywhere.

[2]A metric is (a differentiable choice of) an inner product on each tangent space.

What is implicit here is that the tangent vector $F(x+h)$ at $x+h$ is compared with the tangent vector $F(x)$ at x. More precisely, the tangent vector at $x+h$ is translated to x. This is what we need to make sense of derivatives, i.e. for each path γ on M from p to q, we need an isomorphism

$$\varphi_\gamma : T_pM \to T_qM$$

which we call "parallel transportation". By the well-known existence and uniqueness theorem of an ODE solution, it suffices to give an infinitesimal variation of the isomorphism φ_γ for each direction at any point. This is the notion of *connection* (on the tangent bundle). Given a metric g, there is a unique connection, called the *Levi-Civita connection*, which is compatible with the metric. In particular, the parallel transport φ_γ is always an *isometry* (or an orthogonal transformation if γ is a loop). In the rest of this section a connection means the Levi-Civita connection of a metric.

(1.3) Let us try a simple experiment about parallel transportation. Let $M = S^2$ be the "earth" equipped with the metric induced from the inclusion $S^2 \subset \mathbb{R}^3$. Let's imagine you are on the equator with your right arm pointing East and left arm pointing North. Walk to the East along the equator for 10000 kilometers. Your right arm is still pointing East while your left arm is pointing North. Now walk to the North until you reach the North pole and then walk straight to the point you started. Then you find your right arm is pointing North and your left arm is pointing West! But if you just walk along the equator for 40000 kilometers you get back to the same point and your arms are pointing the same directions as you started. The lesson here is that the parallel transportation *depends* on the choice of the curve γ.[3] For any loop γ based at $x \in M$, we have an isometry $\varphi_\gamma \in O(T_xM)$, which is called the *holonomy* along the loop γ.

(1.4) Definition. (i) The *holonomy group* of a Riemannian manifold (M, g) is the group

$$H = \{\varphi_\gamma \in O(T_xM) \,|\, \gamma \text{ is a loop based at } x\} \subset O(T_xM)$$

of orthogonal transformations. By fixing an orthonormal basis of T_xM, H is identified with a subgroup of $O(n)$ where $n = \dim_{\mathbb{R}} M$.[4]

(ii) A Riemannian manifold (M, g) is called *irreducible* if the representation of the holonomy group H on T_xM is an irreducible representation.

[3] In the Euclidean case, the parallel transportation has nothing to do with the path but this is precisely because the Euclidean space is flat (curvature is zero) and simply connected. Simply connected roughly means that any loop can be continuously deformed to a point.

[4] $H \subset O(n)$ is unique up to conjugation.

(1.5) As any representation of a compact Lie group is a direct sum of irreducible representations, it is reasonable to expect the following theorem of de Rham.

(1.6) Theorem. A compact simply connected Riemannian manifold is a product of irreducible Riemannian manifolds.

(1.7) By Theorem 1.6, we can effectively restrict our concern to irreducible manifolds if we are only interested in simply connected manifolds. Surprisingly the list of irreducible compact simply connected manifolds is quite limited by the following theorem of Berger.

(1.8) Theorem. Let M be an irreducible compact simply connected manifold which is not a symmetric space.[5] Then the holonomy group H of M is one of the following:

(i)	$SO(n)$	where $\dim M = n$	: general Riemannian manifold,
(ii)	$U(m)$	where $\dim M = 2m$	: Kähler manifold,
(iii)	$SU(m)$	where $\dim M = 2m$	: Calabi-Yau manifold,
(iv)	$Sp(r)$	where $\dim M = 4r$	: hyperkähler manifold,
(v)	$Sp(r)Sp(1)$	where $\dim M = 4r$	: quaternionic Kähler manifold,
(vi)	G_2	where $\dim M = 7$,	
(vii)	$Spin(7)$	where $\dim M = 8$.	

(1.9) The above theorem gives us a classification of geometries. For example, the study of general Riemannian manifolds is the Riemannian geometry and the study of Kähler manifolds is the Kähler geometry. Complex algebraic geometry is mostly about (ii), (iii), and (iv). The study of hyperkähler manifolds is the hyperkähler geometry which we shall discuss in the rest of this paper.

2. What is hyperkähler?

(2.1) We've seen above that the holonomy group of an irreducible compact simply connected hyperkähler manifold is $Sp(r)$. But what is hyperkähler anyway? To answer this question we start with the definition of Kähler manifold.

[5]A symmetric space is a manifold of the form G/H where G is a compact Lie group endowed with an involution and H is the subgroup of invariant elements. The symmetric spaces have been all classified and studied with care.

(2.2) Definition. A Riemannian manifold (M, g) is called *Kähler* if it is equipped with a compatible complex structure J i.e. an orthogonal transformation

$$J : TM \to TM$$

such that $J^2 = -1$ and J is parallel.[6] The *Kähler form* ω of (M, g, J) is defined by

$$\omega(X, Y) = g(JX, Y)$$

for any tangent vectors $X, Y \in T_xM$.[7]

(2.3) The simplest example of a Kähler manifold is $\mathbb{C}^n$ with the standard hermitian inner product $\langle x, y\rangle = \sum x_i \overline{y}_i$ for $x = (x_i)$, $y = (y_i)$. The real part of this hermitian inner product is the standard Riemannian metric and its imaginary part is the Kähler form.

If Λ is a full lattice in $\mathbb{C}^n$ then $\mathbb{C}^n/\Lambda$ is a compact Kähler manifold with the induced metric and complex structure from $\mathbb{C}^n$.

The projective space $\mathbb{P}^n = \mathbb{C}^{n+1} - 0/\mathbb{C} - 0$ equipped with the *Fubini-Study metric* is a Kähler manifold. In terms of local coordinates $(1 : z_1 : \cdots : z_n)$, its Kähler form is given by the formula

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(1 + \sum z_i \overline{z}_i).$$

A smooth quasi-projective variety has the induced Kähler structure from $\mathbb{P}^n$. Hence every algebraic manifold is a Kähler manifold.

(2.4) Hyperkähler manifold is a manifold equipped with not just one Kähler structure but an S^2-family of Kähler structures.

(2.5) Definition. A Riemannian manifold (M, g) is *hyperkähler* if there are three complex structures I, J, K ($I^2 = J^2 = K^2 = -1$ and I, J, K are parallel) such that $IJ = K = -JI$. By the equation in (2.2), we get three Kähler forms $\omega_I, \omega_J, \omega_K$.

(2.6) The simplest example of a hyperkähler manifold is of course the quaternionic vector space $\mathbb{H}^n$ where $\mathbb{H}$ is the division algebra of quaternions. Also the quotient of $\mathbb{H}^n$ by a discrete group action is also a hyperkähler manifold. However, the quaternionic projective space $\mathbb{HP}^n$ is not a hyperkähler manifold. (Exercise: Why?)

(2.7) In fact a hyperkähler manifold has infinitely many (S^2-family) of Kähler structures since $aI + bJ + cK$ for (a, b, c) satisfying $a^2 + b^2 + c^2 = 1$ is always a complex structure compatible with the metric. It is possible to

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[6]Parallel means that J commutes with parallel transportation.

[7]It is elementary to show that ω is skew-symmetric and J being parallel implies that ω is a closed 2-form.

construct a complex manifold Z, called the *twistor space*, and a map $Z \to S^2$ whose fiber over $(a, b, c) \in S^2$ is the Kähler manifold with complex structure $aI + bJ + cK$.

(2.8) $Sp(r)$ is the group of orthogonal transformations of $\mathbb{R}^{4n} = \mathbb{H}^n$ which are linear with respect to I, J, K. Hence the holonomy group of a hyperkähler manifold is contained in $Sp(r)$ where $\dim M = 4r$ since the three complex structure I, J, K makes T_xM a quaternionic vector space and I, J, K are parallel. Conversely, if the holonomy group is contained in $Sp(r)$, choose complex structures I_x, J_x, K_x on T_xM which makes T_xM a quaternionic vector space. The parallel transportation of these complex structures I, J, K give us three complex structures I, J, K on M and hence M is a hyperkähler manifold.

(2.9) Another way to think of hyperkähler manifolds is to break the symmetry of the complex structures and choose one particular complex structure, say I. Then (M, g, I) is a Kähler manifold and $\omega_{\mathbb{C}} := \omega_J + I\omega_K$ is a closed complex valued holomorphic 2-form on M which is nondegenerate everywhere. Such a form is called a *holomorphic symplectic form* and a Kähler manifold equipped with a holomorphic symplectic form $\omega_{\mathbb{C}}$ is called a *holomorphic symplectic manifold*. Conversely, a compact holomorphic symplectic manifold M admits a Ricci flat metric for which $\omega_{\mathbb{C}}$ is parallel, by Yau's famous theorem and Bochner's principle. Hence the holonomy group is contained in $Sp(r)$ and M is a hyperkähler manifold.

For a Kähler manifold M, its cotangent bundle T^*M is canonically a holomorphic symplectic manifold and thus a hyperkähler manifold. Indeed, for a complex vector space V of dimension n, $T^*V \cong V \oplus V^*$ and the obvious pairing gives us a holomorphic symplectic form $\omega_{\mathbb{C}}$.[8] For a Kähler manifold M, we have a holomorphic symplectic form on T^*U for each coordinate neighborhood $U \subset \mathbb{C}^n$ and these symplectic forms coincide on the intersections of coordinate neighborhoods. Therefore T^*M is a *noncompact* hyperkähler manifold.

(2.10) Theorem/Definition. The holonomy group of a compact holomorphic symplectic manifold M is exactly $Sp(r)$ if and only if M is simply connected and the holomorphic symplectic form is unique up to a scalar multiple. Such a manifold is called an *irreducible (holomorphic) symplectic manifold*.

(2.11) In summary, the study of manifolds whose holonomy group is contained in $Sp(r)$ is the study of hyperkähler manifolds. The study of compact manifolds whose holonomy is exactly $Sp(r)$ in Berger's list (1.8) is the study of irreducible symplectic manifolds.

[8] V and V^* are Lagrangian subspaces.

Irreducible symplectic manifolds are building blocks for Kähler manifolds with trivial first Chern class[9] by the following theorem, often called the Bogomolov decomposition theorem.

(2.12) Theorem. Let M be a compact Kähler manifold with trivial first Chern class. Then there is a finite étale cover of M, isomorphic to $T \times \prod V_i \times \prod X_j$ where T is a complex torus, V_i are (irreducible) Calabi-Yau and X_j are irreducible symplectic manifolds.

3. Compact hyperkähler manifolds

(3.1) In this section, we consider all the known examples of irreducible symplectic manifolds, i.e. *compact* simply connected Kähler manifold endowed with a closed nondegenerate holomorphic 2 form which spans the space of holomorphic 2 forms. In the subsequent section, we will see some important noncompact examples.

(3.2) The dimension of a hyperkähler manifold is always $4r$, $r \geq 1$. We first consider the case $r = 1$ so that $\dim M = 4$. Observe that $Sp(1) = SU(2)$. Hence M is an irreducible symplectic manifold if and only if M is a simply connected Calabi-Yau which amounts to saying that the canonical bundle $\wedge^2_{\mathbb{C}} TM$ is isomorphic to $\mathcal{O}_M$ and M is simply connected, i.e. M is a *K3 surface*.

There are numerous examples of K3 surfaces: quartic surface X_4 in $\mathbb{P}^3$, complete intersections $X_{(2,3)} \subset \mathbb{P}^4$, $X_{(2,2,2)} \subset \mathbb{P}^5$, double cover of $\mathbb{P}^2$ branched along a sextic, a smooth divisor in $|-K_V|$ on a smooth 3-fold V.

If you don't demand simple connectedness, M being hyperkähler is equivalent to M being either a K3 or a complex torus.

(3.3) For the case of $\dim M = 4r$ with $r \geq 2$, there are two standard series of irreducible symplectic manifolds provided by Beauville. The first series are the *Hilbert schemes* $X^{[r]}$ of r points[10] in a K3 surface X.[11] That $X^{[r]}$ is projective follows from the general theory of Grothendieck's Quot scheme construction and smoothness is proved by deformation theory. Beauville gives a simple proof of the existence of a holomorphic symplectic form by

[9] The first Chern class being trivial is equivalent to the top exterior power of the complex tangent bundle being trivial topologically.

[10] More precisely the Hilbert scheme of zero dimensional subschemes of length r.

[11] Quite surprisingly the cohomology $\oplus_r H^*(X^{[r]})$ of the Hilbert schemes is an irreducible representation space of the Heisenberg superalgebra associated with $H^*(X)$ by a famous work of Nakajima.

using the Hilbert-Chow morphism $X^{[r]} \to S^r X$[12] where $S^r X$ is the symmetric product of X i.e. X^r/S_r.[13]

The second series of irreducible symplectic manifolds also arise from Hilbert schemes. Let A be an abelian surface and let $A^{[r+1]}$ be the Hilbert scheme of $r+1$ points in A. Beauville's argument proves that $A^{[r+1]}$ is a holomorphic symplectic manifold. But this is not simply connected. So we consider the Albanese map $a : A^{[r+1]} \to A$ which is just the sum by the group structure of A. Let $K_r = a^{-1}(0)$. Then K_r is simply connected with induced holomorphic symplectic structure. Hence K_r is a $4r$ dimensional irreducible symplectic manifold, called the *generalized Kummer variety.* If $r = 1$ we recover the classical Kummer variety. (Exercise: Why?)

Until very recently, there were no other examples. To be precise, there were other examples but they were diffeomorphic to Beauville's examples. By Mukai's theorem every moduli space $M(r, c_1, c_2)^s$ of rank r stable sheaves with Chern classes (c_1, c_2) on a K3 surface X is an irreducible symplectic manifold if the moduli space is compact. However it was shown by Mukai, O'Grady and Yoshioka that a compact moduli space of stable sheaves is deformation equivalent to the Hilbert scheme $X^{[r]}$ of the same dimension by Fourier-Mukai transformation and Huybrechts's theorem. In particular, they are diffeomorphic. Hence as a manifold, the compact moduli spaces of stable sheaves are the same as Beauville's examples.

(3.4) Recently (1999 and 2003), two new[14] irreducible symplectic manifolds of dimensions 12 and 20 were discovered by O'Grady. The first one comes from a K3 surface X: let $M_{K3}(2, 0, 2m)$ be the moduli space of rank 2 *semistable* sheaves F on X with Chern classes $c_1(F) = 0$, $c_2(F) = 2m$. This is an irreducible normal *singular* projective variety of (real) dimension $16m - 12$ whose singularities are terminal Gorenstein[15] for $m \geq 2$. O'Grady considered $M_{K3}(2, 0, 4)$ (dimension 20) and constructed a desingularization $\widetilde{M}_{K3}(2, 0, 4)$ which admits a holomorphic symplectic form by blowing up $M_{K3}(2, 0, 4)$ twice and then blowing down once. The hard part is to show that this is indeed simply connected and the second Betti number is greater than or equal to 24. O'Grady used an idea of Jun Li which utilizes the generalized Lefschetz hyperplane theorem and the map to Uhlenbeck compactification constructed also by Jun Li.

The second example comes from an abelian surface A: let $M_{Ab}(2, 0, 2n)$ be the moduli space of rank 2 *semistable* sheaves F on A with Chern classes

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[12] This is just a blow-up along the diagonal if you delete a suitable codimension 2 subset and Hartog's theorem says it is okay to do that.

[13] The holomorphic symplectic form of X induces a holomorphic symplectic form on X^r which induces a holomorphic 2 form on the blow-up of X^r along the diagonal. This is S_r invariant and hence it comes from a holomorphic 2 form on $X^{[r]}$ which turns out to be nondegenerate.

[14] New means that it is not diffeomorphic to Beauville's manifolds.

[15] Just ignore if the words like "normal", "terminal" do not make sense to you.

$c_1(F) = 0$, $c_2(F) = 2n$. O'Grady constructs a symplectic desingularization $\widetilde{M}_{Ab}(2,0,2)$ in exactly the same way as in the K3 case. The trouble is as in Beauville's case that $\widetilde{M}_{Ab}(2,0,2)$ is not simply connected. The way out is to consider the Albanese map. Let

$$a : \widetilde{M}_{Ab}(2,0,2) \to A \times \mathrm{Pic}^0(A)$$

be the product of addition[16] and the determinant map. Then $\widetilde{K} = a^{-1}(0,0)$ is simply connected and equipped the induced holomorphic symplectic form. Hence $\widetilde{K}$ is an irreducible symplectic manifold. By comparing the second Betti numbers O'Grady proved that $\widetilde{K}$ is not homeomorphic to any of the previously known examples.

In this context O'Grady asked if one can find, as O'Grady did, new irreducible symplectic manifolds by constructing a desingularization $\widetilde{M}$ of $M_{K3}(2,0,2m)$ for $m \geq 3$ or $M_{Ab}(2,0,2n)$ for $n \geq 2$ such that $\widetilde{M}$ is holomorphic symplectic. In other words, he raised the following question.

(3.5) Question. Does there exist a holomorphic symplectic desingularization of $M_{K3}(2,0,2m)$ for $m \geq 3$ or $M_{Ab}(2,0,2n)$ for $n \geq 2$?

(3.6) Answer. By using properties of stringy E-function (which is an incarnation of motivic integral) Jaeyoo Choy and I showed that the answer is unfortunately NO for all $m \geq 3$ and all $n \geq 2$. Kaledin, Lehn and Sorger prove the non-existence in a more general context by studying factoriality.

(3.7) Problem. Study the topology and geometry of O'Grady's irreducible symplectic manifolds. For example, compute the Betti numbers and the Picard group of the varieties.[17]

(3.8) Problem. Based on physics arguments, Vafa and Witten says the Euler characteristic of $M_{K3}(2,0,2n)$ must be $e_{4n-3} + \frac{1}{4}e_n$ where e_k is the Euler characteristic of the Hilbert scheme $X^{[k]}$ of k points in a K3 surface X. Find a natural mathematical explanation.

After all, the list of irreducible symplectic manifolds is "embarrassingly" scarce. Hence the following is still an important open problem for geometers.

(3.9) Problem. Find new examples of irreducible symplectic manifolds.

[16]To be precise, this is obtained as follows: first take the second Chern class in the Chow group and then add them up using the group structure of A.

[17]So far, only the Euler characteristic of the 12 dimensional example is known to be 1920.

4. Noncompact hyperkähler manifolds

(4.1) In this section I will discuss several important examples of noncompact hyperkähler manifolds. As we observed in (2.9), the cotangent bundle of a Kähler manifold is a noncompact hyperkähler manifold. The examples discussed in this section are constructed by the technique of hyperkähler quotient which will be considered in the subsequent section.

(4.2) *Moduli space of magnetic monopoles.* Mathematically, a magnetic monopole is a pair of a connection A on a principal $SU(n)$-bundle on $\mathbb{R}^3$ and a Lie algebra valued holomorphic form ϕ (with appropriate decay at infinity) which is critical for the potential function $U = \frac{1}{2}\int_{\mathbb{R}^3} |F_A|^2 + |d_A\phi|^2$ where F_A is the curvature of A.[18] The moduli space $M(n,k)$ of magnetic monopoles is the space of such pairs (A,ϕ) with magnetic charge k (the second Chern class) where the minimum of U is attained,[19] modulo gauge equivalence. Luckily a theorem of Donaldson says this moduli space is an object that looks quite friendly to algebraic geometers. Namely, $M(n,k)$ is isomorphic to the space of rational maps $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1}$ of degree k with $f(\infty) = 0$. For instance we have

$$M(2,k) = \Big\{\frac{a_0 + a_1 z + \cdots + a_{k-1}z^{k-1}}{b_0 + b_1 z + \cdots b_{k-1}z^{k-1} + z^k} \mid \Delta \neq 0\Big\} \subset \mathbb{C}^{2k} \cong \mathbb{H}^k$$

where Δ is the resultant of the numerator and the denominator. As an open subset of $\mathbb{H}^k$, $M(2,k)$ is a noncompact hyperkähler manifold. The topology of $M(n,k)$ has been studied intensively during the past 20 years.

(4.3) *Moduli space of instantons on* $\mathbb{R}^4$*.* The framed moduli space $M(r,n)$ of instantons on $\mathbb{R}^4$ is the space of anti-self-dual $SU(r)$-connections on $\mathbb{R}^4$ with appropriate decay modulo gauge equivalence. Luckily, $M(r,n)$ is also diffeomorphic to an object quite friendly to algebraic geometers: the moduli space of rank r torsion-free sheaves E on $\mathbb{P}^2$ equipped with an isomorphism $E|_{l_\infty} \cong \mathcal{O}_{l_\infty}^{\oplus r}$ with second Chern class n. By a theorem of Barth we have an explicit description of $M(r,n)$ as follows:

$$M(r,n) = \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\}^s / GL(n,\mathbb{C})$$

where B_1, B_2 are $n \times n$ complex matrices, i is an $n \times r$ matrix, and j is an $r \times n$ matrix. The superscript s denotes the locus of *stable* points which means in this case that there is no proper subspace of $\mathbb{C}^n$ which is preserved by B_1, B_2 and contains the image of i.

The set of matrices $\{(B_1, B_2, i, j)\}$ is the cotangent bundle of

$$\mathbf{M} := \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$$

[18] Ignore this sentence if it doesn't make sense to you.

[19] The minimum is attained exactly when the *Bogomolny equation* $F_A = *d_A\phi$ is satisfied.

and $M(r,n)$ is the hyperkähler quotient of $T^*\mathbf{M}$ by the obvious action of $GL(n,\mathbb{C})$. Therefore, there is an induced hyperkähler structure on $M(r,n)$ as we will see in section 6.

(4.4) *Quiver varieties.* For a directed graph Γ with no loop[20] we associate two vector spaces V_k and W_k to each vertex k. For each directed edge h let $in(h)$ (resp. $out(h)$) be the vertex of origin (resp. destination). Let

$$\mathbf{M} := \bigoplus_{h:edge} \mathrm{Hom}(V_{in(h)}, V_{out(h)}) \oplus \bigoplus_{k:vertex} \mathrm{Hom}(W_k, V_k)$$

This is a complex vector space whose cotangent bundle is

$$\begin{aligned} T^*\mathbf{M} = \quad & \textstyle\bigoplus_{h:edge} \big(\mathrm{Hom}(V_{in(h)}, V_{out(h)}) \oplus \mathrm{Hom}(V_{out(h)}, V_{in(h)})\big) \\ & \times \textstyle\bigoplus_{k:vertex} (\mathrm{Hom}(W_k, V_k) \oplus \mathrm{Hom}(V_k, W_k)) \,. \end{aligned}$$

There is an obvious action of $G = \prod U(V_k)$ on $T^*\mathbf{M}$ and the hyperkähler quotient $\mathfrak{M}_\zeta(v,w)$ of $T^*\mathbf{M}$ by the action of G at general $\zeta \in \mathrm{center}(Lie\, G)^*$ is the quiver variety associated with the data (Γ, v, w) where $v = (\dim V_k)$ and $w = (\dim W_k)$.

A famous work of Nakajima says when the graph Γ is of Dynkin type A, D, E or affine, the middle degree cohomology[21]

$$\bigoplus_v H^{mid}(\mathfrak{M}_\zeta(v,w))$$

is an irreducible representation space with highest weight w of the Kac-Moody algebra generated by the Cartan matrix $C = 2I - A$ where A is the adjacency matrix of Γ.[22]

There are many more interesting noncompact hyperkähler manifolds like the hypertoric manifolds (constructed as hyperkähler quotients) and the moduli space of Higgs bundles over a curve.

5. Hyperkähler quotient

(5.1) A useful method of constructing a new hyperkähler manifold from the known examples is to take a quotient by a group action in Hamiltonian fashion. Unfortunately this method doesn't seem to give us any new *compact* hyperkähler manifold because it is very difficult to find a compact hyperkähler manifold equipped with a nontrivial indiscrete group action suitable for hyperkähler quotient construction.

(5.2) To begin with, let us recall *Kähler quotient* first. Let (M, g, I) be a Kähler manifold, equipped with an action of a compact Lie group G which

[20] Think of a Dynkin diagram.

[21] $H^{mid}(X) := H^{\frac{1}{2}\dim_{\mathbb{R}} X}(X)$

[22] The (i,j)-th entry of A is the number of edges joining i and j.

preserves g and I. The Kähler form ω is nondegenerate and hence gives us an isomorphism $TM \cong T^*M$. For any smooth function f on M, df is 1-form, i.e. a section of T^*M. By the isomorphism $TM \cong T^*M$ given by ω, df gives rise to a section of TM, called the *Hamiltonian vector field* of f. A *moment map* for the action of G is a smooth map

$$\mu : M \to Lie(G)^*$$

such that for any $\xi \in Lie(G)$ the vector field X_ξ defined by

$$X_\xi(p) = \frac{d}{dt}|_{t=0} p \cdot \exp(t\xi)$$

is the Hamiltonian vector field of the function $\langle \mu, \xi \rangle : M \to Lie(G)^* \to \mathbb{R}$. The action of G on M is called *Hamiltonian* if there is a moment map μ.

Now the Kähler quotient is defined as

$$M//G := \mu^{-1}(0)/G.$$

This is again a Kähler manifold if the action of G on $\mu^{-1}(0)$ is free. To see this we note that the action of G induces an action of its complexification $G^{\mathbb{C}}$. Consider the open set M^{ss} of points in M whose gradient flow with respect to $-|\mu|^2$ has a limit point in the minimal critical set $\mu^{-1}(0)$. Then Kirwan showed that there is a bijection

$$\mu^{-1}(0)/G \hookrightarrow M^{ss}//G^{\mathbb{C}}.$$

The left hand side $\mu^{-1}(0)/G$ is equipped with a symplectic form induced from ω by a theorem of Marsden and Weinstein while the right hand side $M^{ss}//G^{\mathbb{C}}$ has a complex structure obviously. These symplectic structure and complex structure make $M//G$ a Kähler manifold.

An important result for computing the cohomology of a Kähler quotient $M//G$ is Kirwan's surjectivity which says the restriction map

$$H^*_G(M) \to H^*_G(\mu^{-1}(0))$$

of equivariant cohomology groups[23] is surjective. When G acts freely on $\mu^{-1}(0)$ we have a canonical isomorphism $H^*_G(\mu^{-1}(0)) \cong H^*(M//G)$ and thus a surjective map

$$\kappa : H^*_G(M) \to H^*(M//G)$$

which is called the *Kirwan map.*

(5.3) Now suppose (M, g, I, J, K) is a hyperkähler manifold. Suppose a compact Lie group G acts on M preserving g, I, J, K. Furthermore suppose the group action is Hamiltonian with respect the three symplectic structures $\omega_I, \omega_J, \omega_K$ respectively, i.e. there are three moment maps μ_I, μ_J, μ_K. Let

[23] $H^*_G(X) = H^*(X \times EG/G)$ where EG is a contractible topological space on which G acts freely.

$\mu = (\mu_I, \mu_J, \mu_K) : M \to ((Lie\, G)^*)^{\oplus 3}$ and suppose that G acts freely on $\mu^{-1}(0)$. The *hyperkähler quotient* of M by G is defined by

$$M///G := \mu^{-1}(0)/G.$$

Let us see why $M///G$ is a hyperkähler manifold. As mentioned in (2.9), $\omega_J + I\omega_K$ is holomorphic with respect to the complex structure I and this implies that $f_I := \mu_J + \sqrt{-1}\mu_K$ is a holomorphic map. Hence $M///G = \mu^{-1}(0)/G = \mu_I^{-1}(0) \cap f_I^{-1}(0)/G$ is an analytic subvariety of the Kähler quotient $\mu_I^{-1}(0)/G$. Hence $M///G$ is equipped with a Kähler structure which arose from I, ω_I. Likewise J and K give rise to Kähler structures on $M///G$ which make $M///G$ is a hyperkähler manifold.

(5.4) Problem. As in the Kähler quotient case (5.2), the inclusion $\mu^{-1}(0) \hookrightarrow M$ induces a restriction map

$$\kappa : H^*_G(M) \to H^*_G(\mu^{-1}(0)) \cong H^*(M///G).$$

Is κ surjective?[24]

(5.5) Question. Can you study the cohomology ring of $M///G$ in terms of equivariant Morse theory as Atiyah, Bott, Kirwan did in the Kähler quotient case?[25]

(5.6) Question. If the action of G on $\mu^{-1}(0)$ is not free, then the hyperkähler quotient $M///G$ is a singular variety which is hyperkähler on the smooth locus. Can you find a canonical desingularization?[26]

Last words

I would like to thank Professor Sijong Kwak and the organizers of the KAIST symposium on number theory and algebraic geometry in February 2005 for inviting me. Down below is an *incomplete* list of articles and books from which I've stolen the contents of this paper.

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[24] This is a very important problem as there are many applications. Kirwan has a proof of the surjectivity when G is abelian.

[25] The cohomology of quiver varieties, moduli spaces of sheaves on K3, and so on are not well understood. If you can find a systematic way to study the cohomology of hyperkähler quotients, you can produce numerous papers (if you want to, of course).

[26] I don't recommend this problem to a graduate student because I think I know how to answer this question and I plan to write a paper soon.

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