Chern Classes of Moduli of Vector Bundles over Curves

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1 Introduction

Let Y be a smooth complex projective algebraic curve of genus g with a very ample line bundle $\mathcal{O}_Y(1)$. For a vector bundle E over Y, the *slope* of E is defined as

$$\mu(E) = \frac{\deg E}{\mathrm{rank} E}.$$

We say a vector bundle E over Y is *(semi)-stable* if

$$\mu(F) < \mu(E) \qquad (\leq)$$

for any proper nonzero subbundle F of E. The category of semistable bundles of slope μ is a noetherian and artinian abelian category whose simple objects are stable bundles and thus each semistable bundle E has a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E$$

by semistable subbundles such that E_i/E_{i-1} are stable of slope μ. The filtration may not be unique but the grading

$$gr(E) = \oplus E_i/E_{i-1}$$

is independent of the filtration. We say two semistable bundles E and E' are *s-equivalent* if $gr(E) \cong gr(E')$.

The set of s-equivalence classes of semistable vector bundles of rank r and degree d admits a structure of irreducible projective variety $M_{r,d}(Y)$ by the recipe of geometric invariant theory [MFK94]. In fact, any semistable bundle of rank r and degree d is the quotient of $\mathcal{O}_Y(-m)^{\oplus p}$ for suitable integers p and m which depends only on r and d. Grothendieck's Quot scheme parameterizes such quotient sheaves and two quotient sheaves are isomorphic if and only if they are in the same orbit under the natural action of $PGL(p)$. Now the moduli space $M_{r,d}(Y)$ is the good quotient of the (smooth) subvariety $R_{r,d}$ of semistable points in the Quot scheme.

In this paper, unless stated otherwise, we always assume that the integers r and d are coprime. Then semistable bundles are automatically stable and the stabilizer groups of points in $R_{r,d}$ are all trivial. Hence the moduli space $M_{r,d}(Y)$ is smooth and the universal quotient bundle over $R_{r,d} \times Y$ descends to a universal bundle $\mathcal{E}$ over $M_{r,d}(Y) \times Y$.

By a local study using deformation theory, the tangent space at a point $[E] \in M_{r,d}(Y)$ is isomorphic to $H^1(Y, \mathrm{End}E)$ and the tangent bundle of the moduli space is

$$TM_{r,d}(Y) \cong R^1 p_*(\mathrm{End}\mathcal{E}) \tag{1}$$

where $p : M_{r,d}(Y) \times Y \to M_{r,d}(Y)$ is the projection onto the first component.

The famous theorem of Narasimhan and Seshadri says the moduli space $M_{r,d}(Y)$ is homeomorphic to the moduli space of flat unitary connections on (punctured) Y. In other words,

$$M_{r,d}(Y) \simeq \{(A_i) \in U(r)^{2g} \mid \prod [A_i, A_{i+g}] = exp(\frac{2\pi i d}{r}) I\}/U(r)$$

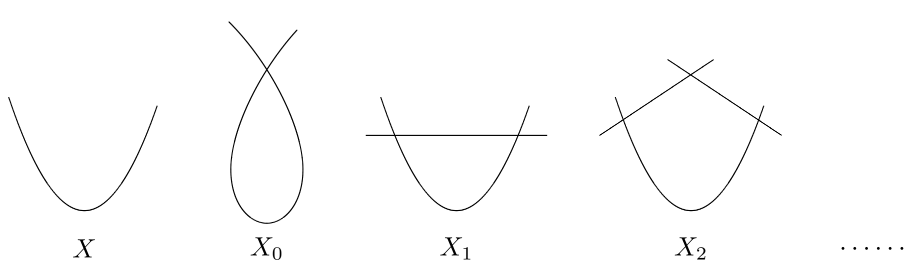
where the action of $U(r)$ is by conjugation.

The purpose of this paper is to explain the results of the joint paper [KL] with Jun Li about the Chern classes of the moduli spaces. In §2, I describe the cohomology ring of the moduli space. In §3, the Newstead-Ramanan conjecture is explained. In §4, I explain Gieseker's proof of the vanishing of Chern classes for the rank 2 case. In §5, our generalization to higher rank cases is discussed.

Notation: We will use the following notations throughout the paper.

- Y = smooth irreducible projective curve of genus g.
- X = smooth irreducible projective curve of genus $g-1$.
- p_1, p_2 = two distinct points in X.
- $X_0 = X/\{p_1 \sim p_2\}$ a curve of genus g with one node.
- X_n = a curve obtained by gluing a chain of n $\mathbb{P}^1$s to X at p_1 and p_2. The rational components are denoted by $C_1, \cdots, C_n$ from left to right.

Pictorially, the curves are as follows:



2 Cohomology ring of the moduli space

We now have a very good understanding of the cohomology ring of $M_{r,d}(Y)$. After pioneering works of Newstead, Atiyah and Bott made a major breakthrough in the seminal paper [AB83]. They applied Morse theory argument to the space $\mathcal{A}$ of unitary connections using the norm square of curvature as the Morse function and showed that the Morse stratification is perfect for the equivariant cohomology with respect to the action of the gauge group $\mathcal{G}$. As a result, they proved that the cohomology of $M_{r,d}(Y)$ is torsion-free. So from now on, we consider only the rational cohomology groups.

Atiyah and Bott furthermore found a set of generators for the cohomology ring in terms of the Chern classes of the universal bundle $\mathcal{E}$ over $M_{r,d}(Y) \times Y$.

Theorem 2.1. *[AB83, Theorem 9.11] Let a_i, b_i^j, f_i be the classes defined by the Künneth decomposition*

$$c_i(\mathcal{E}) = a_i \otimes 1 + \sum_{j=1}^{2g} b_i^j \otimes \sigma_j + f_i \otimes [Y]$$

where $\{\sigma_j, \sigma_{j+g}\}_{j=1}^{g}$ is a symplectic basis of $H^1(Y)$ and $[Y]$ is the fundamental class in $H^2(Y)$. Then the set

$$\{a_i, b_i^j, f_i \,|\, 1 \le i \le r, 1 \le j \le 2g\}$$

generate the cohomology ring $H^*(M_{r,d}(Y))$.

Since a_1 can be expressed in terms of the other classes and f_1 is just the degree d, we may delete a_1 and f_1 from the list of generators. See [AB83, p582] for a normalization of the universal bundle $\mathcal{E}$.

By equivariant Morse theory on the space $\mathcal{A}$ of unitary connections, we have a surjective map

$$H^*_{\overline{\mathcal{G}}}(\mathcal{A}) \to H^*_{\overline{\mathcal{G}}}(\mathcal{A}_{flat})$$

where $\mathcal{A}_{flat}$ is the set of flat unitary connections and $\overline{\mathcal{G}}$ is the quotient of the gauge group $\mathcal{G}$ by $U(1)$. Also we have $H^*_{\overline{\mathcal{G}}}(\mathcal{A}_{flat}) \cong H^*(M_{r,d}(Y))$ by the Narasimhan-Seshadri theorem. The space $\mathcal{A}$ is contractible and thus the equivariant cohomology is

$$H^*_{\overline{\mathcal{G}}}(\mathcal{A}) \cong H^*(B\overline{\mathcal{G}})$$

which is freely generated by some classes $\hat{a}_i, \hat{b}_i^j, \hat{f}_i$. These are mapped to a_i, b_i^j, f_i respectively. This proves the above theorem.

Once we know the generators of a ring, it is natural to ask for the relations. Since $M_{r,d}(Y)$ is a compact smooth oriented manifold, the relations are completely determined by the intersection pairing because of Poincaré duality. The formula for the intersection numbers in the rank 2 case was first deduced by Thaddeus using the Verlinde formula [T92]. Later Witten gave formulas for arbitrary rank cases which were subsequently proved by Jeffrey and Kirwan

[JK98] mathematically by their non-abelian localization theorem. Instead of reproducing the complicated formulas,[1] let me just write down the formula for the rank 2 case to hint a flavor of the formulas;

$$\int_{M_{2,1}(Y)} a_2^j e^{f_2} = \frac{(-1)^{g-1-j}}{2^{g-1}} \mathrm{Res}_{t=0}\left(\frac{1}{t^{2g-2-2j}\sin t}\right).$$

Remark 2.2. In [JKKW], we will generalize these formulas to the cases where the rank r and degree d are no longer coprime.[2] We will provide formulas for the intersection pairing of the *middle perversity intersection cohomology* of $M_{r,d}(Y)$ which turn out to be exactly the same as Witten's formulas with obvious modifications. We will also give formulas for the intersection numbers of equivariant cohomology classes evaluated over the *partial desingularization* of $M_{r,d}(Y)$.

3 Characteristic classes

After learning about the cohomology ring of a manifold, the next natural question is probably about characteristic classes. In the rank 2 case, there are two classical conjectures due to Newstead and Ramanan. The first is about the Chern classes and the second is about the Pontrjagin classes:

1. $c_i(M_{2,1}(Y)) = 0$ for $i > 2(g-1)$.
2. $p_j(M_{2,1}(Y)) = 0$ for $j > g-1$.

Using the description (1) and the Grothendieck-Riemann-Roch theorem, it is in principle possible to express the characteristic classes in terms of the generators in Theorem 2.1. It turns out that the Pontrjagin classes have much simpler expressions. For instance, in the rank 2 case, the total Pontrjagin class is

$$p(TM_{2,1}(Y)) = (1+4a_2)^{2(g-1)}$$

and thus the second conjecture is equivalent to the vanishing of *only one* class

$$a_2^g = 0.$$

This was first proved by Kirwan [K92] by equivariant Morse theory argument, in early 1990's. Recently, in collaboration with Earl, she proved a vanishing result of the Pontrjagin ring for arbitrary rank cases.

Theorem 3.1. *[EK99]* (a) *The Pontrjagin classes are contained in the subring generated by $a_2, \cdots, a_r$.*

(b) *The subring of $H^*(M_{r,d}(Y))$ generated by $a_2, \cdots, a_r$ vanishes in degrees $> 2r(r-1)(g-1)$.*

[1] See §8, §9 of [JK98].

[2] In these cases, the moduli spaces $M_{r,d}(Y)$ are singular and we don't have universal bundles. Furthermore, the equivariant Morse theory is not so much helpful directly.

The proof of (b) is based on Witten's formulas. Earl and Kirwan showed that the intersection number of a class of degree $> 2r(r-1)(g-1)$ in the subring with any class of complementary degree is zero. Since the intersection pairing is perfect, the classes vanish.

The Newstead-Ramanan conjecture about the Chern classes is more difficult because the expressions for the Chern classes are extremely complicated. Even in the rank 2 case, Zagier's formula [Z95] reads as

$$c(TM_{2,1}(Y)) = (1-\beta^2)^{g-1}\exp\left(\frac{2\alpha}{1-\beta} + 2\big(\frac{\tanh^{-1}\sqrt{\beta}}{\beta\sqrt{\beta}} - \frac{1}{\beta(1-\beta)}\big)\gamma^*\right)$$

where $\alpha = 4f_2 + 2\sum_{j=1}^{g} b_1^j b_1^{j+g}$, $\beta = 4a_2$, $\gamma^* = \alpha\beta + 2\sum b_2^j b_2^{j+g}$.

Using his incredible computational skills and insights, Zagier was able to show that

$$\int_{M_{2,1}(Y)} \xi \cdot c(TM_{2,1}(Y)) = 0$$

for any $\xi \in H^{\leq 4g-4}(M_{2,1}(Y))$ by Thaddeus's formula. So he proved the Newstead-Ramanan conjecture for the rank 2 case.

However, in the rank 3 or higher cases, it doesn't seem possible to work out the computation by hand and hence generalization to higher rank cases with this method seems unlikely. Fortunately, there is another way.

4 Gieseker's proof

In [G84], Gieseker provided a geometric proof of the vanishing of the Chern classes of $M_{2,1}(Y)$ by induction on the genus of the Riemann surface. In this section, we explain his proof.

When $g = 1$, $M_{2,1}(Y) \cong Y$ by Atiyah's theorem and hence $c_i(M_{2,1}(Y)) = 0$ for $i > 0$. So from now on, we assume $g \geq 2$. He used then the degeneration argument to complete the induction.

Let W be a flat family of curves over $\mathbb{A}^1 \cong \mathbb{C}$ such that

- $W_s = W|_s$ is a smooth projective curve of genus g for $s \neq 0$,
- $W_0 = W|_0 = X_0$ is the nodal curve,
- W is smooth.

If we delete the central fiber W_0 from W, then by the recipe of geometric invariant theory there is a smooth projective family $\mathcal{M}_{2,1}(W - W_0)$ over $\mathbb{A}^1 - 0$ of moduli spaces whose fiber over s is diffeomorphic to $M_{2,1}(Y)$. The question now is how to fill in the central fiber into the family $\mathcal{M}_{2,1}(W - W_0)$ so that we get a flat projective family of moduli spaces over $\mathbb{A}^1$.

There are two ways, due to Seshadri and Gieseker respectively. The moduli space of stable vector bundles over X_0 is not compact and we need to compactify it. Seshadri's approach is to include torsion-free sheaves on X_0. Namely, the set of s-equivalence classes of semistable torsion-free sheaves on X_0 admits a structure of irreducible projective variety as the good quotient of the Quot scheme. This construction performed for the family W gives us a family of moduli spaces, projective over $\mathbb{A}^1$. This approach is useful for many other purposes but in order to deduce the vanishing result for the general fiber from a suitable vanishing result of the central fiber we want the following properties:

- The total space $\mathcal{M}_{2,1}(W)$ with the central fiber included is smooth.
- The central fiber has normal crossing singularities.

Unfortunately, Seshadri's family does not satisfy either of the properties.

Instead of the Quot scheme, Gieseker used the Hilbert scheme

$$Hilb^P(X_0 \times Gr(p, 2))$$

of curves in $X_0 \times Gr(p, 2)$ where $P(m) = dm + 2 - 2g$, $p = P(1)$ and $Gr(p, 2)$ is the Grassmannian of 2 dimensional quotients of $\mathbb{C}^p$. An embedding

$$X_n \hookrightarrow X_0 \times Gr(p, 2)$$

gives rise to a vector bundle $E \to X_n$ by pulling back the universal quotient bundle over the Grassmannian. Gieseker's central fiber is constructed as the quotient

$$Hilb^P(X_0 \times Gr(p, 2))//PGL(p)$$

and this is the moduli space of vector bundles E over X_n for $n = 0, 1, 2$ with a suitable stability condition. More generally, curves over $\mathbb{A}^1$ in $W \times Gr(p, 2)$ are parameterized by a smooth variety,

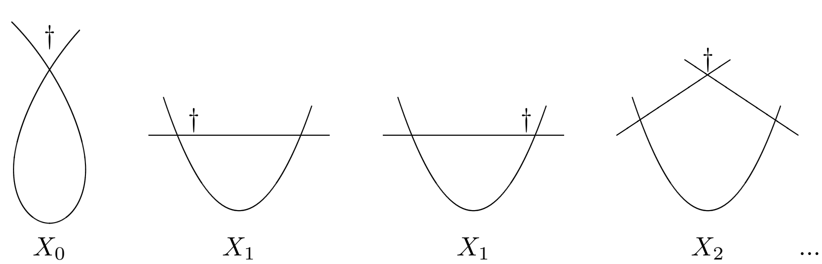
$$Hilb^P_{\mathbb{A}^1}(W \times Gr(p, 2))$$

projective over $\mathbb{A}^1$, whose fiber over 0 has normal crossing singularities. The group $PGL(p)$ acts freely on the semistable points. Hence, the quotient

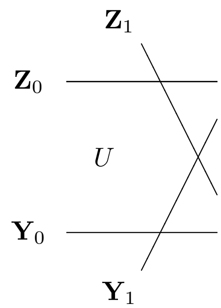
$$\mathcal{M}_{2,1}(W) = Hilb^P_{\mathbb{A}^1}(W \times Gr(p, 2))//PGL(p)$$

is smooth, projective over $\mathbb{A}^1$ and the central fiber has normal crossing singularities. Thus we have a nice degeneration of $M_{2,1}(Y)$ to $M_{2,1}(W_0)$.

Next, we need to relate $M_{2,1}(W_0) = M_{2,1}(X_0)$ with $M_{2,1}(X)$. Recall that the genus of X is $g-1$. Let $\mathbf{M}_0$ be the normalization of $M_{2,1}(X_0)$. This amounts to considering vector bundles E over X_n equipped with a marked node †.



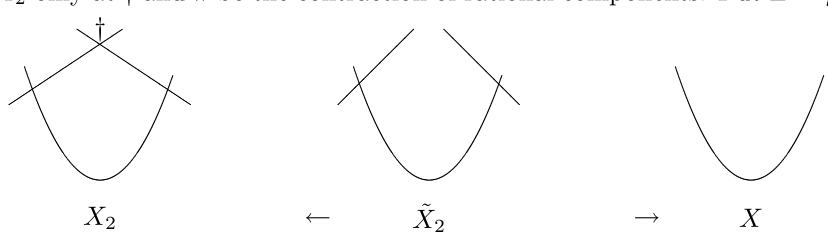
Let U be the open dense subset of vector bundles E over X_0 whose pull-back to the normalization X of X_0 is a stable bundle. Then U is a $GL(2)$-bundle over $M_{2,1}(X)$ where $GL(2)$ accounts for gluing of $E|_{p_1}$ with $E|_{p_2}$. It turns out that the complement of U in $\mathbf{M}_0$ consists of 4 smooth normal crossing divisors, i.e. $\mathbf{M}_0 - U = \mathbf{Y}_0 \cup \mathbf{Z}_0 \cup \mathbf{Y}_1 \cup \mathbf{Z}_1$.



By the moduli property of $M_{2,1}(X)$, we get a morphism $U \to M_{2,1}(X)$ and thus a rational map

$$\mathbf{M}_0 \dashrightarrow M_{2,1}(X). \tag{2}$$

In fact, the morphism $U \to M_{2,1}(X)$ can be extended slightly as follows: Suppose for instance $(E \to X_2, \dagger)$ is an element of $\mathbf{M}_0$. Let ρ be the normalization of X_2 only at $\dagger$ and π be the contraction of rational components. Put $\tilde{E} = \rho^* E$.



Sometimes the sheaf $(\pi_* \tilde{E}^\vee)^\vee$ is a stable bundle over X and the rational map extends to this point.

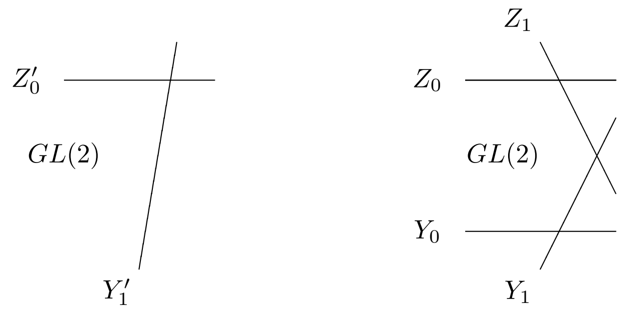
Gieseker proves that the indeterminacy locus of the rational map (2) is precisely a projective bundle $\mathbb{P}W^+$ over the product $B = Jac_0(X) \times Jac_1(X)$ of Jacobians where W^+ is a vector bundle over B. The normal bundle to $\mathbb{P}W^+$ is the pull-back of a vector bundle $W^- \to B$, tensored with $\mathcal{O}_{\mathbb{P}W^+}(-1)$. This is the typical situation for flips in the sense of [T96]. We blow up $\mathbf{M}_0$ along $\mathbb{P}W^+$ and then blow down along the $\mathbb{P}W^+$ direction in the exceptional divisor $\mathbb{P}W^+ \times_B \mathbb{P}W^-$. Let $\mathbf{M}_1$ be the result of this flip. What we achieve by this flip is that the rational map becomes a morphism

$$\mathbf{M}_1 \to M_{2,1}(X).$$

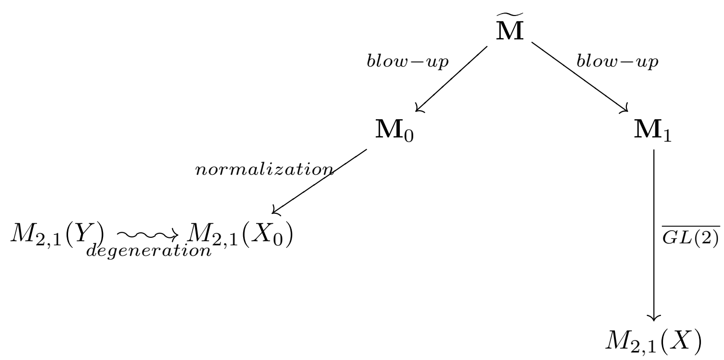
Moreover, the morphism is a fiber bundle over $M_{2,1}(X)$ with fiber $\overline{GL(2)}$, the wonderful compactification of $GL(2)$: The wonderful compactification of $GL(2)$ is constructed as follows. The complement of $GL(2)$ in

$$\mathbb{P}\left(\mathrm{End}(\mathbb{C}^2) \oplus \mathbb{C}\right)$$

consists of two divisors Z_0' (divisor at infinity) and Y_1' (zero locus of determinant). The divisor Y_1' is singular at 0 and so we blow up at 0. Let Y_0 be the exceptional divisor. Next we blow up along $Y_1' \cap Z_0'$ and let Z_1 denote the exceptional divisor. This is the wonderful compactification of $GL(2)$. Let Y_1 and Z_0 be the proper transforms of Y_1' and Z_0' respectively. Let $D = Y_0 + Z_0 + Y_1 + Z_1$.



We are ready to prove the vanishing of Chern classes by degeneration. We have the following diagram.



(3)

The proof is now reduced to a series of very concrete Chern class computations. We use the cotangent bundle Ω instead of the tangent bundle.

$$\begin{array}{lll} & c_i(\Omega_{M_{2,1}(X)}) = 0 & \text{for } i > 2(g-2) \\ \Rightarrow & c_i(\Omega_{\mathbf{M}_1}(\log D)) = 0 & \text{for } i > 2(g-1) \\ \Rightarrow & c_i(\Omega_{\widetilde{\mathbf{M}}}(\log \widetilde{D}) = 0 & \text{for } i > 2(g-1) \\ \Rightarrow & c_i(\Omega_{\mathbf{M}_0}(\log D)) = 0 & \text{for } i > 2(g-1) \\ \Rightarrow & c_i(\Omega_{M_{2,1}(Y)}) = 0 & \text{for } i > 2(g-1). \end{array}$$

5 Generalization

In this section, I will sketch our generalization of Gieseker's proof to higher rank cases. The details will appear in [KL].

We first need to construct a diagram like (3). Let us start with $\mathbf{M}_0$. Nagaraj and Seshadri [NS99] generalized Gieseker's construction of $\mathcal{M}_{2,1}(W)$ to higher rank cases. Using the well-known existence of Seshadri's quotient of the Quot scheme they verified the existence of the good quotient of the Hilbert scheme of curves in $X_0 \times Gr(p,r)$ and the family of moduli spaces satisfies the good properties described in §4. So we take $\mathbf{M}_0$ as the normalization of the central fiber of the family $\mathcal{M}_{r,d}(W)$.

Next, we expect $\mathbf{M}_1$ to be a fiber bundle over $M_{r,d}(X)$ whose fiber is $\overline{GL(r)}$, the wonderful compactification of $GL(r)$. The variety $\overline{GL(r)}$ is constructed from

$$\mathbb{P}\left(\mathrm{End}(\mathbb{C}^r) \oplus \mathbb{C}\right)$$

by blowing up $2(r-1)$ times along smooth subvarieties and the complement of $GL(r)$ consists of $2r$ smooth normal crossing divisors. See [Ka00] for an explicit description. We can define $\mathbf{M}_1$ as the blow-up of

$$\mathbb{P}\left(\mathrm{Hom}(\mathcal{E}|_{p_1}, \mathcal{E}|_{p_2}) \oplus \mathcal{O}\right)$$

along suitable smooth subvarieties.

Having defined $\mathbf{M}_0$ and $\mathbf{M}_1$, it is natural to ask whether we can define $\mathbf{M}_\alpha$ for $0 < \alpha < 1$. If possible, we can study their variations as α moves from 1 to 0 and relate $M_{r,d}(X)$ with $M_{r,d}(X_0)$ and then to $M_{r,d}(Y)$.

Let us define a stability condition for each α. Let E be a vector bundle over X_n of rank r with a marked node † and let $C_1, \cdots, C_n$ denote the rational components. For a subsheaf F of E, we define

$$\mathrm{rank}_\epsilon(F) = (1-\epsilon n)\cdot \mathrm{rank}(F|_X) + \epsilon \sum_{i=1}^{n} \mathrm{rank}(F|_{C_i})$$

$$r^\dagger(F) = \dim \mathrm{Im}(F|_\dagger \to E|_\dagger)$$

$$\mu_\alpha(F) = \frac{\chi(F) - \alpha r^\dagger(F)}{\mathrm{rank}_\epsilon(F)}.$$

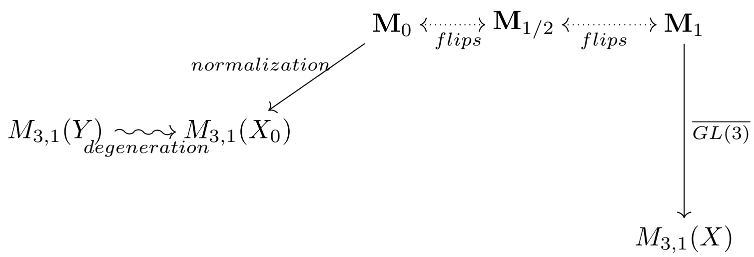
Definition 5.1. *We say $(E, \dagger)$ is α-(semi)stable if $\mu_\alpha(F) < \mu_\alpha(E)$ $(\leq)$ for any proper nonzero subsheaf F of E and arbitrarily small positive number ϵ.*

From now on, we focus on the rank 3 case. Since $M_{3,1}(Y) \cong M_{3,2}(Y)$ by the morphism $[E] \to [E^*]$ and tensoring a line bundle of degree 1, we only need to consider the case when $r = 3, d = 1$. Using the "stack of degeneration" from [L01], we prove in [KL] the following.

Theorem 5.2. *For $\alpha \in [0,1) - \{1/3, 2/3\}$, the set of α-stable vector bundles admits the structure of a proper separated smooth algebraic space $\mathbf{M}_\alpha$.*

In case $\alpha = 0$, $\mathbf{M}_0$ is the biholomorphic to the normalization of the central fiber of the family constructed by Nagaraj and Seshadri. If $2/3 < \alpha < 1$, $\mathbf{M}_\alpha$ is biholomorphic to $\mathbf{M}_1$.

Now we need to study the variation of $\mathbf{M}_\alpha$. By the stability condition, the moduli spaces $\mathbf{M}_\alpha$ vary only at 1/3 and 2/3. We prove in [KL] that $\mathbf{M}_{1/2}$ is obtained from $\mathbf{M}_1$ as the consequence of two flips and similarly $\mathbf{M}_0$ is the consequence of two flips from $\mathbf{M}_{1/2}$. The description is quite explicit and we have the following diagram.



It is a matter of some explicit Chern class computations to verify the vanishing result by induction on genus g although the computation is much more difficult than the rank 2 case.

Theorem 5.3. $c_i(M_{3,1}(Y)) = 0$ *for* $i > 6g - 5$.

As a result, we also have $c_i(M_{3,2}(Y)) = 0$ for $i > 6g - 5$.

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