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INTERSECTION COHOMOLOGY OF SYMPLECTIC QUOTIENTS BY CIRCLE ACTIONS

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Abstract

Let $T = U(1)$ and M be a Hamiltonian T-space with proper moment map $\mu : M \to \mathbb{R}$. When 0 is not a regular value of μ, the symplectic quotient $X = \mu^{-1}(0)/T$ is a singular stratified space. In this paper, we provide a description of the middle perversity intersection cohomology of X as a subspace of the equivariant cohomology $H_T^*(\mu^{-1}(0))$. Our approach is sheaf theoretic.

Contents

1. *Introduction*

Throughout this paper, T denotes the circle group $U(1)$. Let M be a Hamiltonian T-space with proper moment map $\mu : M \to \mathbb{R}$. When 0 is a regular value of μ, $Z = \mu^{-1}(0)$ is a smooth manifold and the quotient $X = Z/T$ has at worst orbifold singularities. In particular, the equivariant cohomology $H_T^*(Z)$ is isomorphic to the ordinary cohomology $H^*(X)$ which in turn is isomorphic to the middle perversity intersection cohomology $IH^*(X)$. By the Atiyah-Bott-Kirwan theory, the equivariant cohomology ring can be computed and hence we can compute the cohomology of the symplectic quotient X.

When 0 is not a regular value of μ, the equivariant cohomology $H_T^*(Z)$ is isomorphic to neither $H^*(X)$ nor $IH^*(X)$. So knowledge of $H_T^*(Z)$ does not directly enable us to compute either the ordinary cohomology or the intersection cohomology of X.

For singular stratified spaces, the middle perversity intersection cohomology has proven to be an important topological invariant. The purpose of this paper is to provide a simple recipe to compute the intersection cohomology of X when 0 is not a regular value of μ.

Let $F_1, \cdots, F_r$ be the T-fixed components in Z. For $1 \leq i \leq r$, the normal space

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of F_i decomposes into positive and negative weight spaces W_i^+, W_i^-. Let

$$d_i = \frac{1}{2}\min\{\dim W_i^+, \dim W_i^-\}.$$

Note that we have a canonical isomorphism $H_T^*(F_i) \cong H^*(F_i)\otimes H_T^*$. Our description of $IH^*(X)$ is as follows.

THEOREM 1.1. *(Theorem 7.1) The intersection cohomology $IH^*(X)$ is isomorphic to the (graded) subspace*

$$V^* = \{\eta \in H_T^*(Z) \mid \eta|_{F_i} \in H^*(F_i) \otimes H_T^{\leq 2d_i - 2}\}.$$

This gives us a very simple recipe to compute the intersection Betti numbers.

COROLLARY 1.2. *(Corollary 7.2) Let $P_t(F_i) = \sum t^k \dim H^k(F_i)$ and*

$$IP_t(X) = \sum_{k=0}^{\infty} t^k \dim IH^k(X)$$

$$P_t^T(Z) = \sum_{k=0}^{\infty} t^k \dim H_T^k(Z).$$

Then

$$IP_t(X) = P_t^T(Z) - \frac{1}{1-t^2}\sum_{1\leq i\leq r} t^{2d_i} P_t(F_i).$$

The intersection cohomology $IH^*(X)$ is equipped with a nondegenerate intersection pairing which is also a topological invariant. We can compute the intersection pairing in terms of the cup product structure of $H_T^*(Z)$.

THEOREM 1.3. *(Theorem 7.6) Let $\nu : IH^*(X) \to V^*$ be the isomorphism in Theorem 7.1. For two classes a, b in $IH^*(X)$ such that* $\deg a + \deg b = \dim X$ *we have*

$$\nu(a) \cup \nu(b) = \langle a, b\rangle\, \nu(e)$$

where e is the unique top degree class representing a point.

Our approach to intersection cohomology is sheaf theoretic, as developed in [**4**], [**2**]. Gysin morphims (§2) play a crucial role in this paper and all the above results are consequences of a decomposition of a sheaf complex (Theorem 6.2).

The circle case is special because the local structure near a singular point is easy to describe (§5). For groups other than $U(1)$, we don't have such a nice local description yet and so we don't know how to generalize our results to more complicated groups.

Now let us discuss some related work. For geometric invariant theory (GIT) quotients, Kirwan [**9**] invented a method to compute the Betti numbers of the intersection cohomology groups by using her partial desingularization [**8**]. However this method works only for the algebraic setting. In [**6**], a theorem similar to Theorem 1.1 is proved for GIT quotients by reductive groups under a technical assumption. (See also [**5**].)

Lerman and Tolman studied the circle quotients [**10**]. They constructed a small resolution by perturbing the moment map and then used this to give a very nice description of the intersection cohomology as a *quotient* of the equivariant cohomology ring $H_T^*(M)$ when M is compact. We provide a new proof of their theorem in §7 (Corollary 7.8).

Unless otherwise stated, all the intersection cohomology groups are with respect to the middle perversity. All the cohomology groups have rational coefficients.

2. *Gysin morphisms*

Let $\alpha : S \to M$ be an inclusion of a closed smooth submanifold into a smooth manifold. Suppose the normal bundle of S is equipped with an almost complex structure and the real codimension of S is $2d$. Then the Thom isomorphism

$$H^{*-2d}(S) \cong H^*(M, M-S)$$

composed with the natural map $H^*(M, M-S) \to H^*(M)$ gives us the *Gysin* homomorphism

$$H^{*-2d}(S) \to H^*(M) \tag{2.1}$$

which fits into the long exact sequence, called the *Gysin sequence*

$$\cdots \to H^{k-2d}(S) \to H^k(M) \to H^k(M-S) \to \cdots .$$

If we compose the Gysin homomorphism (2.1) with the restriction to S, we get

$$c_d : H^{*-2d}(S) \to H^*(S) \tag{2.2}$$

which is just multiplication by the top Chern class of the normal bundle of S.

Suppose $T = U(1)$ acts on M preserving S. Let ET be a contractible free T-space and consider the inclusion

$$\alpha^T : S_T \to M_T \tag{2.3}$$

where $S_T = ET \times_T S$ and $M_T = ET \times_T M$. This induces the *equivariant Gysin* homomorphism

$$H_T^{*-2d}(S) = H^{*-2d}(S_T) \to H^*(M_T) = H_T^*(M) \tag{2.4}$$

whose composition with the restriction $H_T^*(M) \to H_T^*(S)$ is multiplication by the equivariant top Chern class of the normal bundle

$$c_d^T : H_T^{*-2d}(S) \to H_T^*(S). \tag{2.5}$$

This equivariant Gysin homomorphism also fits into the *equivariant Gysin sequence*

$$\cdots \to H_T^{k-2d}(S) \to H_T^k(M) \to H_T^k(M-S) \to \cdots .$$

The fundamental observation for the Atiyah-Bott-Kirwan theory is the following.

Lemma 2.1. *([**1**] 13.4) Let F be a connected T-space on which T acts trivially. Let N be a complex T-vector bundle on F. Assume that the weights of the T-action on the fiber of N are all nonzero. Then the equivariant top Chern class of N is not a zero divisor.*

Suppose S equivariantly retracts onto a T-fixed connected closed submanifold F

in M. Assume that the weights of the action on the normal space of S at a point in F are all nonzero. Then we have $H^*_T(S) \cong H^*_T(F)$ and the Gysin homomorphism for the pair (M, S)

$$H_T^{*-2d}(F) \cong H_T^{*-2d}(S) \to H_T^*(M)$$

is injective because its composition with the restriction to F is injective by Lemma 2.1. Therefore the equivariant Gysin sequence splits into short exact sequences

$$0 \to H_T^{*-2d}(F) \to H_T^*(M) \to H_T^*(M-S) \to 0 \tag{2.6}$$

A useful observation is that the Gysin homomorphism arises from a morphism in the derived category $\mathcal{D}_c^+(M)$ of bounded below constructible sheaves on M. For definitions and basic results on derived category, see [**3**].

Lemma 2.2. *(cf. [**4**] 1.13 (15))* $\alpha^!\mathbb{Q}_M \cong \mathbb{Q}_S[-2d]$.

Proof. Let β denote the inclusion of the complement U of S. Then we have the distinguished triangle (see for instance [**2**] V 5.14)[†]

$$\alpha_!\alpha^!\mathbb{Q}_M \to \mathbb{Q}_M \to \beta_*\beta^*\mathbb{Q}_M \to \alpha_!\alpha^!\mathbb{Q}_M[1]. \tag{2.7}$$

Let $p \in S$. From the associated long exact sequence ([**2**] V 1.8 (7))

$$\cdots \to H^i\left((\alpha^!\mathbb{Q}_M)_p\right) \to H^i\left((\mathbb{Q}_M)_p\right) \to H^i\left((\beta_*\beta^*\mathbb{Q}_M)_p\right) \to \cdots$$

we see that $H^i\left((\alpha^!\mathbb{Q}_M)_p\right)$ is trivial unless $i = 2d$ and $H^{2d}\left((\alpha^!\mathbb{Q}_M)_p\right) \cong \mathbb{Q}$. By [**3**] III 5.2, we get the desired result. $\square$

Since α is a closed immersion, $\alpha_! = \alpha_*$. By composing the isomorphism in Lemma 2.2 with the adjunction morphism $\alpha_!\alpha^!\mathbb{Q}_M \to \mathbb{Q}_M$, we get the *Gysin morphism*

$$\alpha_*\mathbb{Q}_S[-2d] \cong \alpha_!\alpha^!\mathbb{Q}_M \to \mathbb{Q}_M$$

whose hypercohomology is the Gysin homomorphism (2.1). Furthermore, by (2.7) we have the distinguished triangle

$$\alpha_!\mathbb{Q}_S[-2d] \to \mathbb{Q}_M \to \beta_*\beta^*\mathbb{Q}_M \to \alpha_!\mathbb{Q}_S[-2d+1].$$

When T acts on M preserving S, consider the inclusion (2.3). The normal bundle of this embedding has real rank $2d$ and by Lemma 2.2 we get the *equivariant Gysin morphism*

$$(\alpha^T)_*\mathbb{Q}_{S_T}[-2d] \to \mathbb{Q}_{M_T} \tag{2.8}$$

whose hypercohomology is the equivariant Gysin homomorphism (2.4). This also fits into the distinguished triangle

$$\alpha_!^T\mathbb{Q}_{S_T}[-2d] \to \mathbb{Q}_{M_T} \to \beta_*^T(\beta^T)^*\mathbb{Q}_{M_T} \to \alpha_!^T\mathbb{Q}_{S_T}[-2d+1].$$

3. *Hamiltonian circle actions*

Let $T = U(1)$ and M be a Hamiltonian T-space with equivariant *proper* moment map

$$\mu : M \to \mathbb{R}.$$

[†]Following the widely used convention, we write f_* for the derived direct image Rf_*.

Let $Z = \mu^{-1}(0)$ and $X = Z/T$. Consider the gradient flow of $f = -\mu^2$ and define M^{ss} as the opens subset

$$\{p \in M \,|\, p \text{ retracts to a point in } Z \text{ by the gradient flow of } f\}.$$

Let $\phi : Z \to X$ be the quotient map and $\psi : M^{ss} \to X$ be the composition of the retraction to Z and ϕ.

Since Z is compact and T-fixed components are disjoint, there are only finitely many T-fixed components in Z, say $F_1, \cdots, F_r$. Each F_i is a symplectic submanifold and a neighborhood of F_i is T-equivariantly diffeomorphic to a neighborhood of the zero section of the normal bundle of F_i. Pick any $p_i \in F_i$ and let W_i be the normal space to F_i at p_i. Then W_i is a Hamiltonian T-vector space. By choosing a T-equivariant almost complex structure compatible with the symplectic form, we may assume that W_i is a complex vector space on which $T = U(1)$ acts unitarily.

It is well-known that a unitary action of $U(1)$ is completely reducible. Hence we can write

$$W_i = W_i^+ \oplus W_i^-$$

where W_i^+ (resp. W_i^-) is the positive (resp. negative) weight space. Define

$$d_i = \min\{\frac{1}{2}\dim W_i^+, \frac{1}{2}\dim W_i^-\} \quad \text{and} \quad e_i = \max\{\frac{1}{2}\dim W_i^+, \frac{1}{2}\dim W_i^-\}.$$

Let

$$S_i^+ = \{p \in M^{ss} \,|\, p \text{ retracts to a point in } F_i \text{ by the gradient flow of } -\mu\}$$

$$S_i^- = \{p \in M^{ss} \,|\, p \text{ retracts to a point in } F_i \text{ by the gradient flow of } \mu\}.$$

Then S_i^+ is a closed submanifold with codimension $\dim W_i^-$ and S_i^- is a closed submanifold with codimension $\dim W_i^+$. Let

$$S_i = \begin{cases} S_i^+ & \text{if } \dim W_i^- \le \dim W_i^+ \\ S_i^- & \text{otherwise.} \end{cases}$$

In other words, S_i is whichever of $\{S_i^+, S_i^-\}$ has larger dimension such that

$$\operatorname{codim} S_i = 2d_i.$$

Let $\alpha_i : S_i \hookrightarrow M^{ss}$ denote the inclusion.

Since S_i is a closed T-invariant submanifold of M^{ss}, we have the equivariant Gysin morphism (2.8) which fits into the distinguished triangle

$$(\alpha_i^T)_*\mathbb{Q}[-2d_i] \to \mathbb{Q}_{M_T^{ss}} \to (\beta_i^T)_*(\beta_i^T)^*\mathbb{Q}_{M_T^{ss}} \tag{3.1}$$

where α_i^T is the inclusion of $ET \times_T S_i$ in $M_T^{ss} = ET \times_T M^{ss}$ and β_i^T is the inclusion of its complement.

Since F_i is T-fixed in Z, F_i is mapped bijectively onto its image by $\psi : M^{ss} \to X$. By abuse of notation, we denote $\psi(F_i)$ by F_i and the inclusion of F_i in X by σ_i. Since the paths of the steepest descent for $|\mu|$ and μ^2 are identical up to parameterization, $\psi(S_i) = F_i$ and thus

$$\begin{tikzcd} S_i \arrow[r, "\alpha_i"] \arrow[d, "\psi_i"] & M^{ss} \arrow[d, "\psi"] \\ F_i \arrow[r, "\sigma_i"] & X. \end{tikzcd}$$

commutes.

Let $\psi^T : M^{ss}_T = ET \times_T M^{ss} \to X$ and $\psi^T_i : ET \times_T S_i \to F_i$ be the obvious maps induced from ψ and ψ_i respectively. We define the following objects in the derived category $\mathcal{D}^+_c(X)$:

$$\mathcal{C}^\bullet_T(X) = \psi^T_* \mathbb{Q}_{M^{ss}_T}$$

$$\mathcal{C}^\bullet_T(F_i) = (\sigma_i)_*(\psi^T_i)_* \mathbb{Q}_{ET\times_T S_i} = \psi^T_*(\alpha^T_i)_* \mathbb{Q}_{ET\times_T S_i}$$

whose hypercohomology groups are $H^*_T(M^{ss})$ and $H^*_T(S_i)$ respectively. By taking ψ^T_*, the Gysin morphisms in (3.1) give rise to a morphism

$$\delta : \bigoplus_i \mathcal{C}^\bullet_T(F_i)[-2d_i] \to \mathcal{C}^\bullet_T(X) \tag{3.2}$$

whose hypercohomology over X is just the sum of equivariant Gysin homomorphisms

$$\bigoplus_i H^{*-2d_i}_T(S_i) \to H^*_T(M^{ss}). \tag{3.3}$$

Since S_i equivariantly retracts to F_i, (3.3) is equivalent to a homomorphism

$$\delta' : \bigoplus_i H^{*-2d_i}_T(F_i) \to H^*_T(M^{ss}). \tag{3.4}$$

LEMMA 3.1. *δ' is injective.*

Proof. The normal space of S_i at a point in F_i is either W^+_i or W^-_i. In particular, the weights for the T-action are all nonzero. By Lemma 2.1, the equivariant Gysin homomorphism composed with the restriction to F_i

$$H^{*-2d_i}_T(F_i) \to H^*_T(M^{ss}) \to H^*_T(F_i)$$

is injective. By the exactness of the Gysin sequence, the composition

$$H^{*-2d_i}_T(F_i) \to H^*_T(M^{ss}) \to H^*_T(M^{ss} - S_i)$$

is zero. If $j \neq i$, then $F_j \subset M^{ss} - S_i$ and hence

$$H^{*-2d_i}_T(F_i) \to H^*_T(M^{ss}) \to H^*_T(F_j)$$

is zero. Therefore δ' composed with the sum of restrictions

$$\bigoplus_i H^{*-2d_i}_T(F_i) \to H^*_T(M^{ss}) \to \bigoplus_i H^*_T(F_i)$$

is the direct sum of the injective homomorphisms. Hence δ' is injective. □

4. *A decomposition of* $\mathcal{C}^\bullet_T(X)$

For $1 \leq i \leq r$, F_i is a subset of Z and we have a fiber square

$$\begin{tikzcd} F_i \arrow[r, hook, "\iota"] \arrow[d, "\phi_i"'] & Z \arrow[d, "\phi"] \\ F_i \arrow[r, hook, "\sigma_i"'] & X \end{tikzcd}$$

which induces the fiber square

$$\begin{tikzcd} ET \times_T F_i \arrow[r, hook, "\iota^T"] \arrow[d, "\phi_i^T"'] & ET \times_T Z \arrow[d, "\phi^T"] \\ F_i \arrow[r, hook, "\sigma_i"'] & X. \end{tikzcd}$$

Hence by [**4**] 1.13 (13), we have

$$(\sigma_i)^* \phi^T_* \mathbb{Q}_{ET\times_T Z} \cong (\phi_i^T)_* (\iota^T)^* \mathbb{Q}_{ET\times_T Z} \cong (\phi_i^T)_* \mathbb{Q}_{ET\times_T F_i}. \tag{4.1}$$

Notice that since S_i retracts to F_i by the gradient flow of the T-equivariant function μ or $-\mu$, the inclusion

$$\begin{tikzcd} ET \times_T F_i \arrow[rr, hook] \arrow[dr, "\phi_i^T"'] & & ET \times_T S_i \arrow[dl, "\psi_i^T"] \\ & F_i & \end{tikzcd}$$

is a homotopy equivalence and hence

$$\mathcal{C}^\bullet_T(F_i) = (\sigma_i)_* (\psi_i^T)_* \mathbb{Q}_{ET\times_T S_i} \cong (\sigma_i)_* (\phi_i^T)_* \mathbb{Q}_{ET\times_T F_i}. \tag{4.2}$$

Similarly, as M^{ss} is T-equivariantly homotopy equivalent to Z,

$$\mathcal{C}^\bullet_T(X) = (\psi^T)_* \mathbb{Q}_{M^{ss}_T} \cong (\phi^T)_* \mathbb{Q}_{ET\times_T Z}. \tag{4.3}$$

Therefore, by (4.1), (4.2) and (4.3), we get

$$(\sigma_i)_*(\sigma_i)^* \mathcal{C}^\bullet_T(X) \cong (\sigma_i)_*(\sigma_i)^*(\phi^T)_* \mathbb{Q}_{ET\times_T Z} \cong (\sigma_i)_*(\phi_i^T)_* \mathbb{Q}_{ET\times_T F_i} \cong \mathcal{C}^\bullet_T(F_i)$$

When composed with the adjunction morphism

$$\mathcal{C}^\bullet_T(X) \to (\sigma_i)_*(\sigma_i)^* \mathcal{C}^\bullet_T(X),$$

this gives us the morphism

$$\mathcal{C}^\bullet_T(X) \to \mathcal{C}^\bullet_T(F_i) \tag{4.4}$$

Let $\mathcal{C}^\bullet_T(F_i) \to \tau^{\geq 2d_i} \mathcal{C}^\bullet_T(F_i)$ be the truncation morphism ([**4**] 1.14). Then (4.4) composed with the truncation morphism gives us the morphism

$$\rho_i : \mathcal{C}^\bullet_T(X) \to \tau^{\geq 2d_i} \mathcal{C}^\bullet_T(F_i) \tag{4.5}$$

Adding up, we get

$$\rho = \oplus \rho_i : \mathcal{C}^\bullet_T(X) \to \bigoplus_{i=1}^{r} \tau^{\geq 2d_i} \mathcal{C}^\bullet_T(F_i). \tag{4.6}$$

Recall that we have the Gysin morphism δ defined in (3.2).

Proposition 4.1. *$\rho \circ \delta$ is an isomorphism in $\mathcal{D}^+_c(X)$.*

Proof. It suffices to show that $\rho\circ\delta$ induces an isomorphism on stalk cohomology. For $x \in X - \cup F_i$, there is nothing to prove since $\oplus_{i=1}^r \mathcal{C}^\bullet_T(F_i)$ is supported on $\cup F_i$. Let $x \in F_i$. The fiber of

$$\phi_i^T : ET \times_T F_i = BT \times F_i \to F_i$$

is the classifying space $BT = ET/T$ whose cohomology, denoted by H_T^*, is isomorphic to the polynomial ring $\mathbb{Q}[t]$ with $\deg t = 2$. Hence the stalk cohomology of $\mathcal{C}^\bullet_T(F_i)$ and $\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)$ are $H_T^* \cong \mathbb{Q}[t]$ and $H_T^{\geq 2d_i} \cong \mathbb{Q}[t]/\mathrm{span}(1, t, \cdots, t^{d_i-1})$ respectively. The induced homomorphism on stalk cohomology

$$H_T^* \cong \mathbb{Q}[t] \to \mathbb{Q}[t]/\mathrm{span}(1, t, \cdots, t^{d_i-1}) \cong H_T^{\geq 2d_i}$$

is the result of the equivariant Gysin homomorphism for the embedding of 0 into the normal space of S_i, followed by restriction to 0. Hence this is just the multiplication by the product of all weights on the normal space of S_i which is a nonzero multiple of t^{d_i}. This is obviously an isomorphism of $\mathbb{Q}$-vector spaces. So we are done. $\square$

In particular, ρ induces a surjection on stalk cohomology. An immediate consequence is the following decomposition.

Corollary 4.2. *Let $\mathcal{A}^\bullet$ be an object in the triangulated category $\mathcal{D}_c^+(X)$ which fits into the distinguished triangle*

$$\mathcal{A}^\bullet \xrightarrow{\theta} \mathcal{C}^\bullet_T(X) \xrightarrow{\rho} \bigoplus_{i=1}^r \tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i) \longrightarrow \mathcal{A}^\bullet[1].$$

Then

$$\mathcal{A}^\bullet \oplus \left(\bigoplus_{i=1}^r \mathcal{C}^\bullet_T(F_i)[-2d_i]\right) \cong \mathcal{C}^\bullet_T(X).$$

Proof. The morphisms θ and δ give us a morphism

$$\mathcal{A}^\bullet \oplus \left(\bigoplus_{i=1}^r \mathcal{C}^\bullet_T(F_i)[-2d_i]\right) \to \mathcal{C}^\bullet_T(X).$$

This induces an isomorphism on every stalk cohomology and hence we get the isomorphism. $\square$

Such an $\mathcal{A}^\bullet$ always exists because we can simply take the mapping cone of ρ translated by -1 ([**2**] V 5.2). We shall see that $\mathcal{A}^\bullet$ is isomorphic to the intersection cohomology sheaf $\mathcal{IC}^\bullet{}_X$ of X.

5. *Local study of circle quotients*

Recall that, for $1 \leq i \leq r$, W_i is the normal space of F_i in M^{ss} at a point and we have a decomposition into positive and negative weight spaces

$$W_i = W_i^+ \oplus W_i^- \ .$$

By the local normal form theorem ([**12**] 7.4), the normal cone of F_i in X is the symplectic quotient $W_i /\!\!/ T$. As remarked in §3, we may assume $T = U(1)$ acts unitarily so that the symplectic quotient $W_i /\!\!/ T$ is homeomorphic to the good quotient $W_i /\!\!/ \mathbb{C}^*$ in geometric invariant theory [**7**].

Let $\gamma_i : W_i \to W_i /\!\!/ \mathbb{C}^*$ be the quotient map. It is well-known ([**11**] Proposition 2.2) that $\gamma_i^{-1}(\gamma_i(0))$ is the affine cone over the set of unstable points in $\mathbb{P}W_i$ which is exactly $\mathbb{P}W_i^+ \cup \mathbb{P}W_i^-$ by the Hilbert-Mumford criterion ([**11**] Chapter 2 §1). Hence $\gamma_i^{-1}(\gamma_i(0)) = W_i^+ \cup W_i^-$ and thus

$$W_i /\!\!/ \mathbb{C}^* - \gamma_i(0) = \left(W_i - (W_i^+ \cup W_i^-)\right)/\mathbb{C}^*.$$

Since $\mathbb{C}^*$ acts locally freely on $W_i - (W_i^+ \cup W_i^-)$, we get

$$\begin{aligned} H^*\left(W_i /\!\!/ \mathbb{C}^* - \gamma_i(0)\right) &\cong H^*_{\mathbb{C}^*}\left(W_i - (W_i^+ \cup W_i^-)\right) \\ &\cong H^*_T\left(W_i - (W_i^+ \cup W_i^-)\right). \end{aligned} \tag{5.1}$$

LEMMA 5.1. *Let $m_i = d_i + e_i - 2 = \frac{1}{2}\mathrm{codim}_X F_i - 1$. Then we have*

$$H^{\leq m_i}\left(W_i /\!\!/ \mathbb{C}^* - \gamma_i(0)\right) \cong H_T^{\leq 2d_i - 2}.$$

Proof. Without loss of generality we may assume $\dim W_i^+ \leq \dim W_i^-$. Consider first the equivariant Gysin sequence for the pair (W_i, W_i^-):

$$\cdots \longrightarrow H_T^{k-2d_i}(W_i^-) \xrightarrow{g} H_T^k(W_i) \xrightarrow{h} H_T^k(W_i - W_i^-) \longrightarrow \cdots$$

Since W_i and W_i^- are contractible, $H_T^*(W_i) \cong \mathbb{Q}[t]$ and $H_T^*(W_i^-) \cong \mathbb{Q}[t]$. The composition of the equivariant Gysin map g with the restriction to W_i^- is multiplication by the equivariant top Chern class which is just a nonzero multiple, say at^{d_i}, of t^{d_i}. Hence g is injective and the Gysin sequence gives us the short exact sequence

$$0 \longrightarrow \mathbb{Q}[t] \xrightarrow{at^{d_i}} \mathbb{Q}[t] \xrightarrow{h} H_T^*(W_i - W_i^-) \longrightarrow 0.$$

This implies that $H_T^*(W_i - W_i^-) \cong \mathbb{Q}[t]/(t^{d_i}) \cong H_T^{\leq 2d_i - 2}$. In particular, since $m_i \geq 2d_i - 2$,

$$H_T^{\leq m_i}(W_i - W_i^-) \cong H_T^{\leq 2d_i - 2}. \tag{5.2}$$

Since $\mathrm{codim}_{W_i} W_i^+ = 2e_i \geq d_i + e_i \geq m_i + 2$, we deduce from the Gysin sequence for the pair $(W_i - W_i^-, W_i^+ - 0)$ that

$$H_T^{\leq m_i}(W_i - W_i^-) \cong H_T^{\leq m_i}\left(W_i - (W_i^+ \cup W_i^-)\right). \tag{5.3}$$

The lemma follows from (5.1), (5.2) and (5.3). $\square$

6. *The intersection cohomology sheaf*

The middle perversity intersection cohomology of $X = Z/T$ is the hypercohomology of an object $\mathcal{IC}^\bullet{}_X$ in $\mathcal{D}_c^+(X)$ satisfying three axioms, called normalization, the support, and cosupport conditions [**4**] §4. In our case, $U := X - \cup_i F_i$ is a homology manifold because it is a symplectic orbifold and the axioms can be rephrased as follows. Let $\mathcal{A}^\bullet$ be an object in $\mathcal{D}_c^+(X)$ such that $\tau^{\geq 0}\mathcal{A}^\bullet \cong \mathcal{A}^\bullet$. Then $\mathcal{A}^\bullet$ is isomorphic to the intersection cohomology sheaf $\mathcal{IC}^\bullet{}_X$ if it satisfies

(1) normalization: $\mathcal{A}^\bullet|_U \cong \mathbb{Q}_U$
(2) the support condition: for $x \in F_i$, $H^{>m_i}(\mathcal{A}^\bullet{}_x) = 0$ where
$$m_i = \frac{1}{2}\mathrm{codim}_X F_i - 1 = d_i + e_i - 2$$
(3) the cosupport condition: for $x \in F_i$, the adjunction map
$$H^{\leq m_i}(\mathcal{A}^\bullet{}_x) \to H^{\leq m_i}(j_* j^* \mathcal{A}^\bullet{}_x)$$
is an isomorphism where $j : U \hookrightarrow X$ is the inclusion.

LEMMA 6.1. *The object $\mathcal{A}^\bullet$ in Corollary 4.2 satisfies the above axioms. Hence, $\mathcal{A}^\bullet \cong \mathcal{IC}^\bullet{}_X$.*

Proof. Recall that $\mathcal{A}^\bullet$ fits into the distinguished triangle

$$\mathcal{A}^\bullet \longrightarrow \mathcal{C}^\bullet_T(X) \xrightarrow{\rho} \bigoplus_{i=1}^r \tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i). \tag{6.1}$$

(1) normalization: $\bigoplus_{i=1}^r \tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)$ is trivial on U by definition and hence $\mathcal{A}^\bullet|_U \cong \mathcal{C}^\bullet_T(X)|_U$. But T acts locally freely on the smooth manifold $Z - \cup F_i$, so the fibers of

$$ET \times_T (Z - \cup F_i) \to U$$

are the classifying spaces BF for some finite groups F whose rational cohomology are $\mathbb{Q}$ by Macdonald's theorem. Since U is locally contractible, we deduce that $\mathcal{C}^\bullet_T(X)|_U \cong \mathbb{Q}_U$.

(2) support condition: let $x \in F_i$. The distinguished triangle (6.1) gives rise to a long exact sequence

$$\cdots \to H^k(\mathcal{A}^\bullet{}_x) \to H^k(\mathcal{C}^\bullet_T(X)_x) \to H^k(\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)_x) \to \cdots \tag{6.2}$$

The middle term is $H^*(\mathcal{C}^\bullet_T(X)_x) \cong H^*_T \cong \mathbb{Q}[t]$, and the third term is

$$H^*(\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)_x) \cong H_T^{\geq 2d_i} \cong \mathbb{Q}[t]/\mathrm{span}(1, t, \cdots, t^{d_i-1}).$$

The map from the middle to the third in (6.2) is truncation. Hence $H^{>2d_i-2}(\mathcal{A}^\bullet{}_x) = 0$. Since $m_i = d_i + e_i - 2 \geq 2d_i - 2$, we get $H^{>m_i}(\mathcal{A}^\bullet{}_x) = 0$.

(3) cosupport condition: from (6.2), we also deduce that

$$H^*(\mathcal{A}^\bullet{}_x) \cong H^{\leq 2d_i-2}(\mathcal{C}^\bullet_T(X)_x) \cong H_T^{\leq 2d_i-2}.$$

In particular, $H^{\leq m_i}(\mathcal{A}^\bullet{}_x) \cong H_T^{\leq 2d_i-2}$. On the other hand, $j^*\mathcal{A}^\bullet = \mathcal{A}^\bullet|_U \cong \mathbb{Q}_U$ by the normalization condition. Hence

$$H^{\leq m_i}(j_*j^*\mathcal{A}^\bullet{}_x) \cong H^{\leq m_i}\left(\mathbb{R}^{\dim F_i} \times (W_i/\!\!/\mathbb{C}^* - \gamma_i(0))\right) \cong H^{\leq m_i}(W_i/\!\!/\mathbb{C}^* - \gamma_i(0)),$$

which is naturally isomorphic to $H_T^{\leq 2d_i-2}$ by Lemma 5.1. Therefore

$$H^{\leq m_i}(\mathcal{A}^\bullet{}_x) \cong H_T^{\leq 2d_i-2} \cong H^{\leq m_i}(j_*j^*\mathcal{A}^\bullet{}_x).$$

The fact that this isomorphism is equal to the adjunction map is an easy exercise.

Consequently, $\mathcal{A}^\bullet$ satisfies all three axioms and hence $\mathcal{A}^\bullet \cong \mathcal{IC}^\bullet{}_X$. □

The main theorem of this paper now follows from Corollary 4.2.

THEOREM 6.2. $\mathcal{IC}^\bullet{}_X \oplus (\bigoplus_{i=1}^r \mathcal{C}^\bullet_T(F_i)[-2d_i]) \cong \mathcal{C}^\bullet_T(X)$

7. *Cohomological consequences*

Theorem 6.2 enables us to describe the middle perversity intersection cohomology either as a subspace (Theorem 7.1) or as a quotient of $H^*_T(M^{ss})$ (Theorem 7.7).

7.1. *Subspace description*

Notice that the hypercohomology $\mathbb{H}^*(\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i))$ of $\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)$ is

$$H^*(F_i) \otimes H_T^{\geq 2d_i}$$

since the fiber of $\phi_i^T : ET \times_T F_i = BT \times F_i \to F_i$ is BT. By Lemma 6.1 and (6.1), we have the distinguished triangle

$$\mathcal{IC}^\bullet{}_X \longrightarrow \mathcal{C}^\bullet_T(X) \xrightarrow{\rho} \bigoplus_i \tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i). \tag{7.1}$$

This gives us a long exact sequence in hypercohomology

$$\cdots \to IH^k(X) \to H^k_T(M^{ss}) \to \bigoplus_i \mathbb{H}^k(\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i)) \to \cdots$$

By Proposition 4.1, $H^k_T(M^{ss}) \to \bigoplus_i \mathbb{H}^k(\tau^{\geq 2d_i}\mathcal{C}^\bullet_T(F_i))$ is surjective and thus the long exact sequence splits into short exact sequences

$$0 \to IH^*(X) \to H^*_T(M^{ss}) \to \bigoplus_i H^*(F_i) \otimes H_T^{\geq 2d_i} \to 0. \tag{7.2}$$

So we have proved the following.

Theorem 7.1. *The intersection cohomology $IH^*(X)$ is isomorphic to the (graded) subspace*

$$V^* = \{\eta \in H^*_T(M^{ss}) \mid \eta|_{F_i} \in H^*(F_i) \otimes H_T^{\leq 2d_i - 2}\}.$$

This theorem gives us an efficient way to compute the intersection Betti numbers. By the equivariant Morse theory of Kirwan [**7**], we can compute the equivariant Poincaré series

$$P^T_t(M^{ss}) = \sum_{k=0}^{\infty} t^k \dim H^k_T(M^{ss})$$

of the equivariant cohomology. In terms of this, we can easily compute the intersection Betti numbers as follows.

Corollary 7.2. *Let $IP_t(X) = \sum t^k \dim IH^k(X)$ and $P_t(F) = \sum t^k \dim H^k(F)$. Then*

$$IP_t(X) = P^T_t(M^{ss}) - \frac{1}{1-t^2} \sum_{1 \leq i \leq r} t^{2d_i} P_t(F_i).$$

Remark 7.3. Since M^{ss} retracts onto Z equivariantly, $H^*_T(M^{ss}) \cong H^*_T(Z)$. Hence we can replace M^{ss} by Z in Theorem 7.1 and Corollary 7.2.

Example 7.4. We consider a linear circle action on the projective space $\mathbb{P}^n$. Let p, q and s denote the number of positive, negative, and zero weights respectively, so that $n = p + q + s - 1$. Let us assume that $p \leq q$ (the other case being entirely similar). Using equivariant Morse theory, we get

$$\begin{aligned} P^T_t(M^{ss}) &= (P_t(\mathbb{P}^n) - t^{2q+2s}P_t(\mathbb{P}^{p-1}) - t^{2p+2s}P_t(\mathbb{P}^{q-1}))/(1-t^2) \\ &= (1 + t^2 + \cdots + t^{2p+2s-2} - t^{2q+2s} - \cdots - t^{2n})/(1-t^2). \end{aligned}$$

Hence, by the above corollary, the intersection Poincaré polynomial is

$$P^T_t(M^{ss}) - \frac{t^{2p}P_t(\mathbb{P}^{s-1})}{1-t^2} = \frac{(1-t^{2p})(1-t^{2q+2s})}{(1-t^2)^2}$$

which is a palindromic polynomial of degree $2n - 2$.

7.2. *Intersection pairing*

Since X is compact there is an intersection pairing

$$IH^*(X) \otimes IH^*(X) \to \mathbb{Q}.$$

This arises from a morphism

$$\mathcal{IC}^{\bullet \otimes 2}_X \to {}^t\mathcal{IC}^\bullet{}_X \tag{7.3}$$

where the latter is the top perversity intersection cohomology sheaf. Since $\mathcal{IC}^\bullet{}_X$ is a direct summand of $\mathcal{C}^\bullet_T(X)$ by Theorem 6.2, there is a second morphism

$$\mathcal{IC}^{\bullet \otimes 2}_X \to \mathcal{C}^\bullet_T(X)^{\otimes 2} \to \mathcal{C}^\bullet_T(X) \to \mathcal{IC}^\bullet{}_X \to {}^t\mathcal{IC}^\bullet{}_X \tag{7.4}$$

where the morphism $\mathcal{C}^\bullet_T(X)^{\otimes 2} \to \mathcal{C}^\bullet_T(X)$ is the morphism coming from the obvious $\mathbb{Q} \otimes \mathbb{Q} \to \mathbb{Q}$ over $ET \times_T M^{ss}$. This gives us the cup product structure of $H^*_T(M^{ss})$.

Lemma 7.5. *The morphisms* (7.3) *and* (7.4) *are the same.*

Proof. Both are extensions of the obvious $\mathbb{Q} \otimes \mathbb{Q} \to \mathbb{Q}$ over $U = X - \cup F_i$. By [**2**] V 9.14, such an extension is unique. □

Let

$$\nu : IH^*(X) \xrightarrow{\cong} V^* \subset H^*_T(M^{ss})$$

be the isomorphism from Theorem 7.1. Let a, b be two classes in $IH^*(X)$ such that

$$\deg a + \deg b = \dim X = \dim F_i + 2d_i + 2e_i - 2 \geq \dim F_i + 4d_i - 2$$

for every i. By the definition of V^*, $\nu(a)|_{F_i}$ and $\nu(b)|_{F_i}$ lie in $H^*(F_i) \otimes H_T^{\leq 2d_i - 2}$. Hence

$$\nu(a) \cup \nu(b)|_{F_i} \in H^{> \dim F_i}(F_i) \otimes H_T^{\leq 4d_i - 4}$$

which must be zero since $H^{> \dim F_i}(F_i) = 0$. Consequently,

$$\nu(a) \cup \nu(b) \in V^{\dim X} \cong IH^{\dim X}(X).$$

Since X is connected, $IH^{\dim X}(X)$ is one dimensional, generated by the unique class, denoted by e, representing one point. Hence we can write

$$\nu(a) \cup \nu(b) = l\, \nu(e) \tag{7.5}$$

for some rational number l. We claim l is the intersection number $\langle a, b\rangle$ on $IH^*(X)$.

From (7.4) and Lemma 7.5, the intersection pairing is given by

$$IH^*(X)^{\otimes 2} \to H^*_T(M^{ss})^{\otimes 2} \to H^*_T(M^{ss}) \to IH^*(X) \to \mathbb{Q}.$$

By (7.5), if we start with $a \otimes b$, we get l. Therefore, $l = \langle a, b\rangle$. So we have proved the following theorem that enables us to compute the intersection pairing in terms of the cup product of $H^*_T(M^{ss})$.

Theorem 7.6. *For two classes a, b in $IH^*(X)$ such that* $\deg a + \deg b = \dim X$ *we have*

$$\nu(a) \cup \nu(b) = \langle a, b\rangle\, \nu(e).$$

7.3. *Quotient description*

Now we consider the quotient description of $IH^*(X)$. By Theorem 6.2 we have a distinguished triangle

$$\bigoplus_{i=1}^r \mathcal{C}^\bullet_T(F_i)[-2d_i] \xrightarrow{\delta} \mathcal{C}^\bullet_T(X) \longrightarrow \mathcal{IC}^\bullet{}_X \tag{7.6}$$

Upon taking hypercohomology we obtain a short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^r H_T^{*-2d_i}(F_i) \xrightarrow{\delta'} H_T^*(M^{ss}) \longrightarrow IH^*(X) \longrightarrow 0 \tag{7.7}$$

because δ' is injective by Lemma 3.1. Hence $IH^*(X)$ is the quotient of $H_T^*(M^{ss})$ by the images of the equivariant Gysin maps for S_i. The exact sequence (2.6) applied for each S_i, one by one, now tells us the following.

THEOREM 7.7. $IH^*(X) \cong H_T^*(M^{ss})/\operatorname{im}\delta' \cong H_T^*(M^{ss} - \bigcup_i S_i)$

Suppose M is compact. Let $\mathfrak{F}^+$ be the set of T-fixed components in M such that either

(1) $\mu(F) > 0$ or
(2) $F = F_i$ for some i and $\dim W_i^- \leq \dim W_i^+$.

Let $\mathfrak{F}^-$ be the set of all other T-fixed components.

For $F \in \mathfrak{F}^+$, let S_F be the set of points in M which retract to F by the gradient flow of $-\mu$, i.e. S_F is the stable manifold for μ. If $F \in \mathfrak{F}^-$, let S_F denote the unstable manifold for μ, i.e. S_F is the set of points in M which retract to F by the gradient flow of μ. From [**7**], we see that

$$M - M^{ss} = \bigcup_{\mu(F)\neq 0} S_F$$

and thus

$$M - \bigcup_F S_F = M^{ss} - \bigcup_i S_i.$$

Since there are only finitely many fixed components, we can apply (2.6) for each F in order of decreasing absolute value of μ. Therefore, we get a short exact sequence

$$0 \to \bigoplus_F H_T^{*-2d_F}(F) \to H_T^*(M) \to H_T^*(M^{ss} - \bigcup_{i=1}^r S_i) \to 0 \tag{7.8}$$

where $d_F = \frac{1}{2}\operatorname{codim} S_F$. From (7.8) and Theorem 7.7, together with the abelian localization theorem

$$H_T^*(M) \hookrightarrow \bigoplus_F H_T^*(F)$$

we can deduce a theorem of Lerman and Tolman ([**10**] Theorem 1').

COROLLARY 7.8. *[**10**] As a graded ring, $IH^*(X)$ is isomorphic to $H_T^*(M)/K$ where*

$$K = \{\eta \in H_T^*(M) \,|\, \eta|_F = 0 \ \forall\ F \in \mathfrak{F}^+\} \oplus \{\eta \in H_T^*(M) \,|\, \eta|_F = 0 \ \forall\ F \in \mathfrak{F}^-\}.$$

Given the above, the proof is identical to the proof of Theorem 1 in [**13**] for smooth quotients, so we omit it.

Remark 7.9. By Theorem 7.7 and Corollary 7.8, $IH^*(X)$ is equipped with a ring structure but this is not canonical. In the notation of §3, when $\dim S_i^+ = \dim S_i^-$ for some T-fixed component F_i in Z, we could choose either S_i^+ or S_i^- as our S_i. This gives us two different ring structures. For example, consider the action of $\mathbb{C}^*$ on $M = \mathbb{P}^4$ by

$$\lambda \cdot (a_0 : a_1 : \cdots : a_4) = (a_0 : \lambda a_1 : \lambda a_2 : \lambda^{-1} a_3 : \lambda^{-1} a_4).$$

Let S^+ (resp. S^-) be the stable (resp. unstable) manifold by the gradient flow of the moment map, for the fixed point $(1 : 0 : \cdots : 0)$. Then the $\mathbb{C}^*$-orbit spaces of $M^{ss} - S^+$ and $M^{ss} - S^-$ respectively give us two small resolutions and the natural isomorphisms

$$H_T^*(M^{ss} - S^+) \cong IH^*(M/\!\!/T) \cong H_T^*(M^{ss} - S^-)$$

give us two different ring structures. See [**2**] IX Example 1, page 221.

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