INTERSECTION COHOMOLOGY OF SYMPLECTIC QUOTIENTS BY CIRCLE ACTIONS

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Abstract

Let \( T = U(1) \) and \( M \) be a Hamiltonian \( T \)-space with proper moment map \( \mu : M \to \mathbb{R} \). When 0 is not a regular value of \( \mu \), the symplectic quotient \( X = \mu^{-1}(0)/T \) is a singular stratified space. In this paper, we provide a description of the middle perversity intersection cohomology of \( X \) as a subspace of the equivariant cohomology \( H^*_T(\mu^{-1}(0)) \). Our approach is sheaf theoretic.

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1. Introduction

Throughout this paper, \( T \) denotes the circle group \( U(1) \). Let \( M \) be a Hamiltonian \( T \)-space with proper moment map \( \mu : M \to \mathbb{R} \). When 0 is a regular value of \( \mu \), \( Z = \mu^{-1}(0) \) is a smooth manifold and the quotient \( X = Z/T \) has at worst orbifold singularities. In particular, the equivariant cohomology \( H^*_T(Z) \) is isomorphic to the ordinary cohomology \( H^*(X) \) which in turn is isomorphic to the middle perversity intersection cohomology \( IH^*(X) \). By the Atiyah-Bott-Kirwan theory, the equivariant cohomology ring can be computed and hence we can compute the cohomology of the symplectic quotient \( X \).

When 0 is not a regular value of \( \mu \), the equivariant cohomology \( H^*_T(Z) \) is isomorphic to neither \( H^*(X) \) nor \( IH^*(X) \). So knowledge of \( H^*_T(Z) \) does not directly enable us to compute either the ordinary cohomology or the intersection cohomology of \( X \).

For singular stratified spaces, the middle perversity intersection cohomology has proven to be an important topological invariant. The purpose of this paper is to provide a simple recipe to compute the intersection cohomology of \( X \) when 0 is not a regular value of \( \mu \).

Let \( F_1, \ldots, F_r \) be the \( T \)-fixed components in \( Z \). For \( 1 \leq i \leq r \), the normal space...
of $F_i$ decomposes into positive and negative weight spaces $W_i^+, W_i^-$. Let
\[ d_i = \frac{1}{2} \min\{\dim W_i^+, \dim W_i^-\} \]
Note that we have a canonical isomorphism $H^*_T(F_i) \cong H^*(F_i) \otimes H^*_T$. Our description of $IH^*(X)$ is as follows.

**Theorem 1.1.** (Theorem 7.1) The intersection cohomology $IH^*(X)$ is isomorphic to the (graded) subspace
\[ V^* = \{ \eta \in H^*_T(Z) \mid \eta|_{F_i} \in H^*(F_i) \otimes H^*_{\leq 2d_i - 2} \}. \]
This gives us a very simple recipe to compute the intersection Betti numbers.

**Corollary 1.2.** (Corollary 7.2) Let $P_t(F_i) = \sum t^k \dim H^k(F_i)$ and
\[ IP_t(X) = \sum_{k=0}^{\infty} t^k \dim IH^k(X) \]
\[ PT_t(Z) = \sum_{k=0}^{\infty} t^k \dim H^k_T(Z). \]
Then
\[ IP_t(X) = PT_t(Z) - \frac{1}{1-t^2} \sum_{1 \leq i \leq r} t^{2d_i} P_t(F_i). \]
The intersection cohomology $IH^*(X)$ is equipped with a nondegenerate intersection pairing which is also a topological invariant. We can compute the intersection pairing in terms of the cup product structure of $H^*_T(Z)$.

**Theorem 1.3.** (Theorem 7.6) Let $\nu : IH^*(X) \to V^*$ be the isomorphism in Theorem 7.1. For two classes $a, b$ in $IH^*(X)$ such that $\deg a + \deg b = \dim X$ we have
\[ \nu(a) \cup \nu(b) = \langle a, b \rangle \nu(e) \]
where $e$ is the unique top degree class representing a point.

Our approach to intersection cohomology is sheaf theoretic, as developed in [4], [2], Gysin morphims (§2) play a crucial role in this paper and all the above results are consequences of a decomposition of a sheaf complex (Theorem 6.2).

The circle case is special because the local structure near a singular point is easy to describe (§5). For groups other than $U(1)$, we don’t have such a nice local description yet and so we don’t know how to generalize our results to more complicated groups.

Now let us discuss some related work. For geometric invariant theory (GIT) quotients, Kirwan [9] invented a method to compute the Betti numbers of the intersection cohomology groups by using her partial desingularization [8]. However this method works only for the algebraic setting. In [6], a theorem similar to Theorem 1.1 is proved for GIT quotients by reductive groups under a technical assumption. (See also [5].)
Lerman and Tolman studied the circle quotients [10]. They constructed a small resolution by perturbing the moment map and then used this to give a very nice description of the intersection cohomology as a quotient of the equivariant cohomology ring $H_T^*(M)$ when $M$ is compact. We provide a new proof of their theorem in §7 (Corollary 7.8).

Unless otherwise stated, all the intersection cohomology groups are with respect to the middle perversity. All the cohomology groups have rational coefficients.

\section{Gysin morphisms}

Let $\alpha : S \to M$ be an inclusion of a closed smooth submanifold into a smooth manifold. Suppose the normal bundle of $S$ is equipped with an almost complex structure and the real codimension of $S$ is $2d$. Then the Thom isomorphism

$$H^{*-2d}(S) \cong H^*(M, M - S)$$

composed with the natural map $H^*(M, M - S) \to H^*(M)$ gives us the Gysin homomorphism

$$H^{*-2d}(S) \to H^*(M)$$

which fits into the long exact sequence, called the Gysin sequence

$$\cdots \to H^{k-2d}(S) \to H^k(M) \to H^k(M - S) \to \cdots .$$

If we compose the Gysin homomorphism (2.1) with the restriction to $S$, we get

$$c_d : H^{*-2d}(S) \to H^*(S)$$

which is just multiplication by the top Chern class of the normal bundle of $S$.

Suppose $T = U(1)$ acts on $M$ preserving $S$. Let $ET$ be a contractible free $T$-space and consider the inclusion

$$\alpha^T : S_T \to M_T$$

where $S_T = ET \times_T S$ and $M_T = ET \times_T M$. This induces the equivariant Gysin homomorphism

$$H_T^{*-2d}(S) = H^{*-2d}(S_T) \to H^*(M_T) = H_T^*(M)$$

whose composition with the restriction $H_T^*(M) \to H_T^+(S)$ is multiplication by the equivariant top Chern class of the normal bundle

$$c_d^T : H_T^{*-2d}(S) \to H_T^+(S).$$

This equivariant Gysin homomorphism also fits into the equivariant Gysin sequence

$$\cdots \to H_T^{k-2d}(S) \to H_T^k(M) \to H_T^k(M - S) \to \cdots .$$

The fundamental observation for the Atiyah-Bott-Kirwan theory is the following.

\textbf{Lemma 2.1.} ([1] 13.4) Let $F$ be a connected $T$-space on which $T$ acts trivially. Let $N$ be a complex $T$-vector bundle on $F$. Assume that the weights of the $T$-action on the fiber of $N$ are all nonzero. Then the equivariant top Chern class of $N$ is not a zero divisor.

Suppose $S$ equivariantly retracts onto a $T$-fixed connected closed submanifold $F$
in $M$. Assume that the weights of the action on the normal space of $S$ at a point in $F$ are all nonzero. Then we have $H^*_T(S) \cong H^*_T(F)$ and the Gysin homomorphism for the pair $(M, S)$

$$H^{*-2d}_T(F) \cong H^{*-2d}_T(S) \to H^{*_T}_T(M)$$

is injective because its composition with the restriction to $F$ is injective by Lemma 2.1. Therefore the equivariant Gysin sequence splits into short exact sequences

$$0 \to H^{*-2d}_T(F) \to H^*_T(M) \to H^*_T(M - S) \to 0 \quad (2.6)$$

A useful observation is that the Gysin homomorphism arises from a morphism in the derived category $D^+_c(M)$ of bounded below constructible sheaves on $M$. For definitions and basic results on derived category, see [3].

**Lemma 2.2.** (cf. [4] 1.13 (15)) $\alpha^! \mathbb{Q}_M \cong \mathbb{Q}_S[-2d]$.

**Proof.** Let $\beta$ denote the inclusion of the complement $U$ of $S$. Then we have the distinguished triangle (see for instance [2] V 5.14)

$$\alpha_! \alpha^! \mathbb{Q}_M \to \mathbb{Q}_M \to \beta_! \beta^* \mathbb{Q}_M \to \alpha_! \alpha^! \mathbb{Q}_M[1]. \quad (2.7)$$

Let $p \in S$. From the associated long exact sequence ([2] V 1.8 (7))

$$\cdots \to H^i(\alpha^! \mathbb{Q}_M)_p \to H^i(\mathbb{Q}_M)_p \to H^i(\beta_! \beta^* \mathbb{Q}_M)_p \to \cdots$$

we see that $H^i(\alpha^! \mathbb{Q}_M)_p$ is trivial unless $i = 2d$ and $H^{2d}(\alpha^! \mathbb{Q}_M)_p \cong \mathbb{Q}$. By [3] III 5.2, we get the desired result. \hfill \square

Since $\alpha$ is a closed immersion, $\alpha_! = \alpha_*$. By composing the isomorphism in Lemma 2.2 with the adjunction morphism $\alpha_! \alpha^! \mathbb{Q}_M \to \mathbb{Q}_M$, we get the *Gysin morphism*

$$\alpha_* \mathbb{Q}_S[-2d] \cong \alpha_! \alpha^! \mathbb{Q}_M \to \mathbb{Q}_M$$

whose hypercohomology is the Gysin homomorphism (2.1). Furthermore, by (2.7) we have the distinguished triangle

$$\alpha_* \mathbb{Q}_S[-2d] \to \mathbb{Q}_M \to \beta_* \beta^* \mathbb{Q}_M \to \alpha_* \mathbb{Q}_S[-2d + 1].$$

When $T$ acts on $M$ preserving $S$, consider the inclusion (2.3). The normal bundle of this embedding has real rank $2d$ and by Lemma 2.2 we get the *equivariant Gysin morphism*

$$(\alpha^T)_* \mathbb{Q}_S[T][-2d] \to \mathbb{Q}_M[T] \quad (2.8)$$

whose hypercohomology is the equivariant Gysin homomorphism (2.4). This also fits into the distinguished triangle

$$\alpha^T_* \mathbb{Q}_S[T][-2d] \to \mathbb{Q}_M[T] \to \beta^*_T(\beta^T)^* \mathbb{Q}_M[T] \to \alpha^T_* \mathbb{Q}_S[T][-2d + 1].$$

### 3. Hamiltonian circle actions

Let $T = U(1)$ and $M$ be a Hamiltonian $T$-space with equivariant *proper* moment map

$$\mu : M \to \mathbb{R}.$$
Let $Z = \mu^{-1}(0)$ and $X = Z/T$. Consider the gradient flow of $f = -\mu^2$ and define $M^{ss}$ as the opens subset
\[ \{ p \in M \mid p \text{ retracts to a point in } Z \text{ by the gradient flow of } f \}. \]
Let $\phi : Z \rightarrow X$ be the quotient map and $\psi : M^{ss} \rightarrow X$ be the composition of the retraction to $Z$ and $\phi$.

Since $Z$ is compact and $T$-fixed components are disjoint, there are only finitely many $T$-fixed components in $Z$, say $F_1, \cdots, F_r$. Each $F_i$ is a symplectic submanifold and a neighborhood of $F_i$ is $T$-equivariantly diffeomorphic to a neighborhood of the zero section of the normal bundle of $F_i$. Pick any $p_i \in F_i$ and let $W_i$ be the normal space to $F_i$ at $p_i$. Then $W_i$ is a Hamiltonian $T$-vector space. By choosing a $T$-equivariant almost complex structure compatible with the symplectic form, we may assume that $W_i$ is a complex vector space on which $T$ acts unitarily.

It is well-known that a unitary action of $U(1)$ is completely reducible. Hence we can write
\[ W_i = W_i^+ \oplus W_i^- \]
where $W_i^+$ (resp. $W_i^-$) is the positive (resp. negative) weight space. Define
\[ d_i = \min \left\{ \frac{1}{2} \dim W_i^+, \frac{1}{2} \dim W_i^- \right\} \quad \text{and} \quad e_i = \max \left\{ \frac{1}{2} \dim W_i^+, \frac{1}{2} \dim W_i^- \right\}. \]
Let
\[ S_i^+ = \{ p \in M^{ss} \mid p \text{ retracts to a point in } F_i \text{ by the gradient flow of } -\mu \} \]
\[ S_i^- = \{ p \in M^{ss} \mid p \text{ retracts to a point in } F_i \text{ by the gradient flow of } \mu \}. \]
Then $S_i^+$ is a closed submanifold with codimension $\dim W_i^-$ and $S_i^-$ is a closed submanifold with codimension $\dim W_i^+$. Let
\[ S_i = \begin{cases} S_i^+ & \text{if } \dim W_i^- \leq \dim W_i^+ \\ S_i^- & \text{otherwise}. \end{cases} \]
In other words, $S_i$ is whichever of $\{ S_i^+, S_i^- \}$ has larger dimension such that
\[ \text{codim } S_i = 2d_i. \]
Let $\alpha_i : S_i \hookrightarrow M^{ss}$ denote the inclusion.

Since $S_i$ is a closed $T$-invariant submanifold of $M^{ss}$, we have the equivariant Gysin morphism (2.8) which fits into the distinguished triangle
\[ (\alpha_i^T)_* \mathbb{Q}[-2d_i] \rightarrow \mathbb{Q}_{M_i^{ss}} \rightarrow (\beta_i^T)_* (\beta_i^T)^* \mathbb{Q}_{M_i^{ss}} \]
where $\alpha_i^T$ is the inclusion of $ET \times_T S_i$ in $M_i^{ss} = ET \times_T M^{ss}$ and $\beta_i^T$ is the inclusion of its complement.

Since $F_i$ is $T$-fixed in $Z$, $F_i$ is mapped bijectively onto its image by $\psi : M^{ss} \rightarrow X$. By abuse of notation, we denote $\psi(F_i)$ by $F_i$ and the inclusion of $F_i$ in $X$ by $\sigma_i$. Since the paths of the steepest descent for $|\mu|$ and $\mu^2$ are identical up to parameterization, $\psi(S_i) = F_i$ and thus
\[ S_i \xrightarrow{\alpha_i} M^{ss} \]
\[ \xymatrix{ S_i \ar[d]_{\psi_i} \ar[r]^{\alpha_i} & M^{ss} \ar[d]_{\psi} \\ F_i \ar[r]_{\sigma_i} & X. } \]
commutes.

Let \( \psi^T : M_{T}^{ss} \rightarrow X \) and \( \psi_1^T : E \rightarrow T \rightarrow S_i \rightarrow F_i \) be the obvious maps induced from \( \psi \) and \( \psi_i \) respectively. We define the following objects in the derived category \( D^+_{c}(X) \):

\[
C\dot{T}(X) = \psi^T_! \mathbb{Q}_{M_{T}^{ss}}
\]

\[
C\dot{T}(F_i) = (\sigma_i)_* (\psi_i^T)_* \mathbb{Q}_{E \rightarrow T \rightarrow S_i} = \psi_i^T_! (\alpha_i^T)_* \mathbb{Q}_{E \times T \rightarrow S_i}
\]

whose hypercohomology groups are \( H^*_T(M^{ss}) \) and \( H^*_T(S_i) \) respectively. By taking \( \psi^T_* \), the Gysin morphisms in (3.1) give rise to a morphism

\[
\delta : \bigoplus_i C\dot{T}(F_i)[-2d_i] \rightarrow C\dot{T}(X) \tag{3.2}
\]

whose hypercohomology over \( X \) is just the sum of equivariant Gysin homomorphisms

\[
\bigoplus_i H^{* - 2d_i}_T(S_i) \rightarrow H^*_T(M^{ss}). \tag{3.3}
\]

Since \( S_i \) equivariantly retracts to \( F_i \), (3.3) is equivalent to a homomorphism

\[
\delta' : \bigoplus_i H^{* - 2d_i}_T(F_i) \rightarrow H^*_T(M^{ss}). \tag{3.4}
\]

**Lemma 3.1.** \( \delta' \) is injective.

**Proof.** The normal space of \( S_i \) at a point in \( F_i \) is either \( W_i^+ \) or \( W_i^- \). In particular, the weights for the \( T \)-action are all nonzero. By Lemma 2.1, the equivariant Gysin homomorphism composed with the restriction to \( F_i \)

\[
H^{* - 2d_i}_T(F_i) \rightarrow H^*_T(M^{ss}) \rightarrow H^*_T(F_i)
\]

is injective. By the exactness of the Gysin sequence, the composition

\[
H^{* - 2d_i}_T(F_i) \rightarrow H^*_T(M^{ss}) \rightarrow H^*_T(M^{ss} - S_i)
\]

is zero. If \( j \neq i \), then \( F_j \subset M^{ss} - S_i \) and hence

\[
H^{* - 2d_j}_T(F_j) \rightarrow H^*_T(M^{ss}) \rightarrow H^*_T(F_j)
\]

is zero. Therefore \( \delta' \) composed with the sum of restrictions

\[
\bigoplus_i H^{* - 2d_i}_T(F_i) \rightarrow H^*_T(M^{ss}) \rightarrow \bigoplus_i H^*_T(F_i)
\]

is the direct sum of the injective homomorphisms. Hence \( \delta' \) is injective. \( \square \)

4. A decomposition of \( C\dot{T}(X) \)

For \( 1 \leq i \leq r \), \( F_i \) is a subset of \( Z \) and we have a fiber square

\[
\begin{array}{ccc}
F_i & \xrightarrow{i} & Z \\
\phi & \downarrow & \phi \\
F_i & \xrightarrow{\alpha} & X \\
\end{array}
\]
which induces the fiber square

\[
\begin{array}{ccc}
ET \times_T F_i & \overset{\phi^T_i}{\longrightarrow} & ET \times_T Z \\
\downarrow \phi^T & & \downarrow \phi^T \\
F_i & \overset{\sigma_i}{\longrightarrow} & X.
\end{array}
\]

Hence by [4] 1.13 (13), we have

\[
(\sigma_i)^* \phi^T_i \text{Q}_{ET \times_T Z} \cong (\phi^T_i)^*(\psi^T)_* \text{Q}_{ET \times_T Z} \cong (\phi^T_i)^* \text{Q}_{ET \times_T F_i}.
\] (4.1)

Notice that since \( S_i \) retracts to \( F_i \) by the gradient flow of the \( T \)-equivariant function \( \mu \) or \( -\mu \), the inclusion

\[
\begin{array}{ccc}
ET \times_T F_i & \overset{\phi^T_i}{\longrightarrow} & ET \times_T S_i \\
\downarrow \phi^T & & \downarrow \psi^T \\
F_i & \overset{\phi^T_i}{\longrightarrow} & X.
\end{array}
\]

is a homotopy equivalence and hence

\[
C_T(F_i) = (\sigma_i)_*(\psi^T)_* \text{Q}_{ET \times_T S_i} \cong (\sigma_i)_*(\phi^T_i)_* \text{Q}_{ET \times_T F_i}.
\] (4.2)

Similarly, as \( M^{ss} \) is \( T \)-equivariantly homotopy equivalent to \( Z \),

\[
C_T(X) = (\psi^T)_* \text{Q}_{M^{ss}T} \cong (\phi^T_i)_* \text{Q}_{ET \times_T Z}.
\] (4.3)

Therefore, by (4.1), (4.2) and (4.3), we get

\[
(\sigma_i)_*(\sigma_i)^*C_T(X) \cong (\sigma_i)_*(\sigma_i)^*(\phi^T)_* \text{Q}_{ET \times_T Z} \cong (\sigma_i)_*(\phi^T_i)_* \text{Q}_{ET \times_T F_i} \cong C_T(F_i)
\]

When composed with the adjunction morphism

\[
C_T(X) \rightarrow (\sigma_i)_*(\sigma_i)^*C_T(X),
\]

this gives us the morphism

\[
C_T(X) \rightarrow C_T(F_i)
\] (4.4)

Let \( C_T(F_i) \rightarrow \tau^{\geq 2d}C_T(F_i) \) be the truncation morphism ([4] 1.14). Then (4.4) composed with the truncation morphism gives us the morphism

\[
\rho_i : C_T(X) \rightarrow \tau^{\geq 2d}C_T(F_i)
\] (4.5)

Adding up, we get

\[
\rho = \oplus \rho_i : C_T(X) \rightarrow \bigoplus_{i=1}^r \tau^{\geq 2d}C_T(F_i).
\] (4.6)

Recall that we have the Gysin morphism \( \delta \) defined in (3.2).

**Proposition 4.1.** \( \rho \circ \delta \) is an isomorphism in \( D^+_c(X) \).

**Proof.** It suffices to show that \( \rho \circ \delta \) induces an isomorphism on stalk cohomology. For \( x \in X - \bigcup F_i \), there is nothing to prove since \( \oplus_{i=1}^r \tau^{\geq 2d}C_T(F_i) \) is supported on \( \bigcup F_i \). Let \( x \in F_i \). The fiber of

\[
\phi^T_i : ET \times_T F_i = BT \times F_i \rightarrow F_i
\]
is the classifying space $BT = ET/T$ whose cohomology, denoted by $H^*_T$, is isomorphic to the polynomial ring $\mathbb{Q}[t]$ with $\deg t = 2$. Hence the stalk cohomology of $C^\tau_T(F_i)$ and $\tau^{\geq 2d_i}C^\tau_T(F_i)$ are $H^*_T \cong \mathbb{Q}[t]$ and $H^*_T \cong \mathbb{Q}[t]/\text{span}(1, t, \ldots, t^{d_i-1})$ respectively. The induced homomorphism on stalk cohomology
\[
H^*_T \cong \mathbb{Q}[t] \to \mathbb{Q}[t]/\text{span}(1, t, \ldots, t^{d_i-1}) \cong H^*_T^{\geq 2d_i}
\]
is the result of the equivariant Gysin homomorphism for the embedding of $0$ into the normal space of $S_i$, followed by restriction to $0$. Hence this is just the multiplication by the product of all weights on the normal space of $S_i$ which is a nonzero multiple of $t^{d_i}$. This is obviously an isomorphism of $\mathbb{Q}$-vector spaces. So we are done.

In particular, $\rho$ induces a surjection on stalk cohomology. An immediate consequence is the following decomposition.

**Corollary 4.2.** Let $\mathcal{A}'$ be an object in the triangulated category $D^+_c(X)$ which fits into the distinguished triangle
\[
\mathcal{A}' \to C_T^\tau(X) \to \bigoplus_{i=1}^{r} \tau^{\geq 2d_i}C^\tau_T(F_i) \to \mathcal{A}'[1].
\]
Then
\[
\mathcal{A}' \oplus \left( \bigoplus_{i=1}^{r} C^\tau_T(F_i)[-2d_i] \right) \cong C^\tau_T(X).
\]

**Proof.** The morphisms $\theta$ and $\delta$ give us a morphism
\[
\mathcal{A}' \oplus \left( \bigoplus_{i=1}^{r} C^\tau_T(F_i)[-2d_i] \right) \to C^\tau_T(X).
\]
This induces an isomorphism on every stalk cohomology and hence we get the isomorphism.

Such an $\mathcal{A}'$ always exists because we can simply take the mapping cone of $\rho$ translated by $-1$ ([2] V 5.2). We shall see that $\mathcal{A}'$ is isomorphic to the intersection cohomology sheaf $IC^\tau_X$ of $X$.

5. Local study of circle quotients

Recall that, for $1 \leq i \leq r$, $W_i$ is the normal space of $F_i$ in $M^{ss}$ at a point and we have a decomposition into positive and negative weight spaces
\[
W_i = W_i^+ \oplus W_i^-.
\]
By the local normal form theorem ([12] 7.4), the normal cone of $F_i$ in $X$ is the symplectic quotient $W_i/T$. As remarked in §3, we may assume $T = U(1)$ acts unitarily so that the symplectic quotient $W_i/T$ is homeomorphic to the good quotient $W_i/C^*$ in geometric invariant theory [7].

Let $\gamma_i : W_i \to W_i/C^*$ be the quotient map. It is well-known ([11] Proposition 2.2) that $\gamma_i^{-1}(\gamma_i(0))$ is the affine cone over the set of unstable points in $PW_i$ which is exactly $PW_i^+ \cup PW_i^-$ by the Hilbert-Mumford criterion ([11] Chapter 2 §1). Hence $\gamma_i^{-1}(\gamma_i(0)) = W_i^+ \cup W_i^-$ and thus
\[
W_i/C^* - \gamma_i(0) = (W_i - (W_i^+ \cup W_i^-))/C^*.
\]
Since \( C^* \) acts locally freely on \( W_i - (W_i^+ \cup W_i^-) \), we get
\[
H^*(W_i//C^* - \gamma_i(0)) \cong H^*_T(W_i - (W_i^+ \cup W_i^-)) \cong H^*_T(W_i - (W_i^+ \cup W_i^-)).
\] (5.1)

**Lemma 5.1.** Let \( m_i = d_i + e_i - 2 = \frac{1}{2} \codim_x F_i - 1 \). Then we have
\[
H^{\leq m_i}(W_i//C^* - \gamma_i(0)) \cong H^{\leq 2d_i - 2}_T.
\]

**Proof.** Without loss of generality we may assume \( \dim W_i^+ \leq \dim W_i^- \). Consider first the equivariant Gysin sequence for the pair \((W_i, W_i^-)\):
\[
\cdots \to H^{d_i-2i}_T(W_i^-) \to g H^i_T(W_i) \to h H^i_T(W_i - W_i^-) \to \cdots
\]
Since \( W_i \) and \( W_i^- \) are contractible, \( H^*_T(W_i) \cong \mathbb{Q}[t] \) and \( H^*_T(W_i^-) \cong \mathbb{Q}[t] \). The composition of the equivariant Gysin map \( g \) with the restriction to \( W_i^- \) is multiplication by the equivariant top Chern class which is just a nonzero multiple, say \( at^d_i \), of \( t^d_i \). Hence \( g \) is injective and the Gysin sequence gives us the short exact sequence
\[
0 \to \mathbb{Q}[t] \xrightarrow{at^d_i} \mathbb{Q}[t] \xrightarrow{h} H^*_T(W_i - W_i^-) \to 0.
\]
This implies that \( H^*_T(W_i - W_i^-) \cong \mathbb{Q}[t]/(t^{d_i}) \cong H^{\leq 2d_i - 2}_T \). In particular, since \( m_i \geq 2d_i - 2 \),
\[
H^{\leq m_i}(W_i - W_i^-) \cong H^{\leq 2d_i - 2}_T. \tag{5.2}
\]
Since \( \codim W_i^+ = 2e_i \geq d_i + e_i \geq m_i + 2 \), we deduce from the Gysin sequence for the pair \((W_i - W_i^-, W_i^-)\) that
\[
H^{\leq m_i}(W_i - W_i^-) \cong H^{\leq m_i}(W_i - (W_i^+ \cup W_i^-)). \tag{5.3}
\]
The lemma follows from (5.1), (5.2) and (5.3).

6. The intersection cohomology sheaf

The middle perversity intersection cohomology of \( X = Z/T \) is the hypercohomology of an object \( IC^*_X \) in \( D^+_c(X) \) satisfying three axioms, called normalization, the support, and cosupport conditions [4] §4. In our case, \( U := X - \cup_i F_i \) is a homology manifold because it is a symplectic orbifold and the axioms can be rephrased as follows. Let \( A' \) be an object in \( D^+_c(X) \) such that \( \tau^{\geq 0} A' \cong A' \). Then \( A' \) is isomorphic to the intersection cohomology sheaf \( IC^*_X \) if it satisfies

1. normalization: \( A'|_U \cong \mathbb{Q}_U \)
2. the support condition: for \( x \in F_i \), \( H^{> m_i}(A'\cdot x) = 0 \) where
   \[
   m_i = \frac{1}{2} \codim_x F_i - 1 = d_i + e_i - 2
   \]
3. the cosupport condition: for \( x \in F_i \), the adjunction map
   \[
   H^{\leq m_i}(A'\cdot x) \to H^{\leq m_i}(j_* j^* A'\cdot x)
   \]
   is an isomorphism where \( j : U \hookrightarrow X \) is the inclusion.

**Lemma 6.1.** The object \( A' \) in Corollary 4.2 satisfies the above axioms. Hence, \( A' \cong IC^*_X \).
Proof. Recall that $\mathcal{A}'$ fits into the distinguished triangle
\[
\mathcal{A}' \longrightarrow C_T(X) \xrightarrow{\rho} \bigoplus_{i=1}^r \tau^{\geq 2d_i} C_T(F_i).
\] (6.1)

1. **Normalization:** $\bigoplus_{i=1}^r \tau^{\geq 2d_i} C_T(F_i)$ is trivial on $U$ by definition and hence $\mathcal{A}'|_U \cong C_T(X)|_U$. But $T$ acts locally freely on the smooth manifold $Z \cup F_i$, so the fibers of
\[
ET \times_T (Z \cup F_i) \to U
\]
are the classifying spaces $BF$ for some finite groups $F$ whose rational cohomology are $\mathbb{Q}$ by Macdonald's theorem. Since $U$ is locally contractible, we deduce that $C_T(X)|_U \cong \mathbb{Q}U$.

2. **Support condition:** let $x \in F_i$. The distinguished triangle (6.1) gives rise to a long exact sequence
\[
\cdots \to H^k(\mathcal{A}'_x) \to H^k(C_T(X)_x) \to H^k(\tau^{\geq 2d_i} C_T(F_i)_x) \to \cdots
\] (6.2)
The middle term is $H^*(C_T(X)_x) \cong H^*_T \cong \mathbb{Q}[t]$, and the third term is
\[
H^*(\tau^{\geq 2d_i} C_T(F_i)_x) \cong H^{\geq 2d_i} \cong \mathbb{Q}[t]/\text{span}(1, t, \cdots, t^{d_i-1}).
\]
The map from the middle to the third in (6.2) is truncation. Hence $H^{>2d_i-2}(\mathcal{A}'_x) = 0$. Since $m_i = d_i + e_i - 2 \geq 2d_i - 2$, we get $H^{>m_i}(\mathcal{A}'_x) = 0$.

3. **Cosupport condition:** from (6.2), we also deduce that
\[
H^*(\mathcal{A}'_x) \cong H^{\leq m_i-2}(C_T(X)_x) \cong H^{\geq 2d_i-2}_T.
\]
In particular, $H^{\leq m_i}(\mathcal{A}'_x) \cong H^{\leq 2d_i-2}_T$. On the other hand, $j^* \mathcal{A}' = \mathcal{A}'|_U \cong \mathbb{Q}U$ by the normalization condition. Hence
\[
H^{\leq m_i}(j_* j^* \mathcal{A}'_x) \cong H^{\leq m_i}(\mathbb{R}\text{dim } F_i \times (W_i/\mathbb{C}^* - \gamma_i(0))) \cong H^{\leq m_i}(W_i/\mathbb{C}^* - \gamma_i(0)),
\]
which is naturally isomorphic to $H^{\leq 2d_i-2}_T$ by Lemma 5.1. Therefore
\[
H^{\leq m_i}(\mathcal{A}'_x) \cong H^{\leq 2d_i-2}_T \cong H^{\leq m_i}(j_* j^* \mathcal{A}'_x).
\]
The fact that this isomorphism is equal to the adjunction map is an easy exercise. Consequently, $\mathcal{A}'$ satisfies all three axioms and hence $\mathcal{A}' \cong IC_X$.

The main theorem of this paper now follows from Corollary 4.2.

**Theorem 6.2.** $IC_X \oplus (\bigoplus_{i=1}^r C_T(F_i)[-2d_i]) \cong C_T(X)$

7. **Cohomological consequences**

Theorem 6.2 enables us to describe the middle perversity intersection cohomology either as a subspace (Theorem 7.1) or as a quotient of $H^*_T(M^{ss})$ (Theorem 7.7).

7.1. **Subspace description**

Notice that the hypercohomology $H^*(\tau^{\geq 2d_i} C_T(F_i))$ of $\tau^{\geq 2d_i} C_T(F_i)$ is
\[
H^*(F_i) \otimes H^{\geq 2d_i}_T
\]
Hence we can replace similar. Using equivariant Morse theory, we get

$$\text{long exact sequence splits into short exact sequences}$$

Then

$$P^* = \{\eta \in H^*_T(M^s) \mid \eta|_{F_i} \in H^*(F_i) \otimes H_T^{2d_i} \}.$$  

This gives us a long exact sequence in hypercohomology

$$\cdots \rightarrow IH^k(X) \rightarrow H^k_T(M^s) \rightarrow \bigoplus_i \mathbb{H}^k(\tau^{\geq 2d_i}C_T(F_i)) \rightarrow \cdots$$

By Proposition 4.1, $H^k_T(M^s) \rightarrow \bigoplus_i \mathbb{H}^k(\tau^{\geq 2d_i}C_T(F_i))$ is surjective and thus the long exact sequence splits into short exact sequences

$$0 \rightarrow IH^*(X) \rightarrow H^*_T(M^s) \rightarrow \bigoplus_i H^*(F_i) \otimes H_T^{2d_i} \rightarrow 0. \quad (7.2)$$

So we have proved the following.

**Theorem 7.1.** The intersection cohomology $IH^*(X)$ is isomorphic to the (graded) subspace

$$V^* = \{\eta \in H^*_T(M^s) \mid \eta|_{F_i} \in H^*(F_i) \otimes H_T^{2d_i,-2}\}.$$  

This theorem gives us an efficient way to compute the intersection Betti numbers. By the equivariant Morse theory of Kirwan [7], we can compute the equivariant Poincaré series

$$P^T_t(M^s) = \sum_{k=0}^\infty t^k \dim H^k_T(M^s)$$

of the equivariant cohomology. In terms of this, we can easily compute the intersection Betti numbers as follows.

**Corollary 7.2.** Let $IP_t(X) = \sum t^k \dim IH^k(X)$ and $P_t(F) = \sum t^k \dim H^k(F)$. Then

$$IP_t(X) = P^T_t(M^s) - \frac{1}{1 - t^2} \sum_{1 \leq i \leq r} t^{2d_i} P_t(F_i).$$

**Remark 7.3.** Since $M^s$ retracts onto $Z$ equivariantly, $H_T^*(M^s) \cong H_T^*(Z)$. Hence we can replace $M^s$ by $Z$ in Theorem 7.1 and Corollary 7.2.

**Example 7.4.** We consider a linear circle action on the projective space $\mathbb{P}^n$. Let $p, q$ and $s$ denote the number of positive, negative, and zero weights respectively, so that $n = p + q + s - 1$. Let us assume that $p \leq q$ (the other case being entirely similar). Using equivariant Morse theory, we get

$$P^T_t(M^s) = (P_t(\mathbb{P}^p) - t^{2q+2s} P_t(\mathbb{P}^{p-1}) - t^{2p+2s} P_t(\mathbb{P}^{q-1}))/ (1 - t^2)$$

$$= (1 + t^2 + \cdots + t^{2p+2s-2} - t^{2q+2s} - \cdots - t^{2n})/ (1 - t^2).$$

Hence, by the above corollary, the intersection Poincaré polynomial is

$$P^T_t(M^s) - \frac{t^{2p} P_t(\mathbb{P}^{q-1})}{1 - t^2} = \frac{(1 - t^2p)(1 - t^{2q+2s})}{(1 - t^2)^2}$$

which is a palindromic polynomial of degree $2n - 2$. 

**INTERSECTION COHOMOLOGY OF CIRCLE QUOTIENTS**

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7.2. Intersection pairing

Since $X$ is compact there is an intersection pairing

$$IH^*(X) \otimes IH^*(X) \to \mathbb{Q}.$$  

This arises from a morphism

$$IC^\otimes \to IC^\cdot X$$

where the latter is the top perversity intersection cohomology sheaf. Since $IC^\cdot X$ is a direct summand of $C^\cdot T(X)$ by Theorem 6.2, there is a second morphism

$$IC^\otimes \rightarrow C^\cdot T(X) \rightarrow IC^\cdot X \rightarrow IC^\cdot X$$

where the morphism $C^\cdot T(X) \rightarrow IC^\cdot X$ is the morphism coming from the obvious $\mathbb{Q} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ over $ET \times T^{M^{ss}}$. This gives us the cup product structure of $H^\cdot T(M^{ss})$.

**Lemma 7.5.** The morphisms (7.3) and (7.4) are the same.

**Proof.** Both are extensions of the obvious $\mathbb{Q} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ over $U = X - \bigcup F_i$. By [2] V 9.14, such an extension is unique. \qed

Let

$$\nu : IH^*(X) \xrightarrow{\cong} V^* \subset H^*_T(M^{ss})$$

be the isomorphism from Theorem 7.1. Let $a, b$ be two classes in $IH^*(X)$ such that

$$\deg a + \deg b = \dim X = \dim F_i + 2d_i + 2e_i - 2 \geq \dim F_i + 4d_i - 2$$

for every $i$. By the definition of $V^*$, $\nu(a)|_{F_i}$ and $\nu(b)|_{F_i}$ lie in $H^*(F_i) \otimes H^2_{T} - 2$. Hence

$$\nu(a) \cup \nu(b)|_{F_i} \in H^{> \dim F_i(F_i)} \otimes H^{4d_i - 4}_{T}$$

which must be zero since $H^{> \dim F_i(F_i)} = 0$. Consequently,

$$\nu(a) \cup \nu(b) \in V^{\dim X} \cong IH^{\dim X}(X).$$

Since $X$ is connected, $IH^{\dim X}(X)$ is one dimensional, generated by the unique class, denoted by $e$, representing one point. Hence we can write

$$\nu(a) \cup \nu(b) = l \nu(e)$$

for some rational number $l$. We claim $l$ is the intersection number $(a, b)$ on $IH^*(X)$.

From (7.4) and Lemma 7.5, the intersection pairing is given by

$$IH^*(X) \otimes IH^*(X) \to H^*_T(M^{ss}) \otimes H^*_T(M^{ss}) \to IH^*(X) \to \mathbb{Q}.$$ 

By (7.5), if we start with $a \otimes b$, we get $l$. Therefore, $l = (a, b)$. So we have proved the following theorem that enables us to compute the intersection pairing in terms of the cup product of $H^*_T(M^{ss})$.

**Theorem 7.6.** For two classes $a, b$ in $IH^*(X)$ such that $\deg a + \deg b = \dim X$ we have

$$\nu(a) \cup \nu(b) = (a, b) \nu(e).$$
7.3. Quotient description

Now we consider the quotient description of $IH^*(X)$. By Theorem 6.2 we have a distinguished triangle

$$\bigoplus_{i=1}^r C_T^*(F_i)[-2d_i] \xrightarrow{\delta} C_T^*(X) \rightarrow IC^* X \quad (7.6)$$

Upon taking hypercohomology we obtain a short exact sequence

$$0 \rightarrow \bigoplus_{i=1}^r H_T^{*-2d_i}(F_i) \xrightarrow{\delta'} H_T^*(M^{ss}) \rightarrow IH^*(X) \rightarrow 0 \quad (7.7)$$

because $\delta'$ is injective by Lemma 3.1. Hence $IH^*(X)$ is the quotient of $H_T^*(M^{ss})$ by the images of the equivariant Gysin maps for $S_i$. The exact sequence (2.6) applied for each $S_i$, one by one, now tells us the following.

**Theorem 7.7.** $IH^*(X) \cong H_T^*(M^{ss})/\text{im} \, \delta' \cong H_T^*(M^{ss} - \bigcup_i S_i)$

Suppose $M$ is compact. Let $\mathfrak{F}^+$ be the set of $T$-fixed components in $M$ such that either

1. $\mu(F) > 0$ or
2. $F = F_i$ for some $i$ and $\dim W^-_i \leq \dim W^+_i$.

Let $\mathfrak{F}^-$ be the set of all other $T$-fixed components.

For $F \in \mathfrak{F}^+$, let $S_F$ be the set of points in $M$ which retract to $F$ by the gradient flow of $-\mu$, i.e. $S_F$ is the stable manifold for $\mu$. If $F \in \mathfrak{F}^-$, let $S_F$ denote the unstable manifold for $\mu$, i.e. $S_F$ is the set of points in $M$ which retract to $F$ by the gradient flow of $\mu$. From [7], we see that

$$M - M^{ss} = \bigcup_{\mu(F) \neq 0} S_F$$

and thus

$$M - \bigcup_F S_F = M^{ss} - \bigcup_i S_i.$$  

Since there are only finitely many fixed components, we can apply (2.6) for each $F$ in order of decreasing absolute value of $\mu$. Therefore, we get a short exact sequence

$$0 \rightarrow \bigoplus_F H_T^{*-2d_F}(F) \rightarrow H_T^*(M) \rightarrow H_T^*(M^{ss} - \bigcup_i S_i) \rightarrow 0 \quad (7.8)$$

where $d_F = \frac{1}{2} \text{codim} S_F$. From (7.8) and Theorem 7.7, together with the abelian localization theorem

$$H_T^*(M) \rightarrow \bigoplus_F H_T^*(F)$$

we can deduce a theorem of Lerman and Tolman ([10] Theorem 1').

**Corollary 7.8.** [10] As a graded ring, $IH^*(X)$ is isomorphic to $H_T^*(M)/K$ where

$$K = \{ \eta \in H_T^*(M) \mid \eta|_F = 0 \ \forall \ F \in \mathfrak{F}^+ \} \oplus \{ \eta \in H_T^*(M) \mid \eta|_F = 0 \ \forall \ F \in \mathfrak{F}^- \}.$$  

Given the above, the proof is identical to the proof of Theorem 1 in [13] for smooth quotients, so we omit it.
Remark 7.9. By Theorem 7.7 and Corollary 7.8, $IH^\ast(X)$ is equipped with a ring structure but this is not canonical. In the notation of §3, when dim$S_i^+ = \text{dim } S_i^-$ for some $T$-fixed component $F_i$ in $Z$, we could choose either $S_i^+$ or $S_i^-$ as our $S$. This gives us two different ring structures. For example, consider the action of $C^\ast$ on $M = \mathbb{P}^4$ by

$$\lambda \cdot (a_0 : a_1 : \cdots : a_4) = (a_0 : \lambda a_1 : \lambda a_2 : \lambda^{-1}a_3 : \lambda^{-1}a_4).$$

Let $S^+$ (resp. $S^-$) be the stable (resp. unstable) manifold by the gradient flow of the moment map, for the fixed point $(1 : 0 : \cdots : 0)$. Then the $C^\ast$-orbit spaces of $M^{ss} - S^+$ and $M^{ss} - S^-$ respectively give us two small resolutions and the natural isomorphisms

$$H^\ast_T(M^{ss} - S^+ \cong IH^\ast(M//T) \cong H^\ast_T(M^{ss} - S^-)$$

give us two different ring structures. See [2] IX Example 1, page 221.

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