MODULI SPACE OF STABLE MAPS TO PROJECTIVE SPACE VIA GIT

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Abstract. We compare the Kontsevich moduli space $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$ of stable maps to projective space with the quasi-map space $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)\otimes\mathbb{C}^n)//SL(2)$. Consider the birational map

$$\bar{\psi} : \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)\otimes\mathbb{C}^n)//SL(2) \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$$

which assigns to an n-tuple of degree d homogeneous polynomials $f_1, \cdots, f_n$ in two variables, the map $f = (f_1 : \cdots : f_n) : \mathbb{P}^1 \to \mathbb{P}^{n-1}$. In this paper, for $d = 3$, we prove that $\bar{\psi}$ is the composition of three blow-ups followed by two blow-downs. Furthermore, we identify the blow-up/down centers explicitly in terms of the moduli spaces $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$ with $d = 1, 2$. In particular, $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ is the $SL(2)$-quotient of a smooth rational projective variety. The degree two case $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$, which is the blow-up of $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^2 \otimes \mathbb{C}^n)//SL(2)$ along $\mathbb{P}^{n-1}$, is worked out as a warm-up.

1. Introduction

The space of smooth rational curves of given degree d in projective space admits various natural moduli theoretic compactifications via geometric invariant theory (GIT), stable maps, Hilbert scheme or Chow scheme. Since these compactifications give us important but different enumerative invariants, it is an interesting problem to compare the compactifications by a sequence of explicit blow-ups and -downs. We expect all the blow-up centers to be some natural moduli spaces (for lower degrees). Further, we expect the difference between the intersection numbers on any two of the moduli theoretic compactifications should be expressed in terms of the intersection numbers of lower degrees. If the comparison of compactifications is completed, the variation of intersection numbers might be calculated by localization techniques. In this paper, as a first step, we compare the GIT compactification (or quasi-map space) and the Kontsevich moduli space of stable maps. The techniques we use are the Atiyah-Bott-Kirwan theory [13, 14, 15], variation of GIT quotients [4, 19], the blow-up formula [8] and construction of stable maps by elementary modification [11, 2].

Given an n-tuple $(f_1, \cdots, f_n)$ of degree d homogeneous polynomials in homogeneous coordinates t_0, t_1 of $\mathbb{P}^1$, we have a morphism $(f_1 : \cdots : f_n) : \mathbb{P}^1 \to \mathbb{P}^{n-1}$ if $f_1, \cdots, f_n$ have no common zeros. Thus we have an $SL(2)$-invariant rational map $\psi_0 : \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)\otimes\mathbb{C}^n) \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$ which induces a birational map

$$\bar{\psi}_0 : \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)\otimes\mathbb{C}^n)//SL(2) \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d).$$

Our goal is to decompose $\bar{\psi}_0$ into a sequence of blow-ups and blow-downs and describe the blow-up/-down centers explicitly.

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When $d = 1$, $\bar{\psi}_0$ is an isomorphism. When $d = 2$, the following is proved in [11, §4].

Theorem 1.1. *$\bar{\psi}_0$ is the inverse of a blow-up, i.e. $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ is the blow-up of $\mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathbb{C}^n)//\mathsf{SL}(2)$ along $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^2) \times \mathbb{P}^{n-1}//\mathsf{SL}(2) \cong \mathbb{P}^{n-1}$.*

We reproduce the proof in §3 for reader's convenience.

In this paper, our focus is laid on the case where $d = 3$. We prove the following in §5.

Theorem 1.2. *The birational map $\bar{\psi}_0$ is the composition of three blow-ups followed by two blow-downs. The blow-up centers are respectively, $\mathbb{P}^{n-1}$, $\overline{\mathbf{M}}_{0,2}(\mathbb{P}^{n-1}, 1)/\mathsf{S}_2$ (where S_2 interchanges the two marked points) and the blow-up of $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^{n-1}, 2)$ along the locus of three irreducible components. The centers of the blow-downs are respectively the S_2-quotient of a $(\mathbb{P}^{n-2})^2$-bundle on $\overline{\mathbf{M}}_{0,2}(\mathbb{P}^{n-1}, 1)$ and a $(\mathbb{P}^{n-2})^3/\mathsf{S}_3$-bundle on $\mathbb{P}^{n-1}$. In particular, $\bar{\psi}_0$ is an isomorphism when $n = 2$.*

Here of course, S_k denotes the symmetric group on k letters.

Let $\mathbf{P}_0 = \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathbb{C}^n)^s$ be the stable part of $\mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathbb{C}^n)$ with respect to the action of $\mathsf{SL}(2)$ induced from the canonical action on $\mathbb{C}^2$. Let $\mathbf{P}_1$ be the blow-up of $\mathbf{P}_0$ along the locus of n-tuples of homogeneous polynomials having three common zeros (or, *base points*). We get $\mathbf{P}_2$ by blowing up $\mathbf{P}_1$ along the proper transform of the locus of two common zeros. Let $\mathbf{P}_3$ be the blow-up of $\mathbf{P}_2$ along the proper transform of the locus of one common zero. Then we can construct a family of stable maps of degree 3 to $\mathbb{P}^{n-1}$ parameterized by $\mathbf{P}_3$ by using elementary modification. Thus we obtain an $\mathsf{SL}(2)$-invariant morphism

$$\psi_3 : \mathbf{P}_3 \longrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3).$$

The proper transform of the exceptional divisor of the second blow-up turns out to be a $\mathbb{P}^1$-bundle on a smooth variety and the normal bundle is $\mathcal{O}(-1)$ on each fiber. So, we can blow down this divisor and obtain a complex manifold $\mathbf{P}_4$. Further, the proper transform of the exceptional divisor of the first blow-up now becomes a $\mathbb{P}^2$-bundle on a smooth variety and the normal bundle is $\mathcal{O}(-1)$ on each fiber. Hence we can blow down $\mathbf{P}_4$ to obtain a complex manifold $\mathbf{P}_5$. The morphism ψ_3 is constant along the fibers of the blow-downs $\mathbf{P}_3 \to \mathbf{P}_4 \to \mathbf{P}_5$ and hence factors through a holomorphic map $\psi_5 : \mathbf{P}_5 \longrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ which induces

$$\bar{\psi}_5 : \mathbf{P}_5/\mathsf{SL}(2) \longrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3).$$

We can check that this is bijective and therefore $\bar{\psi}_5$ is an isomorphism of projective varieties by the Riemann existence theorem [9, p.442], because $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ is a normal projective variety. In summary, we have the following diagram.

$$\begin{tikzcd}
\mathbf{P}_3 \arrow[r, "\pi_3"] \arrow[d, "\pi_4"'] \arrow[dr, "p_3"] & \mathbf{P}_2 \arrow[r, "\pi_2"] & \mathbf{P}_1 \arrow[r, "\pi_1"] & \mathbf{P}_0 \arrow[d] \\
\mathbf{P}_4 \arrow[d, "\pi_5"'] & \mathbf{P}_3/\mathsf{SL}(2) \arrow[rr] \arrow[d] \arrow[drr, "\bar{\psi}_3"] & & \mathbf{P}_0/\mathsf{SL}(2) \arrow[d, dotted, "\bar{\psi}_0"] \\
\mathbf{P}_5 \arrow[r, "p_5"] & \mathbf{P}_5/\mathsf{SL}(2) \arrow[rr, "\cong", "\bar{\psi}_5"'] & & \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3).
\end{tikzcd}$$

Theorems 1.1 and 1.2 provide us with a new way of calculating the cohomology rings of the moduli spaces of stable maps. By [13, 14, 15], the cohomology ring of the $\mathsf{SL}(2)$-quotient of a projective space is easy to determine. As an application of Theorem 1.1, we prove the following in §3 by the blow-up formula of cohomology rings.

Theorem 1.3. *(1) The rational cohomology ring* $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2))$ *is isomorphic to*

$$\mathbb{Q}[\xi,\alpha^2,\rho]/\left\langle \frac{(\rho+2\alpha+\xi)^n-\xi^n}{\rho+2\alpha}+\frac{(\rho-2\alpha+\xi)^n-\xi^n}{\rho-2\alpha},\right.$$
$$\left.(\rho+2\alpha+\xi)^n+(\rho-2\alpha+\xi)^n,\ \xi^n\rho\right\rangle$$

where ξ*,* ρ*, and* α^2 *are generators of degree* $2,2,4$ *respectively.*

(2) The Poincaré polynomial $P_t(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2))=\sum_{k\geq 0}t^k\dim H^k(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2))$ *is*

$$\frac{(1-t^{2n+2})(1-t^{2n})(1-t^{2n-2})}{(1-t^2)^2(1-t^4)}.$$

(3) The integral *Picard group of* $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2)$ *is*

$$\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2))=\begin{cases}\mathbb{Z}\oplus\mathbb{Z} & \text{for} \quad n\geq 3\\ \mathbb{Z} & \text{for} \quad n=2\end{cases}$$

Behrend-O'Halloran [1, Proposition 4.27] gave a *recursive formula* for a set of generators of the relation ideal but *closed expressions* for a set of generators were unknown. The Poincaré polynomial is equivalent to the calculation of Getzler and Pandharipande [7]. The *rational* Picard group $\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2))\otimes\mathbb{Q}$ was calculated by Pandharipande [18].

In §6, we deduce the following from Theorem 1.2.

Theorem 1.4. *(1) The Poincaré polynomial* $P_t(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3))$ *is*

$$P_t(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3))=\left(\frac{1-t^{2n+8}}{1-t^6}+2\frac{t^4-t^{2n+2}}{1-t^4}\right)\frac{(1-t^{2n})}{(1-t^2)}\frac{(1-t^{2n})(1-t^{2n-2})}{(1-t^2)(1-t^4)}.$$

(2) The rational cohomology ring of $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{\infty},3))=\lim_{n\to\infty}H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3))$ *is isomorphic to*

$$\mathbb{Q}[\xi,\alpha^2,\rho_1^3,\rho_2^2,\rho_3,\sigma]/\langle\alpha^2\rho_1^3,\rho_1^3\sigma,\sigma^2-4\alpha^2\rho_3^2\rangle$$

where ξ,ρ_3 *are degree 2 classes,* σ,ρ_2^2,α^2 *are degree 4 classes, and* ρ_1^3 *is a degree 6 class. The rational cohomology ring of* $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^1,3))$ *is isomorphic to*

$$\mathbb{Q}[\xi,\alpha^2]/\langle\frac{(\xi+\alpha)^2(\xi+3\alpha)^2-(\xi-\alpha)^2(\xi-3\alpha)^2}{2\alpha},\frac{(\xi+\alpha)^2(\xi+3\alpha)^2+(\xi-\alpha)^2(\xi-3\alpha)^2}{2}\rangle.$$

(3) The Picard group of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3)$ *is*

$$\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3))=\begin{cases}\mathbb{Z}\oplus\mathbb{Z} & \text{for} \quad n\geq 3\\ \mathbb{Z} & \text{for} \quad n=2\end{cases}$$

The generators are $\frac{1}{3}(H+\Delta)$ *and* Δ *for* $n\geq 3$ *and* $\frac{1}{3}(H+\Delta)$ *for* $n=2$*, where* Δ *is the boundary divisor of reducible curves and* H *is the locus of stable maps whose images meet a fixed codimension two subspace in* $\mathbb{P}^{n-1}$*.*

It may be possible to find the cohomology ring of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3)$ for all n from Theorem 1.2 but we content ourselves with the $\mathbb{P}^\infty$ case and the $\mathbb{P}^1$ case in this paper. The above description of $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^\infty,3))$ is equivalent to the description of Behrend-O'Halloran [1, Theorem 4.15] and the Poincaré polynomial is equivalent to the calculation given by Getzler and Pandharipande [7]. The *rational* Picard group $\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3))\otimes\mathbb{Q}$ was calculated by Pandharipande [18] but the calculation of the *integral* Picard group seems new.

In a subsequent paper, we shall work out the case for $d=4$ and higher. In [3], we compare the Kontsevich moduli space $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^3,3)$, the Hilbert scheme of twisted cubics and Simpson's moduli space of stable sheaves.

2. Preliminaries

2.1. **Kontsevich moduli space.** The Kontsevich moduli space $\overline{\mathbf{M}}_{0,k}(\mathbb{P}^{n-1},d)$ or the moduli space of stable maps to $\mathbb{P}^{n-1}$ of genus 0 and degree d with k marked points is a compactification of the space of smooth rational curves of degree d in $\mathbb{P}^{n-1}$ with k marked points. It is the coarse moduli space of morphisms $f:C\to\mathbb{P}^{n-1}$ of degree d where C are connected nodal curves of arithmetic genus 0 with k nonsingular marked points $p_1,\cdots,p_k$ on C such that the automorphism groups of $(f,p_1,\cdots,p_k)$ are finite. Here, an automorphism of a stable map means an automorphism $\phi:C\to C$ that satisfies $f\circ\phi=f$ and that fixes the marked points. See [6] for the construction and basic facts on $\overline{\mathbf{M}}_{0,k}(\mathbb{P}^{n-1},d)$.

By [6, 12], $\overline{\mathbf{M}}_{0,k}(\mathbb{P}^{n-1},d)$ is a normal irreducible projective variety with at worst orbifold singularities.

2.2. **Cohomology of blow-up.** We recall a few basic facts on the cohomology ring of a blow-up along a smooth submanifold from [8]. To begin with, we consider the cohomology ring of a projective bundle $\pi:\mathbb{P}N\to Y$ where N is a vector bundle of rank r. Let $\rho=c_1(\mathcal{O}_{\mathbb{P}N}(1))$ and consider the exact sequence

$$0\longrightarrow\mathcal{O}(-1)\longrightarrow\pi^*N\longrightarrow Q\longrightarrow 0$$

where Q is the cokernel of the tautological monomorphism $\mathcal{O}(-1)\to\pi^*N$. By the Whitney formula, $(1-\rho)(1+c_1(Q)+c_2(Q)+\cdots)=1+c_1(N)+\cdots+c_r(N)$. By expanding, we obtain $c_r(Q)=\rho^r+\rho^{r-1}c_1(N)+\cdots+c_r(N)=0$ because Q is a vector bundle of rank $r-1$. By spectral sequence, we see immediately that this is the only relation on $H^*(Y)$ and ρ. In other words,

$$H^*(\mathbb{P}N)=H^*(Y)[\rho]/\langle\rho^r+\rho^{r-1}c_1(N)+\cdots+\rho c_{r-1}(N)+c_r(N)\rangle.$$

Example. Let $Y=B\mathbb{C}^*=\mathbb{P}^\infty$ be the classifying space of $\mathbb{C}^*$ and let $\alpha=c_1(E\mathbb{C}^*\times_{\mathbb{C}^*}\mathbb{C})$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ with weight 1. Let N be a vector space on which $\mathbb{C}^*$ acts with weights $w_1,\cdots,w_r$. Then the rational equivariant cohomology ring of the projective space $\mathbb{P}N$ is

$$H^*_{\mathbb{C}^*}(\mathbb{P}N)=H^*(E\mathbb{C}^*\times_{\mathbb{C}^*}\mathbb{P}N)=\mathbb{Q}[\rho,\alpha]/\langle\prod_{i=1}^r(\rho+w_i\alpha)\rangle.$$

Let X be a connected complex manifold and $\imath:Y\hookrightarrow X$ be a smooth connected submanifold of codimension r. Let $\pi:\tilde{X}\to X$ be the blow-up of X along Y and $\tilde{Y}=\mathbb{P}_YN$ be the exceptional divisor where N is the normal bundle of Y. From [8, p.605], we have an isomorphism

$$H^*(\tilde{X}) \cong H^*(X) \oplus \bigoplus_{k=1}^{r-1} \rho^k H^*(Y)$$

of vector spaces where $\rho = c_1(\mathcal{O}_{\tilde{X}}(-\tilde{Y}))$. Suppose the Poincaré dual $[Y] \in H^{2r}(X)$ of Y in X is nonzero. The ring structure of $H^*(\tilde{X})$ is not hard to describe. First, $H^*(X) \cong \pi^* H^*(X)$ is a subring of $H^*(\tilde{X})$ since π^* is a monomorphism. For $\gamma \in H^*(X)$ and $\rho^k\beta$ ($\beta \in H^*(Y)$, $1 \le k < r$), their product is $\gamma \cdot \rho^k \beta = \iota^*(\gamma)\beta\rho^k$ because ρ is supported in $\tilde{Y}$. Finally, we have the relation

$$\rho^r + c_1(N)\rho^{r-1} + \cdots + c_{r-1}(N)\rho + \pi^*[Y] = 0 \tag{2.1}$$

in the cohomology ring $H^*(\tilde{X})$. For the left hand side of (2.1) restricts to a relation on the projective bundle $\tilde{Y}$ and thus it comes from a class in $H^*(X)$ which has support in Y. By the Thom-Gysin sequence, we have an exact diagram

$$\begin{tikzcd} H^0(Y) \arrow[r] \arrow[dr, "\cup[Y]"'] & H^{2r}(X) \arrow[r] \arrow[d] & H^{2r}(X-Y) \\ & H^{2r}(Y) & \end{tikzcd}$$

Since $[Y] \neq 0$, $\cup[Y]$ is injective and hence we obtain the relation (2.1).

In particular, we have the following

Proposition 2.1. *If* $\iota^* : H^*(X) \to H^*(Y)$ *is surjective and* $[Y] \neq 0$*, then we have an isomorphism of rings*

$$H^*(\tilde{X}) = H^*(X)[\rho]/\langle \rho \cdot \ker(\iota^*), \rho^r + c_1(N)\rho^{r-1} + \cdots + c_{r-1}(N)\rho + \pi^*[Y] \rangle.$$

Remark. Everything in this subsection holds true for equivariant cohomology rings when there is a group action on X preserving Y. For we can simply replace X and Y by the homotopy quotients $X_G = EG \times_G X$ and $Y_G = EG \times_G Y$ respectively, where EG is a contractible free G-space and $BG = EG/G$. Recall that $H^*_G(X) = H^*(X_G)$ by definition.

2.3. **Atiyah-Bott-Kirwan theory.** Let X be a smooth projective variety on which a complex reductive group G acts linearly, i.e. $X \subset \mathbb{P}^N$ for some N and G acts via a homomorphism $G \to GL(N+1)$. We denote by X^s (resp. X^{ss}) the open subset of stable (resp. semistable) points in X. Then there is a stratification $\{S_\beta | \beta \in \mathcal{B}\}$ of X indexed by a partially ordered set $\mathcal{B}$ such that X^{ss} is the open stratum S_0 and the Gysin sequence for the pair $(U_\beta = X - \cup_{\gamma > \beta} S_\gamma, U_\beta - S_\beta)$ splits into an exact sequence in rational equivariant cohomology

$$0 \to H_G^{j-2\lambda(\beta)}(S_\beta) \to H_G^j(U_\beta) \to H_G^j(U_\beta - S_\beta) \to 0$$

where λ_β is the codimension of S_β. As a consequence, we obtain an isomorphism

$$H_G^j(X) \cong H_G^j(X^{ss}) \oplus \bigoplus_{\beta \neq 0} H_G^{j-2\lambda(\beta)}(S_\beta)$$

of vector spaces and an injection of rings

$$H^*_G(X) = H^*_T(X)^W \hookrightarrow H^*_T(X) \xhookrightarrow{\oplus i_F^*} \bigoplus\nolimits_{F \in \mathcal{F}} H^*_T(F) \tag{2.2}$$

where T is the maximal torus of G, W the Weyl group, $\mathcal{F}$ the set of T-fixed components F, $i_F : F \hookrightarrow X$ the inclusion. Hence by finding the image of $H_G^{j-2\lambda(\beta)}(S_\beta)$ for $\beta \neq 0$ in $\bigoplus_{F\in\mathcal{F}} H_T^*(F)$, we can calculate the kernel of the surjective homomorphism

$$\kappa : H_G^*(X) \longrightarrow H_G^*(X^{ss})$$

induced by the inclusion $X^{ss} \to X$. Often $H_G^*(X)$ is easy to calculate and in that case we can calculate the cohomology ring of $H_G^*(X^{ss})$ which is isomorphic to $H^*(X/\!/G)$ when $X^{ss} = X^s$. See [13, 15] for details.

Example. Let $W_d = \mathrm{Sym}^d\mathbb{C}^2$ on which $G = SL(2)$ acts in the canonical way. Then the Poincaré polynomial $P_t(\mathbb{P}(W_d \otimes \mathbb{C}^n)/\!/SL(2)) = \sum_{k\ge 0} t^k \dim H^k(\mathbb{P}(W_d \otimes \mathbb{C}^n)/\!/SL(2))$ is

$$\frac{(1-t^{2mn-2})(1-t^{2mn})}{(1-t^2)(1-t^4)}$$

for $d = 2m-1$ odd. The case $n = 1$ is worked out in [13] and the general case is straightforward. In case $d = 2m$ even, the equivariant Poincaré series $P_t^{SL(2)}(\mathbb{P}(W_d \otimes \mathbb{C}^n)^{ss}) = \sum_{k\ge 0} t^k \dim H^k_{SL(2)}(\mathbb{P}(W_d \otimes \mathbb{C}^n)^{ss})$ is

$$\frac{1-t^{2n(m+1)-2}-t^{2n(m+1)}+t^{2n(2m+1)-2}}{(1-t^2)(1-t^4)}.$$

Example. Let $G = SL(2)$ act on $\mathbb{C}^2$ in the obvious way and trivially on $\mathbb{C}^n$. Let $\mathbf{P}_0$ be the semistable part of $\mathbf{P} = \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^2)\otimes\mathbb{C}^n)$ with respect to the action of G. Let us determine the equivariant cohomology ring $H_G^*(\mathbf{P}_0)$. From the previous subsection, we have an isomorphism of rings

$$H_G^*(\mathbf{P}) = \mathbb{Q}[\xi, \alpha^2]/\langle \xi^n(\xi-2\alpha)^n(\xi+2\alpha)^n\rangle$$

where ξ is the equivariant first Chern class of $\mathcal{O}(1)$ because the weights of the action of the maximal torus $T = \mathbb{C}^*$ on $\mathrm{Sym}^2(\mathbb{C}^2)\otimes\mathbb{C}^n$ are $2, 0, -2$, each with multiplicity n. By the localization theorem, we have an inclusion

$$i^* = (i_2^*, i_0^*, i_{-2}^*) : H_G^*(\mathbf{P}) \hookrightarrow H_T^*(Z_2) \oplus H_T^*(Z_0) \oplus H_T^*(Z_{-2})$$

where $Z_k \cong \mathbb{P}^{n-1}$ for $k = 2, 0, -2$ are the T-fixed components of weight k. With the identification $H_T^*(Z_k) \cong \mathbb{Q}[\xi,\alpha]/\langle\xi^n\rangle$, the homomorphism i^* sends ξ to $(\xi - 2\alpha, \xi, \xi+2\alpha)$ and α^2 to $(\alpha^2, \alpha^2, \alpha^2)$. There is only one unstable stratum S_β in $\mathbf{P}$, namely $GZ_2 = GZ_{-2}$. The composition of the Gysin map

$$j_* : H_G^{*-(4n-2)}(S_\beta) \hookrightarrow H_G^*(\mathbf{P})$$

with i_2^* or i_{-2}^* is the multiplication by the Euler class of the normal bundle to S_β because $Z_2, Z_{-2} \subset S_\beta$. Hence

$$i_{\pm 2}^* \circ j_*(1) = \frac{(\xi \mp 2\alpha)^n(\xi \mp 4\alpha)^n}{\mp 2\alpha}$$

as the normal bundle to $Z_{\pm 2}$ in $\mathbf{P}$ has Euler class $(\xi \mp 2\alpha)^n(\xi\mp 4\alpha)^n$ and the normal bundle to $Z_{\pm 2}$ in S_β has Euler class $\mp 2\alpha$. Since Z_0 is disjoint from S_β, $i_0^* \circ j_*(1) = 0$. It is easy to see that

$$\frac{\xi^n(\xi+2\alpha)^n}{2\alpha} + \frac{\xi^n(\xi-2\alpha)^n}{-2\alpha} \tag{2.3}$$

is the unique element in $H^*_G(\mathbf{P})$ whose image by i^* is $(i_2^*\circ j_*(1), i_0^*\circ j_*(1), i_{-2}^*\circ j_*(1))$. Hence, $j_*(1)$ is (2.3) and similarly any element in the image of j_* is of the form

$$f(\xi,\alpha)\frac{\xi^n(\xi+2\alpha)^n}{2\alpha}+f(\xi,-\alpha)\frac{\xi^n(\xi-2\alpha)^n}{-2\alpha}$$

for a polynomial $f(\xi,\alpha)$. Consequently, we have an isomorphism of rings

$$H^*_G(\mathbf{P}_0)\cong\mathbb{Q}[\xi,\alpha^2]/\langle\xi^n\frac{(\xi+2\alpha)^n-(\xi-2\alpha)^n}{2\alpha},\xi^n\frac{(\xi+2\alpha)^n+(\xi-2\alpha)^n}{2}\rangle. \tag{2.4}$$

Example. Let $\mathbf{P}_0$ be the stable part of $\mathbf{P}=\mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2)\otimes\mathbb{C}^n)$ with respect to the action of $G=SL(2)$. As above, the action on $\mathbb{C}^n$ is trivial and the action of G on $\mathbb{C}^2$ is the obvious one. As in the previous example, we can calculate the cohomology ring

$$H^*_G(\mathbf{P}_0)\cong\mathbb{Q}[\xi,\alpha^2]/\langle\frac{(\xi+\alpha)^n(\xi+3\alpha)^n-(\xi-\alpha)^n(\xi-3\alpha)^n}{2\alpha}, \frac{(\xi+\alpha)^n(\xi+3\alpha)^n+(\xi-\alpha)^n(\xi-3\alpha)^n}{2}\rangle. \tag{2.5}$$

We leave the verification as an exercise.

2.4. **Stability after blow-up.** We recall a few basic results about GIT stability from [14, §3]. Let G be a complex reductive group acting on a smooth variety X with a linearization on an ample line bundle L. Let Y be a nonsingular G-invariant closed subvariety of X and let $\pi:\tilde{X}\to X$ be the blow-up of X along Y with exceptional divisor E. The line bundle $L_d=\pi^*L^{\otimes d}\otimes\mathcal{O}(-E)$ is very ample for d sufficiently large and the action of G on L lifts to an action on L_d. With respect to this linearization, we consider the (semi)stability of points in $\tilde{X}$. We recall the following ([14, (3.2) and (3.3)]):

(1) If $\pi(y)$ is not semistable in X then y is not semistable in $\tilde{X}$;
(2) If $\pi(y)$ is stable in X then y is stable in $\tilde{X}$.

In particular, if $X^s=X^{ss}$, then $\tilde{X}^s=\tilde{X}^{ss}=\pi^{-1}(X^s)$. If $X^s/G=X/\!/G$ is projective, then $\mathrm{bl}_{Y^s}X^s/G=\tilde{X}/\!/G$ is projective as well.

In case $X^s\neq X^{ss}$, $\pi^{-1}(X^{ss})$ is the union of some of the strata $\tilde{S}_\beta$ described in [13]. (See [14, (3.4)].) For $G=SL(2)$, the indexing β are the weights of the actions of the maximal torus $\mathbb{C}^*$, on the fibers of L_d at $\mathbb{C}^*$-fixed points in $\pi^{-1}(X^{ss})$.

2.5. **Variation of GIT quotients.** We recall the variation of Geometric Invariant Theory (GIT) quotients from [4, 19]. Let $G=SL(2)$ and let X be an irreducible smooth projective variety acted on by G. Since G is simple, there exists at most one linearization on any ample line bundle L on X. Let L_0 and L_1 be two ample line bundles on X with G-linearizations for which semistability coincides with stability. Let $L_t=L_0^{1-t}\otimes L_1^t$ for rational $t\in[0,1]$. Then we have the following.

(1) $[0,1]$ is partitioned into subintervals $0=t_0<t_1<\cdots<t_n=1$ such that on each subinterval (t_{i-1},t_i) for $1\le i\le n$, the GIT quotient $X/\!/_tG$ with respect to L_t remains constant.
(2) The walls t_i are precisely those ample line bundles L_t for which there exists $x\in X$ where the maximal torus $\mathbb{C}^*$ of G fixes x and acts trivially on the fiber of L_t over x.

Let $\tau = t_i$ be such a wall and let X^0 be the union of G-orbits of all such x as in (2) for τ. Let $\tau^{\pm} = \tau \pm \delta$ for sufficiently small $\delta > 0$ and let $L^{\pm} = L_{\tau^{\pm}}$, $L^0 = L_\tau$. Further let $X^{ss}(*)$ be the set of semistable points with respect to L^* for $* = 0, +, -$. Let $X^{\pm} = X^{ss}(0) - X^{ss}(\mp)$ and let $X//G(*)$ be the quotient of X with respect to L^* for $* = 0, +, -$. Let $\nu^{\pm}$ be the weight of the $\mathbb{C}^*$ action on the fiber of $L^{\pm}$ at $x \in X^0$. Suppose $(\nu^+, \nu^-) = 1$ and the stabilizer of x is $\mathbb{C}^*$ for $x \in X^0$. Then we have the following.

(3) Let N be the normal bundle to X^0 in X and $N^{\pm}$ be the positive (resp. negative) weight space of N. Then $X^{\pm}//G$ is the locally trivial fibration over $X^0//G$ with fiber weighted projective space $\mathbb{P}(|w_j^{\pm}|)$ where $w_j^{\pm}$ are the weights of $N^{\pm}$ and $X//G(\pm) - X^{\pm}//G = X//G(0) - X^0//G$.
(4) The blow-up of $X//G(*)$ at $X^*//G$ for $* = 0, +, -$ is the fiber product $X//G(-) \times_{X//G(0)} X//G(+)$.

Example. Let $X = \mathbb{P}(\mathrm{Sym}^2\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^n)$ where $G = SL(2)$ acts on $\mathbb{C}^2$ and $\mathrm{Sym}^2\mathbb{C}^2$ in the obvious way and trivially on $\mathbb{C}^n$. Let us study the variation of the GIT quotient $X//_{(1,m)}G$ as we vary the line bundle $\mathcal{O}(1, m)$. The weights of the maximal torus $\mathbb{C}^*$ on $\mathrm{Sym}^2\mathbb{C}^2$ are $2, 0, -2$ and the weights on $\mathbb{C}^2 \otimes \mathbb{C}^n$ are $1, -1$. Hence, there is only one wall, namely at $m = 2$, and X^0 is the union of the G-orbits of $\{t_0^2\} \times \mathbb{P}(t_1 \otimes \mathbb{C}^n) \cong \mathbb{P}^{n-1}$ where t_0, t_1 are homogeneous coordinates of $\mathbb{P}^1$. The normal bundle N to X^0 has rank n and the positive weight space N^+ has rank $n-1$ while the negative weight space N^- has rank 1. Let $L_0 = \mathcal{O}(1, 1)$ and $L_1 = \mathcal{O}(1, 3)$. Then $\nu^+ = -1$ and $\nu^- = 1$. So $X^+//G$ is a $\mathbb{P}^{n-2}$-bundle on $X^0//G = \mathbb{P}^{n-1}$, namely the projective tangent bundle $\mathbb{P}T_{\mathbb{P}^{n-1}}$, and $X^-//G = X^0//G = \mathbb{P}^{n-1}$. Therefore, $X//_{(1,m)}G = X//G(+)$ for $m >> 0$ is the blow-up of $X//_{(1,1)}G = X//G(-)$ along $\mathbb{P}^{n-1}$.

3. Degree two case

In this section, we work out the degree two case as a warm-up. We prove the following.

Theorem 3.1. (1) *$\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ is the blow-up of $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^2 \otimes \mathbb{C}^n)//SL(2)$ along $\mathbb{P}(\mathrm{Sym}^2\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^n)//SL(2) = \mathbb{P}^{n-1}$.*

(2) *The rational cohomology ring $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2))$ is*

$$\mathbb{Q}[\xi, \alpha^2, \rho]/\left\langle \frac{(\rho + 2\alpha + \xi)^n - \xi^n}{\rho + 2\alpha} + \frac{(\rho - 2\alpha + \xi)^n - \xi^n}{\rho - 2\alpha}, \right.$$
$$\left. (\rho + 2\alpha + \xi)^n + (\rho - 2\alpha + \xi)^n, \xi^n\rho \right\rangle$$

where ξ, ρ, and α^2 are generators of degree $2, 2, 4$ respectively.

(3) *The Poincaré polynomial $P_t(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)) = \sum_{k \geq 0} t^k \dim H^k(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2))$ is*

$$\frac{(1 - t^{2n+2})(1 - t^{2n})(1 - t^{2n-2})}{(1 - t^2)^2(1 - t^4)}.$$

(4) *The Picard group of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ is*

$$\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \textit{for} \quad n \geq 3 \\ \mathbb{Z} & \textit{for} \quad n = 2 \end{cases}$$

Item (1) is Theorem 4.1 in [11]. We include the proof of (1) for reader's convenience. Let $W = W_2 = \mathrm{Sym}^2\mathbb{C}^2$ and $V = \mathbb{C}^n$. Let $G = SL(2)$ act on W in the obvious way and trivially on V. An element x in $\mathbb{P}(W \otimes V)$ is represented by an n-tuple of homogeneous quadratic polynomials in two variables t_0, t_1. We call the common zero locus in $\mathbb{P}^1$ of such n-tuple, the *base points* of x.

Let $\mathbf{P}_0$ be the open subset of semistable points in $\mathbb{P}(W \otimes V)$ with respect to the action of G. Let $\Sigma_0^k \subset \mathbf{P}_0$ be the locus of k base points for $k = 0, 1, 2$ so that we have a decomposition

$$\mathbf{P}_0 = \Sigma_0^0 \sqcup \Sigma_0^1 \sqcup \Sigma_0^2.$$

For $x \in \mathbf{P}_0$ to have two base points, the n homogeneous polynomials representing x should be all linearly dependent and hence $\Sigma_0^2 = [\mathbb{P}W \times \mathbb{P}V]^{ss}$ where the superscript ss denotes the semistable part with respect to $\mathcal{O}(1,1)$.

Let $\pi_1 : \mathbf{P}_1 \to \mathbf{P}_0$ be the blow-up along the smooth closed variety Σ_0^2 and let $\mathbf{P}_1^s$ be the stable part of $\mathbf{P}_1$ with respect to the linearization on $\mathcal{O}(1^\epsilon) := \pi_1^*\mathcal{O}(1) \otimes \mathcal{O}(-\epsilon E_1)$ for sufficiently small $\epsilon > 0$ where E_1 is the exceptional divisor of π_1.

We claim that there is a family of stable maps to $\mathbb{P}V$ of degree 2 parameterized by $\mathbf{P}_1^s$, which gives us a G-invariant morphism

$$\psi_1 : \mathbf{P}_1^s \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$$

and thus a morphism $\bar{\psi}_1 : \mathbf{P}_1^s/G \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$, such that $\bar{\psi}_1$ is an isomorphism on Σ_0^0/G. Note that Σ_0^0 is the space of all holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}V$ of degree 2. Since semistability coincides with stability for $\mathbf{P}_1$, $\mathbf{P}_1^s/G$ is irreducible projective normal and so is $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ because $\mathbb{P}^{n-1}$ is convex. Therefore, to deduce that $\bar{\psi}_1$ is an isomorphism, it suffices to show $\bar{\psi}_1$ is injective.

To construct a family of stable maps parameterized by $\mathbf{P}_1^s$, we start with the evaluation map $W^* \otimes (V \otimes W) \longrightarrow V$ which gives rise to

$$H^0(\mathbb{P}^1 \times \mathbf{P}_0, \mathcal{O}(2,1)) = W \otimes (V \otimes W)^* \longleftarrow V^* = H^0(\mathbb{P}V, \mathcal{O}(1)).$$

Hence we have a rational map

$$\varphi_0 : \mathbb{P}^1 \times \mathbf{P}_0 \dashrightarrow \mathbb{P}V = \mathbb{P}^{n-1}$$

which is a morphism on the open set $\mathbb{P}^1 \times \Sigma_0^0$. For $x \in \Sigma_0^1$, we can choose homogeneous coordinates t_0, t_1 of $\mathbb{P}^1$ such that x is represented by an n-tuple of homogeneous quadratic polynomials which are all linear combinations of t_0^2 and $t_0 t_1$. So x is a strictly semistable point in $\mathbf{P}_0$ whose orbit closure intersects with Σ_0^2. Hence x becomes unstable in $\mathbf{P}_1$ (see [14, §6]). The proper transform of Σ_0^1 does not appear in $\mathbf{P}_1^s$.

Let φ_1' be the composition of φ_0 and $\mathrm{id} \times \pi_1 : \mathbb{P}^1 \times \mathbf{P}_1^s \to \mathbb{P}^1 \times \mathbf{P}_0$. The two base points of each $x \in \Sigma_0^2$ are distinct by semistability and thus the locus in $\mathbb{P}^1 \times \mathbf{P}_1^s$ where φ_1' is undefined consists of two sections over $\mathbf{P}_1^s$ because Σ_0^2 is simply connected. Let $\mu_1 : \Gamma_1 \to \mathbb{P}^1 \times \mathbf{P}_1^s$ be the blow-up along the two sections and let $\varphi_1 = \varphi_1' \circ \mu_1$. The evaluation map above gives us a homomorphism $V^* \otimes \mathcal{O}_{\Gamma_1} \to \mu_1^*\mathcal{O}_{\mathbb{P}^1 \times \mathbf{P}_1^s}(2,1)$ which vanishes simply along the exceptional divisor $\mathcal{E}_1$ of μ_1 and hence we obtain a surjective homomorphism

$$V^* \otimes \mathcal{O}_{\Gamma_1} \twoheadrightarrow \mu_1^*\mathcal{O}_{\mathbb{P}^1 \times \mathbf{P}_1^s}(2,1) \otimes \mathcal{O}_{\Gamma_1}(-\mathcal{E}_1).$$

Therefore, we obtain a diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}^{n-1} \\ \mathbf{P}_1^s & \end{tikzcd}$$

The first map is a flat family of semistable curves and the restriction of the second map to each fiber of π is a degree 2 map. Further f factors through the contraction of the middle components in Γ_1 over the exceptional divisor of π_1. So we get a family of stable maps to $\mathbb{P}^{n-1}$ of degree 2 parameterized by $\mathbf{P}_1^s$ and thus a morphism $\psi_1 : \mathbf{P}_1^s \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$. By construction, ψ_1 is G-invariant and factors through a morphism $\bar{\psi}_1 : \mathbf{P}_1^s/G \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$. The locus Σ_0^0 of no base points is the set of all holomorphic maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$ of degree two and $\bar{\psi}_1|_{\Sigma_0^0/G}$ is an isomorphism onto the open subset in $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ of irreducible stable maps. The complement $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2) - \bar{\psi}_1(\Sigma_0^0)$ is the locus of stable maps of degree two with two irreducible components and thus $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2) - \bar{\psi}_1(\Sigma_0^0)$ is a $(\mathbb{P}^{n-2} \times \mathbb{P}^{n-2})/S_2$ bundle on $\mathbb{P}^{n-1}$. $\mathbb{P}^{n-1}$ determines the image of the intersection point of the two irreducible components and $(\mathbb{P}^{n-2} \times \mathbb{P}^{n-2})/S_2$ determines the pair of lines in $\mathbb{P}^{n-1}$ passing through the chosen point.

On the other hand, let $C = (f_j^\lambda = a_j t_0 t_1 + \lambda b_j t_0^2 + \lambda c_j t_1^2)_{1 \le j \le n, \lambda \in \mathbb{C}}$ represent a curve in $\mathbf{P}_0$ passing through $(a_j t_0 t_1)_{1 \le j \le n} \in \Sigma_0^2$. Suppose (a_j) is not parallel to (b_j) and (c_j) in $\mathbb{C}^n$. Then φ_0 restricts to $(f_1^\lambda : \cdots : f_n^\lambda) : \mathbb{P}^1 \times C \dashrightarrow \mathbb{P}^{n-1}$ and Γ_1 over C is the blow-up of $\mathbb{P}^1 \times C$ along $\{0, \infty\} \times \{0\}$. By direct local computation, we see immediately that the morphism constructed above $\Gamma_1 \to \mathbb{P}^{n-1}$ at $\lambda = 0$ is the map of the tree of three $\mathbb{P}^1$'s, whose left (resp. right) component is mapped to the line joining (a_j) and (b_j) (resp. (c_j)), and whose middle component is mapped to the point (a_j). This proves that $\bar{\psi}_1$ is bijective.

Next we study the cohomology ring of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$. By the isomorphism $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2) \cong \mathbf{P}_1^s/G$, we have an isomorphism in rational cohomology

$$H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)) \cong H_G^*(\mathbf{P}_1^s).$$

From §2.3, the Poincaré series of $H_G^*(\mathbf{P}_0) = H_{SL(2)}^*(\mathbb{P}(V \otimes W)^{ss})$ is

$$P_t^{SL(2)} = \frac{1}{(1-t^2)(1-t^4)}(1 - t^{4n-2} - t^{4n} + t^{6n-2})$$

and by (2.4), we have

(3.1)

$$H_G^*(\mathbf{P}_0) = \mathbb{Q}[\xi, \alpha^2] / \left\langle \xi^n \frac{(\xi + 2\alpha)^n - (\xi - 2\alpha)^n}{2\alpha}, \xi^n \frac{(\xi + 2\alpha)^n + (\xi - 2\alpha)^n}{2} \right\rangle$$

where ξ and α^2 are generators of degree 2 and 4 respectively.

By the blow-up formula [8, p.605], we have

$$P_t^{SL(2)}(\mathbf{P}_1) = P_t^{SL(2)}(\mathbf{P}_0) + \frac{1}{1-t^4} \frac{1-t^{2n}}{1-t^2} \frac{t^2 - t^{4n-4}}{1-t^2}$$

as $\Sigma_0^2/G = \mathbb{P}^{n-1}$. To find the ring $H_G^*(\mathbf{P}_1)$, we need to compute the normal bundle N to the blow-up center $(\mathbb{P}^2 \times \mathbb{P}^{n-1})^{ss}$. Let K and C be respectively the kernel and cokernel of the composition

$$\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{O}(0,1) \hookrightarrow \mathcal{O}(1,1)^{\oplus 3}$$

on $\mathbb{P}^2 \times \mathbb{P}^{n-1}$, induced from the tautological homomorphisms $\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{O}(1)$ on $\mathbb{P}^{n-1}$ and $\mathcal{O} \hookrightarrow \mathcal{O}(1)^{\oplus 3}$ on $\mathbb{P}^2$. By a simple diagram chase with the Euler sequences for the projective spaces, we see immediately that the normal bundle N to $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{3n-1}$ is the bundle $\mathrm{Hom}(K, C)$. By definition, the total Chern characters of K and C are respectively

$$ch(K) = n - e^{\xi_2} \quad \text{and} \quad ch(C) = e^{\xi_1+\xi_2+2\alpha} + e^{\xi_1+\xi_2} + e^{\xi_1+\xi_2-2\alpha} - e^{\xi_2}$$

where $\xi_1 = c_1(\mathcal{O}(1,0))$ and $\xi_2 = c_1(\mathcal{O}(0,1))$. Therefore,

$$ch(\mathrm{Hom}(K,C)) = ch(K^*)ch(C) = (n-e^{-\xi_2})(e^{\xi_1+\xi_2+2\alpha}+e^{\xi_1+\xi_2}+e^{\xi_1+\xi_2-2\alpha}-e^{\xi_2}).$$

If we restrict to the semistable part $(\mathbb{P}^2 \times \mathbb{P}^{n-1})^{ss}$ where $\xi_1 = 0$, we obtain

$$ch(\mathrm{Hom}(K,C)) = e^{2\alpha}(ne^{\xi} - 1) + e^{-2\alpha}(ne^{\xi} - 1)$$

with $\xi = \xi_2$ and thus the Chern classes are given by

$$\sum_{k=0}^{2n-2} t^k c_{2n-2-k}(N) = \frac{(t+2\alpha+\xi)^n - \xi^n}{t+2\alpha} \frac{(t-2\alpha+\xi)^n - \xi^n}{t-2\alpha}$$

for a formal variable t. The restriction of the Poincaré dual of the blow-up center to the exceptional divisor is the constant term in t of the above polynomial. Hence the difference between the constant term and the Poincaré dual of the blow-up center is a multiple of ξ^n because ξ^n generates the kernel of the restriction homomorphism

$$H^*_G(\mathbf{P}_0) \twoheadrightarrow H^*_G((\mathbb{P}^2 \times \mathbb{P}^{n-1})^{ss}) \cong \mathbb{Q}[\xi, \alpha^2]/\langle \xi^n \rangle.$$

Therefore, by Proposition 2.1

$$H^*_G(\mathbf{P}_1) = H^*_G(\mathbf{P}_0)[\rho]/\langle \xi^n\rho, \frac{(\rho+2\alpha+\xi)^n - \xi^n}{\rho+2\alpha} \frac{(\rho-2\alpha+\xi)^n - \xi^n}{\rho-2\alpha} + \xi^n q(\rho,\xi,\alpha)\rangle$$

for some homogeneous polynomial $q(\rho, \xi, \alpha)$ of degree $n-2$. Now, we subtract out the unstable part in $\mathbf{P}_1$. By the recipe of [14], we obtain

$$P_t^{SL(2)}(\mathbf{P}_1^s) = P_t^{SL(2)}(\mathbf{P}_1) - \frac{1}{1-t^2} \frac{1-t^{2n}}{1-t^2} \frac{t^{2n-2}(1-t^{2n-2})}{1-t^2}$$

$$= \frac{(1-t^{2n+2})(1-t^{2n})(1-t^{2n-2})}{(1-t^2)^2(1-t^4)}. \tag{3.2}$$

The above normal bundle N splits into the direct sum of two subbundles N^+ and N^- with respect to the weights. Their Chern classes are the coefficients of the polynomials in t

$$\frac{(t+2\alpha+\xi)^n - \xi^n}{t+2\alpha} \quad \text{and} \quad \frac{(t-2\alpha+\xi)^n - \xi^n}{t-2\alpha}.$$

The restrictions, of the normal bundle to the unstable stratum, to fixed point components are given by the pullbacks of N^+ and N^- respectively, tensored with $\mathcal{O}(1)$.

Using the notation of §2.3, the image of $H_G^{j-2\lambda(\beta)}(S_\beta)$ for the unique unstable stratum S_β in $H^*_G(\mathbf{P}_1)$ is generated by

$$\frac{(\rho+2\alpha+\xi)^n - \xi^n}{\rho+2\alpha} + \frac{(\rho-2\alpha+\xi)^n - \xi^n}{\rho-2\alpha}, (\rho+2\alpha+\xi)^n + (\rho-2\alpha+\xi)^n + c\xi^n \tag{3.3}$$

for some $c \in \mathbb{Q}$, because ξ^n generates the kernel of the restriction homomorphism

$$H^*_G(\mathbf{P}_1) \longrightarrow H^*_T(\mathbb{P}N^+) \oplus H^*_T(\mathbb{P}N^-)$$

by direct calculation. It is an elementary exercise to check that the two polynomials in (3.3) and $\xi^n\rho$ are pairwisely coprime and hence the Poincaré polynomial of the quotient ring of $\mathbb{Q}[\xi, \alpha^2, \rho]$ by the ideal generated by the three polynomials coincides with (3.2). Therefore, the three polynomials generate the relation ideal for $H^*_G(\mathbf{P}_1^s) \cong H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2))$. From the condition that the three polynomials generate the relations in (3.1), it is easy to deduce that $c = 0$. So we proved

$$H^*_G(\mathbf{P}_1^s) \cong \mathbb{Q}[\xi, \alpha^2, \rho]/\left\langle \frac{(\rho + 2\alpha + \xi)^n - \xi^n}{\rho + 2\alpha} + \frac{(\rho - 2\alpha + \xi)^n - \xi^n}{\rho - 2\alpha}, \right.$$
$$\left. (\rho + 2\alpha + \xi)^n + (\rho - 2\alpha + \xi)^n, \xi^n\rho \right\rangle$$

as desired.

In [1], Behrend and O'Halloran prove that $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)) = \mathbb{Q}[b, t, k]/(G_n)$ where b, t are degree 2 generators and k is a degree 4 generator. Here, G_n denotes three polynomials defined recursively by the matrix equation $G_n = A^{n-1}G_1$, where

$$A = \begin{pmatrix} b & 0 & 0 \\ 1 & 0 & k \\ 0 & 1 & t \end{pmatrix}, \qquad G_1 = \begin{pmatrix} b(2b - t) \\ 2b - t \\ 2 \end{pmatrix}.$$

This presentation is equivalent to ours by the following change of variables

$$b = \xi, \quad t = 2(\xi + \rho), \quad k = 4\alpha^2 - (\xi + \rho)^2.$$

Finally, the Picard group of $\mathbf{P}_0$ is $\mathbb{Z}$ and the group $\mathrm{Pic}(\mathbf{P}_0)^G$ of equivariant line bundles is a subgroup because the acting group is $G = SL(2)$. Hence, $\mathrm{Pic}(\mathbf{P}_0)^G = \mathbb{Z}$. By the blow-up formula of the Picard group [9, II,§8], we obtain $\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)) \cong \mathrm{Pic}(\mathbf{P}_1^s)^G = \mathbb{Z}^2$ for $n \geq 3$ because the blow-up center is invariant. The first isomorphism comes from Kempf's descent lemma [5]. When $n = 2$, $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 2)$ is $\mathbb{P}^2$ and hence the Picard group is $\mathbb{Z}$.

4. A birational transformation

In this section, we study a birational transformation that will be used in the subsequent section.

Let $V_i = \mathbb{C}^n$ for $i = 1, \cdots, r$ and let $V = \oplus_{i=1}^r V_i$. For $z \in V$, we write $z = (z_1, \cdots, z_r)$ with $z_i \in V_i$. For any subset $I \subset \{1, \cdots, r\}$, we let

$$\bar{\Sigma}^I = \{z \in V \,|\, z_i = 0 \text{ for } i \in I\} \quad \text{and} \quad \bar{\Sigma}_0^k = \cup_{|I|=k} \bar{\Sigma}^I$$

for $k \leq r$. We will blow up V, r times and then blow down $(r-1)$ times to obtain $\prod_{i=1}^r \mathcal{O}_{\mathbb{P}V_i}(-1)$, a rank r vector bundle on $(\mathbb{P}^{n-1})^r$.

The blow-ups are defined inductively as follows. Let $X_0 = V$. For $1 \leq j \leq r$, let $\pi_j : X_j \to X_{j-1}$ be the blow-up of $\bar{\Sigma}_{j-1}^{r-j+1}$ and let $\bar{\Sigma}_j^k$ be the proper transform of $\bar{\Sigma}_{j-1}^k$ for $k \neq r - j + 1$, while $\bar{\Sigma}_j^{r-j+1}$ is the exceptional divisor of π_j. We claim then that X_r can be blown down first along $\bar{\Sigma}_r^2$, next along the proper transform $\bar{\Sigma}_{r+1}^3$ of $\bar{\Sigma}_r^3$, next along the proper transform $\bar{\Sigma}_{r+2}^4$ of $\bar{\Sigma}_r^4$ and so on until the blow-down along the proper transform $\bar{\Sigma}_{2r-2}^r$ of $\bar{\Sigma}_r^r$.

Proposition 4.1. *After the r blow-ups and $(r-1)$ blow-downs as above, V becomes $\prod_{i=1}^r \mathcal{O}_{\mathbb{P}V_i}(-1)$.*

Proof. We use induction on r. For $r = 1$, there is nothing to prove. Suppose it holds true for $r-1$. We have an open covering $X_1 = \cup_{i=1}^r U_i$ of $X_1 = \mathcal{O}_{\mathbb{P}V}(-1)$ with $U_i = X_1 - \bar{\Sigma}_1^{\{i\}}$ where $\bar{\Sigma}_1^{\{i\}}$ is the proper transform of $\bar{\Sigma}^{\{i\}} = \{z \in V \,|\, z_i = 0\}$, since $\cap_{i=1}^r \bar{\Sigma}_1^{\{i\}} = \emptyset$. Then $U_i = \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)n}$. Over an affine open subset $\mathbb{C}^{n-1}$ of $\mathbb{P}^{n-1}$, U_i is $\mathbb{C}^n \times \oplus_{i=1}^{r-1}\mathbb{C}^n$ and the blow-ups described above give us the corresponding blow-ups of $\oplus_{i=1}^{r-1}\mathbb{C}^n$. By considering an affine open cover of $\mathbb{P}^{n-1}$ and by the induction hypothesis, we see that after $(r-1)$ more blow-ups and $(r-2)$ blow-downs, U_i becomes

$$\mathcal{O}_{(\mathbb{P}^{n-1})^r}(-1, \overrightarrow{0}) \oplus \bigoplus_{k=1}^{r-1} \mathcal{O}_{(\mathbb{P}^{n-1})^r}(1, -\overrightarrow{e_k})$$

where $\overrightarrow{e_1}, \cdots, \overrightarrow{e}_{r-1}$ are standard basis vectors. Therefore, after r blow-ups and $(r-2)$ blow-downs, V becomes a smooth projective variety X_{2r-2} which admits an open covering $\cup_{i=1}^r (\mathcal{O}_{(\mathbb{P}^{n-1})^r}(-1, \overrightarrow{0}) \oplus \bigoplus_{k=1}^{r-1} \mathcal{O}_{(\mathbb{P}^{n-1})^r}(1, -\overrightarrow{e_k}))$. By checking the transition maps at general points, we see that X_{2r-2} is the total space of $\mathcal{O}(-1)$ over $\mathbb{P}(\oplus_{i=1}^r \mathcal{O}(-\overrightarrow{e_i}))$ over $(\mathbb{P}^{n-1})^r$. Hence, X_{2r-2} is the blow-up of $\oplus_{i=1}^r \mathcal{O}(-\overrightarrow{e_i})$ over $(\mathbb{P}^{n-1})^r$ along the zero section. Hence we can blow down X_{2r-2} further to obtain

$$\oplus_{i=1}^r \mathcal{O}_{(\mathbb{P}^{n-1})^r}(-\overrightarrow{e_i}) = \prod_{i=1}^r \mathcal{O}_{\mathbb{P}V_i}(-1)$$

as desired. □

Remark 4.2. More generally, let $E \to X$ be a fiber bundle locally the direct sum of r vector bundles of rank n. Then we can similarly define $\bar{\Sigma}_0^k$ and perform r blow-ups and $(r-1)$ blow-downs as above. Proposition 4.1 holds true in this slightly more general setting by the same proof.

5. Main construction

Let $n \geq 2$ and $V = \mathbb{C}^n$. Let $W_d = \mathrm{Sym}^d\mathbb{C}^2 = H^0(\mathbb{P}^1, \mathcal{O}(d))$ be the space of homogeneous polynomials of degree d in two variables $t_0, t_1 \in H^0(\mathbb{P}^1, \mathcal{O}(1))$. We will frequently drop the subscript d for convenience. The identity element $\mathbb{C} \to \mathrm{Hom}(W, W) = W \otimes W^*$ gives us a nontrivial homomorphism

$$H^0(\mathbb{P}^1 \times \mathbb{P}(V \otimes W), \mathcal{O}(d, 1)) = W \otimes (V \otimes W)^* \leftarrow V^* = H^0(\mathbb{P}V, \mathcal{O}(1)) \tag{5.1}$$

and thus we obtain a rational map

$$\mathbb{P}^1 \times \mathbb{P}(V \otimes W) \dashrightarrow \mathbb{P}V. \tag{5.2}$$

Let $\mathbf{P} = \mathbb{P}(V \otimes W_d) \cong \mathbb{P}^{(d+1)n-1}$. The irreducible $SL(2)$-representation on W_d induces a linear action of $SL(2)$ on $\mathbf{P}$ and let $\mathbf{P}^s$ be the stable part of $\mathbf{P}$ with respect to this action in Mumford's GIT sense. Then (5.2) restricts to a rational map

$$\varphi_0 : \mathbb{P}^1 \times \mathbf{P}^s \dashrightarrow \mathbb{P}V \tag{5.3}$$

which gives us a rational map $\psi_0 : \mathbf{P}^s \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$. Since this map is clearly $SL(2)$-invariant, we obtain a rational map

$$\bar{\psi}_0 : \mathbb{P}^{(d+1)n-1}/\!/SL(2) = \mathbf{P}^s/SL(2) \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d). \tag{5.4}$$

In this section, we prove the following.

Theorem 5.1. *For $d = 3$, the birational map $\bar{\psi}_0$ is the composition of three blow-ups followed by two blow-downs. The blow-up centers are respectively, $\mathbb{P}^{n-1}$, $\overline{\mathbf{M}}_{0,2}(\mathbb{P}^{n-1}, 1)/S_2$ (where S_2 interchanges the two marked points) and the blow-up of $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^{n-1}, 2)$ along the locus of three irreducible components. The centers of the blow-downs are respectively the S_2-quotient of a $(\mathbb{P}^{n-2})^2$-bundle on $\overline{\mathbf{M}}_{0,2}(\mathbb{P}^{n-1}, 1)$ and a $(\mathbb{P}^{n-2})^3/S_3$-bundle on $\mathbb{P}^{n-1}$.*

5.1. **Stratification of $\mathbf{P}^s$.** An element $\xi \in \mathbf{P}$ is represented by a choice of n sections of $H^0(\mathbb{P}^1, \mathcal{O}(d))$. If there is no *base point* (i.e. common zero) of ξ, we get a regular morphism $\varphi_0|_{\mathbb{P}^1\times\{\xi\}}$ from $\mathbb{P}^1$ to $\mathbb{P}^{n-1}$ of degree d and thus $\psi_0 : \mathbf{P}^s \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, d)$ is well defined at ξ.

Let us focus on the $d = 3$ case, from now on. The following is immediate from the Hilbert-Mumford criterion for stability [16].

Lemma 5.2. *(1) Semistability coincides with stability, i.e. $\mathbf{P}^{ss} = \mathbf{P}^s$.*

(2) $\mathbf{P}^s$ consists of $\xi \in \mathbf{P}$ which has no base point of multiplicity ≥ 2.

Let $\mathbf{P}_0 = \mathbf{P}^s$. We decompose $\mathbf{P}^s$ by the number of base points:

$$\mathbf{P}_0 = \Sigma_0^0 \sqcup \Sigma_0^1 \sqcup \cdots \sqcup \Sigma_0^d \tag{5.5}$$

where Σ_0^k is the locus in $\mathbf{P}_0$ of ξ with k base points for $k = 0, 1, \cdots, d$. We put the subscript 0 is to keep track of the blow-ups and -downs in what follows. When it is necessary to specify the degree d, we shall write $\mathbf{P}_j(d)$ for $\mathbf{P}_j$ and $\Sigma_j^i(d)$ for Σ_j^i. The rational map φ_0 is well-defined on the open set $\mathbb{P}^1 \times \Sigma_0^0$ and thus we have a family of degree d stable maps to $\mathbb{P}^{n-1}$.

By Lemma 5.2, no element of $\mathbf{P}^s$ admits a base point of multiplicity ≥ 2. The following proposition gives us a local description of the stratification (5.5).

Lemma 5.3. *(1) For $k = 1, 2, 3$, Σ_0^{4-k} are locally closed smooth subvarieties of $\mathbf{P}_0$ whose $SL(2)$-quotients are respectively, $\mathbb{P}^{n-1}$, $\mathbf{M}_{0,2}(\mathbb{P}^{n-1}, 1)/S_2$ and $\mathbf{M}_{0,1}(\mathbb{P}^{n-1}, 2)$.*

(2) The normal cone of Σ_0^3 in the closure $\bar{\Sigma}_0^2$ is a fiber bundle locally the union of three transversal rank $n-1$ subbundles of the normal bundle $N_{\Sigma_0^3/\mathbf{P}_0}$.

(3) The normal cone of Σ_0^3 in the closure $\bar{\Sigma}_0^1$ is a fiber bundle locally the union of three transversal rank $2(n-1)$ subbundles of the normal bundle $N_{\Sigma_0^3/\mathbf{P}_0}$.

(4) The normal cone of Σ_0^2 in the closure $\bar{\Sigma}_0^1$ is a fiber bundle locally the union of two transversal rank $n-1$ subbundles of the normal bundle $N_{\Sigma_0^2/\mathbf{P}_0}$.

The deepest stratum is easy to describe. In order to have three base points, ξ must be represented by a rank 1 homomorphism in $\mathbb{P}(V \otimes W) = \mathbb{P}\operatorname{Hom}(V^*, W)$. Since any rank 1 homomorphism factors through $\mathbb{C}$, the locus of rank 1 homomorphisms in $\mathbb{P}\operatorname{Hom}(V^*, W)$ is $\mathbb{P}V \times \mathbb{P}W$ and hence $\Sigma_0^3 = [\mathbb{P}^3 \times \mathbb{P}^{n-1}]^s$. Since the GIT quotient $\mathbb{P}W/\!/SL(2)$ is just a point,

$$\Sigma_0^3/\!/SL(2) \cong \mathbb{P}^{n-1}. \tag{5.6}$$

It is important to remember that the stabilizer in $PGL(2) = SL(2)/\{\pm 1\}$ of a point ξ in Σ_0^3 is the symmetric group S_3. This group will act nontrivially on the exceptional divisors after blow-ups.

For Σ_0^1 and Σ_0^2, we consider the multiplication morphism

$$\Phi^k : \mathbb{P}^k \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^{4-k}) \longrightarrow \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^4) \tag{5.7}$$

defined by $\Phi^k((a_0:\cdots:a_k),(b_{i,0}:\cdots:b_{i,3-k})_{1\le i\le n})\to(c_{i,j})_{1\le i\le n,0\le j\le 3}$ where $c_{i,j}=\sum_{l=0}^{j}a_l b_{i,j-l}$. Here $a_l=0$, $b_{i,j}=0$ unless $0\le l\le k$, $0\le j\le 3-k$. Then for $k\le l$, we have $(\Phi^k)^{-1}(\Sigma_0^l(3))=[\mathbb{P}^k\times\Sigma_0^{l-k}(3-k)]^s$ where the superscript s denotes the stable part with respect to the $SL(2)$-action on $\mathcal{O}(1,1)$. Furthermore, Φ^k maps $[\mathbb{P}^k\times\Sigma_0^0(3-k)]^s$ bijectively onto $\Sigma_0^k(3)=\Sigma_0^k$. It is easy to see that the tangent map of Φ^k is injective over the open locus of distinct (or no) base points in $\mathbb{P}(\mathbb{C}^n\otimes\mathbb{C}^4)$, and hence we obtain an isomorphism $[\mathbb{P}^k\times\Sigma_0^0(3-k)]^s\cong\Sigma_0^k$ for $k=1,2$. Therefore

$$\Sigma_0^k/SL(2)=[\mathbb{P}^k\times\Sigma_0^0(3-k)]//SL(2)\cong\mathbf{M}_{0,k}(\mathbb{P}^{n-1},3-k)/S_k$$

because $\Sigma_0^0(3-k)$ is the space of holomorphic maps $\mathbb{P}^1\to\mathbb{P}^{n-1}$ of degree $3-k$.

Next, let us determine the normal cones $C_{\Sigma_0^3/\bar\Sigma_0^1}$, $C_{\Sigma_0^3/\bar\Sigma_0^2}$ and $C_{\Sigma_0^2/\bar\Sigma_0^1}$. As observed above, Φ^2 and Φ^1 are immersions into $\mathbf{P}_0$ and they are clearly three to one over Σ_0^3. Since Φ^k are immersions, we see by local computation that the directions of the locus Σ_0^2 of two base points in the normal bundle $N_{\Sigma_0^3/\mathbf{P}_0}$ consists of a fiber bundle locally the union of three transversal subbundles of rank $n-1$ and that the directions of the locus Σ_0^1 of one base point consists of a fiber bundle locally the union of three transversal subbundles of rank $2n-2$ whose mutual intersections are precisely the rank $n-1$ subbundles for Σ_0^2. Similarly by local computation, we obtain that the normal cone to Σ_0^2 in $\bar\Sigma_0^1$ consists of a fiber bundle locally the union of two transversal subbundles of rank $n-1$ of the normal bundle $N_{\Sigma_0^2/\mathbf{P}_0}$.

5.2. **Blow-ups.** As in the previous subsection, we let $\mathbf{P}_0$ be the stable part of $\mathbb{P}(V\otimes W)$ and let $\bar\Sigma_0^k$ be the closure of Σ_0^k in $\mathbf{P}_0$. Let $\mathbf{P}_1$ be the blow-up of $\mathbf{P}_0$ along the smooth subvariety Σ_0^3 and let $\bar\Sigma_1^k$ be the proper transform of $\bar\Sigma_0^k$ for $k=1,2$. Let $\bar\Sigma_1^3$ be the exceptional divisor of the blow-up. Then by Lemma 5.3, $\bar\Sigma_1^2$ is a smooth subvariety of $\mathbf{P}_1$ and in fact $\bar\Sigma_1^2$ is the line bundle $\mathcal{O}(-1)$ over the disjoint union of three projective bundles on Σ_0^3 in a neighborhood of $\bar\Sigma_1^3\cap\bar\Sigma_1^2$. Furthermore, $\bar\Sigma_1^1$ about $\bar\Sigma_1^1\cap\bar\Sigma_1^3$ is the union of three projective subbundles of $\mathbb{P}N_{\Sigma_0^3/\mathbf{P}_0}$ of fiber dimension $2n-3$ whose mutual intersections are the three projective bundles $\bar\Sigma_1^3\cap\bar\Sigma_1^2$.

Next we blow up $\mathbf{P}_1$ along the smooth subvariety $\bar\Sigma_1^2$ and obtain a variety $\mathbf{P}_2$. Let $\bar\Sigma_2^k$ be the proper transform of $\bar\Sigma_1^k$ for $k=1,3$ and let $\bar\Sigma_2^2$ be the exceptional divisor of the blow-up. Then by Lemma 5.3 again, $\bar\Sigma_2^1$ is a smooth subvariety of $\mathbf{P}_2$.

Let $\mathbf{P}_3$ be the blow-up of $\mathbf{P}_2$ along the smooth subvariety $\bar\Sigma_2^1$ and let $\bar\Sigma_3^k$ be the proper transform of $\bar\Sigma_2^k$ for $k\ne 1$. Let $\bar\Sigma_3^1$ be the exceptional divisor of the blow-up.

By [14, Lemma 3.11], blow-up commutes with quotient. Therefore, $\mathbf{P}_3$ is a smooth quasi-projective variety whose quotient by the induced $SL(2)$ action is the blow-up of $\mathbb{P}(V\otimes W)//SL(2)$ along $\Sigma_0^3/SL(2)=\mathbb{P}^{n-1}$, $\bar\Sigma_1^2/SL(2)$ and $\bar\Sigma_2^1/SL(2)$.

Lemma 5.4. *(1) $\bar\Sigma_1^2/SL(2)$ is the blow-up of the GIT quotient $\mathbb{P}^2\times\mathbb{P}(\mathbb{C}^n\otimes\mathbb{C}^2)//_{(1,1)}SL(2)$ with respect to the linearization $\mathcal{O}(1,1)$, along $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^{n-1}//_{(1,1,1)}SL(2)$ with respect to the linearization $\mathcal{O}(1,1,1)$.*

(2) $\bar\Sigma_2^1/SL(2)$ is the blow-up of $\mathbb{P}^1\times\mathbb{P}(\mathbb{C}^n\otimes\mathbb{C}^3)//_{(1,1)}SL(2)$, first along $\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^{n-1}//_{(1,1,1)}SL(2)$ and second along the proper transform of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}(\mathbb{C}^n\otimes\mathbb{C}^2)//_{(1,1,1)}SL(2)$.

Proof. We claim that for $k=1,2$, the morphism

$$\Phi^k:[\mathbb{P}^k\times\mathbb{P}(\mathbb{C}^n\otimes\mathbb{C}^{4-k})]^s\to\bar\Sigma_0^k\subset\mathbf{P}_0$$

becomes an equivariant isomorphism after $3-k$ blow-ups. Since $\bar{\Sigma}^k_{3-k}$ for $k=1,2$ are smooth, it suffices to show that the induced map

$$\Phi^k_{3-k} : [\mathbb{P}^k \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^{4-k})]^s_{3-k} \to \bar{\Sigma}^k_{3-k} \subset \mathbf{P}_{3-k}$$

is bijective where $[\mathbb{P}^k \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^{4-k})]^s_{3-k}$ denotes the result of $3-k$ blow-ups along $[\mathbb{P}^k \times \bar{\Sigma}^{3-k-i}_i(3-k)]^s$ for $i = 0, \cdots, 3-k-1$. By Lemma 5.3, this is a straightforward exercise. For instance, when $k=2$, the fiber of $\pi_1 : \bar{\Sigma}^2_1 \to \bar{\Sigma}^2_0$ over a point in Σ^3_0 is the disjoint union of three copies of $\mathbb{P}^{n-2}$ which is the same as the fiber of $\pi_1 \circ \Phi^2_1 = \Phi^2 \circ \tilde{\pi}_1$ where $\tilde{\pi}_1$ is the blow-up map of $[\mathbb{P}^2 \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^2)]^s$ along the smooth subvariety $[\mathbb{P}^2 \times \Sigma^1_0(1)]^s$ of codimension $n-1$, since Φ^2 is three to one over Σ^3_0.

By the above claim, $\bar{\Sigma}^2_1/\mathsf{SL}(2)$ is isomorphic to the blow-up of the GIT quotient $\mathbb{P}^2 \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^2)/\!/_{(1,1)}\mathsf{SL}(2)$ with respect to the linearization $\mathcal{O}(1,1)$, along $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^{n-1}/\!/_{(1,1,1)}\mathsf{SL}(2)$ with respect to the linearizaiton $\mathcal{O}(1,1,1)$. Similarly, we obtain (2). □

Corollary 5.5. *(1) $\bar{\Sigma}^2_1/\mathsf{SL}(2)$ is $\overline{\mathbf{M}}_{0,2}(\mathbb{P}^{n-1},1)/S_2$. In other words, it is $\mathbb{P}\mathrm{Sym}^2(\mathcal{U})$ over the Grassmannian $\mathsf{Gr}(2,n)$ where $\mathcal{U}$ is the universal rank 2 bundle. In particular, the Poincaré polynomial of $\bar{\Sigma}^2_1/\mathsf{SL}(2)$ is that of $\mathbb{P}^2 \times \mathsf{Gr}(2,n)$.*

(2) $\bar{\Sigma}^1_2/\mathsf{SL}(2)$ is the blow-up of $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^{n-1},2)$ along the locus of three irreducible components. This locus is the S_2-quotient of the fiber product $\mathbb{P}(\mathcal{O}^{\oplus n}/\mathcal{O}(-1)) \times_{\mathbb{P}^{n-1}} \mathbb{P}(\mathcal{O}^{\oplus n}/\mathcal{O}(-1))$. In particular, the Poincaré polynomial of $\bar{\Sigma}^1_2/\mathsf{SL}(2)$ is that of $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-2}$.

Proof. (1) Note that $\mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^2)/\!/\mathsf{SL}(2)$ is the Grassmannian $\mathsf{Gr}(2,n)$ and the quotient $\mathbb{P}^2 \times \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^2)/\!/_{(1,m)}\mathsf{SL}(2)$ with respect to the linearization $\mathcal{O}(1,m)$ for $m > 2$ is $\mathbb{P}\mathrm{Sym}^2(\mathcal{U})$ over the Grassmannian $\mathsf{Gr}(2,n)$ where $\mathcal{U}$ is the universal rank 2 bundle. We study the variation of GIT quotients as we vary our linearization from $\mathcal{O}(1,1)$ to $\mathcal{O}(1,m)$ with $m > 2$. (See [19, 4].) There is only one wall at $m=2$ and the flip consists of only the blow-up along $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^{n-1}/\!/_{(1,1,1)}\mathsf{SL}(2)$ with respect to the linearizaiton $\mathcal{O}(1,1,1)$. See §2.5.

(2) By the degree two case in §3, we have

$$\mathrm{bl}_{\Sigma^2_0(2)}\mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^3)/\!/_{1^\epsilon}\mathsf{SL}(2) \cong \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2)$$

where the linearization is $\pi_1^*\mathcal{O}(1) \otimes \mathcal{O}(-\epsilon E) =: \mathcal{O}(1^\epsilon)$ with E the exceptional divisor of the blow-up π_1 of $\mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^3)$. Let

$$\mathbf{P}_1(2) = \mathrm{bl}_{\Sigma^2_0(2)}\mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^3)^s$$

and $\mathbf{P}^s_1(2)$ be its stable part. Then $\mathbb{P}^1 \times \mathbf{P}^s_1(2)/\!/_{(1,m\cdot 1^\epsilon)}\mathsf{SL}(2)$ for $m >> 0$ is a $\mathbb{P}^1$-bundle over $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2)$. There is only one wall at $m_0 = 1/2\epsilon$ as we vary the linearization from $(1, 1^\epsilon)$ to $(1, m \cdot 1^\epsilon)$ with $m >> 0$. It is straightforward to check that the flip at m_0 is the composition of a blow-up and a blow-down as follows: the blow-up is precisely the quotient of the blow-up

$$\mu_1 : \Gamma_1 \longrightarrow \mathbb{P}^1 \times \mathbf{P}^s_1(2)$$

in §3 while the blow-down contracts the middle components of the curves in Γ_1 lying over $\Sigma^2_1(2)$. The result of the blow-down is obviously the universal curve $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^{n-1},2)$ over $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},2)$. □

5.3. **Blow-downs.** Now we show that $\mathbf{P}_3$ can be blown down twice. First, note that locally the normal bundle to $\bar{\Sigma}_1^2$ is the direct sum of two vector bundles, say $V_1 \oplus V_2$, and the normal cone in $\bar{\Sigma}_1^1$ is $V_1 \cup V_2$. Hence the proper transform of $\bar{\Sigma}_1^2$ in $\mathbf{P}_3$ is the blow-up of $\mathbb{P}(V_1 \oplus V_2)$ along $\mathbb{P}V_1 \sqcup \mathbb{P}V_2$, which is a $\mathbb{P}^1$-bundle on $\mathbb{P}V_1 \times_{\bar{\Sigma}_1^2} \mathbb{P}V_2$. From §4, we can blow down along this $\mathbb{P}^1$-bundle to obtain a complex manifold $\mathbf{P}_4$ with a locally free action of $\mathsf{SL}(2)$. The proper transform of $\bar{\Sigma}_1^2$ in $\mathbf{P}_4$ is a $(\mathbb{P}^{n-2})^2$-bundle on $\bar{\Sigma}_1^2$.

Next, the normal bundle to Σ_0^3 is locally the direct sum of three vector bundles, say $V_1 \oplus V_2 \oplus V_3$ and the normal cone in $\bar{\Sigma}_0^2$ is $V_1 \cup V_2 \cup V_3$ while the normal cone in $\bar{\Sigma}_0^1$ is $\bigcup_{i \neq j}(V_i \oplus V_j)$. Hence the proper transform of Σ_0^3 in $\mathbf{P}_3$ is the blow-up of $\mathbb{P}(V_1 \oplus V_2 \oplus V_3)$ along $\sqcup_i \mathbb{P}V_i$ and then along $\sqcup_{i \neq j} \mathbb{P}(\widetilde{V_i \oplus V_j})$ where $\mathbb{P}(\widetilde{V_i \oplus V_j})$ is the blow-up of $\mathbb{P}(V_i \oplus V_j)$ along $\mathbb{P}V_i \sqcup \mathbb{P}V_j$. From §4, we can blow down $\mathbf{P}_4$ further along the proper transform $\bar{\Sigma}_4^3$ of $\bar{\Sigma}_3^3$, to obtain a complex manifold $\mathbf{P}_5$. The image $\bar{\Sigma}_5^3$ in $\mathbf{P}_5$ of $\bar{\Sigma}_4^3$ is a $(\mathbb{P}^{n-2})^3$-bundle on Σ_0^3.

5.4. **The Kontsevich moduli space as an $\mathsf{SL}(2)$-quotient.** In this subsection, we prove that there is a family of stable maps of degree 3 to $\mathbb{P}^{n-1}$ parameterized by $\mathbf{P}_3$ so that the rational map $\psi_0 : \mathbf{P}_0 \dashrightarrow \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ extends to a morphism $\psi_3 : \mathbf{P}_3 \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$. Furthermore, ψ_3 factors through the blow-downs $\mathbf{P}_3 \to \mathbf{P}_4 \to \mathbf{P}_5$ and we obtain an $\mathsf{SL}(2)$-invariant map $\psi_5 : \mathbf{P}_5 \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$. The induced map $\bar{\psi}_5 : \mathbf{P}_5/\mathsf{SL}(2) \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ is bijective and hence an isomorphism of varieties because $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ is normal projective.

In summary, we have the following diagram.

$$\begin{tikzcd}
\mathbf{P}_3 \arrow[r, "\pi_3"] \arrow[d, "\pi_4"'] \arrow[dr, "p_3"] & \mathbf{P}_2 \arrow[r, "\pi_2"] & \mathbf{P}_1 \arrow[r, "\pi_1"] & \mathbf{P}_0 \arrow[d] \\
\mathbf{P}_4 \arrow[d, "\pi_5"'] & \mathbf{P}_3/\mathsf{SL}(2) \arrow[rr] \arrow[d] \arrow[drr, "\bar{\psi}_3"] & & \mathbf{P}_0/\mathsf{SL}(2) \arrow[d, dotted, "\bar{\psi}_0"] \\
\mathbf{P}_5 \arrow[r, "p_5"] & \mathbf{P}_5/\mathsf{SL}(2) \arrow[rr, "\cong", "\bar{\psi}_5"'] & & \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3).
\end{tikzcd}$$

Proposition 5.6. $\mathbf{P}_5/\mathsf{SL}(2)$ *is isomorphic to* $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$.

Let $\varphi_1' : \mathbb{P}^1 \times \mathbf{P}_1 \to \mathbb{P}^1 \times \mathbf{P}_0 \dashrightarrow \mathbb{P}^{n-1}$ be the composition of $\mathrm{id} \times \pi_1$ and φ_0. The locus of undefined points of φ_1' lying over $\bar{\Sigma}_1^3$ consists of three sections over $\bar{\Sigma}_1^3$ and let $\mu_1 : \Gamma_1 \to \mathbb{P}^1 \times \mathbf{P}_1$ be the blow-up along the union of the three sections which is a codimension 2 subvariety. Let $H_1 = \mu_1^* \mathcal{O}_{\mathbb{P}^1 \times \mathbf{P}_1}(3,1) \otimes \mathcal{O}_{\Gamma_1}(-\mathcal{E}_1)$ where $\mathcal{E}_1$ is the exceptional divisor of the blow-up μ_1. Then the evaluation homomorphism $ev_1' : \mathcal{O}_{\Gamma_1}^{\oplus n} \to \mu_1^* \mathcal{O}_{\mathbb{P}^1 \times \mathbf{P}_1}(3,1)$ factors through $ev_1 : \mathcal{O}_{\Gamma_1}^{\oplus n} \to H_1$ since ev_1' vanishes along $\mathcal{E}_1$ by construction. Obviously $\Gamma_1 \to \mathbf{P}_1$ is a flat family of semistable curves of genus 0. Let $\varphi_1 = \varphi_1' \circ \mu_1$.

Next, we let $\varphi_2' : \Gamma_1 \times_{\mathbf{P}_1} \mathbf{P}_2 \to \Gamma_1 \dashrightarrow \mathbb{P}^{n-1}$ be the composition of φ_1 with the obvious morphism $\Gamma_1 \times_{\mathbf{P}_1} \mathbf{P}_2 \to \Gamma_1$. Over the divisor $\bar{\Sigma}_2^2$, there are two sections on $\Gamma_1 \times_{\mathbf{P}_1} \mathbf{P}_2$ where φ_2' is not defined. Let $\mu_2 : \Gamma_2 \to \Gamma_1 \times_{\mathbf{P}_1} \mathbf{P}_2$ be the blow-up along the union of these two sections. Certainly Γ_2 is a family of semistable curves of genus 0 and the pull-back $ev_2' : \mathcal{O}_{\Gamma_2}^{\oplus n} \to \mu_2^* H_1$ of ev_1 factors through

$ev_2 : \mathcal{O}_{\Gamma_2}^{\oplus n} \to H_2 = \mu_2^* H_1 \otimes \mathcal{O}_{\Gamma_2}(-\mathcal{E}_2)$ where $\mathcal{E}_2$ is the exceptional divisor of μ_2, since ev_2' is vanishing along $\mathcal{E}_2$. Let $\varphi_2 = \varphi_2' \circ \mu_2$.

Similarly, let $\varphi_3' : \Gamma_2 \times_{\mathbf{P}_2} \mathbf{P}_3 \to \Gamma_2 \dashrightarrow \mathbb{P}^{n-1}$ be the composition of φ_2 with the obvious morphism $\Gamma_2 \times_{\mathbf{P}_2} \mathbf{P}_3 \to \Gamma_2$. Let $\mu_3 : \Gamma_3 \to \Gamma_2 \times_{\mathbf{P}_2} \mathbf{P}_3$ be the blow-up of the section of undefined points of φ_3' over $\bar{\Sigma}_3^1$. Then Γ_3 is a family of semistable curves of genus 0 parameterized by $\mathbf{P}_3$ and the pull-back $ev_3' : \mathcal{O}_{\Gamma_3}^{\oplus n} \to \mu_3^* H_2$ of ev_2 factors through $ev_3 : \mathcal{O}_{\Gamma_3}^{\oplus n} \to H_3 = \mu_3^* H_2 \otimes \mathcal{O}_{\Gamma_3}(-\mathcal{E}_3)$ where $\mathcal{E}_3$ is the exceptional divisor of μ_2. We claim ev_3 is surjective and thus $\varphi_3 = \varphi_3' \circ \mu_3$ is a morphism extending φ_0.

Indeed, we can check this by direct local computation. For example, let $(a_j) \neq 0$ in $\mathbb{C}^n$ and let $x = (a_j t_0 t_1 (t_0 + t_1))_{1 \le j \le n}$ be a point in Σ_0^3. For (b_j), (c_j), (d_j) in $\mathbb{C}^n - \{0\}$, not parallel to (a_j), consider the curve

$$C = (f_j^\lambda = a_j t_0 t_1 (t_0 + t_1) + \lambda c_j t_0^3 + \lambda (c_j - b_j + d_j) t_0^2 t_1 + \lambda b_j t_1^3)_{1 \le j \le n, \lambda \in \mathbb{C}}$$

in $\mathbf{P}_0$ passing through x at $\lambda = 0$. If we restrict the above construction to C, then Γ_1 at $\lambda = 0$ is the comb of four $\mathbb{P}^1$'s and the restriction of ev_1 to each irreducible component is respectively given by

$$(a_j t_0 + b_j t_1), \quad (a_j t_0 + c_j t_1), \quad (a_j t_0 + d_j t_1), \quad (a_j)$$

for homogeneous coordinates t_0, t_1 of $\mathbb{P}^1$. Hence there is no base point of ev_1 and the stable map to $\mathbb{P}^{n-1}$ thus obtained depends only on the three points in $\mathbb{P}^{n-2}$ corresponding to (b_j), (c_j) and (d_j).

Next, let $(a_j), (b_j) \neq 0$ and (a_j) is not parallel to (b_j) in $\mathbb{C}^n$. Let $x = ((a_j t_0 + b_j t_1) t_0 t_1)_{1 \le j \le n}$ be a point in Σ_1^2. For (c_j) which not parallel to (a_j) and for (d_j) which is not parallel to (b_j), consider the curve

$$D = ((a_j t_0 + b_j t_1) t_0 t_1 + \lambda c_j t_0^3 + \lambda d_j t_1^3)_{1 \le j \le n, \lambda \in \mathbb{C}}.$$

This represents a curve passing through x at $\lambda = 0$. Then Γ_2 restricted to $\lambda = 0$ is the union of three lines and they are mapped to $\mathbb{P}^{n-1}$ by

$$(a_j t_0 + b_j t_1), \quad (a_j t_0 + c_j t_1), \quad (b_j t_0 + d_j t_1).$$

This obviously is a stable map to $\mathbb{P}^{n-1}$.

Finally, suppose $(c_j) = (0)$ in the above case. This is the case where you choose the normal direction to Σ_1^2 contained in $\bar{\Sigma}_1^1$. Then the fiber corresponding to $\lambda = 0$ in Γ_2 has three irreducible components and the evaluation maps on the components are respectively

$$(a_j t_0 + b_j t_1), \quad (a_j t_0), \quad (b_j t_0 + d_j t_1).$$

So still there is a point (in the second component) at which the map to $\mathbb{P}^{n-1}$ is not well-defined. We choose a curve in $\mathbf{P}_2$ to this point, whose direction is normal to Σ_2^1. This amounts to considering the limits of

$$D_\mu = ((a_j t_0 + b_j t_1) t_0 t_1 + \lambda \mu e_j t_0^3 + \lambda d_j t_1^3)_{1 \le j \le n, \lambda \in \mathbb{C}}, \quad \mu \in \mathbb{C}$$

for some (e_j) not parallel to (a_j). By the previous case, for $\mu \neq 0$, the fiber of D_μ in Γ_2 at $\lambda = 0$ has three components and they are mapped to $\mathbb{P}^{n-1}$ respectively by

$$(a_j t_0 + b_j t_1), \quad (a_j t_0 + \mu e_j t_1), \quad (b_j t_0 + d_j t_1).$$

So we have a family of nodal curves parameterized by $\mu \in \mathbb{C}$ and stable maps from $E_\mu|_{\lambda=0}$ for $\mu \neq 0$. The construction of Γ_3 blows up the only base point

$(t_0 : t_1) = (0 : 1)$ in the second component for $\mu = 0$ and ev_3 becomes

$$(a_j t_0 + b_j t_1), \quad (a_j), \quad (a_j t_0 + e_j t_1), \quad (b_j t_0 + d_j t_1)$$

by the elementary modification. If we contract down the constant component, then we get a stable map of three irreducible components

$$(a_j t_0 + b_j t_1), \quad (a_j t_0 + e_j t_1), \quad (b_j t_0 + d_j t_1).$$

By checking case by case as above, we conclude that ev_3 is surjective and ψ_3 factors through a holomorphic map $\psi : \mathbf{P}_5 \to \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$. The bijectivity of $\bar{\psi}_5$ can be also checked by case by case local computation as above. We omit the cumbersome details.

When $n = 2$, the third blow-up map π_3 is an isomorphism since $\bar{\Sigma}_2^1$ is a divisor and π_2 is canceled with π_4 while π_1 is canceled with π_5. Therefore, $\bar{\psi}_0$ is an isomorphism and we have $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^1, 3) \cong \mathbf{P}_0/G = \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2) \otimes \mathbb{C}^2)//SL(2)$.

Finally, from the Riemann existence theorem [9, p.442] we deduce that $\mathbf{P}_5/SL(2)$ is a projective variety and $\bar{\psi}_5$ is an isomorphism of varieties.

6. Cohomology ring of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$

In this section, we study the cohomology of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ by using Theorem 5.1.

6.1. **Betti numbers.** We compute the Poincaré polynomial of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ in this subsection. We use the notation

$$P_t(X) = \sum t^k \dim H^k(X) \quad \text{and} \quad P_t^G(X) = \sum t^k \dim H_G^k(X)$$

for a topological space X. Let $G = SL(2)$.

From §2.3, the equivariant Poincaré series of $\mathbf{P}_0$ is

$$P_t^G(\mathbf{P}_0) = \frac{(1 - t^{4n-2})(1 - t^{4n})}{(1 - t^2)(1 - t^4)}.$$

By the blow-up formula, we have

$$P_t^G(\mathbf{P}_1) = P_t^G(\mathbf{P}_0) + \frac{t^2 - t^{6n-6}}{1 - t^2} \frac{(1 - t^{2n})}{(1 - t^2)}$$

since the blow-up center Σ_0^3 has quotient $\mathbb{P}^{n-1}$. Similarly, because $\bar{\Sigma}_1^2/G$ is a $\mathbb{P}^2$-bundle over the Grassmannian $Gr(2, n)$, we get

$$P_t^G(\mathbf{P}_2) = P_t^G(\mathbf{P}_1) + \frac{t^2 - t^{4n-4}}{1 - t^2} \left((1 + t^2 + t^4) \frac{(1 - t^{2n})(1 - t^{2n-2})}{(1 - t^2)(1 - t^4)} \right).$$

Since $\bar{\Sigma}_2^1/G$ is the blow-up of $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^{n-1}, 2)$ along $\mathbb{P}^{n-2} \times_{S_2} \mathbb{P}^{n-2}$ bundle on $\mathbb{P}^{n-1}$, we have

$$P_t^G(\mathbf{P}_3) = P_t^G(\mathbf{P}_2) + \frac{t^2 - t^{2n-2}}{1 - t^2} \left((1 + t^2) \frac{(1 - t^{2n})^2 (1 - t^{2n-2})}{(1 - t^2)^3} \right).$$

The map $\mathbf{P}_3/G \to \mathbf{P}_4/G$ contracts the proper transform of the exceptional divisor of the second blow-up, $\bar{\Sigma}_3^2/G$. It is the blow-up of $\bar{\Sigma}_2^2/G$ along $(\bar{\Sigma}_2^1 \cap \bar{\Sigma}_2^2)/G$, and this blow-up center is $\mathbb{P}^{n-2}$ bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $Gr(2, n)$. So we obtain

$$P_t^G(\mathbf{P}_4) = P_t^G(\mathbf{P}_3) - \frac{t^2}{1 + t^2} \left((1 + t^2 + t^4) \frac{(1 - t^{2n})(1 - t^{2n-2})}{(1 - t^2)(1 - t^4)} \frac{(1 - t^{4n-4})}{(1 - t^2)} \right.$$

$$+\frac{t^2-t^{2n-2}}{1-t^2}(1+t^2)^2\frac{(1-t^{2n-2})(1-t^{2n})}{(1-t^2)(1-t^4)}\frac{(1-t^{2n-2})}{(1-t^2)}\Big).$$

One should be careful about the S_2 action. Similarly, after the second blow-down, we obtain

$$P_t^G(\mathbf{P}_5)=P_t^G(\mathbf{P}_4)-\frac{t^2+t^4}{1+t^2+t^4}\frac{(1-t^{2n})^2(1-t^{2n-2})(1-t^{2n+2})}{(1-t^2)^3(1-t^4)}.$$

In summary, we proved

$$\begin{aligned}
P_t(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3)) &= \frac{(1-t^{4n-2})(1-t^{4n})}{(1-t^2)(1-t^4)}+\frac{t^2-t^{6n-6}}{1-t^2}\frac{(1-t^{2n})}{(1-t^2)}\\
&+\frac{t^2-t^{4n-4}}{1-t^2}\left((1+t^2+t^4)\frac{(1-t^{2n})(1-t^{2n-2})}{(1-t^2)(1-t^4)}\right)\\
&+\frac{t^2-t^{2n-2}}{1-t^2}\left((1+t^2)\frac{(1-t^{2n})^2(1-t^{2n-2})}{(1-t^2)^3}\right)\\
&-\frac{t^2}{1+t^2}\Bigg((1+t^2+t^4)\frac{(1-t^{2n})(1-t^{2n-2})}{(1-t^2)(1-t^4)}\frac{(1-t^{4n-4})}{(1-t^2)}+\\
&\quad\frac{t^2-t^{2n-2}}{1-t^2}(1+t^2)^2\frac{(1-t^{2n-2})(1-t^{2n})}{(1-t^2)(1-t^4)}\frac{(1-t^{2n-2})}{(1-t^2)}\Bigg)\\
&-\frac{t^2+t^4}{1+t^2+t^4}\frac{(1-t^{2n})^2(1-t^{2n-2})(1-t^{2n+2})}{(1-t^2)^3(1-t^4)}\\
&=\left(\frac{1-t^{2n+8}}{1-t^6}+2\frac{t^4-t^{2n+2}}{1-t^4}\right)\frac{(1-t^{2n})}{(1-t^2)}\frac{(1-t^{2n})(1-t^{2n-2})}{(1-t^2)(1-t^4)}.
\end{aligned}$$

6.2. **Cohomology ring.** Since we know how to compare the cohomology rings before and after a blow-up and we know $H_G^*(\mathbf{P}_0)$ from (2.2), it should be possible, at least in principle, to calculate the cohomology ring of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1},3)$ explicitly. However it seems extremely difficult to work out the details in reality. We content ourselves with the limit case where $n\to\infty$, i.e. $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^\infty,3))$.

First of all, by (2.5), we have

$$\begin{aligned}
H_G^*(\mathbf{P}_0)\cong\mathbb{Q}[\xi,\alpha^2]/\langle &\frac{(\xi+\alpha)^n(\xi+3\alpha)^n-(\xi-\alpha)^n(\xi-3\alpha)^n}{2\alpha}\\
&\frac{(\xi+\alpha)^n(\xi+3\alpha)^n+(\xi-\alpha)^n(\xi-3\alpha)^n}{2}\rangle.
\end{aligned}$$

for a degree 2 class ξ and a degree 4 class α^2. If we take $n\to\infty$, we obtain $H_G^*(\mathbf{P}_0)=\mathbb{Q}[\xi,\alpha^2]$. The first blow-up gives us a new degree 2 generator ρ_1 and we obtain

$$H_G^*(\mathbf{P}_1)=\mathbb{Q}[\xi,\alpha^2,\rho_1]/\langle\alpha^2\rho_1\rangle$$

because α^2 generates the kernel of the surjective restriction to the center of the first blow-up whose codimension is infinite. Next, it is easy to check that the restriction to the center of the second blow-up is an isomorphism and the codimension is infinite. Hence, we have

$$H_G^*(\mathbf{P}_2)=\mathbb{Q}[\xi,\alpha^2,\rho_1,\rho_2]/\langle\alpha^2\rho_1\rangle.$$

For the third blow-up, the restriction to the blow-up center is not surjective any more. The blow-up center is obtained by blowing up $[\mathbb{P}^1\times\mathbb{P}^{3n-1}]^s$. Let ξ_1 and ξ_2 be the generators of $\mathbb{P}^1$ and $\mathbb{P}^{3n-1}$ respectively. Let ρ_3 be minus the Poincaré dual of the exceptional divisor. Then in addition to ρ_3, we need to include another generator $\rho_3\xi_1$ of degree 4 which we denote by σ. Then by a routine calculation, we obtain

$$H^*_G(\mathbf{P}_3) = \mathbb{Q}[\xi, \alpha^2, \rho_1, \rho_2, \rho_3, \sigma]/\langle \alpha^2\rho_1, \rho_1\sigma, \sigma^2 - 4\alpha^2\rho_3^2\rangle.$$

Now, $H^*_G(\mathbf{P}_4)$ is a subring of $H^*_G(\mathbf{P}_3)$. By the description of the normal bundle to $\bar{\Sigma}_4^2$ in the previous section and the blow-up formula, we see that

$$H^*_G(\mathbf{P}_4) = \mathbb{Q}[\xi, \alpha^2, \rho_1, \rho_2^2, \rho_3, \sigma]/\langle \alpha^2\rho_1, \rho_1\sigma, \sigma^2 - 4\alpha^2\rho_3^2\rangle.$$

Similarly, we obtain

$$H^*_G(\mathbf{P}_5) = \mathbb{Q}[\xi, \alpha^2, \rho_1^3, \rho_2^2, \rho_3, \sigma]/\langle \alpha^2\rho_1^3, \rho_1^3\sigma, \sigma^2 - 4\alpha^2\rho_3^2\rangle.$$

So we proved

$$H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^\infty, 3)) = \mathbb{Q}[\xi, \alpha^2, \rho_1^3, \rho_2^2, \rho_3, \sigma]/\langle \alpha^2\rho_1^3, \rho_1^3\sigma, \sigma^2 - 4\alpha^2\rho_3^2\rangle.$$

where ξ, ρ_3 are degree 2 classes, $\sigma, \rho_2^2, \alpha^2$ are degree 4 classes, and ρ_1^3 is a degree 6 class. This is isomorphic to the description in [1].

When $n = 2$, $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^1, 3)) \cong H^*_G(\mathbf{P}_0)$ because $\bar{\psi}_0$ is an isomorphism. Hence $H^*(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^1, 3))$ is given by (2.5).

6.3. **Picard group.** We use the notation of §5. Since the locus of nontrivial automorphisms in $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$ has codimension at least two, we can delete them when calculating the Picard group. We also delete $\bar{\Sigma}_5^2/G$ and $\bar{\Sigma}_5^3/G$ whose codimensions are at least two. Then on the resulting open set of $\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)$, the birational map $\bar{\psi}^{-1} : \overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3) = \mathbf{P}_5/G \dashrightarrow \mathbf{P}_0/G$ coincides with the blow-up map π_3 since the blow-up/down centers for $\pi_1, \pi_2, \pi_4, \pi_5$ were deleted. Hence $\bar{\psi}^{-1}$ is an honest blow-up along a smooth subvariety. For $n = 2$ only, $\bar{\psi}^{-1}$ is an isomorphism. By the blow-up formula for Picard groups in [9, II,§8], we obtain

$$\mathrm{Pic}(\overline{\mathbf{M}}_{0,0}(\mathbb{P}^{n-1}, 3)) \cong \begin{cases} \pi_3^*\mathrm{Pic}(\mathbf{P}_0/G) \oplus \mathbb{Z}\Delta & \text{for} \quad n \geq 3 \\ \pi_3^*\mathrm{Pic}(\mathbf{P}_0/G) & \text{for} \quad n = 2 \end{cases}$$

where $\Delta = \bar{\Sigma}_5^1/G$ is the boundary divisor of reducible curves. On the other hand, by Kempf's descent lemma [5] and by checking the action of the stabilizers on the fibers of line bundles, we obtain that the Picard group $\mathrm{Pic}(\mathbf{P}_0/G)$ is isomorphic to the equivariant Picard group $\mathrm{Pic}(\mathbf{P}_0)^G = \mathbb{Z}\mathcal{O}(2)$ which is a subgroup of $\mathrm{Pic}(\mathbf{P}_0) \cong \mathrm{Pic}(\mathbb{P}^{4n-1}) \cong \mathbb{Z}\mathcal{O}(1)$.

Lastly, we note that the closure of the codimension one subset of elements in $\mathbf{P}_0$ whose images meet a fixed codimension 2 subspace is $\mathcal{O}(6)$: Given two linear equations for the subspace, we obtain two polynomials of degree three in $t_0, t_1 \in H^0(\mathbb{P}^1, \mathcal{O}(1))$. The condition for a degree 3 curve to meet the subspace is given by the resultant of the two polynomials of degree 3 in t_0, t_1. This divisor is smooth and contains Σ^1. Hence $\pi_3^*\mathcal{O}(6)$ is $H + \Delta$. Therefore, $\pi_3^*\mathcal{O}(2) = \frac{1}{3}(H + \Delta)$. This completes the proof of Theorem 1.4.

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