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Intersection cohomology of GIT quotients

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Chapter 1

Derived categories and intersection cohomology

Our goal in this lecture series is to understand the intersection cohomology of GIT quotients from sheaf-theoretic perspective.

1.1 Motivation for intersection cohomology

1.1.1

Let X be a compact Kähler manifold of (real) dimension $2n$ and let $H^*(X) = H^*(X;\mathbb{C})$. Then the following theorems hold.

1. Poincaré duality
   The cup product $\cup : H^{n-i}(X) \otimes H^{n+i}(X) \to H^{2n}(X) \cong \mathbb{C}$ is perfect.

2. Hodge decomposition
   $H^i(X) = \oplus_{p+q=i} H^{p,q}$ with $H^{p,q} = \overline{H^{q,p}}$.
   In particular, if i is odd, $h^i(X)$ is even.

3. Lefschetz hyperplane theorem
   The restriction map $H^i(X) \to H^i(X \cap H)$ to a hyperplane intersection is an isomorphism for $i < n-1$ and injective for $i = n-1$.

4. Hard Lefschetz theorem
   $\omega^i : H^{n-i}(X) \to H^{n+i}(X)$ is an isomorphism, i.e. $sl(2)$ acts on $H^*(X)$.

5. Hodge index theorem
   The signature of the intersection pairing on $H^n(M;\mathbb{R})$ is $\sum_{p+q \text{ even}} (-1)^p h^{p,q}$.

For surfaces, the signature is $2 - h^{1,1} = 1 - (h^{1,1} - 1)$ i.e. there is a unique positive eigenvalue for $H^{1,1}$.

6. Morse theory.

These are indispensable essential tools in the study of algebraic varieties. But all the above *fail* when X is singular!

1.1.2 Example

(1) Let X be two rational curves meeting at a point.

Then we have $H^0(X) = \mathbb{Z}$, $H^1(X) = 0$, $H^2(X) = \mathbb{Z} \oplus \mathbb{Z}$.
Hence Poincaré duality and Hard Lefschetz fail.

(2) Let X be an irreducible rational curve with one node as its only singular point.

Then we have $H^0(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}$, $H^2(X) = \mathbb{Z}$.
Hence Hodge decomposition fails.

1.1.3 Dreams come true (sometimes!)

So it is highly desirable to have a cohomology theory for projective (possibly singular) varieties with the above 6 theorems. The (middle perversity) intersection cohomology $IH^*(X)$ by Goresky and MacPherson is such a theory.

Recent applications include

1. computation of $H^*(S^{[n]})$ for a surface S
2. representation theory
3. mathematical formulation of Gopakumar-Vafa invariants.

1.2 Stratification and chain complex

1.2.1 Definition

A topological space X equipped with a filtration by closed subsets

$$(\mathfrak{X}) : X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

is a *stratified space* if

1. $X - X_{n-1}$ is dense
2. $S_{n-k} = X_{n-k} - X_{n-k-1}$ is a manifold of codimension $2k$ or empty (This is called a stratum.)
3. for $x \in S_{n-k}$, there exist a stratified space L of dimension $2k-1$ and a stratification preserving homeomorphism between a neighborhood of x in X and $\mathbb{R}^{\dim S_{n-k}} \times cL$ where $cL = L \times [0, \infty)/L \times 0$.

1.2.2 Theorem

1. (Whitney) Any variety over $\mathbb{C}$ admits a stratification.
2. (Sjamaar-Lerman) The symplectic reduction of a proper Hamiltonian space admits a stratification.

1.2.3 Definition

A *geometric chain* ξ is a (not necessarily finite) formal expression

$$\xi = \sum_{\text{locally finite}} \xi_\sigma \sigma$$

for a triangulation T of X. Two geometric chains ξ and ξ' are equivalent if they define the same chain in a common refinement of their triangulations.

1.2.4 Definition

Let $\dim_{\mathbb{R}} X = 2n$. Let

$$C^i(X) = \{\text{geometric chains } \xi \text{ in } X \text{ of codimension } i\}$$

so that we have a chain complex

$$C^{\cdot}(X): \qquad \cdots \xrightarrow{\partial} C^i(X) \xrightarrow{\partial} C^{i+1} \xrightarrow{\partial} \cdots$$

whose cohomology is by definition the Borel-Moore homology

$$H^*(C^{\cdot}(X)) = H^{BM}_{2n-*}(X).$$

1.2.5 Remark

Actually, the geometric chains define a sheaf complex: for an open set U, let $C^i(U)$ be the vector space of geometric chains ξ in U of codimension i. Then C^i is a soft sheaf i.e. the restriction to any closed subset $C^i(X) \to C^i(F)$ is surjective. Hence we have a complex of soft sheaves

$$0 \to C^0 \to C^1 \to \cdots \to C^i \to \cdots$$

whose hypercohomology is $H^{BM}_{2n-*}(X)$.

1.2.6 Definition

The *hypercohomology* of a sheaf complex $\mathcal{F}^{\cdot}$ is $H^*(\mathcal{I}^{\cdot}(X))$ for any injective (soft, c-soft, or fine) resolution $\mathcal{F}^{\cdot} \to \mathcal{I}^{\cdot}$. A chain map $\mathcal{F}^{\cdot} \to \mathcal{I}^{\cdot}$ is called an injective resolution if $\mathcal{I}^k$ are all injective and the induced maps $H^k(\mathcal{F}^{\cdot}) \cong H^k(\mathcal{I}^{\cdot})$ are isomorphisms for all k.

1.3 Intersection cohomology

1.3.1 Definition

A *perversity* is an increasing function $p : \{1, 2, \cdots, n\} \to \{0, 1, \cdots, 2n-2\}$ such that $q(i) = 2i - 2 - p(i)$ is also increasing.

Example:

1. $0(i) = 0$ (zero perversity)
2. $t(i) = 2i - 2$ (top perversity)
3. $m(i) = i - 1$ (middle perversity) – most important.

1.3.2 Definition

$$\begin{aligned} IC_p^i(X) = \{\xi \in C^i(X) \mid \quad & \dim(|\xi| \cap X_{n-c}) \leq 2n - 2c - i + p(c), \\ & \dim(|\partial\xi| \cap X_{n-c}) \leq 2n - 2c - i - 1 + p(c)\} \end{aligned}$$

We are controlling the intersection of a cycle with the singular strata. This is certainly a subcomplex of $C^{\cdot}(X)$. The *perversity p intersection cohomology* is defined as

$$IH_p^*(X) = H^*(IC_p^{\cdot}(X)).$$

Example:

1. $IH_0^*(X) = H^*(X)$
2. $IH_t^*(X) = H_{2n-*}^{BM}(X)$
3. $IH_m^*(X) =: IH^*(X)$ – most important.

1.3.3 Intersection cohomology sheaf

For each open set U of X, the assignment

$$\begin{aligned} IC_{p,X}^i(U) = \{\xi \in C^i(U) \mid \quad & \dim(|\xi| \cap X_{n-c}) \leq 2n - 2c - i + p(c), \\ & \dim(|\partial\xi| \cap X_{n-c}) \leq 2n - 2c - i - 1 + p(c)\} \end{aligned}$$

is a *soft* sheaf and thus

$$IC_{p,X}^{\cdot} : \qquad \cdots \xrightarrow{\partial} IC_{p,X}^i \xrightarrow{\partial} IC_{p,X}^{i+1} \xrightarrow{\partial} \cdots$$

is a complex of soft sheaves whose hypercohomology is $IH_p^*(X)$. Let us call this sheaf complex the intersection cohomology sheaf.

1.3.4 Theorem (Goresky-MacPherson)

$IH_p^*(X)$ is independent of the choice of stratification and thus is a homeomorphism invariant of X.

1.3.5 Corollary

When X is smooth, $IH_p^*(X) \cong H^*(X)$ for any perversity p.

1.3.6 Exercise

1. For the curves in 1.1.2., compute their middle perversity intersection cohomology.[1]

2. Let L be a compact oriented manifold of dimension $2k-1$ and let X be the cone $cL = L \times [0,\infty)/L \times 0$ over L. Show that $IH^*(X) = H^{<k}(L) = \oplus_{i<k} H^i(L)$.[2]

1.3.7 Proposition

If L is a stratified space of dimension $2k-1$, $IH_p^*(cL) = IH_p^{\leq p(k)}(L)$.

[1] In the first case, $IH^0(X) = \mathbb{Z} \oplus \mathbb{Z}$, $IH^1(X) = 0$, and $IH^2(X) = \mathbb{Z} \oplus \mathbb{Z}$.

[2] For instance, if $i \geq k$, $\xi \in IC^i(X)$ is not allowed to pass the vertex of the cone. For $\xi \subset L \times 1$, $\xi \times 1 = \partial(\xi \times [1,\infty))$.

1.4 Derived category of sheaf complexes

1.4.1 Definition

1. Let $\mathcal{S}h_X$ denote the category of sheaves of $\mathbb{C}$-vector spaces on X.
2. Let $\mathcal{K}_X$ be the category given by
   (a) objects: complex of sheaves $\mathcal{F}^{\cdot} : \cdots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots$, $d^2 = 0$
   (b) morphisms: $\{$chain maps $\mathcal{F}^{\cdot} \to \mathcal{E}^{\cdot}\}/\{$homotopically equivalent to zero maps$\}$
3. A chain map $\mathcal{F}^{\cdot} \to \mathcal{E}^{\cdot}$ is called a *quasi-isomorphism* if $H^i(\mathcal{F}^{\cdot}) \cong H^i(\mathcal{E}^{\cdot})$ for all i.

The collection of quasi-isomorphisms form a multiplicative system.[3]

1.4.2 Derived category

1. Let $\mathbf{D}^{\geq 0}(X)$ be the category given by
   (a) objects: bounded below sheaf complexes $(\mathcal{F}^{\cdot}, d)$
   (b) morphisms: $Mor_{\mathcal{K}_X}(\mathcal{F}^{\cdot}, \mathcal{E}^{\cdot})$ localized by quasi-isomorphisms. A morphism is of the form

$$\mathcal{F}^{\cdot} \xleftarrow[q.i.]{} \mathcal{S}^{\cdot} \longrightarrow \mathcal{E}^{\cdot}$$

2. Let $\mathbf{D}_c^{\geq 0}(X)$ be the full subcategory of $\mathbf{D}^{\geq 0}(X)$ whose objects are constructible with respect to a stratification, i.e. cohomology sheaves are locally constant of finite rank on each stratum.

1.4.3 Theorem

The object $IC^{\cdot}_{p,X} \in \mathbf{D}_c^{\geq 0}(X)$ is independent of the choice of stratification.

There is a purely sheaf theoretic construction of intersection cohomology sheaf by Deligne.

[3] closed under composition, completes squares and left zero divisor $\Leftrightarrow$ right zero divisor.

1.5 Deligne's construction

1.5.1 Notation

1. Let X be is a stratified space

$$(\mathfrak{X}) : X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

2. Let $U_k = X - X_{n-k}$ and $j_k : U_k \hookrightarrow U_{k+1} = U_k \cup S_{n-k}$.

1.5.2 Deligne's construction

1. Since U_1 is smooth, $\boxed{IC^{\cdot}_{p,X} \cong \mathbb{C}_{U_1}}$ the constant sheaf.

2. For $x \in S_{n-k}$, a neighborhood V of x in X is $\mathbb{R}^{2n-2k} \times cL$ for some stratified space L of dimension $2k-1$ and thus

$$\begin{aligned} IH^*_p(V) &\cong IH^*_p(cL) &&\cong IH^{\leq p(k)}_p(L) \\ &\cong IH^{\leq p(k)}_p(\mathbb{R}^{2n-2k} \times (cL - v)) &&\cong IH^{\leq p(k)}_p(V \cap U_k) \end{aligned}$$

since $cL - v = \mathbb{R} \times L$ where v is the vertex of the cone cL. Therefore we have $\boxed{IC^{\cdot}_{U_{k+1}} \cong \tau^{\leq p(k)}(j_k)_* IC^{\cdot}_{U_k}}$

1.5.3 Truncation

Truncations of a sheaf complex from below and above

$$(\tau^{<l}\mathcal{F}^{\cdot})^i = \begin{cases} \mathcal{F}^i & \text{for } i < l-1 \\ \mathrm{ker} d_i & \text{for } i = l-1 \\ 0 & \text{for } i > l-1 \end{cases}$$

$$(\tau^{\geq l}\mathcal{F}^{\cdot})^i = \begin{cases} 0 & \text{for } i < l \\ \mathrm{coker} d_{i-1} & \text{for } i = l \\ \mathcal{F}^i & \text{for } i > l \end{cases}$$

1.5.4 Axioms

$IC^{\cdot}_{p,X}$ is the unique (up to isomorphism) object $\mathcal{P}^{\cdot}$ in $\mathbf{D}^{\geq 0}_c(X)$ satisfying

1. normalization: $\mathcal{P}^{\cdot}|_{U_1} \cong \mathbb{C}_{U_1}$

2. support: $H^{>p(k)}(\mathcal{P}^{\cdot}_x) = 0$ for $x \in S_{n-k}$

3. cosupport: $H^{\leq p(k)}(\mathcal{P}^{\cdot}_x) \cong H^{\leq p(k)}(j_* j^* \mathcal{P}^{\cdot}_x)$ for $x \in S_{n-k}$ where $j : X - S_{n-k} \hookrightarrow X$.

1.6 Equivariant intersection cohomology

1. Let a compact Lie group K act on a stratified space X preserving the stratification $\mathfrak{X} = \{X_i\}$. Then $\{EK \times_K X_i\}$ is a stratification of $EK \times_K X =: X_K$.[4]

2. Deligne's construction gives us a sheaf complex $IC^{\cdot}_{p,X_K}$ on X_K.

3. The hypercohomology of $IC^{\cdot}_{p,X_K}$ is defined as the equivariant intersection cohomology $IH^*_{p,K}(X)$.

4. The obvious fibration $X \to EK \times_K X \to BK$ gives rise to a spectral sequence
$$H^i(BK) \otimes IH^j_p(X) \Rightarrow IH^{i+j}_{p,K}(X).$$

[4] EK denotes a contractible K space where K acts freely and $BK = EK/K$. EK can be approximated by a sequence of finite dimensional smooth manifolds EK_r.

Chapter 2

Circle quotients

2.1 Symplectic reduction

2.1.1 Definition

- Let (M, ω) be a symplectic manifold, i.e. $\omega : TM \cong TM^*$. A *Hamiltonian vector field* of a function f is $\omega^{-1}(df)$.
- Let K be a compact Lie group acting on M symplectically (i.e. $g^*\omega = \omega$). For each $\xi \in \mathbf{k} = \mathrm{Lie}K$, let X_ξ be the vector field generated by the infinitesimal action of ξ.
- A *moment map* for the K-action on M is an K-equivariant map $\mu : M \to \mathbf{k}^*$ such that for $\xi \in \mathbf{k}$ X_ξ is the Hamiltonian vector field of $\langle \mu, \xi \rangle : M \to \mathbf{k}^* \to \mathbb{R}$. We say (M, ω, μ) is a Hamiltonian K-space.
- $M /\!\!/ K = \mu^{-1}(0)/K$ is called the symplectic reduction of M by the Hamiltonian action of K.

2.1.2 Theorem (Kirwan, Ness)

Suppose $M \subset \mathbb{P}^n$ is a smooth projective variety and K acts on M via a homomorphism $K \to U(n+1)$. Then the geometric invariant theory (GIT) quotient $M /\!\!/ K^{\mathbb{C}}$ is homeomorphic to the symplectic reduction $M /\!\!/ K$.

2.1.3 Goal

Compute $IH^*(M /\!\!/ K)$. In this chapter, we focus on the case where K is the circle group $U(1)$. The circle case has been studied by Goresky-MacPherson, Hu, Lerman-Tolman. Once again, our perspective is sheaf theoretic.

2.2 Atiyah-Bott-Kirwan theory

2.2.1

Morse theory on M with respect to $|\mu|^2$ enables us to compute the equivariant cohomology ring $H_K^*(\mu^{-1}(0))$.

- If K acts locally freely (i.e. stabilizers are finite), then $X := \mu^{-1}(0)/K$ is an orbifold[1] and hence
$$H_K^*(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/K) \cong IH^*(X).$$
- If the action is not locally free, then $H_K^*(\mu^{-1}(0)) \ncong H^*(\mu^{-1}(0)/K)$ and $H_K^*(\mu^{-1}(0)) \ncong IH^*(X)$.

Our interest lies in the second case. The problem is to deduce $IH^*(X)$ from our knowledge on $H_K^*(\mu^{-1}(0))$.

2.2.2 Example

Suppose $\lambda \in T = U(1)$ acts on $\mathbb{P}^4$ as $\mathrm{diag}(1, \lambda, \lambda, \lambda^{-1}, \lambda^{-1})$ with moment map $\mu(z_0, \cdots, z_4) = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)/\sum |z_i|^2$. The critical set is the fixed point set $\{(1,0,0,0,0)\} \cup \{(0,x,y,0,0)\} \cup \{0,0,0,x,y)\}$. Subtracting the two unstable strata gives rise to a short exact sequence
$$0 \to H_T^{*-6}(\mathbb{P}^1) \oplus H_T^{*-6}(\mathbb{P}^1) \to H_T^*(\mathbb{P}^4) \to H_T^*(\mu^{-1}(0)) \to 0$$
and thus we see that the Poincaré series of $H_T^*(\mu^{-1}(0))$ is
$$\frac{1+t^2+t^4+t^6+t^8}{1-t^2} - 2t^6\frac{1+t^2}{1-t^2} = \frac{1+t^2+t^4-t^6-t^8}{1-t^2}.$$

2.2.3 Gysin sequence

Let $\alpha : S \hookrightarrow M$ be the inclusion of a symplectic submanifold of codimension $2d$. The Thom isomorphism $H^{*-2d}(S) \cong H^*(M, M-S)$ together with the long exact sequence for the pair $(M, M-S)$ gives us the Gysin sequence
$$\cdots \to H^{*-2d}(S) \to H^*(M) \to H^*(M-S) \to \cdots$$

If α is T-equivariant, the induced map $\alpha_T : S_T \hookrightarrow M_T$ ($S_T = ET \times_T S$ and $M_T = ET \times_T M$) gives rise to the equivariant Gysin sequence
$$\cdots \to H_T^{*-2d}(S) \to H_T^*(M) \to H_T^*(M-S) \to \cdots$$

[1] When X is an orbifold, the constant sheaf $\mathbb{C}$ on X satisfies the axioms for intersection cohomology sheaf and hence $H^*(X) \cong IH^*(X)$.

2.3 Functors in derived categories

2.3.1 Functors of sheaves

Let $f : X \to Y$ be a continuous map of locally compact finite dimensional spaces.

1. Let $\mathcal{A}$ be a sheaf on X. Then we have sheaves $f_*\mathcal{A}$ and $f_!\mathcal{A}$:
   (a) $f_*\mathcal{A}(V) = A(f^{-1}(V))$ defines a sheaf $f_*\mathcal{A}$
   (b) $f_!\mathcal{A}(V) = \{s \in \mathcal{A}(f^{-1}(V)) \,|\, \mathrm{supp} s \to V \text{ is proper}\}$
2. Let $\mathcal{B}$ be a sheaf on Y. Then $f^{-1}\mathcal{B}$ is the sheafification of the assignment $V \mapsto \lim_{V \supset f(U)} \mathcal{B}(V)$. Then $(f^{-1}\mathcal{B})_x \cong \mathcal{B}_{f(x)}$.

2.3.2 Adjoint property

f_*, $f_!$ induce functors $\mathbf{D}(X) \to \mathbf{D}(Y)$ and $f^{-1} = f^*$ induces a functor $\mathbf{D}(Y) \to \mathbf{D}(X)$. We have the adjoint property

$$\mathrm{Hom}(f^*\mathcal{B}, \mathcal{A}) = \mathrm{Hom}(\mathcal{B}, f_*\mathcal{A})$$

which gives rise to the natural morphism $\mathcal{B} \to f_*f^*\mathcal{B}$ in $\mathbf{D}(Y)$.

2.3.3 Theorem (Verdier duality)

There exists a functor $f^! : \mathbf{D}(Y) \to \mathbf{D}(X)$ such that

$$\mathrm{Hom}(f_!\mathcal{A}, \mathcal{B}) = \mathrm{Hom}(\mathcal{A}, f^!\mathcal{B})$$

. This gives rise to a natural map $f_!f^!\mathcal{B} \to \mathcal{B}$.

2.3.4 Example

Suppose $f : X \hookrightarrow Y$ is a closed immersion.

1. $f_!\mathcal{A} = f_*\mathcal{A}$ is an extension by zero and $f^*\mathcal{A}$ is the restriction to X.
2. Let $\mathcal{A}^\cdot \to \mathcal{I}^\cdot$ be an injective resolution. Then $f^!\mathcal{A}^\cdot = f^!\mathcal{I}^\cdot$ and $f^!\mathcal{I}^i(V \cap X) = \Gamma_X(\mathcal{I}^i, V)$ is the space of sections on V with support in X.
3. The hypercohomology of $f^!\mathbb{C}_Y$ is $H^*(Y, Y - X)$.

2.4 Distinguished triangles

2.4.1 Remark

A morphism $\mathcal{F}^{\cdot} \to \mathcal{E}^{\cdot}$ in the derived category consists of two chain maps $\mathcal{F}^{\cdot} \leftarrow \mathcal{S}^{\cdot} \to \mathcal{E}^{\cdot}$ where the first one is a quasi-isomorphism. The kernel and cokernel are not well-defined. Hence we don't have a notion of (short) exact sequence in the derived category.

2.4.2 Definition

Let $\phi : \mathcal{F}^{\cdot} \to \mathcal{E}^{\cdot}$ be a chain map. The *cone* of ϕ is defined as

$$C(\phi)^i = \mathcal{F}^{i+1} \oplus \mathcal{E}^i \qquad d^i = \begin{pmatrix} -d_{\mathcal{F}}^{i+1} & 0 \\ \phi^{i+1} & d_{\mathcal{E}}^i \end{pmatrix}$$

so that we have an exact sequence

$$0 \to \mathcal{E}^{\cdot} \to C(\phi)^{\cdot} \to \mathcal{F}^{\cdot}[1] \to 0$$

where $\mathcal{F}^{\cdot}[1]$ is the result of shifting $\mathcal{F}[1]^i = \mathcal{F}^{i+1}$. Let $Cyl(\phi)^{\cdot}$, called the *cylinder*, be the cone of $C(\phi)^{\cdot}[-1] \to \mathcal{F}^{\cdot}$.

Topologically, the cylinder corresponds to the mapping cylinder and the cone to the mapping cone.

2.4.3 Definition

A *triangle* is a sequence of morphisms

$$\mathcal{K}^{\cdot} \to \mathcal{L}^{\cdot} \to \mathcal{M}^{\cdot} \to \mathcal{K}^{\cdot}[1].$$

A triangle is called *distiguished* if it is isomorphic to a natural sequence

$$\mathcal{F}^{\cdot} \to Cyl(\phi)^{\cdot} \to C(\phi)^{\cdot} \to \mathcal{F}^{\cdot}[1]$$

for some $\phi : \mathcal{F}^{\cdot} \to \mathcal{E}^{\cdot}$.

2.4.4 Proposition

Given a distinguished triangle $\mathcal{K}^{\cdot} \to \mathcal{L}^{\cdot} \to \mathcal{M}^{\cdot} \to \mathcal{K}^{\cdot}[1]$ we have a long exact sequence of hypercohomology groups

$$\mathbb{H}^i(\mathcal{K}^{\cdot}) \to \mathbb{H}^i(\mathcal{L}^{\cdot}) \to \mathbb{H}^i(\mathcal{M}^{\cdot}) \to \mathbb{H}^{i+1}(\mathcal{K}^{\cdot}).$$

2.4.5 Example

Let $\imath : X \hookrightarrow Y$is an inclusion of closed subset and $\jmath : U = Y - X \hookrightarrow Y$. We have a distinguished triangle

$$\imath_! \imath^! \mathcal{F}^{\cdot} \to \mathcal{F}^{\cdot} \to \jmath_* \jmath^* \mathcal{F}^{\cdot} \to \imath_! \imath^! \mathcal{F}^{\cdot}[1]$$

since $0 \to \Gamma_X(\mathcal{I}^{\cdot}, V) \to \Gamma(\mathcal{I}^{\cdot}, V) \to \Gamma(\mathcal{I}^{\cdot}, V - X) \to 0$ is exact for any flabby resolution $\mathcal{F}^{\cdot} \to \mathcal{I}^{\cdot}$ and open set V.

2.5 Gysin morphisms

2.5.1

Let $\alpha : S \hookrightarrow M$ be an inclusion of a symplectic submanifold of codimension $2d$ and $\beta : M - S = U \hookrightarrow M$. Then we have the distinguished triangle

$$\alpha_! \alpha^! \mathbb{C}_M \to \mathbb{C}_M \to \beta_* \beta^* \mathbb{C}_M \to \alpha_! \alpha^! \mathbb{C}_M[1]$$

2.5.2 Lemma (Thom isomorphism)

$$\alpha^! \mathbb{C}_M \cong \mathbb{C}_S[-2d]$$

<u>Proof</u>: By taking stalk cohomology of the the above distinguished triangle at any point $p \in S$, we get a long exact sequence

$$\cdots H^i((\alpha^! \mathbb{C}_M)_p) \to H^i(\mathbb{C}_p) \to H^i((\beta_* \beta^* \mathbb{C}_M)_p) \to \cdots .$$

By the Thom isomorphism theorem $H^i((\alpha^! \mathbb{C}_M)_p)$ is trivial unless $i = 2d$ and $H^{2d}((\alpha^! \mathbb{C}_M)_p) \cong \mathbb{C}$. Hence $\alpha^! \mathbb{C}_M \cong \mathbb{C}_S[-2d]$. $\square$

2.5.3 Gysin morphism

The Gysin morphism for α is $\alpha_* \mathbb{C}_S[-2d] \to \mathbb{C}_M$ whose hypercohomology is the Gysin map $H^{*-2d}(S) \to H^*(M)$.

2.5.4 Equivariant case

When T acts on S and M and α is equivariant, $\alpha^T : S_T \to M_T$ is an inclusion of codimension $2d$ where $S_T = ET \times_T S$, $M_T = ET \times_T M$. Hence we have the equivariant Gysin morphism

$$(\alpha^T)_* \mathbb{C}_{S_T}[-2d] \to \mathbb{C}_{M_T}$$

which fits into the distinguished triangle

$$(\alpha^T)_* \mathbb{C}_{S_T}[-2d] \to \mathbb{C}_{M_T} \to \beta^T_* (\beta^T)^* \mathbb{C}_{M_T} \to (\alpha^T)_* \mathbb{C}_{S_T}[-2d+1]$$

whose hypercohomology is the equivariant Gysin sequence.

2.6 Hamiltonian circle action

2.6.1 Definition

- $Z = \mu^{-1}(0)$.
- $F_1, \cdots, F_r = T$-fixed components in Z.
- The normal space to F_i decomposes as the direct sum of positive and negative weight spaces
$$W_i = W_i^+ \oplus W_i^-.$$
- $d_i = \frac{1}{2}\min\{\dim W_i^+, \dim W_i^-\}$.
- $M^{ss} = \{p \in M \,|\, \text{gradient flow of } -\mu^2 \text{ has a limit point in } Z\}$
- $S_i^{\pm} = \{p \in M^{ss} \,|\, p \text{ retracts to a point in } F_i \text{ by } \mp\mu\}$.
- Let S_i be as follows so that $\operatorname{codim} S_i = 2d_i$:
$$S_i = \begin{cases} S_i^+ & \text{if } \dim W_i^- \leq \dim W_i^+ \\ S_i^- & \text{otherwise.} \end{cases}$$
- $\alpha_i : S_i \hookrightarrow M^{ss}$
- $\phi : M^{ss} \to Z/T = X$ retraction followed by quotient
$\psi : M_T^{ss} = ET \times_T M^{ss} \to X$ obvious induced map
$\sigma_i : F_i \hookrightarrow X$ obvious inclusion
- $$\mathcal{C}_T^{\bullet}(M) = \psi_* \mathbb{C}_{M_T^{ss}}$$
$$\mathcal{C}_T^{\bullet}(F_i) = (\sigma_i)_*(\psi_i)_* \mathbb{C}_{ET \times_T S_i} = \psi_*(\alpha_i^T)_* \mathbb{C}_{ET \times_T S_i}$$

2.6.2 Two morphisms

- Gysin morphism
$$\delta : \bigoplus_i \mathcal{C}_T^{\bullet}(F_i)[-2d_i] = \bigoplus_i \psi_*(\alpha_i^T)_* \mathbb{C}_{ET \times_T S_i}[-2d_i] \to \psi_* \mathbb{C}_{M_T^{ss}} = \mathcal{C}_T^{\bullet}(M)$$
- restriction followed by truncation
$$\rho : \mathcal{C}_T^{\bullet}(M) = \psi_* \mathbb{C}_{M_T^{ss}} \to \bigoplus_i \psi_*(\alpha_i^T)_* \mathbb{C}_{ET \times_T S_i} = \bigoplus_{i=1}^r \mathcal{C}_T^{\bullet}(F_i) \to \bigoplus_{i=1}^r \tau^{\geq 2d_i} \mathcal{C}_T^{\bullet}(F_i).$$

2.6.3 Proposition

$\rho \circ \delta$ is an isomorphism.
<u>Proof</u>: multiplication by t^d is an isomorphism $\mathbb{C}[t] \to t^d\mathbb{C}[t]$ and $H_T^* := H^*(BT) \cong \mathbb{C}[t]$. $\square$

2.6.4 Corollary

Let $\mathcal{A}^{\cdot}$ be the cone of ρ shifted by -1 in the triangulated category $\mathbf{D}_c^{\geq 0}(X)$ that fits into the distinguished triangle

$$\mathcal{A}^{\cdot} \xrightarrow{\theta} \mathcal{C}_T^{\bullet}(M) \xrightarrow{\rho} \bigoplus_{i=1}^r \tau^{\geq 2d_i}\mathcal{C}_T^{\bullet}(F_i) \longrightarrow \mathcal{A}^{\cdot}[1].$$

Then

$$\mathcal{A}^{\cdot} \oplus \left(\bigoplus_{i=1}^r \mathcal{C}_T^{\bullet}(F_i)[-2d_i]\right) \cong \mathcal{C}_T^{\bullet}(M).$$

2.6.5 Theorem

$$IC_X^{\cdot} \cong \mathcal{A}^{\cdot}$$

<u>Proof</u>: $\mathcal{A}^{\cdot}$ satisfies the normalization, support and cosupport axioms. In fact, the normal cone of F_i in X is $W_i /\!\!/ T - v = (W_i - W_i^+ \cup W_i^-)/\mathbb{C}^*$ and thus

$$IH^*(W_i /\!\!/ T) \cong H^{\leq 2d_i - 2}((W_i - W_i^+ \cup W_i^-)/\mathbb{C}^*) \cong H_T^{\leq 2d_i - 2}(W_i - W_i^+ \cup W_i^-) \cong H_T^{\leq 2d_i - 2}.$$

2.6.6 Corollary

$$IC_X^{\cdot} \oplus \left(\bigoplus_{i=1}^r \mathcal{C}_T^{\bullet}(F_i)[-2d_i]\right) \cong \mathcal{C}_T^{\bullet}(M)$$

2.7 Cohomological consequences

2.7.1 Theorem

From the splitting distinguished triangle

$$IC^{\cdot}_X \to \mathcal{C}^{\bullet}_T(M) \to \left(\bigoplus_{i=1}^{r} \tau^{\geq 2d_i} \mathcal{C}^{\bullet}_T(F_i) \right),$$

the intersection cohomology $IH^*(X)$ is isomorphic to the (graded) subspace

$$V^* = \{\zeta \in H^*_T(M^{ss}) \mid \zeta|_{F_i} \in H^*(F_i) \otimes H_T^{\leq 2d_i - 2}\}.$$

2.7.2 Example

Suppose $\lambda \in T = U(1)$ acts on $\mathbb{P}^4$ as $\mathrm{diag}(1, \lambda, \lambda, \lambda^{-1}, \lambda^{-1})$. The only fixed point set in $Z = \mu^{-1}(0)$ is $\{(1, 0, 0, 0, 0)\}$ and its normal space $\mathbb{C}^4$ is acted on by T with 2 positive and 2 negative weights. Hence $d = 2$.

The Poincaré series of $H^*_T(\mu^{-1}(0))$ is

$$\frac{1 + t^2 + t^4 - t^6 - t^8}{1 - t^2}.$$

The intersection Poincaré polynomial is obtained by subtracting the Poincaré polynomial of

$$H^*(pt) \otimes H_T^{\geq 2d} = \frac{t^4}{1 - t^2}$$

and hence

$$IP_t(X) = \frac{1 + t^2 - t^6 - t^8}{1 - t^2} = 1 + 2t^2 + 2t^4 + t^6.$$

2.7.3 Theorem

For two classes a, b in $IH^*(X)$ such that $\deg a + \deg b = \dim X$ we have

$$\nu(a) \cup \nu(b) = \langle a, b \rangle \, \nu(e)$$

where $\nu : IH^*(X) \to H^*_T(Z)$ is the hypercohomology of $IC^{\cdot}_X \cong \mathcal{A}^{\cdot} \to \mathcal{C}^{\bullet}_T(M)$ and e is the unique top degree class representing a point.

2.7.4 Example

Continuation of the above example. It is elementary that

$$H_T^*(\mathbb{P}^4) = \mathbb{C}[\xi, \tau]/\langle \xi(\xi - \tau)^2(\xi + \tau)^2 \rangle$$

where ξ is a generator in $H^2(\mathbb{P}^4)$ and τ is a generator in H_T^2.

The equivariant Euler classes for the two unstable strata are $\xi(\xi - \tau)^2$, $\xi(\xi + \tau)^2$ respectively. Therefore,

$$H_T^*((\mathbb{P}^4)^{ss}) = \mathbb{C}[\xi, \tau]/\langle \xi(\xi - \tau)^2,\, \xi(\xi + \tau)^2 \rangle$$

A Gröbner basis for the relation ideal is

$$\{\xi^3 + 3\xi\tau^2, \xi^2\tau, \xi\tau^3\}$$

where $\xi > \tau$. Hence as a vector space,

$$H_T^*((\mathbb{P}^4)^{ss}) = \mathbb{C}\{1, \xi, \tau, \xi^2, \xi\tau, \tau^2, \xi\tau^2, \tau^3, \tau^4, \tau^5, \cdots\}$$

To get V^*, we remove $\mathbb{C}\{\tau^2, \tau^3, \tau^4, \cdots\}$ to get

$$V^* = \oplus_{0 \le i \le 3} V^{2i}$$

$$V^0 = \mathbb{C}, \quad V^2 = \mathbb{C}\{\xi, \tau\}, \quad V^4 = \mathbb{C}\{\xi^2, \xi\tau\}, \quad V^6 = \mathbb{C}\{\xi\tau^2\}.$$

As $\xi^3 = -\xi\tau^2$ and $\xi^2\tau = 0$, the intersection matrix for $IH^2(X) \otimes IH^4(X)$ is up to a constant

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

.

Chapter 3

Kirwan map and its splitting

3.1 Setting

- K compact Lie group, $G = K^{\mathbb{C}}$ reductive group
- $\rho : K \to U(n+1)$ homomorphism, $\rho^{\mathbb{C}} : G \to GL(n+1)$
- $M \subset \mathbb{P}^n$ smooth projective variety acted on by K via ρ
- The moment map $\mu_{\mathbb{P}^n} : \mathbb{P}^n \to u(n+1)^*$ for the action of $U(n+1)$ on $\mathbb{P}^n$ is up to constant x^*Ax/x^*x for $A \in u(n+1)$.
- The composition $\mu : M \hookrightarrow \mathbb{P}^n \to u(n+1)^* \to \mathbf{k}^*$ is a moment map for the K-action.
- $M^{ss} = \{p \in M \,|\, \text{gradient flow from } p \text{ w.r.t. } -|\mu|^2 \text{ has a limit point in } \mu^{-1}(0)\}$ = the set of semistable points in Mumford's sense
- $Z = \mu^{-1}(0) \subset M^{ss}$ deformation retract
- $M /\!\!/ G = \{\text{closed orbits in } M^{ss}\}$ is a projective variety.
- $M /\!\!/ G$ is homeomorphic to the symplectic reduction Z/K.
- The equivariant cohomology ring $H_K^*(M^{ss}) \cong H_K^*(Z)$ is computable by the Atiyah-Bott-Kirwan theory.
- Goal: describe $IH^*(X)$ in terms of $H_K^*(Z)$.

3.2 Partial desingularization

- M^{ss} is a quasi-projective variety; $X = M/\!\!/G$ is projective
- {singular points in X}={closed orbits in M^{ss} with nontrivial stabilizers}.
- $\mathcal{R}(M^{ss})$={conjugacy classes of the identity components of stabilizers R of points whose orbits are closed}
- For $R \in \mathcal{R}(M^{ss})$, $M_R^{ss} = R$-fixed point set in M^{ss} (smooth quasi-projective)
- When $\dim R$ is largest, $GM_R^{ss} \cong G \times_{N^R} M_R^{ss}$ is smooth where N^R is the normalizer of R in G.
- M_1 = blow-up of M^{ss} along GM_R^{ss}
- K action lifts to M_1; M_1^{ss} = the set of semistable points in M_1
- $\mathcal{R}(M_1^{ss}) = \mathcal{R}(M^{ss}) - \{R\}$; $M_1/\!\!/G$ is less singular than $M/\!\!/G$
- Continue this process till there is no infinite stabilizers

$$\widetilde{M}^{ss} = M_r^{ss} \to M_{r-1}^{ss} \to \cdots \to M_1^{ss} \to M^{ss}$$

$\widetilde{X} = \widetilde{M}/\!\!/G$ is an orbifold, called the *partial desingularization.*

3.3 Kirwan map

3.3.1 Equivariant cohomology

- The above $\pi : \widetilde{M}^{ss} \to M^{ss}$ induces a homomorphism

$$\pi^* : H_K^*(M^{ss}) \to H_K^*(\widetilde{M}^{ss}) \cong H_K^*(\widetilde{Z}) \cong H^*(\widetilde{X}) \cong IH^*(\widetilde{X})$$

  where $\widetilde{Z}$ is the zero level set of the moment map for $\widetilde{M}^{ss}$.

- $\psi : M_K^{ss} = EK \times_K M^{ss} \to M/\!\!/G = X$
- $\widetilde{\psi} : \widetilde{M}_K^{ss} = EK \times_K \widetilde{M}^{ss} \to \widetilde{M}/\!\!/G = \widetilde{X}$

$$\begin{tikzcd} M_K^{ss} \arrow[d, "\psi"'] & \widetilde{M}_K^{ss} \arrow[l, "\pi"'] \arrow[d, "\widetilde{\psi}"] \\ X & \widetilde{X} \arrow[l, "p"'] \end{tikzcd}$$

- Since G acts locally freely on $\widetilde{M}^{ss}$, $\widetilde{\psi}_*\mathbb{C}_{\widetilde{M}_K^{ss}} \cong \mathbb{C}_{\widetilde{X}}$ (Bernstein-Lunts)
- $\mathcal{C}_K^{\bullet}(M) = \psi_*\mathbb{C}_{M_K^{ss}} \in \mathbf{D}_c^{\geq 0}(X)$
- $\mathcal{C}_K^{\bullet}(\widetilde{M}) = \psi_*\pi_*\mathbb{C}_{\widetilde{M}_K^{ss}} = p_*\widetilde{\psi}_*\mathbb{C}_{\widetilde{M}_K^{ss}} = p_*\mathbb{C}_{\widetilde{X}} \in \mathbf{D}_c^{\geq 0}(X)$
- The map of equivariant cohomology comes from the morphism

$$\mathcal{C}_K^{\bullet}(M) = \psi_*\mathbb{C}_{M_K^{ss}} \to \psi_*\pi_*\pi^*\mathbb{C}_{M_K^{ss}} = \psi_*\pi_*\mathbb{C}_{\widetilde{M}_K^{ss}} = p_*\mathbb{C}_{\widetilde{X}} = \mathcal{C}_K^{\bullet}(\widetilde{M})$$

  by taking hypercohomology.

3.3.2 Decomposition theorem (BBDG)

$$p_*\mathbb{C}_{\widetilde{X}} \cong IC_X^{\bullet} \oplus \mathcal{F}^{\bullet}$$

for some $\mathcal{F}^{\bullet} \in \mathbf{D}_c^{\geq 0}(X)$ supported on the singular locus. In particular, we have a morphism $p_*\mathbb{C}_{\widetilde{X}} \to IC_X^{\bullet}$.

3.3.3 Kirwan map

The composition

$$\kappa_M^{ss} : \mathcal{C}^\bullet_K(M) \to \mathcal{C}^\bullet_K(\widetilde{M}) \cong p_*\mathbb{C}_{\widetilde{X}} \to IC^\cdot_X$$

gives us a homomorphism $\kappa_M^{ss} : H_K^*(M^{ss}) \to IH^*(X)$.

3.3.4 Theorem (Kirwan, Woolf)

κ_M^{ss} is surjective.

3.3.5 Remark

Kirwan used this map to give an algorithm for the Betti numbers of the intersection cohomology. But this method does not provide us with a strong insight on intersection pairing.

3.3.6 Motivation

Witten gave formulas for the intersection pairing of the moduli space of vector bundles over curves. Jeffrey and Kirwan proved his formulas in the smooth case. For the singular case, it is not obvious how to interpret the formulas because they are really about equivariant cohomology classes. Thanks to the Kirwan map, if we use intersection cohomology, the intersection pairing of the equivariant cohomology classes makes sense. Therefore, the natural question is to generalize Witten's formulas to the singular case by using intersection cohomology!

3.3.7 Strategy

Our strategy to compute the intersection pairing of $IH^*(X)$ is to construct a natural (right) inverse of κ_M^{ss} i.e. splitting of the Kirwan map!

3.4 Placid maps

Unlike ordinary cohomology, intersection cohomology is functorial only for a limited class of maps.

3.4.1 Definition

Let p, q be perversities. A continuous map $f : Y \to X$ is called (p, q)-*placid* if there is a stratification of X such that for each stratum S,

$$q(\operatorname{codim} S) \leq p(\operatorname{codim} f^{-1}(S)).$$

3.4.2 Proposition

If $f : Y \to X$ is (p, q)-placid, then pulling back cycles induces

$$f^* : IH_q^*(X) \to IH_p^*(Y).$$

In fact, this arises from a morphism $IC^{\cdot}_{q,X} \to f_* IC^{\cdot}_{p,Y}$ in the derived category. A similar statement holds for the equivariant case.

3.4.3 Proposition

If $\phi : M^{ss} \to M/\!\!/G = X$ is (t, m)-placid, we have a morphism

$$IC^{\cdot}_X \to \mathcal{C}^{\bullet}_K(M)$$

which induces a map on hypercohomology $\phi^* : IH^*(X) \to H_K^*(M^{ss})$.
<u>Proof</u>: Since M^{ss} is smooth, the equivariant intersection cohomology with respect to any perversity is isomorphic to the ordinary equivariant cohomology. As ϕ is (t, m)-placid, we have a map

$$IH^*(X) \to IH_{t,K}^*(M^{ss}) \cong H_K^*(M^{ss})$$

as desired.

3.5 Kirwan map and its splitting

3.5.1 Theorem

$\phi^* : IH^*(X) \to H_K^*(M^{ss})$ is a right inverse of $\kappa_M^{ss} : H_K^*(M^{ss}) \to IH^*(X)$.
Proof: ϕ^* and κ_M^{ss} are both from morphisms in the derived category $\mathbf{D}_c^{\geq 0}(X)$. Composing them we get

$$IC_X^{\cdot} \to \mathcal{C}_K^{\bullet}(M) \to IC_X^{\cdot}.$$

When restricted to the smooth part, it is the identity map. An extension of the identity map on the smooth part to $IC_X^{\cdot} \to IC_X^{\cdot}$ is unique by Deligne's construction. Hence the composition is the identity. □

3.5.2 Corollary

$IH^*(X)$ is a subsapce of $H_K^*(M^{ss})$ if $\phi : M^{ss} \to X$ is (t, m)-placid.

3.5.3 Theorem

$\phi^* : IH^*(X) \to H_K^*(M^{ss})$ preserves the intersection pairing in the sense that

$$\phi^*(\alpha) \cup \phi^*(\beta) = \langle \alpha, \beta \rangle \phi^*(e)$$

for α, β in $IH^*(X)$ such that $\deg \alpha + \deg \beta = \dim X$ where e is the top degree class representing a point.
Proof: ϕ^* is just pulling back cycles and we can choose cycles for α, β which intersect only on the smooth part and transversely. The intersection of the pull-backs is the pull-back of the intersections. □

3.5.4 Two issues

1. When does $\phi : M^{ss} \to X$ satisfy the (t, m)-placid condition?
2. What is the image of $IH^*(X)$ in $H_K^*(M^{ss})$?

If we can settle these two issues, we have a complete description of $IH^*(X)$ as a vector space with a perfect pairing, in terms of the equivariant cohomology which is in principle computable by the ABK theory.

3.6 Almost balanced action

3.6.1 Stratification of $X = M/\!\!/G$

- $Z = \mu^{-1}(0)$; $Z_H = \{x \in Z \,|\, \mathrm{Stab}x = H\}$
- $Z_{(H)} = KZ_H = K \times_{N^H} Z_H$ where N^H is the normalizer of H.
- $Z = \sqcup_{(H)} Z_{(H)}$; $X_{(H)} = Z_{(H)}/K = Z_H/N^H$ smooth
- $X = \sqcup_{(H)} X_{(H)}$ stratification of X
- For $x \in Z_H$, the *symplectic slice* is

$$W = T_x(Kx)^{\perp\omega}/T_xKx.$$

3.6.2 Local normal form theorem

A neighborhood of $Z_{(H)}$ in M is a fiber bundle

$$\begin{tikzcd} F \arrow[r] & \mathcal{N} \arrow[d] \\ & X_{(H)} \end{tikzcd}$$

where $F = K \times_H ((\mathbf{k}/\mathbf{h})^* \times W)$. A neighborhood of $X_{(H)}$ in X is a fiber bundle

$$\begin{tikzcd} W/\!\!/H \arrow[r] & \mathcal{N}/\!\!/K \arrow[d] \\ & X_{(H)} \end{tikzcd}$$

since $F/\!\!/K = W/\!\!/H$.

3.6.3 (t, m)-placid condition

- $\mathrm{codim}_X X_{(H)} = \dim W/\!\!/H$
- $\mathrm{codim}_{M^{ss}} \phi^{-1}(X_{(H)}) = \mathrm{codim}_W \phi_W^{-1}(v)$ where $\phi_W : W \to W/\!\!/H$ is the quotient map and v is the vertex of the cone $W/\!\!/H$.
- $\phi : M^{ss} \to X$ is (t, m)-placid if and only if $t(\mathrm{codim}_W \phi_W^{-1}(v)) \geq m(\dim W/\!\!/H)$ i.e.

$$\mathrm{codim}_W \phi_W^{-1}(v) > \frac{1}{2} \dim W/\!\!/H$$

3.6.4 Theorem (Kirwan)

$$\operatorname{codim}_W \phi_W^{-1}(v) = \min_{\beta \in \mathcal{B}} \{2n(\beta) - \dim H/\mathrm{Stab}\beta\}$$

where $\mathcal{B}$ is the set of closest points β (in the positive Weyl chamber) to zero in the convex hull of a collection of weights of the H-action on W and $n(\beta)$ is the number of weights α such that $\langle \alpha, \beta \rangle < \langle \beta, \beta \rangle$.

3.6.5 Proposition

$\phi : M^{ss} \to X$ is (t, m)-placid if and only if

$$2n(\beta) - \dim H/\mathrm{Stab}\beta > \frac{1}{2} \dim W /\!\!/ H$$

for all β and H.

3.6.6 Example

Suppose the weights are distributed symmetrically with respect to the origin. Then $2n(\beta) \geq \frac{1}{2} \dim W$ and the inequality is satisfied. This is the case for the moduli space of vector bundles over curves.

3.6.7 Definition

We say the action of K on M is *almost balanced* if the inequality 3.6.5 holds for any β and H.

3.7 Image of $IH^*(X)$ in $H_K^*(M^{ss})$

3.7.1 Definition

The action of K on M is *weakly balanced* if it is almost balanced and so is the action of N^H/H on M_H for each $H^{\mathbb{C}} \in \mathcal{R}(M)$.

3.7.2 Theorem

For $H^{\mathbb{C}} \in \mathcal{R}(M)$, consider the natural map

$$K \times_{N^H} M_H^{ss} \to KM_H^{ss} \subset M^{ss}$$

and the induced map

$$\gamma_H : H_K^*(M^{ss}) \to H_K^*(K \times_{N^H} M_H^{ss}) \cong [H^*_{N_0^H/H}(M_H^{ss}) \otimes H_H^*]^{\pi_0 N^H}$$

where N_0^H is the identity component of the normalizer N^H of H. The image of $IH^*(X)$ in $H_K^*(M^{ss})$ by pull-back is

$$V^* = \{\zeta \in H_K^*(M^{ss}) \,|\, \gamma_H(\zeta) \in H^*_{N_0^H/H}(M_H^{ss}) \otimes H_H^{<\frac{1}{2}\operatorname{codim} X_{(H)}} \quad \forall H\}.$$

3.7.3 Example

Let $K = SU(2)$ act on the space of binary forms $\mathbb{P}^{2n}$. The Poincaré series for the equivariant cohomology is by the ABK theory

$$P_t^K(Z) = \frac{1 + t^2 + \cdots + t^{4n}}{1 - t^4} - \sum_{n<j\leq 2n} \frac{t^{2(j-1)}}{1 - t^2}.$$

Let $H = U(1)$. Then $M_H^{ss} = \{X^n Y^n\}$ and $\operatorname{codim} X_{(H)} = 2n - 3$. Hence to get the Poincaré polynomial for $V^* \cong IH^*(X)$ we have only to subtract out $t^{4[\frac{n}{2}]}/(1 - t^4)$. Therefore, we have

$$IP_t(X) = \begin{cases} \frac{(1-t^{2n})(1-t^{2n})}{(1-t^2)(1-t^4)} & \text{if } n \text{ odd} \\ \frac{(1-t^{2n-2})(1-t^{2n+2})}{(1-t^2)(1-t^4)} & \text{if } n \text{ even} \end{cases}$$

For $\alpha, \beta \in V^*$ of complementary degrees, write $\alpha\beta = q(\xi, \tau^2)$ for some polynomial q where ξ is the degree 2 generator of $H^*(\mathbb{P}^n)$ and τ^2 is the degree 4 generator of H_K^*. Then the intersection pairing of α and β in $IH^*(X)$ is

$$\langle \alpha, \beta \rangle = \operatorname{res}_{X=0} \sum_{0<n-j\leq n} \frac{q((n-2j)X, X^2}{(2X)^{2n-2} \prod_{k\neq j}(k-j)}.$$

This follows from the nonabelian localization plus the observation that $\alpha\beta$ is supported on the smooth part.

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