MODULI SPACES OF WEIGHTED POINTED STABLE RATIONAL CURVES VIA GIT

YOUNG-HOON KIEM AND HAN-BOM MOON

Abstract. We construct the Mumford-Knudsen space $\overline{M}_{0,n}$ of n-pointed stable rational curves by a sequence of explicit blow-ups from the GIT quotient $(\mathbb{P}^1)^n//\mathrm{SL}(2)$ with respect to the symmetric linearization $\mathcal{O}(1,\cdots,1)$. The intermediate blown-up spaces turn out to be the moduli spaces of weighted pointed stable curves $\overline{M}_{0,n\cdot\epsilon}$ for suitable ranges of ϵ. As an application, we provide a new unconditional proof of M. Simpson's Theorem about the log canonical models of $\overline{M}_{0,n}$. We also give a basis of the Picard group of $\overline{M}_{0,n\cdot\epsilon}$.

1. Introduction

Recently there has been a tremendous amount of interest in the birational geometry of moduli spaces of stable curves. See for instance [1, 4, 7, 10, 11, 17, 19, 21] for the genus 0 case only. Most prominently, it has been proved in [1, 4, 21] that the log canonical models for $(\overline{M}_{0,n}, K_{\overline{M}_{0,n}}+\alpha D)$, where D is the boundary divisor and α is a rational number, give us Hassett's moduli spaces $\overline{M}_{0,n\cdot\epsilon}$ of weighted pointed stable curves with *symmetric* weights $n\cdot\epsilon=(\epsilon,\cdots,\epsilon)$. See §2.1 for the definition of $\overline{M}_{0,n\cdot\epsilon}$ and Theorem 1.2 below for a precise statement. The purpose of this paper is to prove that actually all the moduli spaces $\overline{M}_{0,n\cdot\epsilon}$ can be constructed by explicit blow-ups from the GIT quotient $(\mathbb{P}^1)^n//\mathrm{SL}(2)$ with respect to the symmetric linearization $\mathcal{O}(1,\cdots,1)$ where $\mathrm{SL}(2)$ acts on $(\mathbb{P}^1)^n$ diagonally. More precisely, we prove the following.

Theorem 1.1. *There is a sequence of blow-ups*

(1)
$$\overline{M}_{0,n}=\overline{M}_{0,n\cdot\epsilon_{m-2}}\to\overline{M}_{0,n\cdot\epsilon_{m-3}}\to\cdots\to\overline{M}_{0,n\cdot\epsilon_2}\to\overline{M}_{0,n\cdot\epsilon_1}\to(\mathbb{P}^1)^n//\mathrm{SL}(2)$$

where $m=\lfloor\frac{n}{2}\rfloor$ and $\frac{1}{m+1-k}<\epsilon_k\le\frac{1}{m-k}$. Except for the last arrow when n is even, the center for each blow-up is a union of transversal smooth subvarieties of same dimension. When n is even, the last arrow is the blow-up along the singular locus which consists of $\frac{1}{2}\binom{n}{m}$ points in $(\mathbb{P}^1)^n//\mathrm{SL}(2)$, i.e. $\overline{M}_{0,n\cdot\epsilon_1}$ is Kirwan's partial desingularization (see [14]) of the GIT quotient $(\mathbb{P}^1)^{2m}//\mathrm{SL}(2)$.

If the center of a blow-up is the transversal union of smooth subvarieties in a nonsingular variety, the result of the blow-up is isomorphic to that of the sequence of smooth blow-ups along the irreducible components of the center in any order (see §2.3). So each of the above arrows can be decomposed into the composition of smooth blow-ups along the irreducible components.

As an application of Theorem 1.1, we give a new proof of the following theorem of M. Simpson ([21]) without relying on Fulton's conjecture.

Date: March 2010.

Partially supported by an NRF grant.

Theorem 1.2. *Let α be a rational number satisfying $\frac{2}{n-1} < \alpha \le 1$ and let $D = \overline{M}_{0,n} - M_{0,n}$ denote the boundary divisor. Then the log canonical model*

$$\overline{M}_{0,n}(\alpha) = \mathrm{Proj}\left(\bigoplus_{l\ge 0} H^0(\overline{M}_{0,n}, \mathcal{O}(\lfloor l(K_{\overline{M}_{0,n}} + \alpha D)\rfloor))\right)$$

satisfies the following:

(1) *If $\frac{2}{m-k+2} < \alpha \le \frac{2}{m-k+1}$ for $1 \le k \le m-2$, then $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,n\cdot\epsilon_k}$.*
(2) *If $\frac{2}{n-1} < \alpha \le \frac{2}{m+1}$, then $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n /\!/ G$ where the quotient is taken with respect to the symmetric linearization $\mathcal{O}(1, \cdots, 1)$.*

There are already two different *unconditional* proofs of Theorem 1.2 by Alexeev-Swinarski [1] and by Fedorchuk-Smyth [4]. See Remark 5.13 for a brief outline of the two proofs. In this paper we obtain the ampleness of some crucial divisors directly from Theorem 1.1. As another application, we give an explicit basis of the Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ for each k.

It is often the case in moduli theory that adding an extra structure makes a problem easier. Let $0 \le k < n$. A pointed nodal curve $(C, p_1, \cdots, p_n)$ of genus 0 together with a morphism $f : C \to \mathbb{P}^1$ of degree 1 is called k-*stable* if

i. all marked points p_i are smooth points of C;
ii. no more than $n-k$ of the marked points p_i can coincide;
iii. any ending irreducible component C' of C which is contracted by f contains more than $n-k$ marked points;
iv. the group of automorphisms of C preserving f and p_i is finite.

A. Mustata and M. Mustata prove the following in [19].

Theorem 1.3. [19, §1] *There is a fine moduli space F_k of k-stable pointed parameterized curves $(C, p_1, \cdots, p_n, f)$. Furthermore, the moduli spaces F_k fit into a sequence of blow-ups*

$$\mathbb{P}^1[n] = F_{n-2} \xrightarrow{\psi_{n-2}} F_{n-3} \xrightarrow{\psi_{n-3}} \cdots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 = (\mathbb{P}^1)^n \tag{2}$$

whose centers are transversal unions of smooth subvarieties.

The first term $\mathbb{P}^1[n]$ is the Fulton-MacPherson compactification of the configuration space of n points in $\mathbb{P}^1$ constructed in [5]. The blow-up centers are transversal unions of smooth subvarieties and hence we can further decompose each arrow into the composition of smooth blow-ups along the irreducible components in any order. This blow-up sequence is actually a special case of L. Li's inductive construction of a *wonderful compactification* of the configuration space and transversality of various subvarieties is a corollary of Li's result [17, Proposition 2.8]. (See §2.3.) The images of the blow-up centers are invariant under the diagonal action of $SL(2)$ on $(\mathbb{P}^1)^n$ and so this action lifts to F_k for all k. The aim of this paper is to show that the GIT quotient of the sequence (2) by $SL(2)$ gives us (1).

To make sense of GIT quotients, we need to specify a linearization of the action of $G = SL(2)$ on F_k. For $F_0 = (\mathbb{P}^1)^n$, we choose the symmetric linearization $L_0 = \mathcal{O}(1, \cdots, 1)$. Inductively, we choose $L_k = \psi_k^* L_{k-1} \otimes \mathcal{O}(-\delta_k E_k)$ where E_k is the exceptional divisor of ψ_k and $0 < \delta_k << \delta_{k-1} << \cdots << \delta_1 << 1$. Let F_k^{ss}

(resp. F_k^s) be the semistable (resp. stable) part of F_k with respect to L_k. Then by [14, §3] or [8, Theorem 3.11], we have

$$\psi_k^{-1}(F_{k-1}^s) \subset F_k^s \subset F_k^{ss} \subset \psi_k^{-1}(F_{k-1}^{ss}). \tag{3}$$

In particular, we obtain a sequence of morphisms

$$\bar{\psi}_k : F_k /\!/ G \to F_{k-1} /\!/ G.$$

It is well known that a point $(x_1, \cdots, x_n)$ in $F_0 = (\mathbb{P}^1)^n$ is stable (resp. semistable) if $\geq \lfloor \frac{n}{2} \rfloor$ points (resp. $> \lfloor \frac{n}{2} \rfloor$ points) do not coincide ([18, 13]).

Let us first consider the case where n is odd. In this case, $F_0^s = F_0^{ss}$ because $\frac{n}{2}$ is not an integer. Hence $F_k^s = F_k^{ss}$ for any k by (3). Since the blow-up centers of ψ_k for $k \leq m+1$ lie in the unstable part, we have $F_k^s = F_0^s$ for $k \leq m+1$. Furthermore, the stabilizer group of every point in F_k^s is $\{\pm 1\}$, i.e. $\bar{G} = PGL(2)$ acts freely on F_k^s for $0 \leq k \leq n-2$ and thus $F_k /\!/ G = F_k^s / G$ is nonsingular. By the stability conditions, forgetting the degree 1 morphism $f : C \to \mathbb{P}^1$ gives us an invariant morphism $F_{n-m+k}^s \to \overline{M}_{0,n\cdot\epsilon_k}$ which induces a morphism

$$\phi_k : F_{n-m+k} /\!/ G \to \overline{M}_{0,n\cdot\epsilon_k} \quad \text{for } k = 0, \cdots, m-2.$$

Since both varieties are nonsingular, we can conclude that ϕ_k is an isomorphism by showing that the Picard numbers are identical. Since $\bar{G}$ acts freely on F_{n-m+k}^s, the quotient of the blow-up center of $\psi_{n-m+k+1}$ is again a transversal union of $\binom{n}{m-k}$ smooth varieties $\Sigma_{n-m+k}^S /\!/ G$ for a subset S of $\{1, \cdots, n\}$ with $|S| = m-k$, which are isomorphic to the moduli space $\overline{M}_{0,(1,\epsilon_k,\cdots,\epsilon_k)}$ of weighted pointed stable curves with $n-m+k+1$ marked points (Remark 4.4). Finally we conclude that

$$\varphi_k : \overline{M}_{0,n\cdot\epsilon_k} \cong F_{n-m+k} /\!/ G \xrightarrow{\bar{\psi}_{n-m+k}} F_{n-m+k-1} /\!/ G \cong \overline{M}_{0,n\cdot\epsilon_{k-1}}$$

is a blow-up by using a lemma in [14] which tells us that quotient and blow-up commute. (See §2.2.) It is straightforward to check that this morphism φ_k is identical to Hassett's natural morphisms (§2.1). Note that the isomorphism

$$\phi_{m-2} : \mathbb{P}^1[n] /\!/ G \xrightarrow{\cong} \overline{M}_{0,n}$$

was obtained by Hu and Keel ([9]) when n is odd because L_0 is a *typical* linearization in the sense that $F_0^{ss} = F_0^s$. The above proof of the fact that ϕ_k is an isomorphism in the odd n case is essentially the same as Hu-Keel's. However their method does not apply to the even degree case.

The case where n is even is more complicated because $F_k^{ss} \neq F_k^s$ for all k. Indeed, $F_m /\!/ G = \cdots = F_0 /\!/ G = (\mathbb{P}^1)^n /\!/ G$ is singular with exactly $\frac{1}{2}\binom{n}{m}$ singular points. But for $k \geq 1$, the GIT quotient of F_{n-m+k} by G is nonsingular and we can use Kirwan's partial desingularization of the GIT quotient $F_{n-m+k} /\!/ G$ ([14]). For $k \geq 1$, the locus Y_{n-m+k} of closed orbits in $F_{n-m+k}^{ss} - F_{n-m+k}^s$ is the disjoint union of the transversal intersections of smooth divisors Σ_{n-m+k}^S and $\Sigma_{n-m+k}^{S^c}$ where $S \sqcup S^c = \{1, \cdots, n\}$ is a partition with $|S| = m$. In particular, Y_{n-m+k} is of codimension 2 and the stabilizers of points in Y_{n-m+k} are all conjugates of $\mathbb{C}^*$. The weights of the action of the stabilizer $\mathbb{C}^*$ on the normal space to Y_{n-m+k} are $2, -2$. By Luna's slice theorem ([18, Appendix 1.D]), it follows that $F_{n-m+k} /\!/ G$ is smooth along the divisor $Y_{n-m+k} /\!/ G$. If we let $\tilde{F}_{n-m+k} \to F_{n-m+k}^{ss}$ be the blow-up of F_{n-m+k}^{ss} along Y_{n-m+k}, $\tilde{F}_{n-m+k}^{ss} = \tilde{F}_{n-m+k}^s$ and $\tilde{F}_{n-m+k} /\!/ G = \tilde{F}_{n-m+k}^s / G$ is nonsingular. Since blow-up and quotient commute (§2.2), the induced map

$$\tilde{F}_{n-m+k}//G \to F_{n-m+k}//G$$

is a blow-up along $Y_{n-m+k}//G$ which has to be an isomorphism because the blow-up center is already a smooth divisor. So we can use $\tilde{F}^s_{n-m+k}$ instead of F^{ss}_{n-m+k} and apply the same line of arguments as in the odd degree case. In this way, we can establish Theorem 1.1.

To deduce Theorem 1.2 from Theorem 1.1, we note that by [21, Corollary 3.5], it suffices to prove that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample for $\frac{2}{m-k+2} < \alpha \le \frac{2}{m-k+1}$ where $D_k = \overline{M}_{0,n\cdot\epsilon_k} - M_{0,n}$ is the boundary divisor of $\overline{M}_{0,n\cdot\epsilon_k}$ (Proposition 5.6). By the intersection number calculations of Alexeev and Swinarski ([1, §3]), we obtain the nefness of $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ for $\alpha = \frac{2}{m-k+1} + s$ for some (sufficiently small) positive number s. Because any positive linear combination of an ample divisor and a nef divisor is ample, it suffices to show that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample for $\alpha = \frac{2}{m-k+2} + t$ for *any* sufficiently small $t > 0$. We use induction on k. By calculating the canonical divisor explicitly, it is easy to show when $k = 0$. Because φ_k is a blow-up with exceptional divisor D_k^{m-k+1}, $\varphi_k^*(K_{\overline{M}_{0,n\cdot\epsilon_{k-1}}} + \alpha D_{k-1}) - \delta D_k^{m-k+1}$ is ample for small $\delta > 0$ if $K_{\overline{M}_{0,n\cdot\epsilon_{k-1}}} + \alpha D_{k-1}$ is ample. By a direct calculation, we find that these ample divisors give us $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ with $\alpha = \frac{2}{m-k+2} + t$ for any sufficiently small $t > 0$. So we obtain a proof of Theorem 1.2.

For the moduli spaces of *unordered* weighted pointed stable curves

$$\widetilde{M}_{0,n\cdot\epsilon_k} = \overline{M}_{0,n\cdot\epsilon_k}/S_n$$

we can simply take the S_n quotient of our sequence (1) and thus $\widetilde{M}_{0,n\cdot\epsilon_k}$ can be constructed by a sequence of *weighted blow-ups* from $\mathbb{P}^n//G = \left((\mathbb{P}^1)^n//G\right)/S_n$. In particular, $\widetilde{M}_{0,n\cdot\epsilon_1}$ is a weighted blow-up of $\mathbb{P}^n//G$ at its singular point when n is even.

Here is an outline of this paper. In §2, we recall necessary materials about the moduli spaces $\overline{M}_{0,n\cdot\epsilon_k}$ of weighted pointed stable curves, partial desingularization and blow-up along transversal center. In §3, we recall the blow-up construction of the moduli space F_k of weighted pointed parameterized stable curves. In §4, we prove Theorem 1.1. In §5, we prove Theorem 1.2. In §6, we give a basis of the Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ as an application of Theorem 1.1.

Acknowledgement. This paper grew out of our effort to prove a conjecture of Brendan Hassett (passed to us by David Donghoon Hyeon): When n is even, $\widetilde{M}_{0,n\cdot\epsilon_1}$ is the (weighted) blow-up of $\mathbb{P}^n//G$ at the singular point. It is our pleasure to thank Donghoon Hyeon for useful discussions. We are also grateful to David Smyth who kindly pointed out an error in a previous draft.

2. Preliminaries

2.1. **Moduli of weighted pointed stable curves.** We recall the definitions and basic facts on Hassett's moduli spaces of weighted pointed stable curves from [7].

A family of nodal curves of genus g with n marked points over base scheme B consists of

(1) a flat proper morphism $\pi : C \to B$ whose geometric fibers are nodal connected curves of arithmetic genus g and
(2) sections $s_1, s_2, \cdots, s_n$ of π.

An n-tuple $\mathcal{A} = (a_1, a_2, \cdots, a_n) \in \mathbb{Q}^n$ with $0 < a_i \le 1$ assigns a weight a_i to the i-th marked point. Suppose that $2g - 2 + a_1 + a_2 + \cdots + a_n > 0$.

Definition 2.1. [7, §2] A family of nodal curves of genus g with n marked points $(C, s_1, \cdots, s_n) \xrightarrow{\pi} B$ is stable of type $(g, \mathcal{A})$ if

(1) the sections $s_1, \cdots, s_n$ lie in the smooth locus of π;
(2) for any subset $\{s_{i_1}, \cdots, s_{i_r}\}$ of nonempty intersection, $a_{i_1} + \cdots + a_{i_r} \le 1$;
(3) $K_\pi + a_1 s_1 + a_2 s_2 + \cdots + a_n s_n$ is π-relatively ample.

Theorem 2.2. [7, Theorem 2.1] *There exists a connected Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$, smooth and proper over $\mathbb{Z}$, representing the moduli functor of weighted pointed stable curves of type $(g, \mathcal{A})$. The corresponding coarse moduli scheme $\overline{M}_{g,\mathcal{A}}$ is projective over $\mathbb{Z}$.*

When $g = 0$, there is no nontrivial automorphism for any weighted pointed stable curve and hence $\overline{M}_{0,\mathcal{A}}$ is a projective *smooth variety* for any $\mathcal{A}$.

There are natural morphisms between moduli spaces with different weight data. Let $\mathcal{A} = \{a_1, \cdots, a_n\}$, $\mathcal{B} = \{b_1, \cdots, b_n\}$ be two weight data and suppose $a_i \ge b_i$ for all $1 \le i \le n$. Then there exists a birational *reduction* morphism

$$\varphi_{\mathcal{A},\mathcal{B}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}.$$

For $(C, s_1, \cdots, s_n) \in \overline{\mathcal{M}}_{g,\mathcal{A}}$, $\varphi_{\mathcal{A},\mathcal{B}}(C, s_1, \cdots, s_n)$ is obtained by collapsing components of C on which $\omega_C + b_1 s_1 + \cdots + b_n s_n$ fails to be ample. These morphisms between moduli stacks induce corresponding morphisms between coarse moduli schemes.

The exceptional locus of the reduction morphism $\varphi_{\mathcal{A},\mathcal{B}}$ consists of boundary divisors D_{I,I^c} where $I = \{i_1, \cdots, i_r\}$ and $I^c = \{j_1, \cdots, j_{n-r}\}$ form a partition of $\{1, \cdots, n\}$ satisfying $r > 2$,

$$a_{i_1} + \cdots + a_{i_r} > 1 \quad \text{and} \quad b_{i_1} + \cdots + b_{i_r} \le 1.$$

Here D_{I,I^c} denotes the closure of the locus of $(C, s_1, \cdots, s_n)$ where C has two irreducible components C_1, C_2 with $p_a(C_1) = 0$, $p_a(C_2) = g$, r sections $s_{i_1}, \cdots s_{i_r}$ lying on C_1, and the other $n - r$ sections lying on C_2.

Proposition 2.3. [7, Proposition 4.5] *The boundary divisor D_{I,I^c} is isomorphic to $\overline{M}_{0,\mathcal{A}'_I} \times \overline{M}_{g,\mathcal{A}'_{I^c}}$, with $\mathcal{A}'_I = (a_{i_1}, \cdots, a_{i_r}, 1)$ and $\mathcal{A}'_{I^c} = (a_{j_1}, \cdots, a_{j_{n-r}}, 1)$. Furthermore, $\varphi_{\mathcal{A},\mathcal{B}}(D_{I,I^c}) \cong \overline{M}_{g,\mathcal{B}'_{I^c}}$ with $\mathcal{B}'_{I^c} = (b_{j_1}, \cdots, b_{j_{n-r}}, \sum_{k=1}^{r} b_{i_k})$.*

From now on, we focus on the $g = 0$ case. Let

$$m = \lfloor \frac{n}{2} \rfloor, \quad \frac{1}{m-k+1} < \epsilon_k \le \frac{1}{m-k} \quad \text{and} \quad n \cdot \epsilon_k = (\epsilon_k, \cdots, \epsilon_k).$$

Consider the reduction morphism

$$\varphi_{n\cdot\epsilon_k, n\cdot\epsilon_{k-1}} : \overline{M}_{0,n\cdot\epsilon_k} \to \overline{M}_{0,n\cdot\epsilon_{k-1}}.$$

Then D_{I,I^c} is contracted by $\varphi_{n\cdot\epsilon_k, n\cdot\epsilon_{k-1}}$ if and only if $|I| = m - k + 1$. Certainly, there are $\binom{n}{m-k+1}$ such partitions $I \sqcup I^c$ of $\{1, \cdots, n\}$.

For two subsets $I, J \subset \{1, \cdots, n\}$ such that $|I| = |J| = m - k + 1$, $D_{I,I^c} \cap D_{J,J^c}$ has codimension at least two in $\overline{M}_{0,n\cdot\epsilon_k}$. So if we denote the complement of the intersections of the divisors by

$$\overline{M}'_{0,n\cdot\epsilon_k} = \overline{M}_{0,n\cdot\epsilon_k} - \bigcup_{|I|=|J|=n-k+1, I\neq J} D_{I,I^c} \cap D_{J,J^c},$$

we have $\mathrm{Pic}(\overline{M}'_{0,n\cdot\epsilon_k}) = \mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_k})$. The restriction of $\varphi_{n\cdot\epsilon_k, n\cdot\epsilon_{k-1}}$ to $\overline{M}'_{0,n\cdot\epsilon_k}$ is a contraction of $\binom{n}{m-k+1}$ *disjoint* divisors and its image is an open subset whose complement has codimension at least two. Therefore we obtain the following equality of Picard numbers:

$$\rho(\overline{M}_{0,n\cdot\epsilon_k}) = \rho(\overline{M}_{0,n\cdot\epsilon_{k-1}}) + \binom{n}{m-k+1}. \tag{4}$$

It is well known that the Picard number of $\overline{M}_{0,n}$ is

$$\rho(\overline{M}_{0,n}) = \rho(\overline{M}_{0,n\cdot\epsilon_{m-2}}) = 2^{n-1} - \binom{n}{2} - 1. \tag{5}$$

Hence we obtain the following lemma from (4) and (5).

Lemma 2.4. (1) *If n is odd,* $\rho(\overline{M}_{0,n\cdot\epsilon_k}) = n + \sum_{i=1}^{k} \binom{n}{m-i+1}$.
(2) *If n is even,* $\rho(\overline{M}_{0,n\cdot\epsilon_k}) = n + \frac{1}{2}\binom{n}{m} + \sum_{i=2}^{k} \binom{n}{m-i+1}$.

2.2. **Partial desingularization.** We recall a few results from [14, 8] on change of stability in a blow-up.

Let G be a complex reductive group acting on a projective nonsingular variety X. Let L be a G-linearized ample line bundle on X. Let Y be a G-invariant closed subvariety of X, and let $\pi : \widetilde{X} \to X$ be the blow-up of X along Y, with exceptional divisor E. Then for sufficiently large d, $L_d = \pi^* L^d \otimes \mathcal{O}(-E)$ becomes very ample, and there is a natural lifting of the G-action to L_d ([14, §3]).

Let X^{ss}(resp. X^s) denote the semistable (resp. stable) part of X. With respect to the polarizations L and L_d, the following hold ([14, §3] or [8, Theorem 3.11]) :

$$\widetilde{X}^{ss} \subset \pi^{-1}(X^{ss}), \qquad \widetilde{X}^{s} \supset \pi^{-1}(X^{s}). \tag{6}$$

In particular, if $X^{ss} = X^s$, then $\widetilde{X}^{ss} = \widetilde{X}^s = \pi^{-1}(X^s)$.

For the next lemma, let us suppose $Y^{ss} = Y \cap X^{ss}$ is nonsingular. We can compare the GIT quotient of $\widetilde{X}$ by G with respect to L_d with the quotient of X by G with respect to L.

Lemma 2.5. [14, Lemma 3.11] *For sufficiently large d, $\widetilde{X}/\!/G$ is the blow-up of $X/\!/G$ along the image $Y/\!/G$ of Y^{ss}.*

Let $\mathcal{I}$ be the ideal sheaf of Y. In the statement of Lemma 2.5, the blow-up is defined by the ideal sheaf $(\mathcal{I}^m)_G$ which is the G-invariant part of $\mathcal{I}^m$, for some m. (See the proof of [14, Lemma 3.11].) In the cases considered in this paper, the blow-ups always take place along *reduced* ideals, i.e. $\widetilde{X}/\!/G$ is the blow-up of $X/\!/G$ along the subvariety $Y/\!/G$ because of the following.

Lemma 2.6. *Let $G = SL(2)$ and $\mathbb{C}^*$ be the maximal torus of G. Suppose Y^{ss} is smooth. The blow-up $\widetilde{X}/\!/G \to X/\!/G$ is the blow-up of the reduced ideal of $Y/\!/G$ if any of the following holds:*

(1) *The stabilizers of points in* X^{ss} *are all equal to the center* $\{\pm 1\}$, *i.e.* $\bar{G} = SL(2)/\{\pm 1\}$ *acts on* X^{ss} *freely.*
(2) *If we denote the* $\mathbb{C}^*$*-fixed locus in* X^{ss} *by* $Z^{ss}_{\mathbb{C}^*}$, $Y^{ss} = Y \cap X^{ss} = GZ^{ss}_{\mathbb{C}^*}$ *and the stabilizers of points in* $X^{ss} - Y^{ss}$ *are all* $\{\pm 1\}$. *Furthermore suppose that the weights of the action of* $\mathbb{C}^*$ *on the normal space of* Y^{ss} *at any* $y \in Z^{ss}_{\mathbb{C}^*}$ *are* $\pm l$ *for some* $l \geq 1$.
(3) *There exists a smooth divisor* W *of* X^{ss} *which intersects transversely with* Y^{ss} *such that the stabilizers of points in* $X^{ss} - W$ *are all* $\mathbb{Z}_2 = \{\pm 1\}$ *and the stabilizers of points in* W *are all isomorphic to* $\mathbb{Z}_4$.

In the cases (1) and (3), $Y//G = Y^s/G$ *and* $X//G = X^s/G$ *are nonsingular and the morphism* $\widetilde{X}//G \to X//G$ *is the smooth blow-up along the smooth subvariety* $Y//G$.

Proof. Let us consider the first case. Let $\bar{G} = PGL(2)$. By Luna's étale slice theorem [18, Appendix 1.D], étale locally near a point in Y^{ss}, X^{ss} is $\bar{G} \times S$ and Y^{ss} is $\bar{G} \times S^Y$ for some nonsingular locally closed subvariety S and $S^Y = S \cap Y$. Then étale locally $\widetilde{X}^{ss}$ is $\bar{G} \times \mathrm{bl}_{S^Y} S$ where $\mathrm{bl}_{S^Y} S$ denotes the blow-up of S along the nonsingular variety S^Y. Thus the quotients $X//G$, $Y//G$ and $\widetilde{X}//G$ are étale locally S, S^Y and $\mathrm{bl}_{S^Y} S$ respectively. This implies that the blow-up $\widetilde{X}//G \to X//G$ is the smooth blow-up along the reduced ideal of $Y//G$.

For the second case, note that the orbits in Y^{ss} are closed in X^{ss} because the stabilizers are maximal. So we can again use Luna's slice theorem to see that étale locally near a point y in Y^{ss}, the varieties X^{ss}, Y^{ss} and $\widetilde{X}$ are respectively $G \times_{\mathbb{C}^*} S$, $G \times_{\mathbb{C}^*} S^0$ and $G \times_{\mathbb{C}^*} \mathrm{bl}_{S^0} S$ for some nonsingular locally closed $\mathbb{C}^*$-equivariant subvariety S and its $\mathbb{C}^*$-fixed locus S^0. Therefore the quotients $X//G$, $Y//G$ and $\widetilde{X}//G$ are étale locally $S//\mathbb{C}^*$, S^0 and $(\mathrm{bl}_{S^0} S)//\mathbb{C}^*$. Thus it suffices to show

$$(\mathrm{bl}_{S^0} S)//\mathbb{C}^* \cong \mathrm{bl}_{S^0}(S//\mathbb{C}^*).$$

Since X is smooth, étale locally we can choose our S to be the normal space to the orbit of y and S is decomposed into the weight spaces $S^0 \oplus S^+ \oplus S^-$. As the action of $\mathbb{C}^*$ extends to $SL(2)$, the nonzero weights are $\pm l$ by assumption. If we choose coordinates $x_1, \cdots, x_r$ for S^+ and $y_1, \cdots, y_s$ for S^-, the invariants are polynomials of $x_i y_j$ and thus $(I^{2m})_{\mathbb{C}^*} = (I_{\mathbb{C}^*})^m$ for $m \geq 1$ where $I = \langle x_1, \cdots, x_r, y_1, \cdots, y_s \rangle$ is the ideal of S^0. By [6, II Exe. 7.11], we have

$$\mathrm{bl}_{S^0} S = \mathrm{Proj}_S(\oplus_m I^m) \cong \mathrm{Proj}_S(\oplus_m I^{2m})$$

and thus

$$(\mathrm{bl}_{S^0} S)//\mathbb{C}^* = \mathrm{Proj}_{S//\mathbb{C}^*}(\oplus_m I^{2m})_{\mathbb{C}^*} = \mathrm{Proj}_{S//\mathbb{C}^*}(\oplus_m (I_{\mathbb{C}^*})^m) = \mathrm{bl}_{I_{\mathbb{C}^*}}(S//\mathbb{C}^*).$$

Since S is factorial and I is reduced, $I_{\mathbb{C}^*}$ is reduced. (If $f^m \in I_{\mathbb{C}^*}$, then $f \in I$ and $(g \cdot f)^m = f^m$ for $g \in \mathbb{C}^*$. By factoriality, $g \cdot f$ may differ from f only by a constant multiple, which must be an m-th root of unity. Because $\mathbb{C}^*$ is connected, the constant must be 1 and hence $f \in I_{\mathbb{C}^*}$.) Therefore $I_{\mathbb{C}^*}$ is the reduced ideal of S^0 on $S//\mathbb{C}^*$ and hence $(\mathrm{bl}_{S^0} S)//\mathbb{C}^* \cong \mathrm{bl}_{S^0}(S//\mathbb{C}^*)$ as desired.

The last case is similar to the first case. Near a point in W, X^{ss} is étale locally $\bar{G} \times_{\mathbb{Z}_2} S$ where $S = S_W \times \mathbb{C}$ for some smooth variety S_W. $\mathbb{Z}_2$ acts trivially on S_W and by ± 1 on $\mathbb{C}$. Etale locally Y^{ss} is $\bar{G} \times_{\mathbb{Z}_2} S_Y$ where $S_Y = (S_W \cap Y) \times \mathbb{C}$. The quotients $X//G$, $Y//G$ and $\widetilde{X}//G$ are étale locally $S_W \times \mathbb{C}$, $(S_W \cap Y) \times \mathbb{C}$ and $\mathrm{bl}_{S_W \cap Y} S_W \times \mathbb{C}$. This proves our lemma. □

Corollary 2.7. *Suppose that (1) of Lemma 2.6 holds. If* $\mathsf{Y}^{ss} = \mathsf{Y}_1^{ss} \cup \cdots \cup \mathsf{Y}_r^{ss}$ *is a transversal union of smooth subvarieties of* X^{ss} *and if* $\widetilde{\mathsf{X}}$ *is the blow-up of* X^{ss} *along* Y^{ss}*, then* $\widetilde{\mathsf{X}}/\!/\mathsf{G}$ *is the blow-up of* $\mathsf{X}/\!/\mathsf{G}$ *along the reduced ideal of* $\mathsf{Y}/\!/\mathsf{G}$ *which is again a transversal union of smooth varieties* $\mathsf{Y}_i/\!/\mathsf{G}$*. The same holds under the condition (3) of Lemma 2.6 if furthermore* Y_i *are transversal to* W*.*

Proof. Because of the assumption (1), $\mathsf{X}^{ss} = \mathsf{X}^s$. If $\mathsf{Y}^{ss} = \mathsf{Y}_1^{ss} \cup \cdots \cup \mathsf{Y}_r^{ss}$ is a transversal union of smooth subvarieties of X^{ss} and if $\pi : \widetilde{\mathsf{X}} \to \mathsf{X}^{ss}$ is the blow-up along Y^{ss}, then $\widetilde{\mathsf{X}}^s = \widetilde{\mathsf{X}}^{ss} = \pi^{-1}(\mathsf{X}^s)$ is the composition of smooth blow-ups along (the proper transforms of) the irreducible components Y_i^{ss} by Proposition 2.10 below. For each of the smooth blow-ups, the quotient of the blown-up space is the blow-up of the quotient along the reduced ideal of the quotient of the center by Lemma 2.6. Hence $\widetilde{\mathsf{X}}/\!/\mathsf{G} \to \mathsf{X}/\!/\mathsf{G}$ is the composition of smooth blow-ups along irreducible smooth subvarieties which are proper transforms of $\mathsf{Y}_i/\!/\mathsf{G}$. Hence $\widetilde{\mathsf{X}}/\!/\mathsf{G}$ is the blow-up along the union $\mathsf{Y}/\!/\mathsf{G}$ of $\mathsf{Y}_i/\!/\mathsf{G}$ by Proposition 2.10 again.

The case (3) of Lemma 2.6 is similar and we omit the detail. □

Finally we recall Kirwan's partial desingularization construction of GIT quotients. Suppose $\mathsf{X}^{ss} \neq \mathsf{X}^s$ and X^s is nonempty. Kirwan in [14] introduced a systematic way of blowing up X^{ss} along a sequence of nonsingular subvarieties to obtain a variety $\widetilde{\mathsf{X}}$ with linearized G action such that $\widetilde{\mathsf{X}}^{ss} = \widetilde{\mathsf{X}}^s$ and $\widetilde{\mathsf{X}}/\!/\mathsf{G}$ has at worst finite quotient singularities only, as follows:

(1) Find a maximal dimensional connected reductive subgroup R such that the R-fixed locus $\mathsf{Z}_\mathsf{R}^{ss}$ in X^{ss} is nonempty. Then
$$\mathsf{G}\mathsf{Z}_\mathsf{R}^{ss} \cong \mathsf{G} \times_{\mathsf{N}^\mathsf{R}} \mathsf{Z}_\mathsf{R}^{ss}$$
is a nonsingular closed subvariety of X^{ss} where N^R denotes the normalizer of R in G.

(2) Blow up X^{ss} along $\mathsf{G}\mathsf{Z}_\mathsf{R}^{ss}$ and find the semistable part X_1^{ss}. Go back to step 1 and repeat this precess until there are no more strictly semistable points.

Kirwan proves that this process stops in finite steps and $\widetilde{\mathsf{X}}/\!/\mathsf{G}$ is called the *partial desingularization* of $\mathsf{X}/\!/\mathsf{G}$. We will drop "partial" if it is nonsingular.

2.3. **Blow-up along transversal center.** We show that the blow-up along a center whose irreducible components are transversal smooth varieties is isomorphic to the result of smooth blow-ups along the irreducible components in any order. This fact can be directly proved but instead we will see that it is an easy special case of beautiful results of L. Li in [17].

Definition 2.8. [17, §1] (1) For a nonsingular algebraic variety X, an *arrangement* of subvarieties S is a finite collection of nonsingular subvarieties such that all nonempty scheme-theoretic intersections of subvarieties in S are again in S.

(2) For an arrangement S, a subset $\mathsf{B} \subset \mathsf{S}$ is called a *building set* of S if for any $s \in \mathsf{S} - \mathsf{B}$, the minimal elements in $\{b \in \mathsf{B} : b \supset s\}$ intersect transversally and the intersection is s.

(3) A set of subvarieties B is called a *building set* if all the possible intersections of subvarieties in B form an arrangement S (called the induced arrangement of B) and B is a building set of S.

The *wonderful compactification* X_B of $\mathsf{X}^0 = \mathsf{X} - \cup_{b \in \mathsf{B}} b$ is defined as the closure of X^0 in $\prod_{b \in \mathsf{B}} \mathrm{bl}_b \mathsf{X}$. Li then proves the following.

Theorem 2.9. [17, Theorem 1.3] *Let* X *be a nonsingular variety and* $B = \{b_1, \cdots, b_n\}$ *be a nonempty building set of subvarieties of* X*. Let* I_i *be the ideal sheaf of* $b_i \in B$*.*

(1) *The wonderful compactification* X_B *is isomorphic to the blow-up of* X *along the ideal sheaf* $I_1 I_2 \cdots I_n$*.*
(2) *If we arrange* $B = \{b_1, \cdots, b_n\}$ *in such an order that the first* i *terms* $b_1, \cdots, b_i$ *form a building set for any* $1 \le i \le n$*, then* $X_B = \mathrm{bl}_{\tilde{b}_n} \cdots \mathrm{bl}_{\tilde{b}_2} \mathrm{bl}_{b_1} X$*, where each blow-up is along a nonsingular subvariety* $\tilde{b}_i$*.*

Here $\tilde{b}_i$ is the *dominant transform* of b_i which is obtained by taking the proper transform when it doesn't lie in the blow-up center or the inverse image if it lies in the center, in each blow-up. (See [17, Definition 2.7].)

Let X be a smooth variety and let $Y_1, \cdots, Y_n$ be transversally intersecting smooth closed subvarieties. Here, *transversal intersection* means that for any nonempty $S \subset \{1, \cdots, n\}$ the intersection $Y_S := \cap_{i \in S} Y_i$ is smooth and the normal bundle $N_{Y_S/X}$ in X of Y_S is the direct sum of the restrictions of the normal bundles $N_{Y_i/X}$ in X of Y_i, i.e.

$$N_{Y_S/X} = \bigoplus_{i \in S} N_{Y_i/X}|_{Y_S}.$$

If we denote the ideal of Y_i by I_i, the ideal of the union $\cup_{i=1}^n Y_i$ is the product $I_1 I_2 \cdots I_n$. Moreover for any permutation $\tau \in S_n$ and $1 \le i \le n$, $B = \{Y_{\tau(1)}, \cdots, Y_{\tau(i)}\}$ is clearly a building set. By Theorem 2.9 we obtain the following.

Proposition 2.10. *Let* $Y = Y_1 \cup \cdots \cup Y_n$ *be a union of transversally intersecting smooth subvarieties of a smooth variety* X*. Then the blow-up of* X *along* Y *is isomorphic to*

$$\mathrm{bl}_{\tilde{Y}_{\tau(n)}} \cdots \mathrm{bl}_{\tilde{Y}_{\tau(2)}} \mathrm{bl}_{Y_{\tau(1)}} X$$

for any permutation $\tau \in S_n$ *where* $\tilde{Y}_i$ *denotes the proper transform of* Y_i*.*

2.4. **Log canonical model.** Let X be a normal projective variety and $D = \sum a_i D_i$ be a rational linear combination of prime divisors of X with $0 < a_i \le 1$. A *log resolution* of (X, D) is a birational morphism $\pi : Y \to X$ from a smooth projective variety Y to X such that $\pi^{-1}(D_i)$ and the exceptional divisors E_i of π are simple normal crossing divisors on Y. Then the discrepancy formula

$$K_Y + \pi_*^{-1}(D) \equiv \pi^*(K_X + D) + \sum_{E_i:\text{exceptional}} a(E_i, X, D) E_i,$$

defines the *discrepancy* of (X, D) by

$$\mathrm{discrep}(X, D) := \inf\{a(E, X, D) : E : \text{exceptional}\}.$$

Let (X, D) be a pair where X is a normal projective variety and $D = \sum a_i D_i$ be a rational linear combination of prime divisors with $0 < a_i \le 1$. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. A pair (X, D) is *log canonical (abbrev. lc)* if $\mathrm{discrep}(X, D) \ge -1$ and *Kawamata log terminal (abbrev. klt)* if $\mathrm{discrep}(X, D) > -1$ and $\lfloor D \rfloor \le 0$.

When X is smooth and D is a normal crossing effective divisor, (X, D) is always lc and is klt if all $a_i < 1$.

Definition 2.11. For lc pair (X, D), the *canonical ring* is

$$R(X, K_X + D) := \oplus_{l \ge 0} H^0(X, \mathcal{O}_X(\lfloor l(K_X + D) \rfloor))$$

and the *log canonical model* is

$$\mathrm{Proj}\, R(X, K_X + D).$$

In [2], Birkar, Cascini, Hacon and McKernan proved that for any klt pair (X, D), the canonical ring is finitely generated, so the log canonical model always exists.

3. Moduli of weighted parameterized stable curves

Let X be a smooth projective variety. In this section, we decompose the map

$$X[n] \to X^n$$

defined by Fulton and MacPherson ([5]) into a *symmetric* sequence of blow-ups along transversal centers. A. Mustata and M. Mustata already considered this problem in their search for intermediate moduli spaces for the stable map spaces in [19, §1]. Let us recall their construction.

Stage 0: Let $F_0 = X^n$ and $\Gamma_0 = X^n \times X$. For a subset S of $\{1, 2, \cdots, n\}$, we let

$$\Sigma_0^S = \{(x_1, \cdots, x_n) \in X^n \,|\, x_i = x_j \text{ if } i, j \in S\}, \quad \Sigma_0^k = \cup_{|S|=k} \Sigma_0^S$$

and let $\sigma_0^i \subset \Gamma_0$ be the graph of the i-th projection $X^n \to X$. Then $\Sigma_0^n \cong X$ is a smooth subvariety of F_0. For each S, fix any $i_S \in S$.

Stage 1: Let F_1 be the blow-up of F_0 along Σ_0^n. Let Σ_1^n be the exceptional divisor and Σ_1^S be the proper transform of Σ_0^S for $|S| \neq n$. Let us define Γ_1 as the blow-up of $F_1 \times_{F_0} \Gamma_0$ along $\Sigma_1^n \times_{F_0} \sigma_0^1$ so that we have a flat family

$$\Gamma_1 \to F_1 \times_{F_0} \Gamma_0 \to F_1$$

of varieties over F_1. Let σ_1^i be the proper transform of σ_0^i in Γ_1. Note that Σ_1^S for $|S| = n-1$ are all disjoint smooth varieties of same dimension.

Stage 2: Let F_2 be the blow-up of F_1 along $\Sigma_1^{n-1} = \sum_{|S|=n-1} \Sigma_1^S$. Let Σ_2^S be the exceptional divisor lying over Σ_1^S if $|S| = n-1$ and Σ_2^S be the proper transform of Σ_1^S for $|S| \neq n-1$. Let us define Γ_2 as the blow-up of $F_2 \times_{F_1} \Gamma_1$ along the disjoint union of $\Sigma_2^S \times_{F_1} \sigma_1^{i_S}$ for all S with $|S| = n-1$ so that we have a flat family

$$\Gamma_2 \to F_2 \times_{F_1} \Gamma_1 \to F_2$$

of varieties over F_2. Let σ_2^i be the proper transform of σ_1^i in Γ_2. Note that Σ_2^S for $|S| = n-2$ in F_2 are all transversal smooth varieties of same dimension. Hence the blow-up of F_2 along their union is smooth by §2.3.

We can continue this way until we reach the last stage.

Stage $n-1$: Let F_{n-1} be the blow-up of F_{n-2} along $\Sigma_{n-2}^2 = \sum_{|S|=2} \Sigma_{n-2}^S$. Let Σ_{n-1}^S be the exceptional divisor lying over Σ_{n-2}^S if $|S| = 2$ and Σ_{n-1}^S be the proper transform of Σ_{n-2}^S for $|S| \neq 2$. Let us define Γ_{n-1} as the blow-up of $F_{n-1} \times_{F_{n-2}} \Gamma_{n-2}$ along the disjoint union of $\Sigma_{n-1}^S \times_{F_{n-2}} \sigma_{n-2}^{i_S}$ for all S with $|S| = 2$ so that we have a flat family

$$\Gamma_{n-1} \to F_{n-1} \times_{F_{n-2}} \Gamma_{n-2} \to F_{n-1}$$

of varieties over F_{n-1}. Let σ_{n-1}^i be the proper transform of σ_{n-2}^i in Γ_{n-1}.

Nonsingularity of the blown-up spaces F_k are guaranteed by the following.

Lemma 3.1. *Σ_k^S for $|S| \ge n-k$ are transversal in F_k i.e. the normal bundle in F_k of the intersection $\cap_i \Sigma_k^{S_i}$ for distinct S_i with $|S_i| \ge n-k$ is the direct sum of the restriction of the normal bundles in F_k of $\Sigma_k^{S_i}$.*

Proof. This is a special case of the inductive construction of the wonderful compactification in [17]. (See §2.3.) In our situation, the building set is the set of all diagonals $B_0 = \{\Sigma_0^S | S \subset \{1, 2, \cdots, n\}\}$. By [17, Proposition 2.8], $B_k = \{\Sigma_k^S\}$ is a building set of an arrangement in F_k and hence the desired transversality follows. □

By construction, F_k are all smooth and $\Gamma_k \to F_k$ are equipped with n sections σ_k^i. When $\dim X = 1$, Σ_{n-2}^2 is a divisor and thus $F_{n-1} = F_{n-2}$. In [19, Proposition 1.8], Mustata and Mustata prove that the varieties F_k are fine moduli spaces for some moduli functors as follows.

Definition 3.2. [19, Definition 1.7] A family of k-*stable parameterized rational curves* over S consists of a flat family of curves $\pi : C \to S$, a morphism $\phi : C \to S \times \mathbb{P}^1$ of degree 1 over each geometric fiber C_s of π and n marked sections $\sigma^1, \cdots, \sigma^n$ of π such that for all $s \in S$,

(1) no more than $n-k$ of the marked points $\sigma^i(s)$ in C_s coincide;
(2) any ending irreducible curve in C_s, except the parameterized one, contains more than $n-k$ marked points;
(3) all the marked points are smooth points of the curve C_s;
(4) C_s has finitely many automorphisms preserving the marked points and the map to $\mathbb{P}^1$.

Proposition 3.3. [19, Proposition 1.8] *Let $X = \mathbb{P}^1$. The smooth variety F_k finely represents the functor of isomorphism classes of families of k-stable parameterized rational curves. In particular, $F_{n-2} = F_{n-1}$ is the Fulton-MacPherson space $\mathbb{P}^1[n]$.*

4. Blow-up construction of moduli of pointed stable curves

In the previous section, we decomposed the natural map $\mathbb{P}^1[n] \to (\mathbb{P}^1)^n$ of the Fulton-MacPherson space into a sequence

$$\mathbb{P}^1[n] = F_{n-2} \xrightarrow{\psi_{n-2}} F_{n-3} \xrightarrow{\psi_{n-3}} \cdots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 = (\mathbb{P}^1)^n \tag{7}$$

of blow-ups along transversal centers. By construction the morphisms above are all equivariant with respect to the action of $G = SL(2)$. For GIT stability, we use the *symmetric* linearization $L_0 = \mathcal{O}(1, \cdots, 1)$ for F_0. For F_k we use the linearization L_k inductively defined by $L_k = \psi_k^* L_{k-1} \otimes \mathcal{O}(-\delta_k E_k)$ where E_k is the exceptional divisor of ψ_k and $\{\delta_k\}$ is a decreasing sequence of sufficiently small positive numbers. Let $m = \lfloor \frac{n}{2} \rfloor$. In this section, we prove the following.

Theorem 4.1. *(i) The GIT quotient $F_{n-m+k}//G$ for $1 \le k \le m-2$ is isomorphic to Hassett's moduli space of weighted pointed stable rational curves $\overline{M}_{0,n\cdot\epsilon_k}$ with weights $n \cdot \epsilon_k = (\epsilon_k, \cdots, \epsilon_k)$ where $\frac{1}{m+1-k} < \epsilon_k \le \frac{1}{m-k}$. The induced maps on quotients*

$$\overline{M}_{0,n\cdot\epsilon_k} = F_{n-m+k}//G \to F_{n-m+k-1}//G = \overline{M}_{0,n\cdot\epsilon_{k-1}}$$

are blow-ups along transversal centers for $k = 2, \cdots, m-2$.

(ii) If n is odd,

$$F_{m+1}//G = \cdots = F_0//G = (\mathbb{P}^1)^n//G = \overline{M}_{0,n\cdot\epsilon_0}$$

and we have a sequence of blow-ups

$$\overline{M}_{0,n} = \overline{M}_{0,n\cdot\epsilon_{m-2}} \to \overline{M}_{0,n\cdot\epsilon_{m-3}} \to \cdots \to \overline{M}_{0,n\cdot\epsilon_1} \to \overline{M}_{0,n\cdot\epsilon_0} = (\mathbb{P}^1)^n /\!/ G$$

whose centers are transversal unions of equidimensional smooth varieties.

(iii) If n is even, $\overline{M}_{0,n\cdot\epsilon_1}$ is a desingularization of

$$(\mathbb{P}^1)^n /\!/ G = F_0 /\!/ G = \cdots = F_m /\!/ G,$$

obtained by blowing up $\frac{1}{2}\binom{n}{m}$ singular points so that we have a sequence of blow-ups

$$\overline{M}_{0,n} = \overline{M}_{0,n\cdot\epsilon_{m-2}} \to \overline{M}_{0,n\cdot\epsilon_{m-3}} \to \cdots \to \overline{M}_{0,n\cdot\epsilon_1} \to (\mathbb{P}^1)^n /\!/ G.$$

Remark 4.2. (1) When n is even, $\overline{M}_{0,n\cdot\epsilon_0}$ is not defined because the sum of weights does not exceed 2.

(2) When n is even, $\overline{M}_{0,n\cdot\epsilon_1}$ is Kirwan's (partial) desingularization of the GIT quotient $(\mathbb{P}^1)^n /\!/ G$ with respect to the symmetric linearization $L_0 = \mathcal{O}(1, \cdots, 1)$.

Let F_k^{ss} (resp. F_k^s) denote the semistable (resp. stable) part of F_k. By (6), we have

$$\psi_k(F_k^{ss}) \subset F_{k-1}^{ss}, \qquad \psi_k^{-1}(F_{k-1}^s) \subset F_k^s. \tag{8}$$

Also recall from [13] that $x = (x_1, \cdots, x_n) \in (\mathbb{P}^1)^n$ is semistable (resp. stable) if $> \frac{n}{2}$ (resp. $\geq \frac{n}{2}$) of x_i's are not allowed to coincide. In particular, when n is odd, $\psi_k^{-1}(F_{k-1}^s) = F_k^s = F_k^{ss}$ for all k and

$$F_{m+1}^s = F_m^s = \cdots = F_0^s, \tag{9}$$

because the blow-up centers lie in the unstable part. Therefore we have

$$F_{m+1} /\!/ G = \cdots = F_0 /\!/ G = (\mathbb{P}^1)^n /\!/ G. \tag{10}$$

When n is even, ψ_k induces a morphism $F_k^{ss} \to F_{k-1}^{ss}$ and we have

$$F_m^{ss} = F_{m-1}^{ss} = \cdots = F_0^{ss} \quad \text{and} \quad F_m /\!/ G = \cdots = F_0 /\!/ G = (\mathbb{P}^1)^n /\!/ G. \tag{11}$$

Let us consider the case where n is odd first. By forgetting the parameterization of the parameterized component of each member of family $(\Gamma_{m+k+1} \to F_{m+k+1}, \sigma^i_{m+k+1})$, we get a rational map $F_{m+k+1} \dashrightarrow \overline{M}_{0,n\cdot\epsilon_k}$ for $k = 0, 1, \cdots, m-2$. By the definition of the stability in §2.1, a fiber over $\xi \in F_{m+k+1}$ is not stable with respect to $n \cdot \epsilon_k = (\epsilon_k, \cdots, \epsilon_k)$ if and only if, in each irreducible component of the curve, the number a of nodes and the number b of marked points satisfy $b\epsilon_k + a \leq 2$. Obviously this cannot happen on the (GIT) stable part F_{m+k+1}^s. Therefore we obtain a morphism $F_{m+k+1}^s \to \overline{M}_{0,n\cdot\epsilon_k}$. By construction this morphism is G-invariant and thus induces a morphism

$$\phi_k : F_{m+k+1} /\!/ G \to \overline{M}_{0,n\cdot\epsilon_k}.$$

Since the stabilizer groups in G of points in F_0^s are all $\{\pm 1\}$, the quotient

$$\bar{\psi}_{m+k+1} : F_{m+k+1} /\!/ G \to F_{m+k} /\!/ G$$

of ψ_{m+k+1} is also a blow-up along a center which consists of transversal smooth varieties by Corollary 2.7.

Since the blow-up center has codimension ≥ 2, the Picard number increases by $\binom{n}{m-k+1}$ for $k = 1, \cdots, m-2$. Since the character group of $SL(2)$ has no free part, by the descent result in [3], the Picard number of $F_{m+1} /\!/ G = F_0^s / G$ is the same as the Picard number of F_0^s which equals the Picard number of F_0. Therefore $\rho(F_{m+1}/\!/G) = n$ and the Picard number of $F_{m+k+1}/\!/G$ is

$$n + \sum_{i=1}^{k} \binom{n}{m-i+1}$$

which equals the Picard number of $\overline{M}_{0,n\cdot\epsilon_k}$ by Lemma 2.4. Since $\overline{M}_{0,n\cdot\epsilon_k}$ and $F_{m+k+1}/\!/G$ are smooth and their Picard numbers coincide, we conclude that ϕ_k is an isomorphism as we desired. So we proved Theorem 4.1 for odd n.

Now let us suppose n is even. For ease of understanding, we divide our proof into several steps.

Step 1: For $k \geq 1$, $F_{m+k}/\!/G$ are nonsingular and isomorphic to the partial desingularizations $\tilde{F}_{m+k}/\!/G$.

The GIT quotients $F_{m+k}/\!/G$ may be singular because there are $\mathbb{C}^*$-fixed points in the semistable part F_{m+k}^{ss}. So we use Kirwan's partial desingularization of the GIT quotients $F_{m+k}/\!/G$ (§2.2). The following lemma says that the partial desingularization process has no effect on the quotient $F_{m+k}/\!/G$ for $k \geq 1$.

Lemma 4.3. *Let F be a smooth projective variety with linearized $G = SL(2)$ action and let F^{ss} be the semistable part. Fix a maximal torus $\mathbb{C}^*$ in G. Let Z be the set of $\mathbb{C}^*$-fixed points in F^{ss}. Suppose the stabilizers of all points in the stable part F^s are $\{\pm 1\}$ and $Y = GZ$ is the union of all closed orbits in $F^{ss} - F^s$. Suppose that the stabilizers of points in Z are precisely $\mathbb{C}^*$. Suppose further that $Y = GZ$ is of codimension 2. Let $\tilde{F} \to F^{ss}$ be the blow-up of F^{ss} along Y and let $\tilde{F}^s$ be the stable part in $\tilde{F}$ with respect to a linearization as in §2.2. Finally suppose that for each $y \in Z$, the weights of the $\mathbb{C}^*$ action on the normal space to Y is $\pm l$ for some $l > 0$. Then $\tilde{F}/\!/G = \tilde{F}^s/G \cong F/\!/G$ and $F/\!/G$ is nonsingular.*

Proof. Since $\bar{G} = G/\{\pm 1\}$ acts freely on F^s, F^s/G is smooth. By assumption, Y is the union of all closed orbits in $F^{ss} - F^s$ and hence $F/\!/G - F^s/G = Y/G$. By Lemma 2.6 (2), $\tilde{F}^s/G$ is the blow-up of $F/\!/G$ along the reduced ideal of Y/G. By our assumption, Z is of codimension 4 and

$$Y/G = GZ/G \cong G \times_{N^{\mathbb{C}^*}} Z/G \cong Z/\mathbb{Z}_2$$

where $N^{\mathbb{C}^*}$ is the normalizer of $\mathbb{C}^*$ in G. Since the dimension of $F/\!/G$ is $\dim F - 3$, the blow-up center Y/G is nonsingular of codimension 1. By Luna's slice theorem ([18, Appendix 1.D]), the singularity of $F/\!/G$ at any point $[Gy] \in Y/G$ is $\mathbb{C}^2/\!/\mathbb{C}^*$ where the weights are $\pm l$. Obviously this is smooth and hence $F/\!/G$ is smooth along Y/G. Since the blow-up center is a smooth divisor, the blow-up map $\tilde{F}^s/G \to F/\!/G$ has to be an isomorphism. □

Let Z_{m+k} be the $\mathbb{C}^*$-fixed locus in F_{m+k}^{ss} and let $Y_{m+k} = GZ_{m+k}$. Then Y_{m+k} is the disjoint union of

$$\Sigma_{m+k}^{S,S^c} := \Sigma_{m+k}^{S} \cap \Sigma_{m+k}^{S^c} \cap F_{m+k}^{ss} \quad \text{for } |S| = m, S^c = \{1, \cdots, n\} - S$$

which are nonsingular of codimension 2 for $k \geq 1$ by Lemma 3.1. For a point

$$(C, p_1, \cdots, p_n, f : C \to \mathbb{P}^1) \in \Sigma_{m+k}^{S,S^c},$$

the parameterized component of C (i.e. the unique component which is not contracted by f) has two nodes and no marked points. The normal space $\mathbb{C}^2$ to Σ_{m+k}^{S,S^c}

is given by the smoothing deformations of the two nodes and hence the stabilizer $\mathbb{C}^*$ acts with weights 2 and -2.

The blow-up $\tilde{F}_{m+k}$ of F^{ss}_{m+k} along Y_{m+k} has no strictly semistable points by [14, §6]. In fact, the unstable locus in $\tilde{F}_{m+k}$ is the proper transform of $\Sigma^S_{m+k} \cup \Sigma^{S^c}_{m+k}$ and the stabilizers of points in $\tilde{F}^s_{m+k}$ are either $\mathbb{Z}_2 = \{\pm 1\}$ (for points not in the exceptional divisor of $\tilde{F}^s_{m+k} \to F^{ss}_{m+k}$) or $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ (for points in the exceptional divisor). Therefore, by Lemma 4.3 and Lemma 2.6 (3), we have isomorphisms

$$\tilde{F}^s_{m+k}/G \cong F_{m+k}//G \tag{12}$$

and $F_{m+k}//G$ are nonsingular for $k \geq 1$.

Step 2: The partial desingularization $\tilde{F}_m//G$ is a nonsingular variety obtained by blowing up the $\frac{1}{2}\binom{n}{m}$ singular points of $F_m//G = (\mathbb{P}^1)^n//G$.

Note that Y_m in F^{ss}_m is the disjoint union of $\frac{1}{2}\binom{n}{m}$ orbits Σ^{S,S^c}_m for $|S| = m$. By Lemma 2.6 (2), the morphism $\tilde{F}^s_m/G \to F_m//G$ is the blow-up at the $\frac{1}{2}\binom{n}{m}$ points given by the orbits of the blow-up center. A point in Σ^{S,S^c}_m is represented by $(\mathbb{P}^1, p_1, \cdots, p_n, \mathrm{id})$ with $p_i = p_j$ if $i, j \in S$ or $i, j \in S^c$. Without loss of generality, we may let $S = \{1, \cdots, m\}$. The normal space to an orbit Σ^{S,S^c}_m is given by

$$(T_{p_1}\mathbb{P}^1)^{m-1} \times (T_{p_{m+1}}\mathbb{P}^1)^{m-1} = \mathbb{C}^{m-1} \times \mathbb{C}^{m-1}$$

and $\mathbb{C}^*$ acts with weights 2 and -2 respectively on the two factors. By Luna's slice theorem, étale locally near Σ^{S,S^c}_m, F^{ss}_m is $G \times_{\mathbb{C}^*} (\mathbb{C}^{m-1} \times \mathbb{C}^{m-1})$ and $\tilde{F}_m$ is $G \times_{\mathbb{C}^*} \mathrm{bl}_0(\mathbb{C}^{m-1} \times \mathbb{C}^{m-1})$ while $\tilde{F}^s_m$ is $G \times_{\mathbb{C}^*} \left[\mathrm{bl}_0(\mathbb{C}^{m-1} \times \mathbb{C}^{m-1}) - \mathrm{bl}_0\mathbb{C}^{m-1} \sqcup \mathrm{bl}_0\mathbb{C}^{m-1}\right]$. By an explicit local calculation, the stabilizers of points on the exceptional divisor of $\tilde{F}_m$ are $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ and the stabilizers of points over F^s_m are $\mathbb{Z}_2 = \{\pm 1\}$. Since the locus of nontrivial stabilizers for the action of $\bar{G}$ on $\tilde{F}^s_m$ is a smooth divisor with stabilizer $\mathbb{Z}_2$, $\tilde{F}_m//G = \tilde{F}^s_m/G$ is smooth and hence $\tilde{F}^s_m/G$ is the desingularization of $F_m//G$ obtained by blowing up its $\frac{1}{2}\binom{n}{m}$ singular points.

Step 3: The morphism $\bar{\psi}_{m+k+1} : F_{m+k+1}//G \to F_{m+k}//G$ is the blow-up along the union of transversal smooth subvarieties for $k \geq 1$. For $k = 0$, we have $\tilde{F}^s_{m+1} = \tilde{F}^s_m$ and thus

$$F_{m+1}//G \cong \tilde{F}^s_{m+1}/G = \tilde{F}^s_m/G = \tilde{F}_m//G$$

is the blow-up along its $\frac{1}{2}\binom{n}{m}$ singular points.

From Lemma 3.1, we know Σ^S_{m+k} for $|S| \geq m - k$ are transversal in F_{m+k}. In particular,

$$\bigcup_{|S|=m} \Sigma^S_{m+k} \cap \Sigma^{S^c}_{m+k}$$

intersects transversely with the blow-up center

$$\bigcup_{|S'|=m-k} \Sigma^{S'}_{m+k}$$

for $\psi_{m+k+1} : F_{m+k+1} \to F_{m+k}$. Hence, by Proposition 2.10 we have a commutative diagram

$$
\begin{tikzcd}
\tilde{F}_{m+k+1} \arrow[r] \arrow[d] & \tilde{F}_{m+k} \arrow[d] \\
F^{ss}_{m+k+1} \arrow[r] & F^{ss}_{m+k}
\end{tikzcd}
\tag{13}
$$

for $k \geq 1$ where the top horizontal arrow is the blow-up along the proper transforms $\tilde{\Sigma}^{S'}_{m+k}$ of $\Sigma^{S'}_{m+k}$, $|S'| = m-k$. By Corollary 2.7, we deduce that for $k \geq 1$, $\bar{\psi}_{m+k+1}$ is the blow-up along the transversal union of smooth subvarieties $\tilde{\Sigma}^{S'}_{m+k}//G \cong \Sigma^{S'}_{m+k}//G$.

For $k = 0$, the morphism $\tilde{F}_{m+1} \to \tilde{F}_m$ is the blow-up along the proper transforms of Σ^S_m and $\Sigma^{S^c}_m$ for $|S| = m$. But these are unstable in $\tilde{F}_m$ and hence the morphism $\tilde{F}^s_{m+1} \to \tilde{F}^s_m$ on the stable part is the identity map. So we obtain $\tilde{F}^s_{m+1} = \tilde{F}^s_m$ and $\tilde{F}^s_{m+1}/G \cong \tilde{F}^s_m/G$.

Step 4: Calculation of Picard numbers.

The Picard number of $F^{ss}_m = F^{ss}_0 \subset F_0 = (\mathbb{P}^1)^n$ is n and so the Picard number of $\tilde{F}_m$ is $n + \frac{1}{2}\binom{n}{m}$. By the descent lemma of [3] as in the odd degree case, the Picard number of

$$F_{m+1}//G \cong \tilde{F}^s_{m+1}/G = \tilde{F}^s_m/G$$

equals the Picard number $n + \frac{1}{2}\binom{n}{m}$ of $\tilde{F}^s_m$. Since the blow-up center of $\tilde{F}_{m+k}//G \to \tilde{F}_{m+k-1}//G$ has $\binom{n}{m-k+1}$ irreducible components, the Picard number of $\tilde{F}_{m+k}//G \cong F_{m+k}//G$ is

$$n + \frac{1}{2}\binom{n}{m} + \sum_{i=2}^{k} \binom{n}{m-i+1} \tag{14}$$

for $k \geq 2$.

Step 5: Completion of the proof.

As in the odd degree case, for $k \geq 1$ the universal family $\pi_k : \Gamma_{m+k} \to F_{m+k}$ gives rise to a family of pointed curves by considering the linear system $K_{\pi_k} + \epsilon_k \sum_i \sigma^i_{m+k}$. Over the semistable part F^{ss}_{m+k} it is straightforward to check that this gives us a family of $n \cdot \epsilon_k$-stable pointed curves. Therefore we obtain an invariant morphism

$$F^{ss}_{m+k} \to \overline{M}_{0,n\cdot\epsilon_k}$$

which induces a morphism

$$F_{m+k}//G \to \overline{M}_{0,n\cdot\epsilon_k}.$$

By Lemma 2.4, the Picard number of $\overline{M}_{0,n\cdot\epsilon_k}$ coincides with that of $F_{m+k}//G$ given in (14). Hence the morphism $F_{m+k}//G \to \overline{M}_{0,n\cdot\epsilon_k}$ is an isomorphism as desired. This completes our proof of Theorem 4.1.

Remark 4.4. Let $S \subset \{1, 2, \cdots, n\}$ with $|S| = m - k$. On $\overline{M}_{0,n\cdot\epsilon_k}$, the blow-up center for $\overline{M}_{0,n\cdot\epsilon_{k+1}} \to \overline{M}_{0,n\cdot\epsilon_k}$ is the union of $\binom{n}{m-k}$ smooth subvarieties $\Sigma^S_{n-m+k}/\!/G$. Each $\Sigma^S_{n-m+k}/\!/G$ parameterizes weighted pointed stable curves with $m-k$ colliding marked points $s_{i_1}, s_{i_2}, \cdots, s_{i_{m-k}}$ for $i_j \in S$. On the other hand, for any member of $\overline{M}_{0,n\cdot\epsilon_k}$, no $m-k+1$ marked points can collide. So we can replace $m-k$ marked points s_{i_j} with $i_j \in S$ by a single marked point which cannot collide with any other marked points. Therefore, an irreducible component $\Sigma^S_{n-m+k}/\!/G$ of the blow-up center is isomorphic to the moduli space of weighted pointed rational curves $\overline{M}_{0,(1,\epsilon_k,\cdots,\epsilon_k)}$ with $n-m+k+1$ marked points as discovered by Hassett. (See Proposition 2.3.)

Remark 4.5. For the moduli space of *unordered* weighted pointed stable curves $\overline{M}_{0,n\cdot\epsilon_k}/S_n$, we can simply take quotients by the S_n action of the blow-up process in Theorem 4.1. In particular, $\overline{M}_{0,n}/S_n$ is obtained by a sequence of weighted blow-ups from $\left((\mathbb{P}^1)^n/\!/G\right)/S_n = \mathbb{P}^n/\!/G$.

5. Log canonical models of $\overline{M}_{0,n}$

In this section, we give a relatively elementary and straightforward proof of the following theorem of M. Simpson by using Theorem 4.1. Let $M_{0,n}$ be the moduli space of n *distinct* points in $\mathbb{P}^1$ up to Aut($\mathbb{P}^1$).

Theorem 5.1. (M. Simpson [21]) *Let α be a rational number satisfying $\frac{2}{n-1} < \alpha \le 1$ and let $D = \overline{M}_{0,n} - M_{0,n}$ denote the boundary divisor. Then the log canonical model*

$$\overline{M}_{0,n}(\alpha) = \mathrm{Proj}\left(\bigoplus_{l\ge 0} H^0(\overline{M}_{0,n}, \mathcal{O}(\lfloor l(K_{\overline{M}_{0,n}} + \alpha D)\rfloor))\right)$$

satisfies the following:

(1) *If $\frac{2}{m-k+2} < \alpha \le \frac{2}{m-k+1}$ for $1 \le k \le m-2$, then $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,n\cdot\epsilon_k}$.*
(2) *If $\frac{2}{n-1} < \alpha \le \frac{2}{m+1}$, then $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n/\!/G$ where the quotient is taken with respect to the symmetric linearization $\mathcal{O}(1, \cdots, 1)$.*

Remark 5.2. Keel and McKernan prove ([12, Lemma 3.6]) that $K_{\overline{M}_{0,n}} + D$ is ample. Because

$$\overline{M}_{0,n\cdot\epsilon_{m-2}} \cong \overline{M}_{0,n\cdot\epsilon_{m-1}} = \overline{M}_{0,n}$$

by definition, we find that (1) above holds for $k = m-1$ as well.

For notational convenience, we denote $(\mathbb{P}^1)^n/\!/G$ by $\overline{M}_{0,n\cdot\epsilon_0}$ for even n as well. Let Σ^S_k denote the subvarieties of F_k defined in §3 for $S \subset \{1, \cdots, n\}$, $|S| \le m$. Let

$$D^S_k = \Sigma^S_{n-m+k}/\!/G \subset F_{n-m+k}/\!/G \cong \overline{M}_{0,n\cdot\epsilon_k}.$$

Then D^S_k is a divisor of $\overline{M}_{0,n\cdot\epsilon_k}$ for $|S| = 2$ or $m-k < |S| \le m$. Let $D^j_k = (\cup_{|S|=j}\Sigma^S_{n-m+k})/\!/G$ and $D_k = D^2_k + \sum_{j>m-k} D^j_k$. Then D_k is the boundary divisor of $\overline{M}_{0,n\cdot\epsilon_k}$, i.e. $\overline{M}_{0,n\cdot\epsilon_k} - M_{0,n} = D_k$. When $k = m-2$ so $\overline{M}_{0,n\cdot\epsilon_k} \cong \overline{M}_{0,n}$, sometimes we will drop the subscript k. Note that if n is even and $|S| = m$, $D^S_k = D^{S^c}_k = \Sigma^{S,S^c}_{n-m+k}/\!/G$.

By Theorem 4.1, there is a sequence of blow-ups

$$\overline{M}_{0,n} \cong \overline{M}_{0,n\cdot\epsilon_{m-2}} \xrightarrow{\varphi_{m-2}} \overline{M}_{0,n\cdot\epsilon_{m-3}} \xrightarrow{\varphi_{m-3}} \cdots \xrightarrow{\varphi_2} \overline{M}_{0,n\cdot\epsilon_1} \xrightarrow{\varphi_1} \overline{M}_{0,n\cdot\epsilon_0} \tag{15}$$

whose centers are transversal unions of smooth subvarieties, except for φ_1 when n is even. Note that the irreducible components of the blow-up center of φ_k furthermore intersect transversely with D^j_{k-1} for $j > m-k+1$ by Lemma 3.1 and by taking quotients.

Lemma 5.3. *Let* $1 \le k \le m-2$.

(1) $\varphi_k^*(D^j_{k-1}) = D^j_k$ *for* $j > m-k+1$.
(2) $\varphi_k^*(D^2_{k-1}) = D^2_k + \binom{m-k+1}{2} D^{m-k+1}_k$.
(3) $\varphi_{k*}(D^j_k) = D^j_{k-1}$ *for* $j > m-k+1$ *or* $j=2$.
(4) $\varphi_{k*}(D^j_k) = 0$ *for* $j = m-k+1$.

Proof. The push-forward formulas (3) and (4) are obvious. Recall from §4 that $\varphi_k = \bar\psi_{n-m+k}$ is the quotient of $\psi_{n-m+k} : F^{ss}_{n-m+k} \to F^{ss}_{n-m+k-1}$. Suppose n is not even or k is not 1. Since D^S_k for $|S| > 2$ does not contain any component of the blow-up center, $\varphi_k^*(D^S_{k-1}) = D^S_k$. If $|S| = 2$, D^S_{k-1} contains a component $D^{S'}_{k-1}$ of the blow-up center if and only if $S' \supset S$. Therefore we have

$$\varphi_k^*(D^S_{k-1}) = D^S_k + \sum_{S' \supset S, |S'| = m-k+1} D^{S'}_k.$$

By adding them up for all S such that $|S| = 2$, we obtain (2).

When n is even and $k = 1$, we calculate the pull-back before quotient. Let $\pi : \tilde F^s_m \to F^{ss}_m$ be the map obtained by blowing up $\cup_{|S|=m} \Sigma^{S,S^c}_m$ and removing unstable points. Recall that $\tilde F^s_m / G \cong F_{m+1} // G \cong \overline{M}_{0,n\cdot\epsilon_1}$ and the quotient of π is φ_1. Then a direct calculation similar to the above gives us $\pi^* \Sigma^2_m = \tilde\Sigma^2_m + 2\binom{m}{2}\tilde\Sigma^m_m$ where $\Sigma^2_m = \cup_{|S|=2}\Sigma^S_m$ and $\tilde\Sigma^2_m$ is the proper transform of Σ^2_m while $\tilde\Sigma^m_m$ denotes the exceptional divisor. Note that by the descent lemma ([3]), the divisor Σ^2_m and $\tilde\Sigma^2_m$ descend to D^2_0 and D^2_1. However $\tilde\Sigma^m_m$ does not descend because the stabilizer group $\mathbb{Z}_2$ in $\bar G = \mathsf{PGL}(2)$ of points in $\tilde\Sigma^m_m$ acts nontrivially on the normal spaces. But by the descent lemma again, $2\tilde\Sigma^m_m$ descends to D^m_1. Thus we obtain (2). □

Next we calculate the canonical divisors of $\overline{M}_{0,n\cdot\epsilon_k}$.

Proposition 5.4. [20, Proposition 1] *The canonical divisor of* $\overline{M}_{0,n}$ *is*

$$K_{\overline{M}_{0,n}} \cong -\frac{2}{n-1} D^2 + \sum_{j=3}^{m} \left(-\frac{2}{n-1}\binom{j}{2} + (j-2) \right) D^j.$$

Lemma 5.5. *(1) The canonical divisor of* $(\mathbb{P}^1)^n // G$ *is*

$$K_{(\mathbb{P}^1)^n//G} \cong -\frac{2}{n-1} D^2_0.$$

(2) For $1 \le k \le m-2$, *the canonical divisor of* $\overline{M}_{0,n\cdot\epsilon_k}$ *is*

$$K_{\overline{M}_{0,n\cdot\epsilon_k}} \cong -\frac{2}{n-1} D^2_k + \sum_{j \ge m-k+1}^{m} \left(-\frac{2}{n-1}\binom{j}{2} + (j-2) \right) D^j_k.$$

Proof. It is well known by the descent lemma ([3]) that Pic$((\mathbb{P}^1)^n // G)$ is a free abelian group of rank n(See §6). The symmetric group S_n acts on $(\mathbb{P}^1)^n // G$ in the obvious manner, and there is an induced action on its Picard group. Certainly the canonical bundle $K_{(\mathbb{P}^1)^n//G}$ and D^2_0 are S_n-invariant. On the other hand, the S_n-invariant part of the rational Picard group is a one dimensional vector space generated by the quotient D_0^2 of $\mathcal{O}_{(\mathbb{P}^1)^n}(n-1,\cdots,n-1)$ and hence we have $K_{(\mathbb{P}^1)^n//G} \cong cD_0^2$ for some $c \in \mathbb{Q}$.

Suppose n is odd. The contraction morphisms φ_k are all compositions of smooth blow-ups for $k \geq 1$. From the blow-up formula of canonical divisors ([6, II Exe. 8.5]) and Lemma 5.3, we deduce that

$$K_{\overline{M}_{0,n\cdot\epsilon_k}} = cD_k^2 + \sum_{j\geq m-k+1}^{m} \left(c\binom{j}{2} + (j-2)\right) D_k^j.$$

Since $\overline{M}_{0,n} \cong \overline{M}_{0,n\cdot\epsilon_{m-2}}$, we get $c = -\frac{2}{n-1}$ from Proposition 5.4.

When n is even, $\varphi_1^*(K_{(\mathbb{P}^1)^n//G}) = cD_1^2 + c\binom{m}{2}D_1^m$ by Lemma 5.3. We write $K_{\overline{M}_{0,n\cdot\epsilon_1}} = cD_1^2 + (c\binom{m}{2} + a)D_1^m$. By the blow-up formula of canonical divisors ([6, II Exe. 8.5]) again, we deduce that

$$K_{\overline{M}_{0,n\cdot\epsilon_k}} = cD_k^2 + \sum_{j\geq m-k+1}^{m-1} \left(c\binom{j}{2} + (j-2)\right) D_k^j + (c\binom{m}{2} + a)D_k^m.$$

From Proposition 5.4 again, we get $c = -\frac{2}{n-1}$ and $a = m-2$. □

We are now ready to prove Theorem 5.1. By [21, Corollary 3.5], the theorem is a direct consequence of the following proposition.

Proposition 5.6. *(1)* $K_{\overline{M}_{0,n\cdot\epsilon_0}} + \alpha D_0$ *is ample if* $\frac{2}{n-1} < \alpha \leq \frac{2}{m+1}$.
(2) For $1 \leq k \leq m-2$, $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ *is ample if* $\frac{2}{m-k+2} < \alpha \leq \frac{2}{m-k+1}$.

Since any positive linear combination of an ample divisor and a nef divisor is ample [16, Corollary 1.4.10], it suffices to show the following:

(a) Nefness of $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ for $\alpha = \frac{2}{m-k+1} + s$ where s is some (small) positive number;
(b) Ampleness of $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ for $\alpha = \frac{2}{m-k+2} + t$ where t is *any* sufficiently small positive number.

We will use Alexeev and Swinarski's intersection number calculation in [1] to achieve (a) (See Lemma 5.12.) and then (b) will immediately follow from our Theorem 4.1.

Definition 5.7. ([21]) Let $\varphi = \varphi_{n\cdot\epsilon_{m-2},n\cdot\epsilon_k} : \overline{M}_{0,n} \to \overline{M}_{0,n\cdot\epsilon_k}$ be the natural contraction map (§2.1). For $k = 0, 1, \cdots, m-2$ and $\alpha > 0$, define $A(k,\alpha)$ by

$$\begin{aligned} A(k,\alpha) &:= \varphi^*(K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k) \\ &= \sum_{j=2}^{m-k} \binom{j}{2}\left(\alpha - \frac{2}{n-1}\right) D^j + \sum_{j\geq m-k+1}^{m} \left(\alpha - \frac{2}{n-1}\binom{j}{2} + j - 2\right) D^j. \end{aligned}$$

Notice that the last equality is an easy consequence of Lemma 5.3.

By [10], there is a birational morphism $\pi_{\vec{x}} : \overline{M}_{0,n} \to (\mathbb{P}^1)^n//_{\vec{x}}G$ for any linearization $\vec{x} = (x_1,\cdots,x_n) \in \mathbb{Q}_+^n$. Since the canonical ample line bundle $\mathcal{O}_{(\mathbb{P}^1)^n}(x_1,\cdots,x_n)//G$ over $(\mathbb{P}^1)^n//_{\vec{x}}G$ is ample, its pull-back $L_{\vec{x}}$ by $\pi_{\vec{x}}$ is certainly nef.

Definition 5.8. [1, Definition 2.3] Let x be a rational number such that $\frac{1}{n-1} \le x \le \frac{2}{n}$. Set $\vec{x} = \mathcal{O}(x, \cdots, x, 2-(n-1)x)$. Define

$$V(x,n) := \frac{1}{(n-1)!} \bigotimes_{\tau \in S_n} L_{\tau\vec{x}}.$$

Obviously the symmetric group S_n acts on $\vec{x}$ by permuting the components of $\vec{x}$.

Notice that $V(x,n)$ is nef because it is a positive linear combination of nef line bundles.

Definition 5.9. [1, Definition 3.5] Let $C_{a,b,c,d}$ be *any* vital curve class corresponding to a partition $S_a \sqcup S_b \sqcup S_c \sqcup S_d$ of $\{1, 2, \cdots, n\}$ such that $|S_a| = a, \cdots, |S_d| = d$.
(1) Suppose $n = 2m+1$ is odd. Let $C_i = C_{1,1,m-i,m+i-1}$, for $i = 1, 2, \cdots, m-1$.
(2) Suppose $n = 2m$ is even. Let $C_i = C_{1,1,m-i,m+i-2}$ for $i = 1, 2, \cdots, m-1$.

By [12, Corollary 4.4], the following computation is straightforward.

Lemma 5.10. *The intersection numbers* $C_i \cdot A(k,\alpha)$ *are*

$$C_i \cdot A(k,\alpha) = \begin{cases} \alpha & \text{if } i < k \\ \left(2 - \binom{m-k}{2}\right)\alpha + m - k - 2 & \text{if } i = k \\ \left(\binom{m-k+1}{2} - 1\right)\alpha - m + k + 1 & \text{if } i = k+1 \\ 0 & \text{if } i > k+1. \end{cases}$$

This lemma is in fact a slight generalization of [1, Lemma 3.7] where the intersection numbers for $\alpha = \frac{2}{m-k+1}$ only are calculated.

The S_n-invariant subspace of Neron-Severi vector space of $\overline{M}_{0,n}$ is generated by D^j for $j = 2, 3, \cdots, m$ ([12, Theorem 1.3]). Therefore, in order to determine the linear dependency of S_n-invariant divisors, we find $m-1$ linearly independent curve classes, and calculate the intersection numbers of divisors with these curves classes. Let U be an $(m-1)\times(m-1)$ matrix with entries $U_{ij} = (C_i \cdot V(\frac{1}{m+j}, n))$ for $1 \le i, j \le m-1$. Since $V(\frac{1}{m+j}, n)$'s are all nef, all entries of U are nonnegative.

Lemma 5.11. [1, §3.2, §3.3] *(1) The intersection matrix* U *is upper triangular and if* $i \le j$*, then* $U_{ij} > 0$*. In particular,* U *is invertible.*
(2) Let $\vec{a} = ((C_1 \cdot A(k, \frac{2}{m-k+1})), \cdots, (C_{m-1} \cdot A(k, \frac{2}{m-k+1})))^t$ *be the column vector of intersection numbers. Let* $\vec{c} = (c_1, c_2, \cdots, c_{m-1})^t$ *be the unique solution of the system of linear equations* $U\vec{c} = \vec{a}$*. Then* $c_i > 0$ *for* $i \le k+1$ *and* $c_i = 0$ *for* $i \ge k+2$*.*

This lemma implies that $A(k, \frac{2}{m-k+1})$ is a positive linear combination of $V(\frac{1}{m+j}, n)$ for $j = 1, 2, \cdots, k+1$. Note that $A(k, \frac{2}{m-k+2}) = A(k-1, \frac{2}{m-(k-1)+1})$ and that for $\frac{2}{m-k+2} \le \alpha \le \frac{2}{m-k+1}$, $A(k,\alpha)$ is a nonnegative linear combination of $A(k, \frac{2}{m-k+2})$ and $A(k, \frac{2}{m-k+1})$. Hence by the numerical result in Lemma 5.11 and the convexity of the nef cone, $A(k,\alpha)$ is nef for $\frac{2}{m-k+2} \le \alpha \le \frac{2}{m-k+1}$. Actually we can slightly improve this result by using continuity.

Lemma 5.12. *For each* $k = 0, 1, \cdots, m-2$*, there exists* $s > 0$ *such that* $A(k,\alpha)$ *is nef for* $\frac{2}{m-k+2} \le \alpha \le \frac{2}{m-k+1} + s$*. Therefore,* $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ *is nef for* $\frac{2}{m-k+2} \le \alpha \le \frac{2}{m-k+1} + s$.

Proof. Let $\vec{a}^\alpha = ((C_1\cdot A(k,\alpha)), \cdots, (C_{m-1}\cdot A(k,\alpha)))^t$ and let $\vec{c}^\alpha = (c_1^\alpha, \cdots, c_{m-1}^\alpha)^t$ be the unique solution of equation $U\vec{c}^\alpha = \vec{a}^\alpha$. Then by continuity, the components $c_1^\alpha, c_2^\alpha, \cdots, c_{k+1}^\alpha$ remain positive when α is slightly increased. By Lemma 5.10 and the upper triangularity of U, c_i^α for $i > k+1$ are all zero. Hence $A(k,\alpha)$ is still nef for $\alpha = \frac{2}{m-k+1} + s$ with sufficiently small $s > 0$. □

With this nefness result, the proof of Proposition 5.6 is obtained as a quick application of Theorem 4.1.

Proof of Proposition 5.6. We prove that in fact $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample for $\frac{2}{m-k+2} < \alpha < \frac{2}{m-k+1} + s$ where s is the small positive rational number in Lemma 5.12. Since a positive linear combination of an ample divisor and a nef divisor is ample by [16, Corollary 1.4.10], it suffices to show that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample when $\alpha = \frac{2}{m-k+2} + t$ for any sufficiently small $t > 0$ by Lemma 5.12.

We use induction on k. It is certainly true when $k = 0$ by Lemma 5.5 because D_0^2 is ample as the quotient of $\mathcal{O}(n-1, \cdots, n-1)$. Suppose $K_{\overline{M}_{0,n\cdot\epsilon_{k-1}}} + \alpha D_{k-1}$ is ample for $\frac{2}{m-k+3} < \alpha < \frac{2}{m-k+2} + s'$ where s' is the small positive number in Lemma 5.12 for $k-1$. Since φ_k is a blow-up with exceptional divisor D_k^{m-k+1},

$$\varphi_k^*(K_{\overline{M}_{0,n\cdot\epsilon_{k-1}}} + \alpha D_{k-1}) - \delta D_k^{m-k+1}$$

is ample for any sufficiently small $\delta > 0$ by [6, II 7.10]. A direct computation with Lemmas 5.3 and 5.5 provides us with

$$\varphi_k^*(K_{\overline{M}_{0,n\cdot\epsilon_{k-1}}} + \alpha D_{k-1}) - \delta D_k^{m-k+1}$$

$$= K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k + \left(\binom{m-k+1}{2}\alpha - \alpha - (m-k-1) - \delta\right) D_k^{m-k+1}.$$

If $\alpha = \frac{2}{m-k+2}$, $\binom{m-k+1}{2}\alpha - \alpha - (m-k-1) = 0$ and thus we can find $\alpha > \frac{2}{m-k+2}$ satisfying $\binom{m-k+1}{2}\alpha - \alpha - (m-k-1) - \delta = 0$. If δ decreases to 0, the solution α decreases to $\frac{2}{m-k+2}$. Hence $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample when $\alpha = \frac{2}{m-k+2} + t$ for any sufficiently small $t > 0$ as desired. □

Remark 5.13. There are already two different proofs of M. Simpson's theorem (Theorem 5.1) given by Fedorchuk–Smyth [4], and by Alexeev–Swinarski [1] without relying on Fulton's conjecture. Here we give a brief outline of the two proofs.

In [21, Corollary 3.5], Simpson proves that Theorem 5.1 is an immediate consequence of the ampleness of $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ for $\frac{2}{m-k+2} < \alpha \leq \frac{2}{m-k+1}$ (Proposition 5.6). The differences in the proofs of Theorem 5.1 reside solely in different ways of proving Proposition 5.6.

The ampleness of $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ follows if the divisor $A(k,\alpha) = \varphi^*(K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k)$ is nef and its linear system contracts only φ-exceptional curves. Here, $\varphi : \overline{M}_{0,n} \to \overline{M}_{0,n\cdot\epsilon_k}$ is the natural contraction map (§2.1). Alexeev and Swinarski prove Proposition 5.6 in two stages: First the nefness of $A(k,\alpha)$ for suitable ranges is proved and next they show that the divisors are the pull-backs of ample line bundles on $\overline{M}_{0,n\cdot\epsilon_k}$. Lemma 5.12 above is only a negligible improvement of the nefness result in [1, §3]. In [1, Theorem 4.1], they give a criterion for a line bundle to be the pull-back of an ample line bundle on $\overline{M}_{0,n\cdot\epsilon_k}$. After some rather sophisticated combinatorial computations, they prove in [1, Proposition 4.2] that $A(k,\alpha)$ satisfies the desired properties.

On the other hand, Fedorchuk and Smyth show that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is ample as follows. Firstly, by applying the Grothendieck-Riemann-Roch theorem, they represent $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ as a linear combination of boundary divisors and tautological ψ-classes. Secondly, for such a linear combination of divisor classes and for a complete curve in $\overline{M}_{0,n\cdot\epsilon_k}$ parameterizing a family of curves with smooth general member, they perform brilliant computations and get several inequalities satisfied by their intersection numbers ([4, Proposition 3.2]). Combining these inequalities, they prove in particular that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ has positive intersection with any complete curve on $\overline{M}_{0,n\cdot\epsilon_k}$ with smooth general member ([4, Theorem 4.3]). Thirdly, they prove that if the divisor class intersects positively with any curve with smooth general member, then it intersects positively with all curves by an induction argument on the dimension. Thus they establish the fact that $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ has positive intersection with all curves. Lastly, they prove that the same property holds even if $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ is perturbed by any small linear combination of boundary divisors. Since the boundary divisors generate the Neron-Severi vector space, $K_{\overline{M}_{0,n\cdot\epsilon_k}} + \alpha D_k$ lies in the interior of the nef cone and the desired ampleness follows.

6. The Picard groups of $\overline{M}_{0,n\cdot\epsilon_k}$

As a byproduct of our GIT construction of the moduli spaces of weighted pointed curves, we give a basis of the *integral* Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ for $0 \le k \le m-2$.

Let e_i be the i-th standard basis vector of $\mathbb{Z}^n$. For notational convenience, set $e_{n+1} = e_1$. For $S \subset \{1, 2, \cdots, n\}$, let $D_k^S = \Sigma_{n-m+k}^S /\!/ G \subset F_{n-m+k} /\!/ G \cong \overline{M}_{0,n\cdot\epsilon_k}$. Note that if $m-k < |S| \le m$ or $|S| = 2$, D_k^S is a divisor of $\overline{M}_{0,n\cdot\epsilon_k}$.

Theorem 6.1. *(1) If n is odd, then the Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ is*

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_k}) \cong \bigoplus_{m-k<|S|\le m} \mathbb{Z}D_k^S \oplus \bigoplus_{i=1}^{n} \mathbb{Z}D_k^{\{i,i+1\}}$$

for $0 \le k \le m-2$.
(2) If n is even, then the Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ is

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_k}) \cong \bigoplus_{m-k<|S|<m} \mathbb{Z}D_k^S \oplus \bigoplus_{1\in S, |S|=m} \mathbb{Z}D_k^S \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}D_k^{\{i,i+1\}} \oplus \mathbb{Z}D_k^{\{1,n-1\}}.$$

for $1 \le k \le m-2$.

Proof. Since the codimensions of unstable strata in $(\mathbb{P}^1)^n$ are greater than 1,

$$\mathrm{Pic}(((\mathbb{P}^1)^n)^{ss}) = \mathrm{Pic}((\mathbb{P}^1)^n) \cong \oplus_{1\le i\le n} \mathbb{Z}\mathcal{O}(e_i).$$

For all $x \in ((\mathbb{P}^1)^n)^s$, $G_x \cong \{\pm 1\}$. If n is even and x is strictly semistable point with closed orbit, then $G_x \cong \mathbb{C}^*$. Since G is connected, G acts on the discrete set $\mathrm{Pic}((\mathbb{P}^1)^n)$ trivially. By Kempf's descent lemma ([3, theorem 2.3]) and by checking the actions of the stabilizers on the fibers of line bundles, we deduce that $\mathcal{O}(a_1, a_2, \cdots, a_n)$ descends to $((\mathbb{P}^1)^n)^{ss}/\!/G$ if and only if 2 divides $\sum a_i$.

Consider the case when n is odd first. It is elementary to check that the subgroup $\{(a_1, \cdots, a_n) \in \mathbb{Z}^n | \sum a_i \in 2\mathbb{Z}\}$ is free abelian of rank n and $\{e_i + e_{i+1}\}$ for $1 \le i \le n$

form a basis of this group. Furthermore, for $S = \{i, j\}$ with $i \neq j$, the big diagonal $(\Sigma^S_{m+1})^s = (\Sigma^S_0)^s$ satisfies $\mathcal{O}((\Sigma^S_0)^s) \cong \mathcal{O}_{F^s_0}(e_i + e_j)$. Hence in $F_{m+1}//G = F_0//G$, $\mathcal{O}(\Sigma^S_0//G) \cong \mathcal{O}(e_i + e_j)$. Therefore we have

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_0}) = \mathrm{Pic}(F_{m+1}//G) = \bigoplus_{i=1}^{n} \mathbb{Z}D_0^{\{i,i+1\}}.$$

By Theorem 4.1, the contraction morphism $\varphi_k : \overline{M}_{0,n\cdot\epsilon_k} \to \overline{M}_{0,n\cdot\epsilon_{k-1}}$ is the blow-up along the union of transversally intersecting smooth subvarieties. By §2.3, this is a composition of smooth blow-ups. In $\overline{M}_{0,n\cdot\epsilon_k}$, the exceptional divisors are D^S_k for $|S| = m - k + 1$. So the Picard group of $\overline{M}_{0,n\cdot\epsilon_k}$ is

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_k}) \cong \varphi_k^* \mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_{k-1}}) \oplus \bigoplus_{|S|=m-k+1} \mathbb{Z}D^S_k.$$

by [6, II Exe. 8.5]. For any S with $|S| = 2$, D^S_{k-1} contains the blow-up center $D^{S'}_{k-1}$ if $S \subset S'$. So $\varphi_k^*(D^S_{k-1})$ is the sum of D^S_k and a linear combination of $D^{S'}_k$ for $S' \supset S, |S'| = m-k+1$. If $|S| > 2$, then $\varphi_k^* D^S_{k-1} = D^S_k$ since it does not contain any blow-up centers. After obvious basis change, we get the desired result by induction.

Now suppose that n is even. Still the group $\{(a_1, \cdots, a_n) \in \mathbb{Z}^n | \sum a_i \in 2\mathbb{Z}\}$ is free abelian of rank n and $\{e_i + e_{i+1}\}_{1\le i\le n-1} \cup \{e_1 + e_{n-1}\}$ form a basis. In $F_m//G = F_0//G$, $\mathcal{O}(\Sigma^S_m//G) \cong \mathcal{O}(e_i + e_j)$ when $S = \{i, j\}$ with $i \neq j$. Hence

$$\mathrm{Pic}(F_m//G) = \bigoplus_{i=1}^{n-1} \mathbb{Z}D_0^{\{i,i+1\}} \oplus \mathbb{Z}D_0^{\{1,n-1\}}.$$

In $\tilde{F}_m$, the unstable loci have codimension two. Therefore we have

$$\mathrm{Pic}(\tilde{F}^s_m) = \mathrm{Pic}(\tilde{F}_m) = \pi_m^* \mathrm{Pic}(F^{ss}_m) \oplus \bigoplus_{1\in S, |S|=m} \mathbb{Z}\tilde{\Sigma}^S_m,$$

where $\pi_m : \tilde{F}_m \to F^{ss}_m$ is the blow-up morphism, and $\tilde{\Sigma}^S_m = \pi_m^{-1}(\Sigma^S_m \cap \Sigma^{S^c}_m)$ for $|S| = m$.

By Kempf's descent lemma, $\mathrm{Pic}(F_{m+k}//G)$ is a subgroup of $\mathrm{Pic}(F^{ss}_{m+k})$ and $\mathrm{Pic}(\tilde{F}^s_{m+k})$ for $0 \le k \le m-2$. From our blow-up description, all arrows except possibly $\tilde{\psi}^*_{m+1}$ in following commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\tilde{F}^s_{m+1}) \arrow[r, equal, "\tilde{\psi}^*_{m+1}"] & \mathrm{Pic}(\tilde{F}^s_m) \\
\mathrm{Pic}(F^{ss}_{m+1}) \arrow[u, "\pi^*_{m+1}"] & \mathrm{Pic}(F^{ss}_m) \arrow[u, "\pi^*_m"'] \arrow[l, "\psi^*_{m+1}"'] \\
\mathrm{Pic}(F_{m+1}//G) \arrow[u] & \mathrm{Pic}(F_m//G) \arrow[u] \arrow[l, "\bar{\psi}^*_{m+1}"']
\end{tikzcd}$$

are injective, and thus the bottom arrow $\bar{\psi}^*_{m+1}$ is also injective. Hence $\mathrm{Pic}(F_{m+1}//G)$ contains the pull-back of $\mathrm{Pic}(F_m//G)$ as a subgroup. Also, for the quotient map $p : \tilde{F}^s_{m+1} \to F_{m+1}//G$, $p^* D^S_1 = \tilde{\Sigma}^S_{m+1}$ for $|S| = m$. Let H be the subgroup of $\mathrm{Pic}(\tilde{F}^s_{m+1})$ generated by the images of $\bar{\psi}^*_{m+1}\mathrm{Pic}(F_m//G)$ and the divisors D^S_1 with $|S| = m$. By definition, the image of $\mathrm{Pic}(F_{m+1}//G)$ contains H. Now by checking the action of stabilizers on the fibers of line bundles, it is easy to see that no line bundles in $\mathrm{Pic}(\tilde{F}^s_{m+1}) - H$ descend to $F_{m+1}//G$. Hence we have

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_1}) = \mathrm{Pic}(F_{m+1}//G) = \bar{\psi}^*_{m+1}\mathrm{Pic}(F_m//G) \oplus \bigoplus_{1\in S, |S|=m} \mathbb{Z}D_1^S. \tag{16}$$

For an S with $|S| = 2$, $\Sigma^S_m//G$ contains the blow-up center $\Sigma^{S'}_m \cap \Sigma^{S'^c}_m//G$ if $S \subset S'$ or $S \subset S'^c$. So $\bar{\psi}^*_{m+1}(D^S_0)$ is the sum of D^S_1 and a linear combination of divisors $D^{S'}_1$ for $S' \supset S$ or $S'^c \supset S$ with $|S'| = m$. From this and (16), we get the following by an obvious basis change:

$$\mathrm{Pic}(\overline{M}_{0,n\cdot\epsilon_1}) = \bigoplus_{1\in S, |S|=m} \mathbb{Z}D^S_1 \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}D^{\{i,i+1\}}_1 \oplus \mathbb{Z}D^{\{1,n-1\}}_1.$$

The rest of the proof is identical to the odd n case and so we omit it. □

References

1. V. Alexeev and D. Swinarski. *Nef divisors on $\overline{M}_{0,n}$ from GIT.* arXiv:0812.0778.
2. C. Birkar, P. Cascini, C. Hacon and J. McKernan. *Existence of minimal models for varieties of log general type.* arXiv:0610203.
3. J.-M. Drezet and M. Narasimhan. *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques.* Invent. Math. 97 (1989), 53–94. MR0999313.
4. M. Fedorchuk and D. Smyth. *Ample divisors on moduli spaces of weighted pointed rational curves, with applications to log MMP for $\overline{M}_{0,n}$.* arXiv:0810.1677.
5. W. Fulton and R. MacPherson. *A compactification of configuration spaces.* Ann. of Math. (2) 139 (1994), no. 1, 183–225. MR1259368.
6. R. Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, 1977. MR0463157.
7. B. Hassett. *Moduli spaces of weighted pointed stable curves.* Adv. Math. 173 (2003), no. 2, 316–352. MR1957831.
8. Y. Hu. *Relative Geometric Invariant Theory and Universal Moduli Spaces.* Internat. J. of Math. 7 (1996), 151–181. MR1382720.
9. Y. Hu and S. Keel. *Mori dream spaces and GIT.* Michigan Math. J. 48 (2000), 331–348. MR1786494.
10. M. Kapranov. *Chow quotients of Grassmannians. I.* I. M. Gelfand Seminar, 29–110, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993. MR1237834.
11. S. Keel. *Intersection theory of moduli space of stable n-pointed curves of genus zero.* Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574. MR1034665.
12. S. Keel and J. McKernan. *Contractible extreamal rays on $\overline{M}_{0,n}$.* arXiv:9607009.
13. F. Kirwan. *Cohomology of quotients in algebraic geometry and symplectic geometry.* Mathematical Notes, 31. Princeton University Press, Princeton, NJ, 1984. MR0766741.
14. F. Kirwan. *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers.* Ann. of Math. (2) 122 (1985), no. 1, 41–85. MR0799252.
15. F. Knudsen. *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$.* Math. Scand. 52 (1983), no. 2, 161–199. MR0702953.
16. R. Lazarsfeld. *Positivity in Algebraic Geometry I.* Springer 2004.
17. L. Li. *Wonderful compactification of an arrangement of subvarieties.* to appear in Michigan J. Math. arXiv:0611412.
18. D. Mumford, J. Fogarty and F. Kirwan. *Geometric Invariant Theory.* Ser. Modern Surveys Math. 34 Third Edition 1994. MR1304906.
19. A. Mustata and M. Mustata. *Intermediate moduli spaces of stable maps.* Invent. Math. 167 (2007), no. 1, 47–90. MR2264804.
20. R. Pandharipande, *The canonical class of $\overline{M}_{0,n}(\mathbb{P}^r, d)$.* Internat. Math. Res. Notices 1997, no. 4, 173–186. MR1436774.
21. M. Simpson. *On Log canonical models of the moduli space of stable pointed curves.* arXiv:0709.4037.

Department of Mathematics and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail address: kiem@math.snu.ac.kr

Department of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail address: spring-1@snu.ac.kr