Desingularizations of moduli spaces of rank 2 sheaves with trivial determinant

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I. Vector Bundles over Curves
II. Higgs Bundles over Curves
III. Sheaves on K3 and Abelian Surfaces
IV. Questions

I. Vector Bundles over Curves

1. Moduli space of bundles M_0

- X = smooth proj. curve of genus $g \geq 3$.

- $F \to X$ rank 2 bundle with $\det F = \mathcal{O}_X$.
  F is *polystable* if F is stable or
  $F \cong L \oplus L^{-1}$ for $L \in \mathrm{Pic}^0(X) =: J$.

- $M_0 := \{\text{polystable } F\}/\text{isom}$
  admits a scheme structure such that
  for any vector bundle $\mathcal{F} \to S \times X$
  with $\mathcal{F}|_{\{s\}\times X}$ semistable for $\forall\ s \in S$,
  the obvious map $S \to M_0$ which maps
  $$s \mapsto [\mathrm{gr}(\mathcal{F}|_{\{s\}\times X})]$$
  is a morphism of schemes.

- M_0 is a projective irreducible normal variety
  of dimension $3g-3$.

2. Stratification of M_0

- A polystable bundle $F \in M_0$ is one of the following;

(a) F stable

(b) $F \cong L \oplus L^{-1}$ with $L \ncong L^{-1}$

(c) $F \cong L \oplus L$ with $L \cong L^{-1}$

- $M_0 = M_0^s \sqcup (J/\mathbb{Z}_2 - J_0) \sqcup J_0$: stratification

(a) M_0^s= open subset of stable bundles

(b) $J/\mathbb{Z}_2 = \{L \oplus L^{-1} \,|\, L \in J\}$

(c) $J_0 = \mathbb{Z}_2^{2g} = \{L \oplus L \,|\, L \cong L^{-1}\}$

3. Singularities of M_0

- (Luna's slice theorem)
  For polystable F, the analytic type of singularity of $F \in M_0$ is

$$H^1(\mathcal{E}nd_0(F))//\mathrm{Aut}(F)$$

(a) If F is stable, then $\mathrm{Aut}(F) = \mathbb{C}^*$ acts trivially on $H^1(\mathcal{E}nd_0(F))$. Hence M_0 is smooth at $F \in M_0$ and

$$T_F M_0 = H^1(X, \mathcal{E}nd_0(F))$$

(b) If $F = L \oplus L^{-1}$ with $L \not\cong L^{-1}$, then

$$\mathrm{Aut}(F)/\mathbb{C}^* = \mathbb{C}^* \qquad \text{and}$$

$$H^1(\mathcal{E}nd_0(F)) \cong H^1(\mathcal{O}_X) \oplus H^1(L^2) \oplus H^1(L^{-2})$$

where $\mathbb{C}^*$ acts with weight $0, 2, -2$ respectively.

$$\Downarrow$$

M_0 is singular at $F \in M_0$ and the analytic type of the singularity is

$$H^1(L^2) \oplus H^1(L^{-2}) /\!/ \mathbb{C}^*$$

which is the affine cone over

$$\mathbb{P}\left(H^1(L^2) \oplus H^1(L^{-2})\right) /\!/ \mathbb{C}^* = \mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$$

$$\Downarrow$$

By blowing up at the vertex we get a desingularization

$$\mathcal{O}_{\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}}(-1, -1)$$

(c) If $F = L \oplus L$ with $L \cong L^{-1}$, then

$$\mathrm{Aut}(F)/\mathbb{C}^* = \mathbb{P}GL(2) \qquad \text{and}$$

$$H^1(\mathcal{E}nd_0(F)) \cong H^1(\mathcal{O}_X) \otimes \mathfrak{sl}(2)$$

where $\mathbb{P}GL(2)$ acts by conjugation on $\mathfrak{sl}(2)$.

$$\Downarrow$$

M_0 is singular at $F \in M_0$ and the analytic type of the singularity is

$$H^1(\mathcal{O}) \otimes \mathfrak{sl}(2) // \mathbb{P}GL(2) = \mathbb{C}^g \otimes \mathfrak{sl}(2) // SL(2)$$

$$\Downarrow$$

Need three blow-ups to desingularize

- Three blow-ups before quotient
  - $W_0 = \mathbb{C}^g \otimes \mathfrak{sl}(2) \cong \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^g)$
  - W_1= blow-up of W_0 at 0
    line bundle $\mathcal{O}(-1)$ over $\mathbb{P}\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^g)$
  - W_2= blow-up of W_1 along the proper transform of $\mathrm{Hom}_1(\mathbb{C}^3, \mathbb{C}^g)$
  - W_3= blow-up of W_2 along the proper transform Δ of $\mathbb{P}\mathrm{Hom}_2(\mathbb{C}^3, \mathbb{C}^g)$.

- W_3 is a nonsingular quasi-projective variety acted on by $SL(2)$

- Locus of nontrivial stabilizers in $W_3^{ss} = W_3^s$ is a divisor

- $W_3 /\!\!/ SL(2)$ is nonsingular, i.e.
  $\pi : W_3 /\!\!/ SL(2) \to W_0 /\!\!/ SL(2)$
  is a desingularization

- π is the composition of three blow-ups

$$\pi : W_3/\!\!/SL(2) \to W_2/\!\!/SL(2) \to$$

$$\to W_1/\!\!/SL(2) \to W_0/\!\!/SL(2)$$

- D_i= proper transform of exceptional divisor of i-th blow-up in $W_3/\!\!/SL(2)$:
smooth normal crossing divisors

- $\mathcal{A} \to Gr(2, g)$ tautological rank 2 bundle
$\mathcal{B} \to Gr(3, g)$ tautological rank 3 bundle

- D_1= blow-up of projective bundle $\mathbb{P}(S^2\mathcal{B})$ along the locus of rank 1 conics

- D_3= $\mathbb{P}^2 \times \mathbb{P}^{g-2}$ bundle over $Gr(2, g)$

- D_2= $[\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$-bundle over $\mathrm{bl}_0\mathbb{C}^g]/\mathbb{Z}_2$

- normal bundle of D_3
  $= \mathcal{O}(-1)$ along $\mathbb{P}^2$-direction

$\Rightarrow$ can blow down along $\mathbb{P}^2$-direction of D_3

- D_1 becomes $\mathbb{P}^5$-bundle over $Gr(3, g)$
  normal bundle is $\mathcal{O}(-1)$ along $\mathbb{P}^5$

$\Rightarrow$ can blow down along the $\mathbb{P}^5$-direction

- three desingularizations of $W_0 /\!\!/ SL(2)$

4. Kirwan's desingularization

- M_0 can be desingularized by 3 blow-ups along

  i) $J_0 = \mathbb{Z}_2^{2g}$

  ii) proper transform of $J/\mathbb{Z}_2$

  iii) nonsingular subvariety Δ lying in the exceptional divisor of the first blow-up.

- $\pi : \widehat{M} \to M_0$ Kirwan desingularization
  Explicit description of exceptional divisors

- $\widehat{M}$ can be blown down twice to give us three desingularizations of M_0:

$$\begin{array}{ccccc} \widehat{M} & \longrightarrow & \overline{M} & \longrightarrow & \widetilde{M} \\ & \searrow & \downarrow & \swarrow & \\ & & M_0 & & \end{array}$$

5. Applications

- Can compute the cohomology of $\overline{M}$ and $\widetilde{M}$ by using Kirwan's computation of $H^*(\widehat{M})$.

- discrepancy divisor :

$$\widehat{M} \xrightarrow[2D_3]{\mathbb{P}^2} \overline{M} \xrightarrow[5D_1]{\mathbb{P}^5} \widetilde{M} \xrightarrow[(g-2)\widetilde{D}_2]{} M_0$$

$$K_{\widetilde{M}} - \pi^* K_{M_0} = 2D_3 + 5D_1 +$$

$$+(g-2)(D_2 + 3D_1 + 2D_3)$$

$$= (3g-1)D_1 + (g-2)D_2 + (2g-2)D_3$$

Hence M_0 has terminal singularities.

- (Kiem-Li) Stringy E-function :

$$E_{st}(M_0) = \frac{(1-u^2v)^g(1-uv^2)^g - (uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} - \frac{(uv)^{g-1}}{2}\left(\frac{(1-u)^g(1-v)^g}{1-uv} - \frac{(1+u)^g(1+v)^g}{1+uv}\right).$$

- (Kirwan) E-polynomial of $IH^*(M_0)$

$$IE(M_0) = \sum_{k,p,q}(-1)^k h^{p,q}(IH^k(M_0))u^p v^q$$

$$= \frac{(1-u^2v)^g(1-uv^2)^g - (uv)^{g+1}(1-u)^g(1-v)^g}{(1-uv)(1-(uv)^2)} - \frac{(uv)^{g-1}}{2}\left(\frac{(1-u)^g(1-v)^g}{1-uv} + (-1)^{g-1}\frac{(1+u)^g(1+v)^g}{1+uv}\right).$$

- The stringy Euler number is

$$\frac{1}{4}\cdot\chi(J_0) = \frac{1}{4}\cdot 2^{2g}$$

6. Seshadri's desingularization

- Fix $x_0 \in X$.
  E = rank 4 bundle with $\det E \cong \mathcal{O}_X$
  $0 \neq s \in E^*|_{x_0}$ quasi-parabolic structure
  $0 < a_1 < a_2 \ll 1$ parabolic weights.

- (Mehta-Seshadri)
  $\exists$ fine moduli space P of stable parabolic bundles of rank 4;
  P is a smooth projective variety.

- Seshadri's desingularization $\mathbf{S}$ is a nonsingular closed subvariety of P.

- **Proposition** (Seshadri)
  Let E be a semistable rank 4 bundle.
  (1) [$\exists 0 \neq s \in E^*|_{x_0}$ s.t. (E, s) is stable]
  $\Leftrightarrow$ [$\nexists L \in \mathrm{Pic}^0(X)$ s.t. $L \oplus L \hookrightarrow E$]
  (2) Let $(E_1, s_1), (E_2, s_2) \in P$.
  Suppose $\dim \mathrm{End} E_1 = \dim \mathrm{End} E_2 = 4$. Then $(E_1, s_1) \cong (E_2, s_2) \Leftrightarrow E_1 \cong E_2$

- **Corollary** $\imath : M_0^s \hookrightarrow P$
  [$\because$ for $F \in M_0^s$, $E = F \oplus F$ does not contain $L \oplus L$ for any $L \in \mathrm{Pic}^0(X)$ and $\mathrm{End}(F) = \mathfrak{gl}(2)$.]

- **Theorem** (Seshadri)
  (1) $\mathbf{S} = \overline{\imath(M_0^s)}$ is the locus of (E, s), $\det E = \mathcal{O}_X$ and $\mathrm{End} E$ is a specialization of the algebra $M(2) = \mathfrak{gl}(2)$ of 2×2 matrices.

  (2) $\mathbf{S}$ is a desingularization of M_0, i.e. $\mathbf{S}$ is smooth and $\exists$ morphism $\pi_S : \mathbf{S} \to M_0$ such that $\pi_S = \imath^{-1}$ on M_0^s.

▶ **Theorem** (Kiem-Li)
(1) $\exists$ birational morphism $\rho_S : \widehat{M} \to \mathbf{S}$
(2) $\mathbf{S} \cong \widetilde{M}$ and ρ_S is the composition of two blow-ups $\widehat{M} \to \overline{M} \to \widetilde{M}$.

▶ **Remark**
(1) is essential. (2) follows from Zariski's main theorem.

▶ **Strategy**
Construct a suitable family of rank 4 semistable bundles near each point of $\widehat{M}$. Then use the universal property of $\mathbf{S}$.

7. Moduli space of Hecke cycles

- $M_x = \{\text{stable } F \text{ of rank } 2, \det F \cong \mathcal{O}(-x)\}/\text{iso}$

$$\begin{array}{ccc} M_x & \hookrightarrow & M_X = \bigsqcup_{x \in X} M_x \\ \downarrow & & \downarrow \text{det} \\ x & \hookrightarrow & X \end{array}$$

- For $F \in M_0^s$ and $\nu \in \mathbb{P}F^*|_x$, let

$$F^\nu := \ker(F \longrightarrow F|_x \overset{\nu}{\twoheadrightarrow} \mathbb{C}) \in M_x$$

Define $\theta_x : \mathbb{P}F_x^* \hookrightarrow M_x$ by $\theta_x(\nu) = F^\nu$

$$\begin{array}{ccc} \mathbb{P}F^* & \xrightarrow{\theta} & M_X \\ & \searrow \quad \swarrow_{\text{det}} & \\ & X & \end{array}$$

- $\Phi : M_0^s \to \text{Hilb}(M_X)$, $\quad \Phi(F) = \theta(\mathbb{P}F^*)$ is an open immersion with Hilbert polynomial
  $P(n) = (4n+1)(4n-1)(g-1)$
  $\mathcal{O}_{M_X}(1) = K_{\text{det}}^* \otimes (\text{det})^* K_X$: ample on M_X

▶ **Definition** (Narasimhan-Ramanan)
$\mathbf{N} := \overline{\Phi(M_0^s)}$ = irreducible component of Hilb(M_X) containing $\Phi(M_0^s)$. A cycle in $\mathbf{N}$ is called a *Hecke cycle* and $\mathbf{N}$ is called the *moduli of Hecke cycles*.

▶ **Theorem** (Narasimhan-Ramanan)
$\mathbf{N}$ is a nonsingular variety and $\exists\ \pi_N : \mathbf{N} \to M_0$, which is an isomorphism over M_0^s.

▶ **Theorem** (Choe-Choy-Kiem)
(1) $\exists$ birational morphism $\rho_N : \widehat{M} \to \mathbf{N}$
(2) $\mathbf{N} \cong \overline{M}$ and ρ_N is $\widehat{M} \to \overline{M}$.

▶ **Strategy**
Construct a family of Hecke cycles near each point of $\widehat{M}$. Then use the universal property of $\mathbf{N}$.

II. Higgs Bundles over Curves

1. Higgs pairs

- V = rank 2 bundle with $\det V \cong \mathcal{O}_X$
  $\phi \in H^0(\mathrm{End}_0 V \otimes K_X)$
  (V, ϕ) = an $SL(2)$-Higgs bundle

- (V, ϕ) is polystable if stable or
  $(V, \phi) = (L, \psi) \oplus (L^{-1}, -\psi)$ for $(L, \psi) \in T^*J$

- $\mathbf{M} = \{$polystable pairs $(V, \phi)\}$/isom
  admits a structure of irreducible normal quasi-projective variety of dimension $6g - 6$

- stratification of $\mathbf{M}$
  $\mathbf{M} = \mathbf{M}^s \sqcup (T^*J/\mathbb{Z}_2 - J_0) \sqcup J_0$

2. Singularities of $\mathbf{M}$

(a) $\mathbf{M}^s$ is smooth, equipped with a (holomorphic) symplectic form, i.e. $\mathbf{M}^s$ is hyperkähler.

- (Kiem-Yoo) can compute $E(\mathbf{M}^s)$ by carefully working out the subvarieties corresponding to all possible types of V

(b) (Simpson) Singularities along $T^*J/\mathbb{Z}_2 - J_0$

$$\mathbb{H}^{g-1} \otimes_{\mathbb{C}} \mathbb{C}^2 /\!/\!/ \mathbb{C}^*$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ with weights $1, -1$

- desingularized by blowing up at the vertex of the cone:

$$\mathcal{O}(-1) \to \mathbb{P}(T^*\mathbb{P}^{g-2})$$

where $\mathbb{P}(T^*\mathbb{P}^{g-2})$ is $\mathbb{P}^{g-3}$-bundle on $\mathbb{P}^{g-2}$; a holomorphic contact manifold

(c) (Simpson) Singularities along J_0 is

$$\mathbb{H}^g \otimes_{\mathbb{C}} \mathfrak{sl}(2) /// SL(2)$$

- (O'Grady) desingularized by 3 blow-ups

  - three exceptional divisors of the desingularization are smooth normal crossing
  - can describe the divisors and their intersections explicitly

3. Desingularizations of $\mathbf{M}$

- $\mathbf{M}$ is desingularized by three blow-ups along

  i) J_0

  ii) proper transform of $T^*J/\mathbb{Z}_2$

  iii) nonsingular subvariety lying in the exceptional divisor of the first blow-up

  $\Rightarrow$ Kirwan desingularization $\pi : \widehat{\mathbf{M}} \to \mathbf{M}$.

- can blow down $\widehat{\mathbf{M}}$ twice to give three desingularizations of $\mathbf{M}$

$$\begin{array}{ccccc} \widehat{\mathbf{M}} & \longrightarrow & \overline{\mathbf{M}} & \longrightarrow & \widetilde{\mathbf{M}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbf{M} & & \end{array}$$

4. Application

- The discrepancy divisor is ($g \geq 3$)

  $K_{\widehat{\mathbf{M}}} = (6g-7)D_1 + (2g-4)D_2 + (4g-6)D_3.$

- Question
  Does there exist a (holomorphic) symplectic desingularization of $\mathbf{M}$?

- Kontsevich's theorem: If there is a crepant (=symplectic) resolution of $\mathbf{M}$, $E_{st}(\mathbf{M})$ is a polynomial with integer coefficients.

- (Kiem-Yoo) can give an explicit formula of $E_{st}(\mathbf{M})$ and prove that it is not a polynomial with integer coefficients for $g \geq 3$.
  $\Rightarrow \nexists$ symplectic desingularization for $g \geq 3$

- For $g = 2$, $\exists$ symplectic desing.

III. Sheaves on K3 and Abelian Surfaces

1. Moduli space of rank 2 sheaves

- S= K3 or Abelian surface, generic $\mathcal{O}_S(1)$

- F = rank 2 torsion-free sheaf with $c_1(F) = 0$ and $c_2(F) = 2n$ for $n \geq 2$

- $\mathcal{M} = \mathcal{M}_S(2, 0, 2n) = \{\text{polystable sheaves } F\}/\sim$ admits a structure of irreducible normal projective variety of dimension $8n - 6$ (K3) or $8n + 2$ (Abelian)

- stratification of $\mathcal{M}$
  $\mathcal{M} = \mathcal{M}^s \sqcup (\Sigma - \Omega) \sqcup \Omega$
  where Ω= $S^{[n]}$, Σ= $\mathrm{Sym}^2(S^{[n]})$ (K3 case)
  or $\Omega = S^{[n]} \times \widehat{S}$, $\Sigma = \mathrm{Sym}^2(S^{[n]} \times \widehat{S})$ (Abelian)

2. Singularities of $\mathcal{M}$

(a) (Mukai) $\mathcal{M}^s$ is smooth, equipped with a (holomorphic) symplectic form, i.e.
$\mathcal{M}^s$ is hyperkähler.

(b) (O'Grady) Singularities along $\Sigma - \Omega$

$$\mathbb{H}^{n-1} \otimes_{\mathbb{C}} \mathbb{C}^2 /// \mathbb{C}^*$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ with weights $1, -1$

(c) Singularities along Ω is

$$\mathbb{H}^n \otimes_{\mathbb{C}} \mathfrak{sl}(2) /// SL(2)$$

- desingularized by 3 blow-ups

- - three exceptional divisors of the desingularization are smooth normal crossing
  - can describe the divisors and their intersections explicitly

3. Desingularizations of $\mathcal{M}$

- $\mathcal{M}$ is desingularized by three blow-ups
  $\Rightarrow$ Kirwan desingularization $\pi : \widehat{\mathcal{M}} \to \mathcal{M}$.

- can blow down $\widehat{\mathcal{M}}$ twice to give three desingularizations of $\mathcal{M}$

$$\begin{array}{ccccc} \widehat{\mathcal{M}} & \longrightarrow & \overline{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{M}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{M} & & \end{array}$$

- (O'Grady) When $\dim \mathcal{M} = 10$, $\widetilde{\mathcal{M}}$ is a symplectic desingularization of $\mathcal{M}$.
  $\Rightarrow$ 2 new irreducible symplectic manifolds!

- Question (O'Grady)
  Does there exist a symplectic (or crepant) desingularization of $\mathcal{M}$ when $\dim \mathcal{M} > 10$?

- (Choy-Kiem) can give an explicit formula of $E_{st}(\mathcal{M}) - E(\mathcal{M}^s)$ and prove that $E_{st}(\mathcal{M})$ is not a polynomial when $\dim \mathcal{M} > 10$.
  $\Rightarrow \nexists$ symplectic desingularization when $\dim \mathcal{M} > 10$ by Kontsevich's theorem.

- Kaledin-Lehn-Sorger proved this nonexistence result by showing $\mathbb{Q}$-factoriality of $\mathcal{M}$.

IV. Questions

- Are the desingularizations

$$\overline{\mathbf{M}}, \widetilde{\mathbf{M}} \text{ of } \mathbf{M} \quad \text{and} \quad \overline{\mathcal{M}}, \widetilde{\mathcal{M}} \text{ of } \mathcal{M}$$

  moduli spaces of some natural classes of objects as in the curve case?

  [Choy proved that $\widetilde{\mathcal{M}}$ is the moduli space analogous to Seshadri's.]

- When does the stringy E-function $E_{st}(Y)$ of a projective (singular) variety Y coincide with the E-polynomial $IE(Y)$ of intersection cohomology $IH^*(Y)$?

- What is the equivariant version $E_{st}(Y, G)$ of stringy E-function when a reductive group G is acting on a (singular) variety Y?

`Thank you!!`