ON THE COHOMOLOGY OF HYPERKÄHLER QUOTIENTS

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Abstract. This paper gives a partial desingularisation construction for hyperkähler quotients and a criterion for the surjectivity of an analogue of the Kirwan map to the cohomology of hyperkähler quotients. This criterion is applied to some linear actions on hyperkähler vector spaces.

1. Introduction

Hyperkähler manifolds are manifolds M equipped with a Riemannian metric g and three independent complex structures $\mathbf{i}, \mathbf{j}, \mathbf{k}$ compatible with the metric which satisfy $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$. They correspondingly have three symplectic forms $\omega_1, \omega_2, \omega_3$, or one real symplectic form ω_1 and one complex symplectic form $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$. Suppose a compact connected Lie group K acts on M preserving the metric and the symplectic forms. We say the action is *Hamiltonian* if there are moment maps μ_i for each ω_i. It has been an outstanding problem how much of the package of properties of Hamiltonian group actions on symplectic manifolds extends to hyperkähler quotients

$$M/\!/\!/K := \mu^{-1}(0)/K$$

where $\mu = (\mu_1, \mu_2, \mu_3)$.

The first result of this paper is that the partial desingularisation construction of [9] extends to hyperkähler quotients. For this, we fix a complex structure, say $\mathbf{i}$, which gives us a holomorphic moment map $\mu_{\mathbb{C}} = \mu_2 + \sqrt{-1}\mu_3 : M \to \mathfrak{k}^* \otimes_{\mathbb{R}} \mathbb{C}$, and note that with respect to this complex structure $M/\!/\!/K$ can be identified with the Kähler quotient of the complex subvariety $W = \mu_{\mathbb{C}}^{-1}(0) = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ of M. If, as in [9], we blow up along the locus of most singular points, i.e. the locus of largest stabilizers, we get a strictly less singular quotient and by repeating this process, we obtain an analytic variety $\widetilde{M/\!/\!/K}$ which has at worst orbifold singularities and a proper birational holomorphic surjection $\widetilde{M/\!/\!/K} \to M/\!/\!/K$. Note that when the hyperkähler moment map μ has a central regular value then it is possible to obtain a partial desingularization $= \mu^{-1}(\epsilon)/K$ of the hyperkähler quotient $M/\!/\!/K = \mu^{-1}(0)/K$ by perturbing the moment map, and this partial desingularisation is itself hyperkähler. The partial desingularisation $\widetilde{M/\!/\!/K}$, on the other hand, does not inherit a hyperkähler structure, but it does always exist and depends only on the choice of complex structure $\mathbf{i}$.

Next, we consider the restriction map, often called the Kirwan map,

$$\kappa : H_K^*(M) \longrightarrow H_K^*(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/K)$$

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from the equivariant cohomology of M to the ordinary cohomology of the hyperkähler quotient $M///K = \mu^{-1}(0)/K$, when 0 is a regular value of μ. (All cohomology in this paper has rational coefficients). In the setting of symplectic quotients the analogous map κ is surjective ([8] Theorem 5.4), at least when the moment map is proper, so a natural question is whether κ is surjective or not in the hyperkähler setting. In the preprint [10], the third author attempted to study this by modifying the ideas and techniques in [8], but there is a crucial sign error in the proof of [10] Lemma 4.2 which invalidates the approach of [10, §4]. Here we follow the approach in §5 of [10] and attempt to prove surjectivity in two stages:

(1) surjectivity of the restriction $H_K^*(M) \to H_K^*(W)$, where $W = \mu_{\mathbb{C}}^{-1}(0)$, by using equivariant Morse theory with respect to $-\|\mu_{\mathbb{C}}\|^2$,
(2) surjectivity of the restriction $H_K^*(W) \to H_K^*(\mu^{-1}(0))$ by using equivariant Morse theory with respect to $-\|\mu_1\|^2$.

Here the norm is induced from a fixed K-invariant inner product on the Lie algebra $\mathfrak{k}$ of K, which we will use to identify $\mathfrak{k}$ with its dual throughout.

There are two difficulties in this approach. The first is the convergence issue of the gradient flows of the norm squares of the moment maps (that is, whether the norm squares are *flow-closed* in the sense of Definition 4.6 below), and the second is that W is in general singular. We avoid the second difficulty by using a generic rotation of the hyperkähler frame of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The three complex structures $\mathbf{i}, \mathbf{j}, \mathbf{k}$ give us in fact an S^2-family of complex structures and the triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is nothing but a choice of an orthonormal frame. We show that $W = \mu_{\mathbb{C}}^{-1}(0)$ is smooth if we use a frame which is general in the sense of Definition 4.3 and 0 is a regular value of μ; moreover, even when 0 is not a regular value of μ, for a general frame there are no non-minimal critical points of $\|\mu_1\|^2$ in W. Together with the results of [10, §5], this enables us to formulate a criterion for surjectivity of κ.

Theorem 1.1. *For a general choice of hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *if*

(1) $-\|\mu_{\mathbb{C}}\|^2$ *on* M *is flow-closed, and*
(2) $-\|\mu_1\|^2$ *on* W *is flow-closed*

then the retriction map

$$H_K^*(M) \to H_K^*(\mu^{-1}(0))$$

is surjective.

Here, a function f is flow-closed if the gradient flow of f from any point is contained in a compact set.

When M is a quaternionic vector space $V = \mathbb{H}^n$ and the action is linear, the flow-closedness of $-||\mu_1||^2$ follows from work of Sjamaar [14]. So we obtain

Proposition 1.2. *Let* K *act linearly on* $V = \mathbb{H}^n$, *and let* $\mu_{\mathbb{C}}$ *be the holomorphic moment map corresponding to a general choice of hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. *If* $-\|\mu_{\mathbb{C}}\|^2$ *on* V *is flow-closed, then the restriction map*

$$H_K^*(V) = H_K^* \longrightarrow H_K^*(\mu^{-1}(0))$$

is surjective. Here $H_K^* = H^*(BK)$ *denotes the equivariant cohomology of a point.*

Therefore, in the situation of Proposition 1.2 the surjectivity of κ follows from the following

Conjecture 1.3. *For a Hamiltonian linear hyperkähler action of a compact Lie group* K *on a quaternionic vector space,* $-\|\mu_{\mathbb{C}}\|^2$ *is flow-closed.*

Although it is purely a calculus problem in this linear case, the question of flow-closedness seems to be difficult. But we prove

Proposition 1.4. *Conjecture 1.3 is true when K is the circle group $U(1)$.*

The layout of this paper is as follows. In §2 we review general results on hyperkähler quotients. In §3 we explain how to construct a partial desingularisation of a hyperkähler quotient by a series of blow-ups. In §4 we outline criteria for the surjectivity of the map κ from the equivariant cohomology of a hyperkähler manifold to the ordinary cohomology of the hyperkähler quotient. In §5 we apply this to hyperkähler quotients of linear actions on quaternionic vector spaces preserving the hyperkähler structure. In the appendix we reproduce the argument of [10, §5] which shows that $-\|\mu_{\mathbb{C}}\|^2$ is equivariantly perfect if it is flow-closed.

2. Hyperkähler quotients

In this section, we recall basic definitions and facts about hyperkähler quotients.

Definition 2.1. *A* hyperkähler manifold *is a Riemannian manifold (M, g) equipped with three complex structures* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *such that*

1. $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$,
2. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are orthogonal with respect to g,*
3. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are parallel with respect to the Levi-Civita connection of g.*

Remark 2.2.
1. $T_pM \cong \mathbb{H}^n$ and so $\dim_{\mathbb{R}} M = 4n$ for some $n \in \mathbb{Z}$.
2. There exists an S^2-family of complex structures on M. Indeed, for any $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$, it is easy to see that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a complex structure on M. There exists a complex manifold Z (the 'twistor space' of M) and a holomorphic map $\pi : Z \to \mathbb{P}^1 \cong S^2$ such that the fiber over (a, b, c) is the complex manifold $(M, a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$.
3. There exists an S^2-family of (real) symplectic forms on M. Indeed, for each complex structure I in the S^2-family described above, $\omega_I(-, -) = g(-, I-)$ defines a symplectic form on M which is Kähler with respect to I. We let $\omega_1, \omega_2, \omega_3$ denote the symplectic forms defined by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively.

Remark 2.3. The complex valued two-form $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$ is holomorphic symplectic, so a hyperkähler manifold is always a holomorphic symplectic manifold. A holomorphic symplectic manifold which is compact always admits a hyperkähler metric, but this is not true in general for non-compact M.

Examples 2.4. Examples of hyperkähler manifolds include

1. $\mathbb{H}^n = T^*\mathbb{C}^n$ and quotients of $\mathbb{H}^n$ by discrete group actions;
2. K3 surfaces, Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

When X is a Kähler manifold, the cotangent bundle of X is holomorphic symplectic but is not necessarily hyperkähler.

Let K be a compact Lie group acting on a hyperkähler manifold M. Suppose that this action preserves the hyperkähler structure $(g, \mathbf{i}, \mathbf{j}, \mathbf{k})$.

Definition 2.5. *A* moment map *for the K-action on M is a differentiable map*

$$\mu = (\mu_1, \mu_2, \mu_3) : M \longrightarrow \mathfrak{k}^* \otimes_{\mathbb{R}} \mathbb{R}^3$$

satisfying

(1) *μ is K-equivariant, i.e. $\mu(kx) = Ad_k^*\mu(x)$ for $k \in K$, $x \in X$;*
(2) *$\langle \mathrm{d}\mu(v), \xi\rangle = (\omega_1(\xi_m, v), \omega_2(\xi_m, v), \omega_3(\xi_m, v))$ for $m \in M$, $v \in T_mM$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{k}^* \otimes_{\mathbb{R}} \mathbb{R}^3$, where*

$$\langle \mathrm{d}\mu(v), \xi\rangle = (\langle \mathrm{d}\mu_1(v), \xi_1\rangle, \langle \mathrm{d}\mu_2(v), \xi_2\rangle, \langle \mathrm{d}\mu_3(v), \xi_3\rangle).$$

We say the action of K is Hamiltonian *if a moment map exists. We call μ_1 the* real moment map *and $\mu_{\mathbb{C}} := \mu_2 + \sqrt{-1}\mu_3$ the* complex moment map *or* holomorphic moment map *with respect to the frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.*

Remark 2.6. (1) If μ is a moment map then any translation of μ by central elements of $\mathfrak{k}$ is again a moment map.
(2) It is straightforward to see that $\mu_{\mathbb{C}} : M \to \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is a holomorphic function with respect to $\mathbf{i}$.

Theorem 2.7. ([5]) *The smooth part of $M/\!/\!/K := \mu^{-1}(0)/K$ inherits a hyperkähler structure from M.*

Examples 2.8. (1) Suppose that $M = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ and that the action of $K = U(1)$ is given by $\lambda(x, y) = (\lambda x, \lambda^{-1}y)$ for $x, y \in \mathbb{C}^n$. Then up to the addition of constants we have $\mu_1(x, y) = \frac{\sqrt{-1}}{2}(x^\dagger x - y^\dagger y)$ and $\mu_{\mathbb{C}}(x, y) = x^T y$. Hence $W = \mu_{\mathbb{C}}^{-1}(0)$ is an affine quadric hypersurface in $\mathbb{C}^{2n}$ and the variation of $\mu^{-1}(c, 0, 0)/U(1)$ around $c = 0$ is the *Mukai flop.*
(2) Suppose $M = T^*\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \times \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$ and $K = U(n)$ acts on M in the natural way. Then the action is Hamiltonian with $\mu_1(x, y) = \frac{\sqrt{-1}}{2}(xx^\dagger - y^\dagger y)$ and $\mu_{\mathbb{C}}(x, y) = xy$.
(3) Suppose $M = T^*\mathrm{End}(\mathbb{C}^n) = \mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n)$ and $K = U(n)$ acting by conjugation. Then up to the addition of central constants we have $\mu_1(B_1, B_2) = \frac{\sqrt{-1}}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger])$ and $\mu_{\mathbb{C}}(B_1, B_2) = [B_1, B_2]$.
(4) Suppose K acts on M with moment map $\mu : M \to \mathfrak{k}^* \otimes \mathbb{R}^3$. Let $\rho : H \to K$ be a homomorphism and let $\mathfrak{h}$ be the Lie algebra of H. A moment map for the induced action of H on M is the composition $M \to \mathfrak{k}^* \otimes \mathbb{R}^3 \to \mathfrak{h}^* \otimes \mathbb{R}^3$ of μ and the dual $\mathfrak{k}^* \to \mathfrak{h}^*$ of the tangent map of ρ.
(5) Let M_1, M_2 be two hyperkähler manifolds acted on by K in Hamiltonian fashion. Then the sum of the moment maps for M_1 and M_2 is a moment map for the diagonal action on the product $M_1 \times M_2$.
(6) (ADHM spaces) Let $M = T^*\mathrm{End}\mathbb{C}^n \times T^*\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and $K = U(n)$. Then

$$\mu_1(B_1, B_2, x, y) = \frac{\sqrt{-1}}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + xx^\dagger - y^\dagger y)$$

and

$$\mu_{\mathbb{C}}(B_1, B_2, x, y) = [B_1, B_2] + xy.$$

The quiver variety construction is similar [1, 12].
(7) (coadjoint orbits) If O is an orbit in $\mathfrak{g}^* = \mathfrak{k}^* \otimes_{\mathbb{R}} \mathbb{C}$, then $\omega_\alpha(\xi_\alpha, \eta_\alpha) = \langle \alpha, [\xi, \eta]\rangle$ defines a holomorphic symplectic form on O and $\mu_{\mathbb{C}} : O \hookrightarrow \mathfrak{g}^*$ is the complex moment map (cf. [4, §26]).

3. Partial desingularization

Our first result in this paper allows us to construct a partial resolution of singularities of a hyperkähler quotient through a sequence of explicit blow-ups, which is closely related to the partial desingularization construction in [9].

Let $\mu : M \to \mathfrak{k}^* \otimes \mathbb{R}^3$ be a moment map for a Hamiltonian K-action on a hyperkähler manifold M. Let $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/K = M////K$ be the quotient map. If $x \in \mu^{-1}(0)$ has trivial stabilizer, then $\pi(x)$ is a smooth point of $M////K$ [5]. We fix a preferred complex structure $\mathbf{i}$ so that

$$\mu = (\mu_1, \mu_{\mathbb{C}}) : M \to \mathfrak{k}^* \otimes (\mathbb{R} \oplus \mathbb{C})$$

and view M as a complex manifold with respect to $\mathbf{i}$, equipped with a holomorphic symplectic two-form $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$. The function

$$\mu_{\mathbb{C}} = \mu_2 + i\mu_3 : M \to \mathfrak{k} \otimes \mathbb{C}$$

is holomorphic with respect to the complex structure $\mathbf{i}$ on M, and thus $W = {\mu_2}^{-1}(0) \cap {\mu_3}^{-1}(0)$ is a complex analytic subvariety of M. Moreover $\mu^{-1}(0)/K = W \cap {\mu_1}^{-1}(0)/K$ is the Kähler quotient of W by the action of K.

We can apply the partial desingularization construction of [9] to the Kähler quotient ${\mu_1}^{-1}(0)/K$ of M by K with respect to the Kähler structure corresponding to $\mathbf{i}$. This gives us a complex orbifold with a birational holomorphic surjection to ${\mu_1}^{-1}(0)/K$ which restricts to an isomorphism over the open subset of ${\mu_1}^{-1}(0)/K$ where the derivative of μ_1 is surjective. The proper transform $\widetilde{W//K}$ of $W//K = W \cap {\mu_1}^{-1}(0)/K = \mu^{-1}(0)/K$ in ${\mu_1}^{-1}(0)/K$ is the result of applying the partial desingularization construction of [9] to the complex subvariety W of M instead of M itself, and thus if W were nonsingular then it would follow immediately that $\widetilde{W//K}$ is a complex orbifold. However in general W is singular (indeed, if W is nonsingular then 0 is a regular value of μ and so no partial desingularization for $M////K = \mu^{-1}(0)/K$ is needed). Nonetheless, we will see in this section that $\widetilde{W//K}$ is always a complex orbifold: the blow-ups in its construction resolve the singularities coming from W at the same time as reducing the singularities created by the quotient construction to orbifold singularities.

The construction of $\widetilde{W//K}$ given in [9] is as follows. Let H_0 be the identity component of the stabilizer of a point in $\mu^{-1}(0) = W \cap {\mu_1}^{-1}(0)$ whose dimension is maximal possible among such, and let Z_{H_0} be the fixed point set of H_0 in W. First we blow $W = \mu_{\mathbb{C}}^{-1}(0)$ up along the closure in W of GZ_{H_0}, and give the resulting blow-up a Kähler structure which is a small perturbation of the pull-back of the Kähler structure on M restricted to W. Then we repeat this process until, after finitely many steps, the points in the blow-up where the real moment map vanishes all have 0-dimensional stabilizers, and the resulting quotient is $\widetilde{W//K}$. Equivalently, by [9, Lemma 3.11], we can construct $\widetilde{W//K}$ directly from the quotient $W//K$ by a sequence of blow-ups.

In order to prove that $\widetilde{W//K}$ has only orbifold singularities even though W is in general singular, we begin with a holomorphic version of the equivariant Darboux theorem.

Theorem 3.1. *Suppose ω_0, ω_1 are holomorphic symplectic two-forms on a complex manifold X and Y is a complex submanifold of X such that ω_0 and ω_1 coincide* *on $TX|_Y$. Suppose further that there exists an open neighborhood U of Y and a differentiable family of holomorphic maps $\varphi_t : U \to U$ for $0 \le t \le 1$ such that*

$$\varphi_1 = \mathrm{id}_U, \quad \varphi_0(U) = Y, \quad \varphi_t|_Y = \mathrm{id}_Y \quad \forall t.$$

Then there exists an open neighborhood U' contained in U and a biholomorphic map $f : U' \to f(U')$ where $f(U')$ is an open subset of X such that

$$f|_Y = \mathrm{id}_Y \quad \text{and} \quad f^*\omega_1 = \omega_0.$$

If a Lie group K acts on X with ω_0, ω_1, Y, U and φ_t all K-invariant, then f can be chosen to commute with the action of K.

Proof. The proof is an obvious modification of Weinstein's in [4]. Let $\omega_t = \omega_0 + t\sigma$ where $\sigma = \omega_1 - \omega_0$. By assumption, σ is trivial on Y and $\mathrm{d}\sigma = 0$. Let ξ_t denote the vector field of φ_t. Then

$$\sigma - \varphi_0^*\sigma = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^*\sigma)\mathrm{d}t = \int_0^1 \varphi_t^*(\imath(\xi_t)\mathrm{d}\sigma)\mathrm{d}t + \mathrm{d}\int_0^1 \varphi_t^*(\imath(\xi_t)\sigma)\mathrm{d}t = \mathrm{d}\beta$$

where $\beta = \int_0^1 \varphi_t^*(\imath(\xi_t)\sigma)\mathrm{d}t$. By our assumption, $\beta|_Y = 0$. We define a *holomorphic* vector field η_t on U by the equation $\imath(\eta_t)\omega_t = -\beta$. By shrinking U to a smaller open neighborhood U' if necessary, we can integrate η_t to obtain *biholomorphic* maps $f_t : U' \to X$ for $0 \le t \le 1$. Then

$$f_1^*\omega_1 - \omega_0 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f_t^*\omega_t)\mathrm{d}t = \int_0^1 f_t^*(\sigma + \mathrm{d}\imath(\eta_t)\omega_t)\mathrm{d}t = 0$$

as desired. Moreover if K acts on X with ω_0, ω_1, Y, U and φ_t all K-invariant, then f_1 is also K-invariant. □

Let $x \in \mu^{-1}(0)$, and note that if $a, b, c, d \in \mathfrak{k}$ then $a_x, \mathbf{i}b_x, \mathbf{j}c_x$ and $\mathbf{k}d_x$ are mutually orthogonal, since, for example,

$$g(\mathbf{j}c_x, \mathbf{k}d_x) = g(\mathbf{ki}c_x, \mathbf{k}d_x) = g(\mathbf{i}c_x, d_x) = d\mu_1(d_x).c = [d, \mu_1(x)].c = 0$$

because $\mu_1(x) = 0$. We next find a local model for a neighborhood of x in M. Let H be the stabilizer of x in K, so that the complexification $H^{\mathbb{C}}$ is the stabilizer of x in $G = K^{\mathbb{C}}$, and let $Y = Gx \cong G/H^{\mathbb{C}}$ be its G-orbit. Further, let B be a ball in $T_x(Gx)^\perp = \mathbf{j}(V_x^{\mathbb{C}}) \oplus \mathcal{W}$ where $V_x = T_x(Kx)$ and

$$\mathcal{W} = (V_x \oplus \mathbf{i}V_x \oplus \mathbf{j}V_x \oplus \mathbf{k}V_x)^\perp.$$

Note that $\mathcal{W}$ is a quaternionic vector space with a linear action of $H^{\mathbb{C}}$. Let $S = H^{\mathbb{C}}B \subset T_xM$. Then the holomorphic slice theorem in [13] tells us that

$$G \times_{H^{\mathbb{C}}} S$$

is biholomorphic to an open neighborhood U of Y in M. Now consider

$$X = G \times_{H^{\mathbb{C}}} (\mathbf{j}V_x^{\mathbb{C}} \oplus \mathcal{W})$$

which is a hyperkähler manifold since it is a hyperkähler quotient of $T^*G \times \mathcal{W}$ by the action of H. By this quotient construction, we obtain a holomorphic symplectic form $\omega'_{\mathbb{C}}$ and a complex moment map

$$\mu'_{\mathbb{C}}(g, \mathbf{j}b, w) = Ad_g^*(b + \mu_{\mathbb{C}}^{\mathcal{W}}(w)) \qquad g \in G, b \in V_x^{\mathbb{C}}, w \in \mathcal{W} \tag{3.1}$$

for the action of K on X where $\mu_{\mathbb{C}}^{\mathcal{W}}$ is the complex moment map for the action of H on $\mathcal{W}$. If we take the hyperkähler quotient of X by the action of K, then we obtain the hyperkähler quotient $\mathcal{W}///H$. Since $X \supset U$, the pull-back of the holomorphic symplectic form $\omega_{\mathbb{C}}$ of M is also holomorphic symplectic, and this coincides with $\omega'_{\mathbb{C}}$ on Y. On the other hand, X is a vector bundle over $Y = G/H^{\mathbb{C}}$ with fibers $\mathbf{j}V_x^{\mathbb{C}} \oplus \mathcal{W}$ and hence by shrinking the fibers, we obtain a family of holomorphic maps $\varphi_t : X \to X$ satisfying the assumptions of Theorem 3.1. Thus by Theorem 3.1, we obtain

Theorem 3.2. *There exists a biholomorphic map from a neighborhood of $G/H^{\mathbb{C}}$ in X to a neighborhood of the orbit Gx in M such that the pull-backs of $\omega_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ are $\omega'_{\mathbb{C}}$ and $\mu'_{\mathbb{C}}$ as at (3.1) above. Furthermore, locally at x, the embedding $W = \mu_{\mathbb{C}}^{-1}(0) \hookrightarrow M$ is biholomorphic to*

$$G \times_{H^{\mathbb{C}}} (0 \times (\mu_{\mathbb{C}}^{\mathcal{W}})^{-1}(0)) \hookrightarrow G \times_{H^{\mathbb{C}}} (\mathbf{j}V_x^{\mathbb{C}} \oplus \mathcal{W}).$$

In particular, the singularity of $M/\!/\!/G$ at $\pi(x)$ is precisely the hyperkähler quotient $\mathcal{W}/\!/\!/H$ of the vector space $\mathcal{W}$.

In the simplest case when $K = U(1)$ is the circle group, then it follows immediately from this theorem that the blow-up of W along the fixed point set of K is nonsingular, and hence $\widetilde{W/\!/K}$ has only orbifold singularities.

Corollary 3.3. *Suppose that $K = U(1) = H$. Then the blow-up of $W = \mu_{\mathbb{C}}^{-1}(0)$ along Z_H where Z_H is the fixed point set of H in W is nonsingular, and $\widetilde{W/\!/K}$ has only orbifold singularities.*

Proof. The hyperkähler moment map μ is locally constant on the K-fixed point set in M, so that Z_H is a union of some of its connected components and hence is nonsingular. In the local model X, we can identify $GZ_H = Z_H$ with $0 \times \mathcal{W}^H$ where $\mathcal{W}^H$ is the subspace of vectors fixed by H. Also W in this local model is $0 \times (\mu_{\mathbb{C}}^{\mathcal{W}})^{-1}(0)$. Since $\mu_{\mathbb{C}}^{\mathcal{W}}$ is homogeneous quadratic, $(\mu_{\mathbb{C}}^{\mathcal{W}})^{-1}(0)$ is the product of an affine quadric cone and $\mathcal{W}^H$ and so after the blow-up along Z_H we obtain a complex manifold whose Kähler quotient $\widetilde{W/\!/K}$ has only orbifold singularities. $\square$

In general for any compact group K we can describe the local structure of the first blow-up in the partial desingularization process.

Proposition 3.4. *Let H_0 be the identity component of the stabilizer of a point in $\mu^{-1}(0)$, whose dimension is maximal possible among such. Then GZ_{H_0}, where Z_{H_0} is the fixed point set of H_0 in W, is closed and nonsingular in a neighborhood of $\mu^{-1}(0)$. If we blow up $W = \mu_{\mathbb{C}}^{-1}(0)$ along the closure of GZ_{H_0}, then over a neighborhood of $\mu^{-1}(0)$ the exceptional divisor is normally nonsingular, i.e. it has a neighborhood isomorphic to a holomorphic line bundle.*

Proof. That GZ_{H_0} is closed and nonsingular in a neighborhood of $\mu^{-1}(0)$ follows from applying [9] Corollary 5.10 and Lemma 5.11 to the action of G on M, together with the local model of Theorem 3.2 and the fact that the hyperkähler moment map for the action of H_0 is locally constant on Z_{H_0}. In the local model X, we can identify GZ_{H_0} with $G \times H^{\mathbb{C}}(0 \times \mathcal{W}^{H_0})$ where $\mathcal{W}^{H_0}$ is the subspace of vectors fixed by H_0. Also W in this local model is $G \times_{H^{\mathbb{C}}} (0 \times (\mu_{\mathbb{C}}^{\mathcal{W}})^{-1}(0))$. Since $\mu_{\mathbb{C}}^{\mathcal{W}}$ is homogeneous quadratic, $(\mu_{\mathbb{C}}^{\mathcal{W}})^{-1}(0)$ is the product of an affine quadric cone and $\mathcal{W}^{H_0}$ and so after the blow-up along GZ_{H_0} we obtain a line bundle along the exceptional divisor. $\square$

It follows that if we blow up the subvariety $W = \mu_{\mathbb{C}}^{-1}(0)$ along GZ_{H_0} where H_0 is as in Proposition 3.4, then the singular locus of the blow-up W_1 of W is the proper transform of the singular locus of W minus GZ_{H_0} and no new singularities near $\mu^{-1}(0)$ are produced by the blow-up. Similarly the local model in Theorem 3.2 tells us that this is true for each of the blow-ups in the construction of $\widetilde{W/\!/K}$, and hence at the end of the construction the proper transform $\tilde{W}$ of W is nonsingular, and the quotient $\widetilde{W/\!/K} = \tilde{W}^{ss}/G$ has only orbifold singularities. Thus $\widetilde{W/\!/K}$ is a complex orbifold with a birational holomorphic surjection to $W/\!/K = W \cap {\mu_1}^{-1}(0)/K = \mu^{-1}(0)/K$ which restricts to an isomorphism over the open subset of $\mu^{-1}(0)/K$ where the derivative of μ is surjective. We therefore call it a *partial desingularization* $\widetilde{M/\!/\!/K}$ of the hyperkähler quotient $M/\!/\!/K = \mu^{-1}(0)/K$.

Remark 3.5. Note however that the construction of this partial desingularization $\widetilde{M/\!/\!/K}$ depends on the choice of preferred complex structure $\mathbf{i}$, and $\widetilde{M/\!/\!/K}$ does not inherit a hyperkähler structure from that of M.

Remark 3.6. Another way to (partially) resolve the singularities of a hyperkähler quotient is to perturb the moment map by a small central element in $\mathfrak{k}^*$. This is better in the sense that it gives us a partial desingularization which is hyperkähler again. However this resolution is possible only when the center contains regular values of the hyperkähler moment map and it does not apply to some examples, such as the moduli of Higgs bundles, or $\mathbb{H}^n \otimes sl(2)/\!/\!/SL(2)$. In fact, there are no hyperkähler resolutions for the latter singularity [6, 7].

4. Surjectivity criterion

In this section, we give a criterion for the surjectivity of the hyperkähler Kirwan map $H^*_K(M) \to H^*(M/\!/\!/K)$.

Let M be a hyperkähler manifold on which a compact connected Lie group K acts preserving the hyperkähler structure. Suppose we have a hyperkähler moment map $\mu = (\mu_1, \mu_{\mathbb{C}})$ which takes values in $\mathfrak{k} \otimes (\mathbb{R} \oplus \mathbb{C}) = \mathfrak{k} \otimes \mathbb{R}^3$ for the action of K. Let T be a maximal torus of K. The following is a simple consequence of Mostow's theorem [11].

Lemma 4.1. *Consider the action of T on M and the stabilizers of points in M. Let $\mathcal{T}$ be the set of nontrivial subtori T' of T which are the identity components of the stabilizers of points in M. Then $\mathcal{T}$ is countable.*

As a consequence, the set of connected components of the fixed point sets of T' for $T' \in \mathcal{T}$ is countable and we write it as $\{Z_j \mid j \in \mathbb{Z}_{\geq 0}\}$. For each Z_j let T_j be the identity component of the stabilizer in T of a general point in Z_j.[1]

Lemma 4.2. *For any $\gamma \in \mathrm{Lie}(T_j)$, $\langle \mu(Z_j), \gamma \rangle$ consists of a single point in $\mathbb{R}^3$.*

The proof of this lemma is elementary and we omit it.

Now for each j we choose $\gamma_j \in \mathrm{Lie}(T_j)$ such that $\langle \mu(Z_j), \gamma_j \rangle \neq 0$ whenever

$$\rho_j := \sup\{\|\langle \mu(Z_j), \gamma \rangle\| \;:\; \|\gamma\| = 1, \gamma \in \mathrm{Lie}(T_j)\} > 0.$$

We require no condition for γ_j when $\rho_j = 0$, in which case we have

$$\langle \mu(Z_j), \gamma \rangle = 0 \quad \text{for any} \quad \gamma \in \mathrm{Lie}(T_j).$$

Since $\langle \mu(Z_j), \gamma_j \rangle \in \mathbb{R}^3$ is nonzero whenever $\rho_j > 0$, a general choice of hyperkähler frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (being uncountably many) satisfies the following condition:

[1] This makes sense because all the orbit type strata have even real codimension.

$\langle \mu(Z_j), \gamma_j \rangle$ *are not of the form* $(a, 0, 0) \in \mathbb{R}^3$ *with* $a \neq 0$ *for all* j *with* $\rho_j > 0$, *i.e.*

$$\langle \mu_{\mathbb{C}}(Z_j), \gamma_j \rangle \neq 0. \tag{4.1}$$

Definition 4.3. *We say a hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ *is* general *in the group of frames* $SO(3)$ *if the above condition is satisfied.*

Definition 4.4. *As in* [8] *we let*

$$W^{ss} = \{x \in W \mid \textit{the gradient flow from } x \textit{ with respect to} - \|\mu_1\|^2$$
$$\textit{has a limit point in } \mu_1^{-1}(0)\}.$$

We say that $x \in W$ *is* semistable *if* $x \in W^{ss}$.

Proposition 4.5. *For a general hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *if the gradient flow from each* $x \in W = \mu_{\mathbb{C}}^{-1}(0)$ *with respect to* $-\|\mu_1\|^2$ *is contained in a compact set, then* $W = W^{ss}$. *If* $\mu^{-1}(0)$ *is smooth,* $W = \mu_{\mathbb{C}}^{-1}(0)$ *is smooth.*

Proof. A point $x \in W$ is a critical point of $\|\mu_1\|^2$ if and only if $\mu_1(x)_x = 0$. Suppose $\mu_1(x) \neq 0$. Without loss of generality, replacing x with kx for some $k \in K$, we may assume $\mu_1(x) \in \mathrm{Lie}(T)$ and $x \in Z_j$ for some j with $\mathrm{Lie}(\mathrm{Stab}_T x) = \mathrm{Lie}(T_j)$. Let $\gamma = \mu_1(x) \in \mathrm{Lie}(T_j)$. Then $\langle \mu_1(Z_j), \gamma \rangle$ is a nonzero constant $\|\gamma\|^2$. Hence $\rho_j > 0$, so $\mu_{\mathbb{C}}(x) \neq 0$ because $\langle \mu_{\mathbb{C}}(x), \gamma_j \rangle = \langle \mu_{\mathbb{C}}(Z_j), \gamma_j \rangle \neq 0$ by (4.1). This is a contradiction to $x \in W$. Therefore $\mu_1(x) = 0$ and all the points in W are semistable. If $\mu^{-1}(0)$ is smooth, all the stabilizers in $G = K^{\mathbb{C}}$ of points in W are finite and hence $d\mu_{\mathbb{C}}$ is surjective and so W is smooth. $\square$

Definition 4.6. *Let* $f : M \to \mathbb{R}$ *be a smooth function defined on a manifold. We say* f *is* flow-closed *if the gradient flow of* f *from any* $x \in M$ *is contained in a compact set.*

Corollary 4.7. *For a general hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *if* $-\|\mu_1\|^2$ *on* $W = \mu_{\mathbb{C}}^{-1}(0)$ *and* $-\|\mu_{\mathbb{C}}\|^2$ *on* M *are flow-closed, then the restriction map* $H_K^*(M) \to H_K^*(\mu^{-1}(0))$ *is surjective.*

Proof. It is proved in [10, §5] that $-\|\mu_{\mathbb{C}}\|^2$ is equivariantly perfect if it is flow-closed. Because [10] contains errors and is unpublished, we have reproduced an edited version of this section in an Appendix (see below).

Therefore the restriction map $H_K^*(M) \to H_K^*(W)$ is surjective, and by Proposition 4.5 above we have $W = W^{ss}$ and so $H_K^*(W) \cong H_K^*(\mu^{-1}(0))$. Thus we obtain the surjectivity. $\square$

We have proved

Theorem 4.8. (surjectivity criterion) *For a general hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *if*

(1) $-\|\mu_{\mathbb{C}}\|^2$ *on* M *is flow-closed, and*
(2) $-\|\mu_1\|^2$ *on* $W = \mu_{\mathbb{C}}^{-1}(0)$ *is flow-closed*

then the restriction map

$$H_K^*(M) \to H_K^*(\mu^{-1}(0))$$

is surjective. If furthermore 0 is a regular value of the hyperkähler moment map μ, *then* $W = \mu_{\mathbb{C}}^{-1}(0)$ *is smooth and the map*

$$\kappa : H_K^*(M) \to H_K^*(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/K)$$

is surjective.

Remark 4.9. Suppose that $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a general hyperkähler frame satisfying conditions (1) and (2) of Theorem 4.8, so that the restriction map

$$H_K^*(M) \to H_K^*(W) = H_K^*(W^{ss}) \cong H_K^*(\mu^{-1}(0)) \tag{4.2}$$

is surjective, but suppose that 0 is not a regular value of the hyperkähler moment map μ. Then as in §3 we can construct a partial desingularization $\widetilde{M///K} = \tilde{W}^{ss}/G$ of the hyperkähler quotient $M///K$ with respect to the preferred complex structure $\mathbf{i}$ (where $G = K^{\mathbb{C}}$ is the complexification of K and $\tilde{W}^{ss}$ is an open subset of a blow-up of $W = \mu_{\mathbb{C}}^{-1}(0)$), and the blow-down map induces a map on K-equivariant cohomology

$$H_K^*(W^{ss}) \to H_K^*(\tilde{W}^{ss}) \cong H^*(\tilde{W}^{ss}/G) = H^*(\widetilde{M///K}). \tag{4.3}$$

The intersection cohomology $IH^*(M///K)$ of $M///K$ (with respect to the middle perversity and rational coefficients) is a direct summand of $H^*(\widetilde{M///K})$ by the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [2, 6.2.5]. This direct decomposition is not in general canonical, but Woolf [15] shows that the partial desingularization construction (and more generally any *G-stable resolution* in the sense of [15, §3]) can be used to define a projection

$$H^*(\widetilde{M///K}) \to IH^*(M///K), \tag{4.4}$$

and also a projection

$$IH_K^*(W^{ss}) \to IH^*(W//K) = IH^*(M///K) \tag{4.5}$$

such that the composition of (4.5) with the canonical map from $H_K^*(W^{ss})$ to $IH_K^*(W^{ss})$ agrees with the composition of (4.4) with (4.3). The argument of [15] uses the decomposition theorem at the level of the equivariant derived category (see [3, §5]) as a decomposition of complexes of sheaves up to quasi-isomorphism, and it can be applied to M and restricted to its complex subvariety $W = \mu_{\mathbb{C}}^{-1}(0)$ to give a projection from the hypercohomology of the restriction to $W^{ss} = W$ of $\mathcal{IC}_K^\bullet(M^{ss})$ (which is just $H_K^*(W^{ss})$ since M is nonsingular) to the hypercohomology of the restriction to $W//K = M///K$ of $\mathcal{IC}^\bullet(M//K)$. We thus find that the partial desingularization construction gives us a map

$$H_K^*(M) \to IH^*(M///K)$$

which is the composition of (4.2), (4.3) and the projection (4.4) from $H^*(\widetilde{M///K})$ onto $IH^*(M///K)$, and factorises through a surjection from $H_K^*(M)$ to the hypercohomology of the restriction to $W//K = M///K$ of $\mathcal{IC}^\bullet(M//K)$.

5. Hyperkähler quotients of quaternionic vector spaces

In this section we consider linear actions on hyperkähler vector spaces.

Let $V = \mathbb{C}^{2n} = T^*\mathbb{C}^n$ be a hyperkähler vector space. Suppose we have a Hamiltonian hyperkähler linear action of a compact Lie group K on V. Fix a *general hyperkähler frame* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Let μ (resp. $\mu_{\mathbb{C}}$) be the hyperkähler (resp. complex) moment map for this action. From the surjectivity criterion in the previous section, we obtain the following.

Proposition 5.1. *Suppose $-\|\mu_{\mathbb{C}}\|^2$ on V is flow-closed. If 0 is a regular value of μ, the restriction map*

$$H_K^*(V) = H_K^* \longrightarrow H_K^*(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/K)$$

is surjective.

This proposition is relevant to Hilbert schemes of points in $\mathbb{C}^2$, ADHM spaces, hypertoric manifolds and quiver varieties, etc.

Proof. Sjamaar has proved that $-\|\mu_1\|^2$ is flow-closed on V and hence on $W = \mu_{\mathbb{C}}^{-1}(0)$ ([14, (4.6)]), so the hypotheses of Theorem 4.8 are satisfied. □

We believe that it is likely that the assumption on flow-closedness of $-\|\mu_{\mathbb{C}}\|^2$ in Proposition 5.1 is always satisfied.

Conjecture 5.2. *For a linear hyperkähler action on a hyperkähler vector space of a compact Lie group K, $-\|\mu_{\mathbb{C}}\|^2$ is flow-closed.*

Although it is purely a calculus problem, we do not know how to prove this conjecture, though we have managed to prove the circle case.

Remark 5.3. Certain other cases follow from the circle case. For example if the conjecture is true for the action of K_1 on V_1 and for the action of K_2 on V_2 then it is true for the action of $K_1 \times K_2$ on $V_1 \oplus V_2$, so the conjecture follows for suitable torus actions.

Proposition 5.4. *Conjecture 5.2 is true for the circle group $U(1)$.*

Let $x, y \in \mathbb{C}^n$. We denote by $\bar{x}, \bar{y}$ the complex conjugates of x, y. By a suitable change of variables and ignoring the variables for which the weights are zero, we may assume

$$\mu_{\mathbb{C}}(x, y) = x \cdot y - c = \sum_{i=1}^{n} x_i y_i - c$$

for some $c \in \mathbb{C}$. Let

$$f(x, y) = |\mu_{\mathbb{C}}(x, y)|^2 \geq 0$$

and let $\gamma(t)$ be the integral curve of the vector field $-\mathrm{grad} f$ from a given $(x_0, y_0) \notin \mu_{\mathbb{C}}^{-1}(0)$.

Lemma 5.5. *$\{\gamma(t)\}$ is contained in a compact subset of $\mathbb{C}^{2n}$.*

Proof. Let

$$\rho(x, y) = |x|^2 + |y|^2 = \sum_{i=1}^{n} (|x_i|^2 + |y_i|^2) \geq 0.$$

By direct computation, we know

$$\mathrm{grad} f = 2\mu_{\mathbb{C}} \begin{pmatrix} \bar{y} \\ \bar{x} \end{pmatrix} \quad \text{and} \quad \mathrm{grad}\rho = 2 \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $\gamma(t) \in \mu_{\mathbb{C}}^{-1}(0)$ for some t then γ is constant and there is nothing to prove. So we assume $\gamma(t) \notin \mu_{\mathbb{C}}^{-1}(0)$ for all $t \geq 0$. On $\mathbb{C}^{2n} - \mu_{\mathbb{C}}^{-1}(0)$, $\sqrt{f} = |\mu_{\mathbb{C}}| \geq 0$ is differentiable and

$$\mathrm{grad}\sqrt{f} = \frac{1}{2\sqrt{f}} \mathrm{grad} f.$$

If we denote by $\langle , \rangle$ the ordinary inner product of $\mathbb{R}^{4n} \cong \mathbb{C}^{2n}$, we have

$$\langle \mathrm{grad}\sqrt{f}, \mathrm{grad}f \rangle = \frac{1}{2\sqrt{f}}|\mathrm{grad}f|^2 = \frac{4|\mu_{\mathbb{C}}|^2}{2\sqrt{f}}(|x|^2+|y|^2) = 2\rho\sqrt{f},$$

$$\begin{aligned}\langle \mathrm{grad}\rho, \mathrm{grad}f \rangle &= 8\,\mathrm{Re}\,\mu_{\mathbb{C}}(\bar{x}\cdot\bar{y}) = 8\,\mathrm{Re}\,\mu_{\mathbb{C}}(\bar{\mu}_{\mathbb{C}}+\bar{c}) \\ &= 8\,\mathrm{Re}\,(f+\bar{c}\mu_{\mathbb{C}}) \geq 8f - 8|c|\sqrt{f}.\end{aligned}$$

Therefore,

$$\langle \mathrm{grad}(\rho+\sqrt{f}), \mathrm{grad}f \rangle \geq 8f + 2\sqrt{f}(\rho - 4|c|)$$

and hence $(\rho+\sqrt{f})(\gamma(t))$ is decreasing whenever $\rho = |x|^2+|y|^2 \geq 4|c|$.

Now we claim, for all $t \geq 0$,

$$(\rho+\sqrt{f})(\gamma(t)) \leq \max\{4|c|+\sqrt{f_0}, \rho_0+\sqrt{f_0}\}$$

where $\rho_0 = \rho(x_0,y_0)$ and $f_0 = f(x_0,y_0)$. The proposition follows from this claim because $\rho = |x|^2+|y|^2$ is bounded along the curve $\gamma(t)$ since $\rho+\sqrt{f}$ is bounded and $\sqrt{f} \geq 0$.

It remains to show the claim above. Since f is decreasing along $\gamma(t)$ by the definition of $\gamma(t)$, $\sqrt{f}$ is also decreasing for all $t \geq 0$. If $\rho(\gamma(t)) \leq 4|c|$ for some $t \geq 0$, $\rho+\sqrt{f} \leq 4|c|+\sqrt{f_0}$ at $\gamma(t)$.

If $\rho(\gamma(t)) > 4|c|$ for some $t \geq 0$, then there are two possibilities.

(1) γ on $[0,t]$ stays outside of the sphere $\rho = 4|c|$: in this case, since $\rho+\sqrt{f}$ is decreasing on $[0,t]$, $\rho+\sqrt{f} \leq \rho_0+\sqrt{f_0}$.

(2) γ on $[0,t]$ moves into and out of the sphere $\rho = 4|c|$: then let τ be the greatest element of $[0,t]$ satisfying $\rho(\gamma(\tau)) \leq 4|c|$. Then $\tau < t$ and because $\rho(\gamma) > 4|c|$ on the interval $(\tau,t]$, it follows that $\rho+\sqrt{f}$ at $\gamma(t)$ is less than $\rho+\sqrt{f}$ at $\gamma(\tau)$, which is less than or equal to $4|c|+\sqrt{f_0}$ since $\rho(\gamma(\tau)) \leq 4|c|$ and $\sqrt{f}$ is decreasing. Hence $\rho+\sqrt{f} \leq 4|c|+\sqrt{f_0}$.

So we are done. □

Examples 5.6. Consider the action of $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ with weights 1 and -1 as usual. For $x, y \in \mathbb{C}^n$, $\mu_{\mathbb{C}}(x,y) = \sum x_i y_i - c$ for $c \neq 0$. There is only one non-minimal critical point which is the origin. By direct computation, the Hessian at the origin has $2n$ positive and $2n$ negative eigenvalues. Hence we deduce that the Poincaré polynomial of the hyperkähler quotient is

$$\frac{1}{1-t^2} - \frac{t^{2n}}{1-t^2} = P_t(\mathbb{P}^{n-1}) = P_t(T^*\mathbb{P}^{n-1}).$$

Appendix A. Morse flow of the norm square of the complex moment map

The following is a corrected version of part of the unpublished preprint [10] (mainly [10] §5). The proof of Lemma A.5 differs from the original only because a sign error in [10] is corrected, and as a result the matrix in (A.5) differs from the matrix in [10] (first equation p. 35). The conclusion of §5 of [10] (that the norm square of the complex moment map is an equivariantly perfect Morse function) remains valid after these modifications. The proof of [10] Lemma 4.2 is, however, incorrect because of a fatal sign error (originally pointed out by Simon Donaldson), and the conclusions of [10, §4] are therefore invalid.

Consider the function $f_{23} : M \to \mathbb{R}$ defined by

$$f_{23}(x) = \|\mu_2(x)\|^2 + \|\mu_3(x)\|^2 = \|\mu_{\mathbb{C}}\|^2.$$

For all $x \in M$ we have

$$\text{grad} f_{23}(x) = 2\mathbf{j}(\mu_2(x)_x - \mathbf{i}\mu_3(x)_x) = 2\mathbf{k}(\mu_3(x)_x + \mathbf{i}\mu_2(x)_x). \tag{A.1}$$

Suppose that $\text{grad} f_{23}(x) = 0$. Then the Hessian H_{23} of f_{23} at x is represented via the metric as the self-adjoint endomorphism of T_xM given by

$$\frac{1}{2}H_{23}(\xi) = (\mathbf{j}\beta_2 + \mathbf{k}\beta_3)(\xi) + \mathbf{j}d\mu_2(x)(\xi)_x + \mathbf{k}d\mu_3(x)(\xi)_x \tag{A.2}$$

for $\xi \in T_xM$. Here $\beta_2 = \mu_2(x)$ and $\beta_3 = \mu_3(x)$ and $(\mathbf{j}\beta_2+\mathbf{k}\beta_3)(\xi)$ denotes the action of $\mathbf{j}\beta_2+\mathbf{k}\beta_3$ on T_xM induced from the action of K and the fact that $(\mathbf{j}\beta_2+\mathbf{k}\beta_3)_x = 0$.

Lemma A.1. *If* $\text{grad} f_{23}(x) = 0$ *and* $\beta_2 = \mu_2(x)$ *and* $\beta_3 = \mu_3(x)$ *then*

$$[\beta_2, \beta_3] = 0$$

Proof. Recall that

$$\mu_{\mathbb{C}} = \mu_2 + \sqrt{-1}\mu_3 : M \to \mathfrak{k}^* \otimes \mathbb{C}$$

is K-invariant with respect to the coadjoint action on $\mathfrak{k}^*$ and holomorphic with respect to the complex structure $\mathbf{i}$ on M. By (A.1) we have $(\beta_3)_x + \mathbf{i}(\beta_2)_x = 0$, and so applying the derivative of $\mu_{\mathbb{C}}$ at x to this we obtain

$$0 = [\beta_3 + \sqrt{-1}\beta_2, \mu_{\mathbb{C}}(x)] = [\beta_3 + \sqrt{-1}\beta_2, \beta_2 + \sqrt{-1}\beta_3] = 2[\beta_3, \beta_2].$$

□

Lemma A.2. *If* $x \in X$ *then* $\text{grad} f_{23}(x) = 0$ *if and only if*

$$\mu_2(x)_x = \mu_3(x)_x = 0.$$

Proof. Suppose $\text{grad} f_{23}(x) = 0$. By Lemma A.1 we have $[\beta_2, \beta_3] = 0$, so

$$\begin{aligned} \mu_2(x)_x \cdot \mathbf{i}\mu_3(x)_x &= d\mu_1(\mu_2(x)_x) \cdot \mu_3(x) \\ &= [\mu_2(x), \mu_1(x)] \cdot \mu_3(x) \\ &= \mu_1(x) \cdot [\beta_2, \beta_3] = 0 \end{aligned}$$

Then by (A.1) the result follows. □

As before let T be a maximal torus of K with Lie algebra $\mathfrak{t}$ and for $j = 2, 3$ let $\mathcal{B}_j$ be the set of $\beta \in \mathfrak{t}^*$ such that there is some $x \in X$ with $\mu_j(x) = \beta$ and $\beta_x = 0$.

Remark A.3. When M is compact then any $\beta \in \mathcal{B}_j$ is the closest point to zero of the convex hull of a nonempty subset of the finite subset of $\mathfrak{t}^*$ which is the image under the T-moment map μ_j^T of the T-fixed point set in M (see [8] Lemma 3.13), and in particular $\mathcal{B}_j$ is finite. In general it follows from the fact that $\|\mu_j\|^2$ is a minimally degenerate Morse function in the sense of [8] that $\mathcal{B}_j$ is at most countable.

Let $\mathcal{B}_{23}$ be a set of representatives of the Weyl group orbits in $\mathcal{B}_2 \times \mathcal{B}_3$. For each $(\beta_2, \beta_3) \in \mathcal{B}_{23}$ let

$$Stab(\beta_2, \beta_3) = Stab(\beta_2) \cap Stab(\beta_3)$$

where $Stab(\beta)$ denotes the stabilizer of β under the coadjoint action of K on $\mathfrak{k}^* \cong \mathfrak{k}$. Let $T_{\beta_2\beta_3}$ be the subtorus of T generated by β_2 and β_3, and let

$$Z_{\beta_2,\beta_3} = \{x \in M \mid T_{\beta_2,\beta_3} \text{ fixes } x \text{ and } \mu_l(x) \cdot \beta_l = \|\beta_l\|^2.\ l = 2, 3\}$$

The argument of [8] Lemma 3.15 shows the following.

Lemma A.4. *The set of critical points for f_{23} is the disjoint union of the closed subsets $\{C_{\beta_2,\beta_3} \mid (\beta_2,\beta_3) \in \mathcal{B}_{23}\}$ of M, where*

$$C_{\beta_2,\beta_3} = K(Z_{\beta_2,\beta_3} \cap {\mu_2}^{-1}(\beta_2) \cap {\mu_3}^{-1}(\beta_3)).$$

Proof. Suppose $\mathrm{grad} f_{23}(x) = 0$. By Lemma A.1 and Lemma A.2 we have

$$[\mu_2(x), \mu_3(x)] = 0$$

and $\mu_2(x)_x = 0 = \mu_3(x)_x$. Hence there exists $k \in K$ such that $\mu_2(kx)$ and $\mu_3(kx)$ both lie in $\mathfrak{t}$ and

$$\mu_2(kx)_{kx} = 0 = \mu_3(kx)_{kx}.$$

Thus we have $\mu_2(kx) = \beta_2$ and $\mu_3(kx) = \beta_3$ for some $(\beta_2,\beta_3) \in \mathcal{B}_2 \times \mathcal{B}_3$, and multiplying k by a suitable element of the normalizer of T in K we may assume that $(\beta_2,\beta_3) \in \mathcal{B}_{23}$. The result follows. □

To show that f_{23} is a minimally degenerate Morse function in the sense of §11 of [8], it is enough to find for each $(\beta_2,\beta_3) \in \mathcal{B}_{23}$ a locally closed submanifold R_{β_2,β_3} of M which contains C_{β_2,β_3} and has the following properties. R_{β_2,β_3} must be closed in a neighbourhood of C_{β_2,β_3}. Moreover there should be a smooth subbundle U of $TM|_{R_{\beta_2,\beta_3}}$ such that $TM|_{R_{\beta_2,\beta_3}} = U + TR_{\beta_2,\beta_3}$ and for each $x \in C_{\beta_2,\beta_3}$ the restriction to U_x of the Hessian H_{23} of f_{23} at x is nondegenerate. Then the normal bundle to R_{β_2,β_3} in M is isomorphic to a quotient of U and U splits near C_{β_2,β_3} as $U^+ + U^-$ where the restriction of the Hessian H_{23} to U_x^+ is positive definite and to U_x^- is negative definite for all x. One finds that the intersection of a small neighbourhood of C_{β_2,β_3} in M with the image of U^+ under the exponential map $Exp : TM \to M$ satisfies the conditions for a minimizing manifold for f_{23} along C_{β_2,β_3}. By [8] (4.21) for $l = 2,3$ we have

$$K(Z_{\beta_l} \cap \mu_l^{-1}(\beta_l)) \cong K \times_{Stab\beta_l} (Z_{\beta_l} \cap \mu_l^{-1}(\beta_l))$$

and $KZ_{\beta_l} \cong K \times_{Stab\beta_l} Z_{\beta_l}$ near $K(Z_{\beta_l} \cap \mu_l^{-1}(\beta_l)$. Therefore

$$C_{\beta_2,\beta_3} = K \times_{Stab(\beta_2,\beta_3)} (Z_{\beta_2,\beta_3} \cap \mu_3^{-1}(\beta_3)) \tag{A.3}$$

and $KZ_{\beta_2,\beta_3} = K \times_{Stab(\beta_2,\beta_3)} Z_{\beta_2,\beta_3}$ near C_{β_2,β_3}. In particular KZ_{β_2,β_3} is smooth near C_{β_2,β_3} (cf. [8] Cor 4.11). If $x \in Z_{\beta_2,\beta_3}$ then $\mu_l(x) \cdot \beta_l = \|\beta_l\|^2$ for $l = 2,3$ so $f_{23}(x) \geq \|\beta_2\|^2 + \|\beta_3\|^2$ with equality if and only if $\mu_2(x) = \beta_2$ and $\mu_3(x) = \beta_3$. Hence the restriction of f_{23} to KZ_{β_2,β_3} takes its minimum value precisely on C_{β_2,β_3}. Thus to prove that f_{23} is minimally degenerate it suffices to prove

Lemma A.5. *For any $x \in Z_{\beta_2,\beta_3} \cap \mu_2^{-1}(\beta_2) \cap \mu_3^{-1}(\beta_3)$ there is a complement U_x to $T_xKZ_{\beta_2,\beta_3}$ in T_xM varying smoothly with x such that H_{23} restricts to a nondegenerate bilinear form on U_x and $U_{sx} = s(U_x)$ for all $s \in Stab(\beta_2,\beta_3)$.*

Proof. From (A.2) it is easy to check that the subspace

$$T_xZ_{\beta_2,\beta_3} + \mathfrak{k}_x + \mathbf{j}\mathfrak{k}_x + \mathbf{k}\mathfrak{k}_x$$

of T_xM is invariant under the action of H_{23} regarded as a self-adjoint endomorphism of T_xM. Hence so is its orthogonal complement V_x say, in T_xM. If $\xi \in V_x$ then

$$d\mu_2(x)(\xi) \cdot b = \xi \cdot \mathbf{j}b_x = 0$$

for all $b \in \mathfrak{k}$, so $d\mu_2(x)(\xi) = 0$ and similarly $d\mu_3(x)(\xi) = 0$. Hence

$$\frac{1}{2}H_{23}(\xi) = \mathbf{j}\beta_2(\xi) + \mathbf{k}\beta_3(\xi)$$

so that the restriction of $\frac{1}{2}H_{23}$ to V_x coincides with the restriction of the Hessian of the function $\mu_2^{(\beta_2)} + \mu_3^{(\beta_3)}$ on M, where $\mu_l^{(\beta_l)}(x)$ is defined as $\mu_l(x) \cdot \beta_l$. But Z_{β_2,β_3} is a union of components of the set of critical points for this function. Because this function is a nondegenerate Morse function on M by the following lemma, the restriction of H_{23} to V_x is nondegenerate.

Lemma A.6. [10, Lemma 3.9] *The function $\mu^{(\beta)} := \mu_2^{(\beta_2)} + \mu_3^{(\beta_3)}$ is a nondegenerate Morse-Bott function*

Proof. The gradient of $\mu^{(\beta)}$ at $x \in X$ is

$$\mathrm{grad}\mu^{(\beta)}(x) = \mathbf{j}(\beta_2)_x + \mathbf{k}(\beta_3)_x.$$

Since $\mathbf{j}(\beta_2)_x \cdot \mathbf{k}(\beta_3)_x = \mathbf{i}(\beta_2)_x \cdot (\beta_3)_x = d\mu_1((\beta_3)_x) \cdot \beta_2 = [\beta_3, \mu_1(x)] \cdot \beta_2 = \mu_1(x) \cdot [\beta_2, \beta_3] = 0$, the set of critical points is precisely the fixed point set of the subtorus T_β which is the closure of the subgroup generated by $\exp \mathbb{R}\beta_2$ and $\exp \mathbb{R}\beta_3$. Thus every connected component is a submanifold of M. So it suffices to prove that its Hessian at any critical point is nondegenerate in directions orthogonal to the critical set. Let H_2, H_3, H be the Hessians at x for $\mu_2^{(\beta_2)}$, $\mu_3^{(\beta_3)}$ and $\mu^{(\beta)}$. Choose local coordinates near x such that T_β acts linearly and the metric is given by a matrix $\rho_{pq}(y)$ with $\rho_{pq}(x) = \delta_{pq}$. Since

$$0 = \mathbf{j}(\beta_2)_y \cdot \mathbf{k}(\beta_3)_y = \mathrm{grad}\mu_2^{(\beta_2)}(y) \cdot \mathrm{grad}\mu_3^{(\beta_3)}(y)$$

for all $y \in M$, we have

$$\sum_{p,q} \frac{\partial \mu_2^{(\beta_2)}}{\partial y_q}(y)\rho_{pq}^{-1}(y)\frac{\partial \mu_3^{(\beta_3)}}{\partial y_p}(y)$$

for all y, and since

$$\mathrm{grad}\mu_2^{(\beta_2)}(x) = 0 = \mathrm{grad}\mu_3^{(\beta_3)}(x),$$

differentiating twice gives

$$H_2H_3 + H_3H_2 = 0$$

which gives us $H^2 = (H_2 + H_3)^2 = H_2^2 + H_3^2$. If $H\xi = 0$,

$$0 = \|H\xi\|^2 = \langle H^2\xi, \xi\rangle = \langle H_2^2\xi, \xi\rangle + \langle H_3^2\xi, \xi\rangle = \|H_2\xi\|^2 + \|H_3\xi\|^2$$

and thus $H_2\xi = 0 = H_3\xi$. Therefore, if $H\xi = 0$, ξ is tangent to the fixed point set of T_β because $\mu_2^{(\beta_2)}$ and $\mu_3^{(\beta_3)}$ are nondegenerate. □

Therefore it suffices to show that there is a complement W_x to $T_xKZ_{\beta_2,\beta_3}$ in

$$T_xZ_{\beta_2,\beta_3} + \mathfrak{k}_x + \mathbf{j}\mathfrak{k}_x + \mathbf{k}\mathfrak{k}_x$$

varying smoothly with x such that H_{23} is nondegenerate on W_x and $W_{sx} = s(W_x)$ for all $s \in Stab(\beta_2, \beta_3)$. For then we may take $U_x = V_x + W_x$.

From (A.2), we obtain

$$\frac{1}{2}H_{23}(\mathbf{j}a_x) = -[\beta_2, a]_x + \mathbf{j}A_x(a)_x,$$

$$\frac{1}{2}H_{23}(\mathbf{k}a_x) = -[\beta_3, a]_x + \mathbf{k}A_x(a)_x,$$

where $A_x : \mathfrak{k} \to \mathfrak{k}$ is the self-adjoint endomorphism defined by

$$A_x(a) \cdot b = a_x \cdot b_x$$

for all $a, b \in \mathfrak{k}$. Hence, via the projection

$$\mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \longrightarrow \mathfrak{k}_x + \mathbf{j}\mathfrak{k}_x + \mathbf{k}\mathfrak{k}_x, \tag{A.4}$$

$\frac{1}{2}H_{23}$ lifts to the endomorphism of $\mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$ given by the matrix

$$\mathcal{M} = \begin{bmatrix} 0 & -Ad\beta_2 & -Ad\beta_3 \\ 0 & A_x & 0 \\ 0 & 0 & A_x \end{bmatrix} \tag{A.5}$$

acting as left multiplication on column vectors. Since $A_x(a) = 0$ implies $a_x = 0$, the image of the kernel of $\mathcal{M}$ by (A.4) is simply $\mathfrak{k}_x$, which is certainly contained in $T_x KZ_{\beta_2,\beta_3}$. Therefore, $\frac{1}{2}H_{23}$ is nondegenerate on the orthogonal complement W_x of $T_x KZ_{\beta_2,\beta_3}$ in $T_x Z_{\beta_2,\beta_3} + \mathfrak{k}_x + \mathbf{j}\mathfrak{k}_x + \mathbf{k}\mathfrak{k}_x$. Thus $U_x = V_x \oplus W_x$ is the desired family of subspaces. □

This completes the proof that f_{23} is a minimally degenerate Morse function and so we deduce that f_{23} is an equivariantly perfect Morse function if flow-closed.

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