Localization of virtual cycles by cosections

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§1. What is a curve?

$\mathbf{k} = \bar{\mathbf{k}}$, char$\mathbf{k} = 0$, $\mathbf{k}^n = \mathbf{k} \times \cdots \times \mathbf{k}$.

[1] Curves in $\mathbf{k}^n$?

(a) By ideals.

- circle of radius 1 $\Leftrightarrow x^2 + y^2 = 1$
- line through O, direction (1,2,4) $\Leftrightarrow y = 2x, z = 4x$
  But some would say $y = 2x, z = 2y$,
  or others $y + z = 6x, 8x + 2y = 3z$.
- Reconciliation? Ideal!
  They are different sets of generators of the same ideal.

Algebraic variety in $\mathbf{k}^n$

$=$ zero set of a finite number of polynomials in $\mathbf{k}[x_1, \cdots, x_n]$

$=$ zero set of an ideal in $\mathbf{k}[x_1, \cdots, x_n]$

Hilbert's Nullstellensatz says there is a 1-1 correspondence

$$\{\text{algebraic varieties in } \mathbf{k}^n\} \Leftrightarrow \{\text{radical ideals in } \mathbf{k}[x_1, \cdots, x_n]\}$$

given by $V \to I(V) = \{f \in \mathbf{k}[x_1, \cdots, x_n] \,|\, f(V) = 0\}$,

$Z(J) = \{x \in \mathbf{k}^n \,|\, f(x) = 0 \forall f \in J\} \leftarrow J$.

Def: A curve in $\mathbf{k}^n$ is an algebraic variety of dimension 1.

. $\text{tr.deg}_{\mathbf{k}}\, \mathbf{k}[x_1, \cdots, x_n]/I(V) = 1$.

(b) By maps (parameterized curves).

- line through O, direction (1,2,4) $\Leftrightarrow x = t, y = 2t, z = 4t$

- circle of radius 1 $\Leftrightarrow x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$
  But some would say $x = \cos\theta, y = \sin\theta$,
  or others $x = \cos(\log u), y = \sin(\log u)$.

- Reconciliation? Reparametrization!
  A curve should be thought of as an equivalence class of parameterized curves modulo reparametrization.

Def: A curve in $\mathbf{k}^n$ is an equivalence class of maps $f : \mathbf{k} \to \mathbf{k}^n$ modulo the automorphism group of the domain Aut($\mathbf{k}$).

(c) By modules.

$J =$ ideal of a curve $\Rightarrow M = \mathbf{k}[x_1, \cdots, x_n]/J$ is a module over the ring $\mathbf{k}[x_1, \cdots, x_n]$.

Def: A curve in $\mathbf{k}^n$ is a module M over the ring $\mathbf{k}[x_1, \cdots, x_n]$ whose associated prime ideals have dimension 1 and ...

(d) There are many other ways to define curves.

[2] Curves in $\mathbb{P}^n$?

$\mathbb{P}^n = \mathbf{k}^{n+1} - \{0\}/\mathbf{k}^* = \{(x_0 : \cdots : x_n) \,|\, \text{not all zero}\}$

$\mathbb{P}^n = \cup_{i=0}^n U_i, \quad U_i = \{\frac{x_0}{x_i} : \cdots : \frac{x_n}{x_i})\} \cong \mathbf{k}^n.$

A projective variety in $\mathbb{P}^n$ is the common zero locus of homogeneous polynomials.

(a) By ideals : A projective curve in $\mathbb{P}^n$ is defined as curves in U_i which coincide on intersections $U_i \cap U_j$, i.e. compatible ideals J_i for each i, i.e. an ideal sheaf$\mathcal{J}$.

(b) By maps : A projective curve in $\mathbb{P}^n$ is defined as an equivalence class of polynomial maps $f : C \to \mathbb{P}^n$ modulo Aut(C) where C is an abstract curve.

(c) By modules : A projective curve in $\mathbb{P}^n$ is defined as the sheaf of modules $\mathcal{O}_{\mathbb{P}^n}/\mathcal{J}$.

§2. Curve counting invariants.

Question: How many conics (=degree 2 curves) in $\mathbb{P}^2$ pass through five general points in $\mathbb{P}^2$?
Answer: 1

A conic is given by a quadratic polynomial

$$a_0z_0^2 + a_1z_1^2 + a_2z_2^2 + a_3z_0z_1 + a_4z_1z_2 + a_5z_0z_2 = 0.$$

$\{\text{conics in } \mathbb{P}^2\} = \{(a_0 : a_1 : a_2 : a_3 : a_4 : a_5)\} \cong \mathbb{P}^5$.
$\{\text{conics through a point } (z_0 : z_1 : z_2) \in \mathbb{P}^2\} = \text{hyperplane in } \mathbb{P}^5$.
$\{\text{conics passing through five general points in } P\} =$ intersection of five hyperplanes in $\mathbb{P}^5$.

How to define a curve counting invariant?

Step 1: Construct the moduli space of all curves of given numerical type

Step 2: Constraints $\Rightarrow$ cycles in the moduli space

Step 3: Find the intersection numbers of the cycles.

E.g. Step 1: $\mathbb{P}^5$.

Step 2: 5 hyperplanes.

Step 3: Intersection number=1.

Delicate issues

(1) The moduli space should be compactified!
Intersection theory is ill behaved if not compact.

(2) Want the invariant to be deformation invariant.
Remain constant under smooth deformation of the target variety.
Expected dimension $\neq$ actual dimesion of the moduli space.
Solution? Use virtual intersection theory!

Compactified moduli spaces of curves

X = fixed smooth projective variety in $\mathbb{P}^n$.

(a) By ideals: Hilbert scheme (Grothendieck 1960s)

$Hilb^f(X) = \{$ideal sheaves of $\mathcal{O}_X$ with Hilb poly $f\}$ compact

Virtual int. numbers on $Hilb^f(X)$ =: Donaldson-Thomas inv.

(b) By maps: Kontsevich moduli (Kontsevich-Manin 1994)

$\overline{\mathcal{M}}_{g,n}(X,d) = \{f : C \to X \,|\, C$ nodal genus g curve,

n marked points $p_1, \cdots, p_n, f_*[C] = d, |\mathrm{Aut}(f)| < \infty\}/\cong$ compact

$(f : C \to X) \cong (f' : C' \to X)$ iff $\exists \eta \in \mathrm{Isom}(C, C')$, $f' \circ \eta = f$.

Virtual int. numbers on $\overline{\mathcal{M}}_{g,n}(X,d)$ =: Gromov-Witten inv.

(c) By modules: Simpson moduli (C.Simpson 1994)
$Simp^f(X) = \{$semistable sheaves on X, Hilb poly $f\}/\sim$ cpt
A pure sheaf F is (semi)stable iff $\forall F' < F$, $\frac{\chi(F'(m))}{r(F')} < (\leq) \frac{\chi(F(m))}{r(F)}$.
Virtual intersection theory makes sense when X is CY ($\wedge^3 T_X \cong \mathcal{O}_X$) 3-fold and stability=semistability $\Rightarrow$ Donaldson-Thomas inv.
Joyce-Song found a generalization to s$\neq$ss case.

(d) Several other compactifications and invariants by stable quotients (Marian-Oprea-Pandharipande), stable pairs (Pandharipande-Thomas), log stable maps (Kim-Kresch-Oh) and so on.

• For $g = 0$, $d \leq 3$ and X homogenous, K. Chung will explain (in this meeting) how the compactified moduli spaces are related by explicit blow-ups. H. Moon will show us nice birational results comparing compactied moduli spaces of $M_{0,n}$.

GW=DT=PT conjecture.

• All these curve counting inv. are expected to be equivalent.

There are precise conjectures comparing these curve counting invariants: Maulik-Nekrasov-Okounkov-Pandharipande, S. Katz, Pandharipande-Thomas, ...

Wall crossing in the derived category
= key for recent progress by Toda, Bridgeland, Thomas, ...

§3. Virtual intersection theory.

Perfect obstruction theory $\Rightarrow$ virtual fund. class $\Rightarrow$ invariants

A perf obstr th on M refers to a morphism $\phi : E^\bullet \to L^\bullet_M$ in the derived category $D^b(M)$ of coherent sheaves on M such that
(i) étale locally $E^\bullet \cong$2-term complex of loc free sheaves
(ii) $h^0(\phi)$ isom and $h^{-1}(\phi)$ surjective.

If $M \hookrightarrow Y$ smooth, the cone of $\phi^\vee : (L^\bullet_M)^\vee \to [E_0 \to E_1]$ equals $T_Y|_M \hookrightarrow N_{M/Y} \oplus E_0 \to E_1$ which induces
$\mathcal{C} = C_{M/Y} \oplus E_0/T_Y|_M \hookrightarrow N_{M/Y} \oplus E_0/T_Y|_M \hookrightarrow E_1$.
Virtual fundamental class is defined as $[M]^{\mathrm{vir}} = 0^!_{E_1}[\mathcal{C}]$.
Deformation invariant if per ob th extends.

How to calculate virtual intersection numbers?

• Virtual int. number = [cohomology class] $\cap [M]^{\text{vir}}$
where M is a compactified moduli space

How to calculate virtual fundamental class?

(1) Localization by torus action
(Kontsevich, Givental, Graber-Pandharipande, ...):
If M has a torus action and $M^T = \sqcup M_i$, then

$$[M]^{\text{vir}} = \imath_* \sum \frac{[M_i]^{\text{vir}}}{e(N_i^{\text{vir}})}.$$

(2) Quantum Lefschetz and Grothedieck-Riemann-Roch
(Givental, Kim, Lian-Liu-Yau, Coates, ...):
If $X \subset \mathbb{P}^4$ is a quitic 3-fold, $\overline{\mathcal{M}}_{0,n}(X,d) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)$ is the zero locus of a section of vector bundle $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)$

$$[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)] \cap c_{top}(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)).$$

(1)+(2) gave proofs of the Mirror conjecture.

(3) Degeneration formula (J.Li)
If X degenerates to $Y_1 \cup Y_2$,
$[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}} = \sum (\mathrm{coeff}) [\overline{\mathcal{M}}^{rel}_{g_1,n_1}(Y_1,d_1)]^{\mathrm{vir}} * [\overline{\mathcal{M}}^{rel}_{g_2,n_2}(Y_2,d_2)]^{\mathrm{vir}}$.

Okounkov-Pandharipande calculated all GW invariants for curves by induction on genus using the degeneration formula.

(4) Behrend function and Milnor numbers (Behrend 2005)
If perf ob th $E^\bullet \to L_M^\bullet$ is symmetric ($\theta : E^\bullet \cong (E^\bullet)^\vee[1]$, $\theta^\vee[1] = \theta$), there is a constructible function f on M such that

$$\deg[M]^{\mathrm{vir}} = \sum_k k \cdot \chi(f^{-1}(k)).$$

(5) Localization by cosection (K.-J.Li, arXiv 1007.3085)

- For a perf obs th $[E^{-1} \xrightarrow{\alpha} E^0] \to L_M^\bullet$,

its obstruction sheaf is defined as $Ob_M = \mathrm{coker}(E_0 \xrightarrow{\alpha^\vee} E_1)$.

- If $Ob_M|_U \twoheadrightarrow \mathcal{O}_U$ for open $U \subset M$,

$[M]^{\mathrm{vir}}$ is a cycle with support in $M - U$.

Applications of Localization by Cosection.

(1) GW inv. of general type surfaces (K.-J.Li, Lee-Parker)
For a family of stable maps $f : \mathcal{C} \to X$, $\pi : \mathcal{C} \to \mathcal{M} = \overline{\mathcal{M}}_{g,n}(X,d)$ and $\omega \in H^0(X, \Omega_X^2)$, we have

$$Ob_{\mathcal{M}} = \text{coker}\left(\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/\mathcal{M}}, \mathcal{O}_{\mathcal{C}}) \to R^1\pi_* f^* T_X\right)$$

$$R^1\pi_* f^* T_X \to R^1\pi_* f^*\Omega_X \to R^1\pi_*\Omega_{\mathcal{C}/\mathcal{M}} \to R^1\pi_*\omega_{\mathcal{C}/\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}$$

This induces a cosection $Ob_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$.

Reduces the calculation to the curve where ω degenerates.
Proof of Maulik-Pandharipande formula on low deg GW inv.

(2) Proof of Katz-Klemm-Vafa conjecture which counts curves in K3 (Maulik-Pandharipande-Thomas, 2010)

Localization by cosection enables us to push the counting on K3 surface to an open CY 3-fold. Then degeneration + toric calculation prove the formula.

(3) A theory of spin curve counting (H.Chang-J.Li)

$\bar{M}_g^{1/2}$ = moduli of spin curves (C, L), $L^2 \cong \omega_C$.

Perf obs th on $\pi_*\mathcal{L}$ where $\pi : \mathcal{C} \to \bar{M}_g^{1/2}$ and $\mathcal{L}$ denote universal family. At a point $\xi = (C, L, s) \in \pi_*\mathcal{L}$, the obstruction space is $H^1(L)$ and tensoring with s gives a cosection $Ob_\xi = H^1(L) \to H^1(L^2) \cong H^1(\omega_C) = \mathbb{C}$.

(4) A wall crossing formula without Chern-Simons functional (K.-J.Li, August 2010)

$$\mathsf{deg}[M_+]^{\mathsf{vir}} - \mathsf{deg}[M_-]^{\mathsf{vir}} = (-1)^{\chi(A,B)-1} \cdot \chi(A,B) \cdot \mathsf{deg}[M(A)]^{\mathsf{vir}} \cdot \mathsf{deg}[M(B)]^{\mathsf{vir}}.$$

(5) A theory of generalized DT invariants via Kirwan blow-ups (K.-JLi, in progress) : different from Joyce-Song approach; equipped with perfect obstruction theory; wall crossing formula.

`Thank you!`