CURVE COUNTING AND BIRATIONAL GEOMETRY OF COMPACTIFIED MODULI SPACES OF CURVES

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1. Introduction

Although curve counting has been studied for thousands of years, it still remains an active area of research. Recent introduction of virtual intersection theory by Li-Tian ([37]) and Behrend-Fantechi ([3, 4]) provided an efficient way of constructing a curve counting invariant which remains constant under deformation. Examples include Gromov-Witten invariants, Donaldson-Thomas invariants, and Pandharipande-Thomas invariants. All these invariants are defined as the virtual intersection numbers on compactified moduli spaces of curves in a given smooth projective variety X.

The purpose of this paper is to discuss two major issues for curve counting invariants.

(1) Comparison of the compactified moduli spaces of curves.
(2) Methods of calculating virtual intersection numbers.

For item (1), the moduli space of smooth curves in X can be compactified by taking closures in the Hilbert scheme, the stable map space, or the moduli space of semistable sheaves. We will see recent comparison results of these compactifications in two directions; one by Mori theory and the other by elementary modification.

For item (2), we will itemize all known methods for calculating virtual intersection numbers. Special emphasis will be laid on the method of localization by cosection due to J. Li and the author ([24]). Various applications are discussed as well.

The layout of this paper is as follows. Quick introductions on curve counting invariants and virtual intersection theory are given in §2 and §4 respectively. In §3, we recall several well-known compactified moduli spaces of curves. In §5, we state the comparison problem for rational curves and survey Mori theoretic results. In §6, comparison results by elementary modification are discussed. In §7, a closely related problem of comparing compactifications of $M_{0,n}$ is discussed.

2. Curve counting invariants

In this paper, all schemes are defined over $\mathbb{C}$.

To motivate, let us start with a well-known classical example.

2.1. **Example.** Find the number of lines in a quintic hypersurface X in $\mathbb{P}^4$.

We can solve this problem in three steps.

Step 1. The space of all lines in $\mathbb{P}^4$ is the Grassmannian $Gr(2,5)$ of two dimensional subspaces in $\mathbb{C}^5$. If we denote the universal rank 2 bundle over $Gr(2,5)$ by $\mathcal{U}$, we have the following diagram

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^4}(5) \arrow[d] \\ \mathbb{P}\mathcal{U} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}^4 \\ Gr(2,5). & \end{tikzcd}$$

The quintic hypersurface X is defined as the vanishing of a section s of $\mathcal{O}_{\mathbb{P}^4}(5)$ and $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ is a vector bundle of rank 6. The section s induces a section $\tilde{s}$ of $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$.

Step 2. The constraint that a line l in $\mathbb{P}^4$ lies in X is equivalent to l, as a point $Gr(2,5)$, lies in $\tilde{s}^{-1}(0)$.

Step 3. The number of lines in X is the number of points in $\tilde{s}^{-1}(0)$ which is

$$0^![\tilde{s}] = [Gr(2,5)] \cap c_6(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)) = 2875$$

by Grothendieck-Riemann-Roch and Schubert calculus.

2.2. **Defining a curve counting invariant.** As this example suggests, a curve counting invariant is defined in three steps:

(1) construct moduli of curves;
(2) find cycles for constraints;
(3) find intersection numbers of the cycles of constraints.

2.3. **Two issues.** In actually defining curve counting invariants, there arise several delicate issues. Let me mention two.

(1) Intersection numbers are not well defined if the moduli space is not compact. Here compact means proper and separated. Hence we need to find a *compactified moduli space* of curves. But there are many ways to compactify and hence we may get many curve counting invariants. So the first issue is to

compare compactified moduli spaces and
the intersection numbers on them.

(2) We expect the curve counting invariant to be deformation invariant. In other words, if we have a smooth family of projective varieties

$$\begin{tikzcd} X_t \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ t \arrow[r, hook] & T \end{tikzcd}$$

we want the invariant of X_t to be independent of t. There is a standard way of achieving deformation invariance: a *perfect obstruction theory* on the moduli space M enables us to define a *virtual fundamental cycle* which makes it possible to define *virtual intersection numbers* as integration over the virtual fundamental cycle of cohomology classes. (These will be explained later.) A natural choice of perfect obstruction theory on X_t extends to the family $\mathcal{X}$ and deformation invariance is obtained automatically from the construction of a virtual fundamental cycle. By now this construction is standard but the calculation of virtual intersection numbers is very difficult. So the second issue is to find good techniques to

calculate the virtual intersection numbers.

3. Compactified moduli spaces

In this setion we recall known compact moduli spaces of curves with perfect obstruction theory. Throughout this section, X is a fixed smooth projective variety with fixed embedding $X \subset \mathbb{P}^r$. Let P be a fixed linear polynomial. There are various perspectives to think of a curve which give us different compactifications.

3.1. **Hilbert scheme.** One can think of a smooth curve C in X as the ideal sheaf I_C of defining equations of C. In 1960s, Grothendieck proved that the collection of ideals sheaves with fixed Hilbert polynomial P form a projective scheme $Hilb^P(X)$ and called it the Hilbert scheme. Furthermore, there is a universal family $\mathcal{C} \hookrightarrow Hilb^P(X) \times X$, flat over X. In his thesis [52] and in [53], R. Thomas proved that if the canonical divisor of X is antiample or trivial and $\dim X \leq 3$, $Hilb^P(X)$ admits a perfect obstruction theory and thus obtained virtual intersection numbers on $Hilb^P(X)$ are called the *Donaldson-Thomas invariants.*

3.2. **Stable map space.** One can think of a smooth curve C in X as a map $f : C \to X$ from an abstract curve C to X modulo reparametrization. Two curves $f : C \to X$ and $f' : C' \to X$ are said to be equivalent if they are related by reparameterizations, i.e. there exists an isomorphism $\eta : C \to C'$ such that $f' \circ \eta = f$. In [32], Kontsevich and Manin showed that the collection of all morphisms $f : C \to X$ with $p_i \in C$ satisfying

(1) C is a nodal curve of genus g and $p_1, \cdots, p_n$ are distinct smooth points of C;
(2) $f_*[C] = d \in H_2(X, \mathbb{Z})$;

(3) the automorphism group of f fixing the marked points p_i is finite

form a proper separated Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X,d)$. Subsequently, Li-Tian ([37]) and Behrend-Fantechi ([3, 4]) showed that there is a natural perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,d)$. The virtual intersection numbers on $\overline{\mathcal{M}}_{g,n}(X,d)$ are called the *Gromov-Witten invariants*.

3.3. **Stable sheaf space.** One can think of a smoooth curve C in X as a module $\mathcal{O}_C$ of $\mathcal{O}_X$. We say a coherent sheaf F on X is *pure* if for every nontrivial subsheaf F', $\dim F' = \dim F$. A pure sheaf F is called *stable* if for every nontrivial $F' \leq F$,

$$\frac{\chi(F'(m))}{r(F')} < \frac{\chi(F(m))}{r(F)} \quad \text{for } m >> 0$$

where $r(F)$ is the leading coefficient of the Hilbert polynomial $\chi(F(m))$. We get semistability if $<$ is replaced by $\leq$ above. In [49], C. Simpson proved that the collection of semistable sheaves modulo S-equivalence[1] with fixed Hilbert polynomial P form a projective scheme $Simp^P(X)$. Thomas proved in [53] that when $K_X \leq 0$, $\dim X \leq 3$ and stability coincides with semistability, there exists a natural perfect obstruction theory and thus we obtain virtual intersection numbers which are also called the *Donaldson-Thomas invariants.*

Recently, Joyce-Song ([21]) generalized the notion of Donaldson-Thomas invariants to the case where stability is different from semistability.

3.4. **Stable pair space.** One can think of a curve C in X as a coherent sheaf $\mathcal{O}_C$ together with a section $\mathcal{O}_X \to \mathcal{O}_C$ which is the restriction morphism. By Le Potier's result in [35], the collection of all pairs (F,s) satisfying

(1) F is a purely 1-dimensional sheaf on X with Hilbert polynomial P and $s \in H^0(F)$;
(2) $\dim \operatorname{coker}(s) = 0$

form a projective scheme $Pairs^P(X)$. In [48], Pandharipande-Thomas showed that when X is a Calabi-Yau 3-fold, there is a perfect obstruction theory and thus obtained virtual intersection numbers are called the *Pandharipande-Thomas invariants.*

3.5. **Stable quotient space.** One can think of a curve C in Grassmannian $Gr(k,r)$ as a vector bundle F of rank $r-k$ with an epimorphism $\mathcal{O}_C^{\oplus r} \to F$. It was proved by Marian-Oprea-Pandharipande in [40] that the collection of (C, p_i, Q, q) satisfying

(1) C is nodal curve and $p_1, \cdots, p_n$ are smooth points of C;
(2) Q is a coherent sheaf of degree d and rank $r-k$ on C and $q : \mathcal{O}_C^{\oplus r} \to Q$ is an epimorphism;

[1] Two semistable sheaves are S-equivalent if the Jordan-Hölder factors and the multiplicities are the same.

(3) Q is locally free at nodes and marked points;
(4) $\omega_C(\sum p_i) \otimes \det Q$ is ample

form a proper separated Deligne-Mumford stack $\overline{Q}_{g,n}(Gr(k,r),d)$. They also proved that $\overline{Q}_{g,n}(Gr(k,r),d)$ admits a perfect obstruction theory and thus obtained virtual intersection numbers are called the *Marian-Oprea-Pandharipande invariants*.

3.6. **Comparison problem.** All the above curve counting invariants are expected to be equivalent. The equivalence of Gromov-Witten and Donaldson-Thomas (with Hilbert scheme) has been proved for toric 3-folds in [41]. The equivalence of Donaldson-Thomas and Pandharipande-Thomas was proved by Toda ([54]) and Bridgeland ([5]) by using the wall crossing in the derived category and Hall algebra.

However for full comparision, it seems that we need

(1) comparison of compactified moduli spaces;
(2) comparison of virtual intersection numbers under certain birational transformations (wall crossing formula).

4. Virtual intersection theory

In this section, we discuss the issue of calculation method of virtual intersection numbers. As mentioned above, virtual intersection numbers are defined through

perfect obstruction theory $\Rightarrow$ virtual fundamental class $\Rightarrow$ invariants.

Recall that a Deligne-Mumford stack M is equipped with a cotangent complex $L^\bullet_M$ which retains all information about the deformation theory of M ([20]).

4.1. **Definition.** A *perfect obstruction theory* on a Deligne-Mumford stack M refers to a morphism

$$\phi : E^\bullet \to L^\bullet_M$$

in the derived category $D^b(M)$ of coherent sheaves on M such that

(i) étale locally $E^\bullet \cong$ 2-term complex $[E^{-1} \to E^0]$ of locally free sheaves
(ii) $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

If we let $(E^\bullet)^\vee = [E_0 \to E_1]$ denote the dual of $E^\bullet$, (ii) implies that $H^0((E^\bullet)^\vee)$ is the Zariski tangent sheaf of M and $H^1((E^\bullet)^\vee)$ is an obstruction sheaf of M.

4.2. **Virtual fundamental cycle.** Let $M \hookrightarrow Y$ be an étale local embedding into a smooth variety. The cone of

$$\phi^\vee : (L^\bullet_M)^\vee \to [E_0 \to E_1]$$

equals

$$T_Y|_M \hookrightarrow N_{M/Y} \oplus E_0 \to E_1$$

which induces

$$\mathcal{C} = C_{M/Y} \oplus E_0/T_Y|_M \hookrightarrow N_{M/Y} \oplus E_0/T_Y|_M \hookrightarrow E_1.$$

The virtual fundamental class is now defined as

$$[M]^{\mathrm{vir}} = 0^!_{E_1}[\mathcal{C}].$$

This local construction can be globalized by either finding a global two term complex of locally free sheaves $E^\bullet$ or using the intersection theory on Artin stacks developed by Kresch in [33]. Obviously, when M is smooth and $E^\bullet \cong L^\bullet_M = \Omega_M$, $[M]^{\mathrm{vir}}$ is the ordinary fundamental class $[M]$.

Virtual intersection numbers on M are defined as

$$[M]^{\mathrm{vir}} \cap (\text{cycles}).$$

Suppose $M = M_0$ is a fiber of a smooth family $\mathfrak{M} \to T$ and $0 \in T$. If the perfect obstruction theory extends to $\mathfrak{M}$, the virtual intersection numbers on M_t are independent of t.

4.3. **Methods of calculation.** Calculating of virtual intersection numbers is usually very hard. There are only a few methods of calculation and each of the methods has limited range of applications. Hence new techniques should be developed. The following is a list of known methods of calculation.

(1) Localization by torus action (Kontsevich [31], Givental [16]), Graber-Pandharipande [17]): If M has a torus action and $M^T = \sqcup M_i$, then

$$[M]^{\mathrm{vir}} = \imath_* \sum \frac{[M_i]^{\mathrm{vir}}}{e(N_i^{\mathrm{vir}})}.$$

This method is very effective for Gromov-Witten invariants of toric varieties or homogeneous varieties.

(2) Quantum Lefschetz and Grothedieck-Riemann-Roch (Givental [16], Kim [28], Lian-Liu-Yau [38], Coates [10]): For example, if $X \subset \mathbb{P}^4$ is a quitic 3-fold, $\overline{\mathcal{M}}_{0,n}(X,d) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)$ is the zero locus of a section of vector bundle $\pi_* f^*\mathcal{O}_{\mathbb{P}^4}(5)$ on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)$

$$[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4,d)] \cap c_{top}(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)).$$

The combination of (1) and (2) gave proofs of the Mirror conjecture by Givental ([16]) and Lian-Liu-Yau ([38]).

(3) Degeneration formula (J.Li [36]): If X degenerates to $Y_1 \cup Y_2$, the virtual invariants of X can be calculated from those of Y_1 and Y_2. For Gromov-Witten invariants, the formula is of the form

$$[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}} = \sum (\text{coeff})[\overline{\mathcal{M}}^{rel}_{g_1,n_1}(Y_1,d_1)]^{\mathrm{vir}} * [\overline{\mathcal{M}}^{rel}_{g_2,n_2}(Y_2,d_2)]^{\mathrm{vir}}.$$

Okounkov-Pandharipande ([45, 46, 47]) calculated all Gromov-Witten invariants for curves by induction on genus using the degeneration formula.

(4) Behrend function and Milnor numbers (Behrend [2]): If the perfect obstruction theory $E^\bullet \to L^\bullet_M$ on M is symmetric in the sense that there exists an isomorphism $\theta : E^\bullet \cong (E^\bullet)^\vee[1]$ with $\theta^\vee[1] = \theta$, there is a constructible function f on M such that

$$\deg\, [M]^{\mathrm{vir}} = \sum_k k \cdot \chi(f^{-1}(k)).$$

When a Deligne-Mumford stack M has a symmetric obstruction theory, Behrend showed that étale locally M is given by the vanishing locus of an 1-form $\alpha \in H^0(Y, \Omega_Y)$ for some smooth variety Y. If α is the differential of a regular function ξ on Y, we say ξ is a local Chern-Simons functional. If such ξ exists, Behrend's function f can be interpreted in terms of the Milnor numbers of ξ. Joyce-Song ([21]) used this to prove a wall crossing formula for Donaldson-Thomas invariants.

(5) Localization by cosection ([24]): For a perfect obstruction theory

$$[E^{-1} \xrightarrow{\alpha} E^0] \to L^\bullet_M,$$

its obstruction sheaf is defined as

$$Ob_M = \mathrm{coker}(E_0 \xrightarrow{\alpha^\vee} E_1).$$

If there exists a cosection

$$Ob_M|_U \twoheadrightarrow \mathcal{O}_U$$

of the obstruction sheaf over an open set $U \subset M$, $[M]^{\mathrm{vir}}$ is a cycle with support in $M - U$.

The key point is that the virtual normal cone $C|_U$ in $E_1|_U$ lies in the kernel of

$$E_1|_U \to Ob_U \to \mathcal{O}_U.$$

Because the trivial bundle has a nowhere vanishing section, we find that $0^!_{E_1}[C] = 0$ on U.

4.4. **Applications of localization by cosection.** Recently the method of localization by cosection found several interesting applications as listed below.

(1) Gromov-Witten invariants of general type surfaces ([23], Lee-Parker [34]): For a family of stable maps

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ \mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, d) & \end{tikzcd}$$

and $\omega \in H^0(X, \Omega^2_X)$, we have

$$Ob_{\mathcal{M}} = \mathrm{coker}\left(\mathcal{E}xt^1_\pi(\Omega_{\mathcal{C}/\mathcal{M}}, \mathcal{O}_{\mathcal{C}}) \to R^1\pi_* f^* T_X\right).$$

The cosection $Ob_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ is induced from the following natural homomorphisms

$$R^1\pi_*f^*T_X \to R^1\pi_*f^*\Omega_X \to R^1\pi_*\Omega_{\mathcal{C}/\mathcal{M}} \to R^1\pi_*\omega_{\mathcal{C}/\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}$$

This cosection is surjective on the locus where ω is nondegenerate. In particular, if X is holomorphic symplectic, all Gromov-Witten invariants vanish.

The degeneracy locus of ω is usually a curve and hence the calculation is reduced to the curve case which is in principle well understood thanks to the work of Okounkov-Pandharipande ([45, 46, 47]). As a consequence, we proved Maulik-Pandharipande's formula on low degree Gromov-Witten invariants.

(2) Proof of Katz-Klemm-Vafa conjecture which counts curves in a K3 surface (Maulik-Pandharipande-Thomas [42]): The method of localization by cosection enables us to push the counting on a K3 surface to an open CY 3-fold. Then degeneration plus toric calculation (GW=PT) prove the formula.

(3) A theory of spin curve counting (H.Chang-J.Li): Let $\overline{M}_g^{1/2}$ be the moduli of spin curves (C, L) where C is a quasi-stable curve of genus g and L is a line bundle satisfying $L^2 \cong \omega_C$. Let $\mathcal{M}$ be the moduli space of triples (C, L, s) where $(C, L) \in \overline{M}_g^{1/2}$ and $s \in H^0(C, L)$. Since $\mathcal{M}$ is not compact, virtual intersection theory may not seem well defined. However $\overline{M}_g^{1/2}$ is certainly contained in $\mathcal{M}$ by taking $s = 0$ and is compact. So if we can show that virtual fundamental cycle $[\mathcal{M}]^{\mathrm{vir}}$ has support in $\overline{M}_g^{1/2}$, it is still possible to define the virtual intersection numbers on $\mathcal{M}$.

A perfect obstruction theory on $\mathcal{M}$ is obtained as follows. For $(C, L) \in \overline{M}_g^{1/2}$, the obstruction is trivial and the only obstruction for the triple (C, L, s) arises from the deformation of s whose obstruction is $H^1(C, L)$. Tensoring with s gives a cosection

$$Ob_{(C,L,s)} = H^1(L) \to H^1(L^2) \cong H^1(\omega_C) = \mathbb{C}.$$

By Serre duality, this is zero if and only if $s = 0$. Hence the virtual fundamental cycle has support in the compact $\overline{M}_g^{1/2}$ and thus defined are the spin curve counting invariants.

(4) A wall crossing formula without local Chern-Simons functionals for Donaldson-Thomas type invariants ([25]): When the stability condition varies, the moduli space of stable objects in the derived category on a Calabi-Yau 3-fold varies from M_- to M_+, both of which carry symmetric obstruction theories. Suppose the wall is simple in the sense that $M_- - M_+$ (resp. $M_+ - M_-$) is the locus of nonsplit extensions of objects in $M(A)$ (resp. $M(B)$) by objects in $M(B)$ (resp. $M(A)$) where $M(A)$ and $M(B)$ are moduli spaces for which stabilities coincide with semistabilities. Now the difference in the virtual

degrees is

$$\deg[M_+]^{\rm vir} - \deg[M_-]^{\rm vir} = (-1)^{\chi(A,B)-1}\cdot\chi(A,B)\cdot\deg[M(A)]^{\rm vir}\cdot\deg[M(B)]^{\rm vir}.$$

This formula was proved by Joyce-Song for moduli of sheaves using the existence of Chern-Simons functional in [21]. However for derived category objects, it is not known whether a Chern-Simons functional exists or not. Since we removed the use of Chern-Simons functionals, we can apply this wall crossing formula for moduli of derived category objects.

For the proof, we constructed an intrinsic blow-up of Deligne-Mumford stacks with $\mathbb{C}^*$ action. Then we constructed a master space for the variation of stability condition on the intrinsic blow-up to which the method of localization by $\mathbb{C}^*$ action is applied. This gives us the wall crossing formula. We believe that these techniques should be useful for more general wall crossing formulas and generalized Donaldson-Thomas invariants.

5. Comparison problem of compactified moduli spaces of curves

In this section we discuss some comparison results of compactified moduli spaces of curves in a smooth projective variety. It should be mentioned on the outset that we are at the beginning stage of understanding the relationships of the compactified moduli spaces. In the absence of general strategy, it seems reasonable to restrict our concern to the simple case of genus 0 and X a homogeneous projective variety.

5.1. **Notation.** Let $X \subset \mathbb{P}^r$ be a smooth projective variety with fixed embedding. Let R be the space of all smooth rational curves of degree d in X. For $d \geq 2$, R is not compact and the following are natural compactifications:

(1) R can be thought of as a subvariety of the Hilbert scheme $Hilb^{dx+1}(X)$ and upon taking its closure we obtain a compactification

$$\mathbf{H} := \overline{R} \subset Hilb^{dx+1}(X)$$

which we call the Hilbert compactification.

(2) R can be thought of as a subvariety of the stable map space $\overline{\mathcal{M}}_{0,0}(X,d)$ and its closure

$$\mathbf{M} := \overline{R} \subset \overline{\mathcal{M}}_{0,0}(X,d)$$

is a compactification called the Kontsevich compactification.

(3) R can be thought of as a subvariety of the moduli space of stable sheaves $Simp^{dx+1}(X)$ and its closure

$$\mathbf{S} := \overline{R} \subset Simp^{dx+1}(X)$$

is a compactificaiton called the Simpson compactification.

5.2. **Problem.** Compare **H**, **M**, and **S** explicitly.

There are of course other compactifications obtained by taking closures in

(1) the Chow scheme of 1-dimensional subschemes of degree d;
(2) the moduli space of stable pairs of Pandharipande-Thomas;
(3) the moduli space of stable quotients of Marian-Oprea-Pandharipande when X is a Grassmannian.

But let us focus on the comparison of **H**, **M**, and **S** because they admit perfect obstruction theories and seem more fundamental.

There are at least two approaches to this problem:

(1) Mori theoretic approach;
(2) elementary modification (and GIT) approach.

5.3. **Mori theoretic approach.** Mori theory studies the birational geometry of a variety by

(1) finding the effective cone, chamber structure, nef cone and moving cone;
(2) finding log minimal models for each chamber.

In [6], D. Chen studied the case of $X = \mathbb{P}^3$ and $d = 3$ so that $\mathbf{M} = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is a 12-dimensional variety. The Picard number of $\mathbf{M}$ is 2 and $\mathrm{Pic}(X) \otimes \mathbb{Q}$ is generated by divisors

$$H = \{f : C \to \mathbb{P}^3 \,|\, f(C) \cap \text{fixed line} \neq \emptyset\},$$
$$\Delta = \{f : C \to \mathbb{P}^3 \,|\, C \text{ is reducible}\}.$$

Then the effective cone of $\mathbf{M}$ is generated by Δ and $H - \frac{1}{2}\Delta$. The nef cone is generated by H and $H + \Delta$. The moving cone is generated by $H + \Delta$ and $H - \frac{1}{5}\Delta$. Let

$$\mathbf{M}(\alpha) = \mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(\mathbf{M}, m(H + \alpha\Delta))\right).$$

Chen proved that

$$\mathbf{M}(\alpha) = \begin{cases} \mathbf{M} & 0 < \alpha < 1 \\ \mathbf{H} & -\frac{1}{5} < \alpha < 0 \end{cases}$$

In particular, **H** is a log flip of **M**. Then Chen could show that **M** is a Mori dream space, which means we can find all birational morphisms from **M** once we know the Cox ring of **M** via variation of GIT by the work of Hu-Keel ([19]). However, we do not know how to find the Cox ring of **M**.

As a consequence, the two chambers of the moving cone of **M** separated by the ray of H give us **M** and **H** respectively. By a theorem in [26], the chamber generated by Δ and $H + \Delta$ gives the quasi-map space

$$\mathbb{P}(\mathbb{C}^4 \otimes \mathrm{Sym}^3(\mathbb{C}^2))//SL(2)$$

where $SL(2)$ acts on $\mathrm{Sym}^3(\mathbb{C}^2)$ in the standard fashion.

In [7], D. Chen and I. Coskun studied the case of $X = Gr(2,4)$ and $d = 2$. They proved that $\mathbf{M}$ and $\mathbf{H}$ are related by a blow-up followed by a blow-down and $\mathbf{H}$ can be further blown down to the Grassmannain $Gr(3,6)$.

$$\begin{tikzcd}
 & \tilde{\mathbf{M}} \arrow[dl] \arrow[dr] & & \\
\mathbf{M} & & \mathbf{H} \arrow[dr] & \\
 & & & Gr(3,6)
\end{tikzcd}$$

Note that the structure sheaves of all conics are stable sheaves and hence $\mathbf{H} \cong \mathbf{S}$ when $d = 2$.

In the next section, we will consider the other approach of using elementary modificaiton and GIT.

6. Elementary modification Approach

In this section we compare $\mathbf{M}$, $\mathbf{H}$ and $\mathbf{S}$ when $X \subset \mathbb{P}^r$ is a homogeneous projective variety and $d \leq 3$. We let $r \geq 3$. Note that when $d = 1$, R is already projective and hence all the compactifications coincide with R.

6.1. **Degree 2 case.** Suppose first of all that $X = \mathbb{P}^r$. As mentioned above, for $d = 2$, $\mathbf{H} \cong \mathbf{S}$. It is an elementary fact that every conic curve $C \in \mathbf{H}$ is contained in a unique plane $\mathbb{P}^2$. Hence $\mathbf{H}$ is $\mathbb{P}(\mathrm{Sym}^2\mathcal{U})$ where $\mathcal{U}$ is the universal rank 3 bundle over the Grassmannian $Gr(3, r+1)$. In particular, $\mathbf{H}$ is a $\mathbb{P}^5$-bundle over $Gr(3, r+1)$. The locus of conics supported on lines is the locus of rank 1 symmetric forms in $\mathbb{P}(\mathrm{Sym}^2\mathcal{U})$ which is exactly a $\nu_2(\mathbb{P}^2)$-bundle over $Gr(3,6)$ where $\nu_2(\mathbb{P}^2)$ is the Veronese surface in $\mathbb{P}^5$. Hence the blow-up of $\mathbf{H}$ along the locus of linear conics is

$$\tilde{\mathbf{M}} := CC(\mathbb{P}\mathcal{U})$$

over $Gr(3, r+1)$ where $CC(\mathbb{P}\mathcal{U})$ denotes the fiber bundle over $Gr(3, r+1)$ whose fibers are the variety of complete conics $CC(\mathbb{P}^2)$ which is the blow-up of $\mathbb{P}^5$ along the Veronese surface.

On the other hand, by GIT methods, in [22] the author proved that $\mathbf{M}$ is the partial desingularization of the quasi-map space

$$Q = \mathbb{P}(\mathbb{C}^{r+1} \otimes \mathrm{Sym}^2(\mathbb{C}^2))/\!/SL(2)$$

which means in this case that $\mathbf{M}$ is the blow-up of Q along the fixed point locus $\mathbb{P}^r$ of the subgroup $\mathbb{C}^*$ of diagonal matrices. Then $\mathbf{M}$ has only $\mathbb{Z}_2$-quotient singularity along the locus of stable maps to lines. It was further proved that if we blow up along this locus of stable maps with linear image, we obtain $\tilde{\mathbf{M}} = CC(\mathbb{P}\mathcal{U})$. In summary, we find that $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{S}$ are related by the following diagram of blow-ups.

$$\begin{tikzcd}[column sep=small] & & \tilde{\mathbf{M}} = CC(\mathbb{P}\mathcal{U}) \arrow[dl] \arrow[dr] & \\ & \mathbf{M} \arrow[dl] & & \mathbf{H} \cong \mathbf{S}. \\ Q & & & \end{tikzcd}$$

For general homogeneous projective varieties $X \subset \mathbb{P}^r$, Chung, Hong and the author observed that the diagram restricts to X so that we still find $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{S}$ are related by the following diagram of blow-ups ([8]).

$$\begin{tikzcd} & \tilde{\mathbf{M}} \arrow[dl] \arrow[dr] & \\ \mathbf{M} & & \mathbf{H} \end{tikzcd}$$

The blow-up centers are the loci of linear image or linear support.

6.2. **Degree 3 case: Comparison of H and S.** For the degree 3 case, we first compare $\mathbf{H}$ and $\mathbf{S}$. Our result is the following.

__**Theorem**__. ([9, 8]) For $d = 3$ and a homogeneous projective variety $X \subset \mathbb{P}^r$, there exists a morphism $\psi : \mathbf{H} \to \mathbf{S}$ which is the smooth blow-up along Δ_S which is the (smooth) locus of stable sheaves which are planar (i.e. the support is contained in a plane).

When $X = \mathbb{P}^3$, it is easy to check that Δ_S is a divisor and hence $\mathbf{H} \cong \mathbf{S}$ which is a result proved by Freirmuth-Trautmann by a very different method.

Here comes an idea of the proof. The universal family $\mathcal{Z} \subset \mathbf{H} \times X$ defines a family of sheaves $\mathcal{O}_{\mathcal{Z}}$ on $\mathbf{H} \times X$ which is flat over $\mathbf{H}$. Therefore we have a birational map

$$\mathbf{H} \dashrightarrow \mathbf{S} \qquad C \mapsto \mathcal{O}_C.$$

__Step 1: Finding undefined locus__. The locus of unstable sheaves is a smooth divisor Δ_H and each point C in Δ_H is a singular planar cubic C together with a torsion at the singular point p. In particular, for $C \in \Delta_H$, we have an exact sequence

$$0 \to \mathbb{C}_p \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0. \tag{6.1}$$

These quotient sheaves $\mathcal{O}_{C'}$ form a flat family A over Δ_H.
__Step 2: Elementary modification__. We then apply the elementary modification

$$\mathcal{F} := \ker(\mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}}|_{\Delta_H \times X} \to A)$$

over $\mathbf{H} \times X$ and check that $\mathcal{F}$ is now a flat family of stable sheaves. The effect of elementary modification is the interchange of the sub and quotient sheaves so that (6.1) becomes an exact sequence

$$0 \to \mathcal{O}_{C'} \to F \to \mathbb{C}_p \to 0.$$

For instance, if C' is a nodal cubic plane curve with node at p, then the result of the elementary modification is $\nu_* \mathcal{O}_{\tilde{C}'}$ where $\nu : \tilde{C}' \to C'$ is the normalization of the node. Hence we obtain a morphism $\psi : \mathbf{H} \to \mathbf{S}$.
Step 3: Analysis of ψ. We can easily check that Δ_H maps to Δ_S which is the locus of stable sheaves with planar support. This is actually a projective bundle: We forget the normal direction at the singular point. Furthermore we can check that the normal bundle $N_{\Delta_H/\mathbf{H}}$ restricted to a projective fiber is $\mathcal{O}(-1)$. The Fujiki-Nakano criterion for blow-ups ([15]) then guarantees that ψ is a blow-up along Δ_S.

6.3. **Degree 3 case: Comparison of M and S.** For any family of stable maps

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathbf{f}"] \arrow[d, "\pi"'] & X \\ Z & \end{tikzcd}$$

parameterized by a reduced scheme Z, we can associate a coherent sheaf $(\mathrm{id}_Z \times \mathbf{f})_* \mathcal{O}_{\mathcal{C}}$ on $Z \times \mathbb{P}^n$, flat over Z. Hence we have a sheaf $\mathcal{E}_0$ on $\mathbf{M} \times X$ which is flat over $\mathbf{M}$. So we have a birational map

$$\mathbf{M} \dashrightarrow \mathbf{S} \qquad (f : C \to X) \mapsto f_* \mathcal{O}_C.$$

If $f : \mathbb{P}^1 \to L \subset \mathbb{P}^n$ is a d-fold covering onto a line L, then the direct image sheaf $f_* \mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^{d-1}$ is unstable. Our strategy for decomposing the birational map $\mathbf{M} \dashrightarrow \mathbf{S}$ into blow-ups and -downs is as follows:

(1) Find the locus of unstable sheaves in $\mathbf{M}$.
(2) Blow up a component of the indeterminacy locus and apply elementary modification.
(3) Repeat (2) until all sheaves become stable so that we have a diagram

$$\mathbf{M} \leftarrow \widetilde{\mathbf{M}} \to \mathbf{S}.$$

(4) Factorize the morphism $\widetilde{\mathbf{M}} \to \mathbf{S}$ into a sequence of blow-ups.

For the last item, a very useful tool is the variation of GIT quotients by Thaddeus ([51]) and Dolgachev-Hu ([11]).

The locus of unstable sheaves has now two irreducible components

$$\Gamma^1 = \{ f \in \mathbf{M} \,|\, \mathrm{im}(f) \text{ is a line} \},$$

$$\Gamma^2 = \{ f \in \mathbf{M} \,|\, \mathrm{im}(f) \text{ is a union of two lines} \}.$$

For a point $f \in \Gamma^1$, $f_* \mathcal{O}_C = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^2$ and the normal space of Γ^1 to $\mathbf{M}$ at f is canonically

$$\mathrm{Hom}(\mathbb{C}^2, \mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1))).$$

Let $\pi_1 : \mathbf{M}_1 \to \mathbf{M}$ be the blow-up along Γ^1. The destabilizing quotients $f_*\mathcal{O}_C = \mathcal{O}_L \oplus \mathcal{O}_L(-1)^2 \to \mathcal{O}_L(-1)^2$ form a flat family A over the exceptional divisor Γ^1_1 of π_1 and so we can apply the elementary modification

$$\mathcal{E}_1 = \ker((\pi_1 \times \mathrm{id}_X)^*\mathcal{E}_0 \to (\pi_1 \times \mathrm{id}_X)^*\mathcal{E}_0|_{\Gamma^1_1 \times X} \to A).$$

By direct calculation, we find that the locus of unstable sheaves in $\mathcal{E}_1$ still has two irreducible components. One is the proper transform Γ^2_1 of Γ^2 and the other is a subvariety Γ^3_1 of the exceptional divisor Γ^1_1 which is the $\mathbb{P}\mathrm{Hom}_1(\mathbb{C}^2, \mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1)))$-bundle over Γ^1 where

$$\mathbb{P}\mathrm{Hom}_1(\mathbb{C}^2, \mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1))) \cong \mathbb{P}^1 \times \mathbb{P}^{k-1}$$

is the locus of rank 1 homomorphisms with $k = \dim \mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L(-1))$.

Let $\pi_2 : \mathbf{M}_2 \to \mathbf{M}_1$ be the blow-up along Γ^2_1. Apply elementary modification with respect to the first term in the Harder-Narasimhan filtration along the exceptional divisor Γ^2_2 to obtain a family $\mathcal{E}_2$ of sheaves on $\mathbf{M}_2 \times X$. Let Γ^j_2 be the proper transform of Γ^j_1 for $j = 1, 3$. It turns out that the locus of unstable sheaves in $\mathcal{E}_2$ is precisely Γ^3_2. We repeat the same. Let $\pi_3 : \mathbf{M}_3 \to \mathbf{M}_2$ be the blow-up along Γ^3_2 and apply elementary modification along the exceptional divisor Γ^3_3. After this, all sheaves become stable and so we obtain a morphism $\mathbf{M}_3 \to \mathbf{S}$.

To analyze the morphism $\mathbf{M}_3 \to \mathbf{S}$, we keep track of analytic neighborhoods of Γ^1 and Γ^2 through the sequence of blow-ups (and -downs). It turns out that the local geometry is completely determined by variation of GIT quotients. For instance, a neighborhood of Γ^1_1 is a bundle over the space of lines in X with fiber the geometric invariant theory quotient of $\mathcal{O}_{\mathbb{P}^7 \times \mathbb{P}^{2k-1}}(-1, -1)$ by $SL(2)$ with respect to the linearization $\mathcal{O}(1, \alpha)$ for $0 < \alpha << 1$. As we vary α from 0^+ to ∞, the GIT quotient goes through two flips, or two blow-ups followed by two blow-downs. The two blow-ups correspond to our two blow-ups $\mathbf{M}_3 \to \mathbf{M}_2 \to \mathbf{M}_1$ and we can blow down twice $\mathbf{M}_3 \to \mathbf{M}_4 \to \mathbf{M}_5$ in the neighborhoods of Γ^1_j. For $\alpha >> 1$, the GIT quotient of $\mathbb{P}^7 \times \mathbb{P}^{2k-1}$ by $SL(2)$ is a $\mathbb{P}^7$-bundle which can be contracted in the open neighborhood. A similar analysis for a neighborhood of Γ^2 tells us that we can blow down $\mathbf{M}_3$ three times

$$\mathbf{M}_3 \to \mathbf{M}_4 \to \mathbf{M}_5 \to \mathbf{M}_6$$

and the morphism $\mathbf{M}_3 \to \mathbf{S}$ is constant on the fibers of the blow-downs. Hence we obtain an induced morphism $\mathbf{M}_6 \to \mathbf{S}$ which turns out to be injective. So we conclude that $\mathbf{M}_6 \cong \mathbf{S}$.

We can summarize the above discussion as follows.

Theorem. For a projective homogeneous variety $X \subset \mathbb{P}^r$ and $d = 3$, $\mathbf{S}$ is obtained from $\mathbf{M}$ by blowing up along Γ^1, Γ^2_1, Γ^3_2 and then blowing down along Γ^2_3, Γ^3_4, Γ^1_5 where Γ^j_i is the proper transform of Γ^j_{i-1} if Γ^j_{i-1} is not the blow-up/-down center and the image/preimage of Γ^j_{i-1} otherwise.

The following diagram summarizes the comparison results for homogeneous $X \subset \mathbb{P}^r$ and $d = 3$:

$$\begin{tikzcd}
 & & & \mathbf{M}_3 \arrow[dl, "\Gamma_2^3"'] \arrow[dr, "\Gamma_4^2"] & & & & \\
 & & \mathbf{M}_2 \arrow[dl, "\Gamma_1^2"'] & & \mathbf{M}_4 \arrow[dr, "\Gamma_5^3"] & & & \\
 & \mathbf{M}_1 \arrow[dl, "\Gamma^1"'] & & & & \mathbf{M}_5 \arrow[dr, "\Gamma_6^1"] & & \mathbf{H} \arrow[d, "\Delta_S"] \\
\mathbf{M} & & & & & & \mathbf{M}_6 \arrow[r, "\cong"] & \mathbf{S}
\end{tikzcd}$$

All the arrows are blow-ups and the blow-up centers are indicated above the arrows.

As an application, we calculated the Betti numbers of **H** and **S** for Grassmannians. The Betti numbers of **M** for Grassmannians were calculated by A. López-Martín in [39].

7. Birational geometry of $\overline{M}_{0,n}$

In this last section, we discuss a problem which is very similar to that discussed in the previous section.

Let $M_{0,n} = \{(p_1, \cdots, p_n) \in (\mathbb{P}^1)^n \,|\, \text{all distinct}\}/\mathrm{Aut}(\mathbb{P}^1)$ be the moduli space of n distinct points of $\mathbb{P}^1$ modulo isomorphisms. This is not compact for $n \geq 4$ and there are many compactifications. The most famous compactification $\overline{M}_{0,n}$ is due to Knudsen and Mumford ([30]) and is obtained by adding nodal curves $(C, p_1, \cdots, p_n)$ with p_i smooth distinct points of C with finite automorphism group $\mathrm{Aut}(C, p_1, \cdots, p_n) = \{f \in \mathrm{Aut}(C) \,|\, f(p_i) = p_i\}$. Finiteness of the automorphism group is equivalent to saying that each irreducible component has at least 3 nodal or marked points. Another obvious way to compactify $M_{0,n}$ is to take the GIT quotient

$$(\mathbb{P}^1)^n /\!/ SL(2)$$

with respect to the linearization $\mathcal{O}(1, 1, \cdots, 1)$.

In [18], Hassett generalized this construction by introducing the notion of weighted pointed stable curves. We let $w = (w_1, \cdots, w_n) \in \mathbb{Q}^n$ with $0 < w_i \leq 1$ and assign weight w_i to each marked point p_i. Then an n-pointed nodal curve $(C, p_1, \cdots, p_n)$ of arithmetic genus 0 is called w-stable if

(1) all p_i are smooth points.
(2) $p_{i_1} = p_{i_2} = \cdots = p_{i_k}$ implies $\sum_{j=1}^k w_{i_j} \leq 1$;
(3) for each irreducible component C_1 of C, the sum of weights of the marked points in C_1 and the number of nodes is greater than 2.

Hassett in [18] proves that

(1) there is a projective fine moduli space of w-stable curves $\overline{M}_{0,w}$ for each choice of weights w;
(2) if $w \ge w'$ in the sense that $w_i \ge w_i'$ for all i, then there is a contraction morphism $\overline{M}_{0,w} \to \overline{M}_{0,w'}$.

In analogy with the comparison problem we discussed above, it seems reasonable to consider the following.

Problem: Study the birational geometry of $\overline{M}_{0,w}$.

As above, there are two approaches for this problem as well: one by Mori theory and the other by explicit geometry and geometric invariant theory.

7.1. **Mori theoretic approach.** Let us denote $(\epsilon, \cdots, \epsilon)$ by $n \cdot \epsilon$. Let $\Delta = \overline{M}_{0,n} - M_{0,n}$ be the boundary divisor of the Knudsen-Mumford space. From Mori theoretic point of view, it seems natural to consider the log canonical models

$$\overline{M}_{0,n}(\alpha) = \mathrm{Proj}\left(\bigoplus_{m\ge 0} H^0(\overline{M}_{0,n}, m(K_{\overline{M}_{0,n}} + \alpha\Delta))\right)$$

of $\overline{M}_{0,n}$. In [50], M. Simpson proved the following beautiful theorem.

Theorem.

(1) If $\frac{2}{m-k+2} < \alpha \le \frac{2}{m-k+1}$ for $1 \le k \le m-2$, then $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,n\cdot\epsilon_k}$ where $m = \lfloor \frac{n}{2} \rfloor$ and $\frac{1}{m+1-k} < \epsilon_k \le \frac{1}{m-k}$.
(2) If $\frac{2}{n-1} < \alpha \le \frac{2}{m+1}$, then $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n /\!/ G$ where the quotient is taken with respect to the symmetric linearization $\mathcal{O}(1, \cdots, 1)$.

Actually M. Simpson proved this theorem under the assumption that Fulton's conjecture about extremal rays of $\overline{M}_{0,n}$ holds. Subsequently, two unconditional proofs were given by Fedorchuk-Smyth ([13]) and Alexeev-Swinarski ([1]). In [27], H. Moon and the author improved the proof of Alexeev-Swinarski and gave a quick proof.

With Simpson's theorem at hand, it seems natural to ask if all the other moduli spaces of Hassett's can be realized as the log canonical models with respect to some divisors. In [12], Fedorchuk gave such divisors for all weights w in terms of the boundary divisors. Recently, H. Moon ([43]) discovered that if we use ψ classes instead of the boundary divisors, we obtain all Hassett's moduli spaces as follows. Recall that ψ_i for $i = 1, \cdots, n$ are defined by

$$\psi_i = c_1(s_i^* \omega_{\mathcal{C}/\overline{M}_{0,n}})$$

where s_i is the section of ith marked point of the universal family $\mathcal{C} \to \overline{M}_{0,n}$ and $\omega_{\mathcal{C}/\overline{M}_{0,n}}$ is the relative dualizing sheaf.

Theorem. (Moon [43]) Assume Fulton's conjecture. For any $w = (w_1, \cdots, w_n)$ with $\sum w_i > 2$, we have

$$\overline{M}_{0,w} = \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(\overline{M}_{0,n}, m(K_{\overline{M}_{0,n}} + \sum_i w_i\psi_i))\right)$$

The following interesting questions are open.

Question. Is $\overline{M}_{0,n}$ a Mori dream space?

Question. Does $\overline{M}_{0,n}$ admit a tilting generator?

7.2. **Geometric invariant theory approach.** By using the line of ideas of the previous sections, we proved the following. In [44], Mustata-Mustata proved that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ is obtained from $(\mathbb{P}^1)^n$ by a sequence of blow-ups:

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) = F_{n-2} \to F_{n-3} \to \cdots \to F_1 \to F_0 = (\mathbb{P}^1)^n.$$

The first blow-up is along the small diagonal or the locus where all n points coincide. The second blow-up is along the proper transform of the locus where $n-1$ points coincide, and so on. The blow-up center for each blow-up is the transversal union of smooth subvarieties. Note that the blow-up along the union of transversal smooth varieties is the same as the consequence of smooth blow-ups along the irreducible components of the blow-up center in any order. In particular, each blow-up above is the composition of smooth blow-ups. Let $m = \lfloor \frac{n}{2} \rfloor$ and $\frac{1}{m+1-k} < \epsilon_k \leq \frac{1}{m-k}$.

Theorem. ([27])

(1) $F_{n-k}//SL(2) \cong \overline{M}_{0,\epsilon_{m-k}}$ for $k = 2, 3, \cdots, m-1$.
(2) There is a sequence of blow-ups

$$\overline{M}_{0,n} = \overline{M}_{0,n\cdot\epsilon_{m-2}} \to \overline{M}_{0,n\cdot\epsilon_{m-3}} \to \cdots \to \overline{M}_{0,n\cdot\epsilon_2} \to \overline{M}_{0,n\cdot\epsilon_1} \to (\mathbb{P}^1)^n//SL(2).$$

Except for the last arrow when n is even, the center for each blow-up is a union of transversal smooth subvarieties of the same dimension. When n is even, the last arrow is the blow-up along the singular locus which consists of $\frac{1}{2}\binom{n}{m}$ points in $(\mathbb{P}^1)^n//SL(2)$, i.e. $\overline{M}_{0,n\cdot\epsilon_1}$ is Kirwan's partial desingularization ([29]) of the GIT quotient $(\mathbb{P}^1)^{2m}//SL(2)$.

For the moduli spaces of *unordered* weighted pointed stable curves

$$\widetilde{M}_{0,n\cdot\epsilon_k} = \overline{M}_{0,n\cdot\epsilon_k}/S_n$$

we can simply take the S_n quotient of the sequence in the above theorem and thus $\widetilde{M}_{0,n\cdot\epsilon_k}$ can be constructed by a sequence of *weighted blow-ups* from $\mathbb{P}^n//G = \left((\mathbb{P}^1)^n//SL(2)\right)/S_n$. In particular, $\widetilde{M}_{0,n\cdot\epsilon_1}$ is a weighted blow-up of $\mathbb{P}^n//G$ at its singular point when n is even.

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