Étale fundamental group and $D$-modules in characteristic $p > 0$

Xiaotao Sun

Institute of Mathematics
Academy of Mathematics and System Science,
Beijing, China

2013-2-18
Let $X$ be a smooth complex projective variety, $\pi_1 = \pi^{\text{top}}$, and

$\tilde{X} \to X$, $X = \tilde{X}/\pi_1$

- Given a $r$-dimensional representation $\rho : \pi_1 \to \text{GL}(V)$, we get a vector bundle $V_{\rho} := \frac{\tilde{X} \times V}{\pi_1}$ of rank $r$ on $X$:

  $\alpha \cdot (\tilde{x}, v) = (\alpha \cdot \tilde{x}, \rho(\alpha) \cdot v)$, $\forall \alpha \in \pi_1$, $\forall (\tilde{x}, v) \in \tilde{X} \times V$

- $V_{\rho}$ is a $D$-module, $D = D_X$ is the sheaf of differential operators.
The correspondence $\rho \mapsto V_\rho$ defines an equivalence

$$\text{Rep}_k(\pi^{\text{top}}) \cong \text{DM}(X)$$

The category $\text{Rep}_k(\pi^{\text{ét}}) \subset \text{Rep}_k(\pi^{\text{top}})$:

$$\rho \in \text{Rep}_k(\pi^{\text{ét}}) \iff \rho(\pi^{\text{top}}) \text{ finite}$$

is the category of representations of étale fundamental group of $X$.

How much does $\pi^{\text{ét}}_1$ determines the category

$$\text{Rep}_k(\pi^{\text{top}}) \cong \text{DM}(X)$$


Malcev (1940)-Grothendieck (1970):

\[ \pi_1^{\text{ét}} = \{1\} \iff \pi^{\text{top}} = \{1\} \]

The proof of Malcev and Grothendieck depends on the fact:

\[ \pi^{\text{et}} \text{ is the algebraic completion of } \pi^{\text{top}} \]

Malcev (1940): On isomorphic matrix representations of infinite groups

Grothendieck (1970): Représentations linéaires et compactifications profinies des groupes discrets
Theorem (Gieseker, 1975)

Let $X$ be a smooth projective variety over $k$ of $\text{char}(k) = p > 0$

(i) If every $D$-module on $X$ is trivial, then $\pi_1$ is trivial.

(ii) If all $D$-module are rank 1, then $[\pi_1, \pi_1]$ is a pro-$p$-group.

(iii) If every $D$-module is a direct sum of rank 1 $D$-modules, then $\pi_1$ is abelian with no $p$-power order quotient.

Conjecture (Gieseker, 1975)

The converses of above statements might be true.

- Mehta-Esnault: The converse of (i) is true!
- Esnault-Sun: The converse of (ii) and (iii) are true!
$k$: an algebraically closed field of characteristic $p > 0$, 
$X$: a smooth projective variety over $k$, with a fixed point 

$$(a : \text{Spec}(k) \to X) \in X(k).$$

$F$: Frobenius morphism $F = F_X : X \to X$. 
$\pi_1$: $\pi_1 = \pi^{\text{ét}}_1(X, a)$ the étale fundamental group of $X$. 
$\mathcal{D}_X$: the sheaf of differential operators. 

**Definition**

A $D$-module is a coherent $\mathcal{O}_X$-module $E$ with a morphism

$$\nabla : \mathcal{D}_X \to \mathcal{E}nd_k(E)$$

of $\mathcal{O}_X$-algebras.
By a theorem of Katz, it is equivalent to the following definition

**Definition**

A *D*-module $E$ on $X$ is a sequence of bundles

$$E = \{ E_0 := E, E_1, E_2, \cdots, \sigma_0, \sigma_1, \cdots \} = \{ E_i, \sigma_i \}_{i \in \mathbb{N}}$$

where $\sigma_i : F^*E_{i+1} \to E_i$ is a $\mathcal{O}_X$-linear isomorphism.

A morphism $\alpha = \{ \alpha_i \} : E = \{ E_i, \sigma_i \} \to E' = \{ E'_i, \tau_i \}$ is a set of morphisms $\alpha_i : E_i \to E'_i$ of $\mathcal{O}_X$-modules such that

$$F_X^*E_{i+1} \xrightarrow{F_X^*\alpha_{i+1}} F_X^*E'_i$$

is commutative.
Let $E = (E_i, \sigma_i)_{i \in \mathbb{N}}$ be a $D$-module of rank $r$ on $X$. Then

- $E(m) = (E_{i+m}, \sigma_{i+m})_{i \in \mathbb{N}}$ is also a $D$-module.

- There is a $n_0$ such that $E_n$ is semi-stable of slope 0 for $n \geq n_0$ ($\mu(F) := \deg(F)/\text{rk}(F) \leq \mu(E_n), \forall F \subset E_n$).

- If $E = (E_i, \sigma_i)_{i \in \mathbb{N}}$ is an irreducible $D$-module, then $E_n$ is stable for $n \geq n_0$ ($\mu(F) < \mu(E_n)$).

Thus, if there exists an irreducible $D$-module

$$E = (E_n, \sigma_n)_{n \in \mathbb{N}}$$

of rank $r$ on $X$, we can assume that all $E_n$ are stable bundles of rank $r$. 
Theorem (Lange-Stuhler, 1977)

Let $Y$ be a smooth projective variety over $k$ of $\text{char}(k) = p > 0$, $F : Y \to Y$ be the Frobenius morphism. If there is a vector bundle $\mathcal{E}$ on $Y$ and an integer $m > 0$ such that

$$(F^m)^* \mathcal{E} \cong \mathcal{E}$$

then there exists a geometrically connected étale finite cover

$$f : Z \to Y$$

such that $f^* \mathcal{E} \cong \mathcal{O}_Z \oplus \text{rk}(\mathcal{E})$. This gives a representation

$$\pi_1^{\text{ét}}(Y \otimes_k \bar{k}) \to \text{GL}(V)$$

whose associated bundle is $\mathcal{E} \otimes_k \bar{k}$ on $Y \otimes_k \bar{k}$.
If \( \{ E_n \}_{n \in \mathbb{N}} \) is a finite set, there is a \( E \coloneqq E_n \) and an integer \( m \) satisfies

\[
(F^m)^* (E_n) \cong E_n
\]

which gives an irreducible representation

\[
\rho : \pi_1 \rightarrow GL(V)
\]

of dimension \( r \) by Lange-Stuhler’s theorem.

If \( E = \{ E_n \}_{n \in \mathbb{N}} \) is an infinite set, let

\[
A(E(n)) = \overline{\{ E_n, E_{n+1}, \cdots \}} \subset M
\]

be the Zariski closure in the moduli space \( M \) of stable bundles on \( X \) with the same Hilbert polynomial of \( E_n \).
Since $A(E(n)) \supseteq A(E(n + 1))$, there is an $n_0$ such that

$$N := A(E(n_0)) = A(E(n))$$

for all $n \geq n_0$

Thus there is an irreducible component $N_0$ of $N$ such that

$$f = (F^*)^a : N_0 \rightarrow N_0$$

is dominant since $F^* E(n) = \{E_{n-1}, E_n, \cdots\} := E(n - 1)$.

If $k = \bar{F}_p$, the subset of periodic points of

$$f : N_0 \rightarrow N_0$$

is dense in $N_0$. Thus there is a point $x = [\mathcal{E}] \in N_0(k)$ and a $m \in \mathbb{N}$ such that $f^m(x) = x$, i.e., a stable bundle $\mathcal{E}$ of rank $r$ on $X$ such that $(F^m)^*(\mathcal{E}) \cong \mathcal{E}$, which defines an irreducible representation $\pi_1 \rightarrow GL(V)$ of dimension $r$. 
Theorem (Corollary of twisted Lang-Weil estimate)

Let $Y \subset \mathbb{A}^n_{\mathbb{F}_q}$ be an affine variety, $\Gamma \subset Y \times Y$ be an irreducible subvariety over $\overline{\mathbb{F}}_q$. Assume the two projections $\Gamma \rightarrow X$ are dominant. Then, for any closed subvariety $W \subset Y$, there exists

$$x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q), \quad x^{q^m} := (x_1^{q^m}, ..., x_n^{q^m})$$

such that $(x, x^{q^m}) \in \Gamma$ and $x \notin W$ for $m \gg 0$.

- $Y := N_0 \subset \mathbb{A}^n_{\mathbb{F}_q}$, $\Gamma = \text{graph}(f) \subset Y \times Y$, $W \subset Y$ where $f$ is not well-defined. Let $f = (f_1, ..., f_n)$, $f_i \in \mathbb{F}_q(Y)$.

- $\exists \ x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q)$ such that $x \in W$, $(x, x^{q^m}) \in \Gamma$.

- $f(x) = (x_1^{q^m}, ..., x_n^{q^m})$, $f(f(x)) = f(x^{q^m}) = f(x)^{q^m}$

  $\Rightarrow \ f^2(x) = x^{2q^m}, ..., \ f^m(x) = x, \ (m \gg 0)$
If \( k \neq \overline{F}_p \), consider a model \( X_S \to S \) of \( X \), \( S \) smooth affine over \( \overline{F}_p \), with a model \( M_S \to S \) of moduli of stable bundles, and a good model \( N_S \subset M_S \) with rational map

\[
f_S : N_S \dashrightarrow N_S
\]

such that \( f_s := f_S|_{N_s} : N_s \dashrightarrow N_s \), \( s \in S \), is dominant, which coincides with the Frobenius pullback \((F^*_X)_a\)

There is a good reduction \( X_s \) of \( X \) and a stable bundle \( E \) on \( X_{\overline{s}} \) (over \( \overline{F}_p \)), which is Frobenius periodic. Thus we have irreducible representation

\[
\pi_1^{\text{ét}}(X_{\overline{s}}, \overline{x}) \to GL_r(\overline{F}_p)
\]

one has irreducible representation

\[
\pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(X_{\overline{s}}, \overline{x}) \to GL_r(\overline{F}_p)
\]
Let $X$ be a smooth projective variety over an algebraically closed field $k$. If there is an irreducible $D$-module

$$E = (E_n, \sigma_n)_{n \in \mathbb{N}}$$

of rank $r$ on $X$, then there exists an irreducible representation

$$\rho : \pi_1 \rightarrow GL(V)$$

with finite monodromy, where $V$ is a $r$-dimensional vector space over $\overline{F}_p$. 

Xiaotao Sun

Étale fundamental group and $D$-modules in characteristic $p > 0$
Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p > 0$. Then

all irreducible $D$-modules have rank 1

\[ \Leftrightarrow \]

the commutator $[\pi_1, \pi_1]$ of $\pi_1$ is a pro-$p$-group

Let $E = (E_i, \sigma_i)_{i \in \mathbb{N}}$ be an irreducible $D$-module of rank $r$, then there is an irreducible representation

$\rho : \pi_1 \to \text{GL}(V)$

with finite $G := \rho(\pi_1)$, where $V$ has dimension $r$ over $\overline{\mathbb{F}}_p$. 
When the commutator $[\pi_1, \pi_1]$ of $\pi_1$ is a pro-$p$-group,

$$[\pi_1, \pi_1] \rightarrow [\rho(\pi_1), \rho(\pi_1)] = [G, G]$$

is a $p$-group.

For any finite $p$-group $H \subset GL(V)$, $V^H \neq \{0\}$. Thus

$$V^{[G,G]} \neq \{0\}$$

Let $0 \neq v \in V^{[G,G]}$, then, for any $g_1, g_2 \in G$,

$$g_1^{-1} g_2^{-1} g_1 g_2 \cdot v = v,$$

$$g_1 g_2 \cdot v = g_2 g_1 \cdot v.$$

$V$ is an irreducible $G$-module $\Rightarrow G$ is abelian, thus $r = 1$. 
Theorem

Let $X$ be a smooth projective connected variety, and $\pi_1$ be abelian without non-trivial $p$-power order quotient. Then any extension

$$0 \to \mathbb{L} \to E \to \mathbb{L}' \to 0$$

of D-modules is split when $\mathbb{L}$ and $\mathbb{L}'$ are rank one D-modules.

Corollary

All D-modules on $X$ are direct sum of rank one D-modules

$\Updownarrow$

$\pi_1$ is abelian with no non-trivial $p$-power quotient
By twisting with $(\mathbb{L}')^{-1}$, we may assume that $\mathbb{L}' = \mathbb{I}$ (the trivial $D$-module). Then, by definition, a nontrivial extension

$$0 \to \mathbb{L} \to E \to \mathbb{I} \to 0$$

in $\text{DM}(X)$, means that we have a set

$$\Sigma = \{ e_i = (0 \to L_i \to E_i \to \mathcal{O}_X \to 0) \} \in \mathbb{N}$$

of non-trivial extensions such that $e_i \cong F^*(e_{i+1})$.

- If $\Sigma$ is a finite set, then there is a $e_i \in \Sigma$ such that

  $$(F_X^*)^a e_i = e_i$$

  $$(F_X^*)^a (L_i \hookrightarrow E_i \rightarrow \mathcal{O}_X) \cong (L_i \hookrightarrow E_i \rightarrow \mathcal{O}_X)$$
Proposition

Let $L$ be a line bundle on $X$ with $(F_X^*)^a L = L$. Then

$$(F_X^*)^a : H^1(X, L) \rightarrow H^1(X, L)$$

is nilpotent if $\pi_1$ is abelian without non-trivial $p$-power quotient.

- Let $\phi : Y \rightarrow X$ be étale cover such that $\phi^* L = \mathcal{O}_Y$
- $\pi^\text{ét}(Y, y_0)$ is abelian without non-trivial $p$-power quotient, $\Rightarrow (F_Y^*)^N : H^1(\mathcal{O}_Y) \rightarrow H^1(\mathcal{O}_Y)$ is a zero map.
- $\phi^* \cdot (F_X^*)^{Na} = (F_Y^*)^{Na} \cdot \phi^* : H^1(L) \rightarrow H^1(\mathcal{O}_Y)$ is a zero map
- $\phi^*$ injective $\Rightarrow (F_X^*)^{Na} : H^1(L) \rightarrow H^1(L)$ is a zero map
If $\Sigma$ is an infinite set, we will show:

There is a good reduction $X_s$ of $X$ over $\overline{F}_p$, and a non-trivial extension (on $X_s$)

$$e = (0 \to L \to E \to \mathcal{O}_{X_s} \to 0)$$

such that $(F^*_{X_s})^a e = e$ for some integer $a > 0$.

Since $\pi^{\text{ét}}(X, x_0) \to \pi^{\text{ét}}(X_s, (x_0)_s)$ is surjective,

"$\pi^{\text{ét}}(X, x_0)$ abelian, without $p$-power quotient"

$\Rightarrow$ "so is $\pi^{\text{ét}}(X_s, (x_0)_s)$"
Theorem

If the set \( \Sigma = \{ e_i = (0 \to L_i \to E_i \to \mathcal{O}_X \to 0) \}_{i \in \mathbb{N}} \) exists, then there is a nontrivial extension

\[ e = (0 \to L \to E \to \mathcal{O}_X \to 0) \]

and an integer \( a > 0 \) such that \((F^*)^a(e) = e\)

- there is a reduced scheme \( M \) such that

\[ M(k) = \{ e = (0 \to L \to E \to \mathcal{O}_X \to 0) \} \]

- there is a rational map \( f : M \to M \) over \( k \) such that

\[ f(e) = (0 \to F^*L \to F^*E \to \mathcal{O}_X \to 0) := F^*(e) \]
Let $\Sigma(m) := \{ e_{i+m} \in \Sigma \}_{i \in \mathbb{N}} \subset \Sigma$, then

$$f(\Sigma(m)) = \Sigma(m - 1), \quad f(e_{i+1}) = f(e_i)$$

Let $Z \subset M$ be the Zariski closure of $\Sigma \subset M$, then

$$f = f|_Z : Z \dashrightarrow Z$$

is a dominant rational map.

Let $Z' \subset Z$ be the union of irreducible components $Z_i$ with $\text{dim}(Z_i) > 0$. Thus there is an irreducible component $Z_{i_0}$ and an integer $a_1 > 0$ such that

$$f^{a_1} : Z_{i_0} \dashrightarrow Z_{i_0}$$

is a dominant rational map.
In $\Sigma = \{ e_i = (L_i \hookrightarrow E_i \twoheadrightarrow \mathcal{O}_X) \}_{i \in \mathbb{N}}$, we can assume

$$H^1(L_i) \neq 0, \quad H^0(L_i) = 0$$

Let $\mathcal{L}$ be the universal line bundle on $X \times \text{Pic}^\tau_X$, define

$$\mathcal{N}_0 = \{ t \in \text{Pic}^\tau_X \mid H^1(\mathcal{L}_t) \neq 0 \}$$

If $\mathcal{N}_i$ is defined, let $\mathcal{N}_{i+1} = \{ t \in \mathcal{N}_i \mid H^1(\mathcal{L}_t^{p^{i+1}}) \neq 0 \}$. Then, there is a $k_0 \geq 0$ such that

$$\mathcal{N}_i = \mathcal{N}_{k_0}, \quad \forall \ i \geq k_0.$$  

The line bundles $\{ L_{i+p^{k_0}} \}_{i \in \mathbb{N}}$ occurring in $\Sigma(p^{k_0})$ are $k$-points of $\mathcal{N}_{k_0} = \mathcal{N}_i$ for all $i \geq k_0$.

Let $T \subset \mathcal{N}_{k_0}$ be the sub-scheme

$$T = \{ t \in \mathcal{N}_{k_0} \mid H^0(\mathcal{L}_t) = 0 \}.$$
Let $\mathcal{L}$ be the universal line bundle on $X \times T$. In general,

$$R^1p_T^*(\mathcal{L})$$

does not commute with base change. But

$$\mathcal{E} = R^{n-1}p_T^*(\mathcal{L}^\vee \otimes \omega_X)$$

do commute with base change, but may not be locally free.

Let $\pi : M = \mathbb{P}(\mathcal{E}) \to T$ be quotient scheme, whose closed points are quotients

$$H^{n-1}(L^\vee \otimes \omega_X) \to k \to 0, \quad [L] \in T$$

Thus

$$M(k) = \{ e = (0 \to L \to E \to \mathcal{O}_X \to 0) \}$$
Let $\pi^* E \to \mathcal{O}_M(1) \to 0$ be the universal quotient, $(X \times T)'$ be the $F_T$-twist (where $F_T : T \to T$), and $\mathcal{L}'$ be the pullback of $\mathcal{L}$ on $p'_T : (X \times T)' \to T$.

Let $F : X \times T \to (X \times T)'$ be the relative Frobenius, then there is a homomorphism

$$\phi : R^{n-1} p_{T*}(\omega \otimes F^* \mathcal{L}'^\vee) \to R^{n-1} p'_{T*}(\omega' \otimes \mathcal{L}'^\vee)$$

such that, for any $t \in T$, the homomorphism

$$\phi_t : H^{n-1}(\omega_X \otimes F^* \mathcal{L}'_t^\vee) \to H^{n-1}(\omega'_X \otimes \mathcal{L}'_t^\vee)$$

is induced (through Serre duality) by

$$F^* : H^1(X', \mathcal{L}'_t) \to H^1(X, F^* \mathcal{L}'_t)$$
\( F^*(\mathcal{L}') = \mathcal{L}^p \) defines a rational map \( \nu : T \to T \) such that 
\[
\nu^* \mathcal{E} = R^{n-1} \rho_{T*}(\omega \otimes F^* \mathcal{L}'^\vee), \quad F_T^* \mathcal{E} = R^{n-1} \rho'_T(\omega' \otimes \mathcal{L}'^\vee)
\]

Thus the homomorphism \( \phi : \nu^* \mathcal{E} \to F^*_T \mathcal{E} \) induces
\[
\varphi : \pi^* \nu^* \mathcal{E} \to \pi^* F_T^* \mathcal{E} = F_M^* \pi^* \mathcal{E} \to F_M^* \mathcal{O}_M(1)
\]

\( \varphi : \pi^* \nu^* \mathcal{E} \to F_M^* \mathcal{O}_M(1) \) defines the rational map 
\[
f : M \to M
\]
as required.
Thank You!