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대수기하학 특강
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기하학적 불변량이론

서울대학교 수리과학부

김 영 훈

Warm-up Homework
대수기하학특강

• You may discuss the problems with your friends or look up some books on algebraic geometry. But try to fully understand the solutions and then write them in your own language.

(1) Let X be an affine variety and Z_1, Z_2 be two closed disjoint subsets of X. Show that there is a regular function ϕ on X such that $\phi(Z_1) = 0$, $\phi(Z_2) = 1$.

(2) (a) Prove that the complement of a hypersurface (= the zero locus of a homogeneous polynomial) in $\mathbb{P}^n$ is an affine variety.
(b) Show that $\mathbb{P}GL(n) = GL(n, \mathbb{C})/\mathbb{C}^*$ is an affine variety.

(3) Let X be an affine variety and $\mathcal{O}(X)$ be its ring of regular functions. Let Y be any variety (not necessarily affine). Prove that there is a bijection between the set $\mathrm{Hom}_{\mathbb{C}-alg}(\mathcal{O}(X), \mathcal{O}(Y))$ of $\mathbb{C}$-algebra homomorphisms and the set $\mathrm{Hom}_{var}(Y, X)$ of morphisms of varieties.

(4) Let X be any variety. Verify that there is a bijection between the following two sets;
(a) { invertible sheaves $\mathcal{L}$ on X together with global sections $s_0, s_1, \cdots, s_n \in H^0(X, \mathcal{L})$ which generate $\mathcal{L}$ }
(b) { morphisms from X to $\mathbb{P}^n$ }

(5) Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves over a projective variety X. Prove that there is a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to \cdots$$

대수기하 특강 – 1강

Geometric Invariant Theory and Moduli Problems

- **Professor**: Young-Hoon Kiem
- **Office**: 27동 327호
- **Email**: kiem@math.snu.ac.kr
- **Course webpage**: www.math.snu.ac.kr/ kiem/git02.html
- **Textbook**: Peter Newstead, *Introduction to moduli problems and orbit spaces*, Tata Lecture Note, 1978.
- **References**:
  (1) Mumford, Fogarty, Kirwan, *Geometric Invariant Theory*, 3rd ed., Springer-Verlag, 1994
  (2) F. Kirwan, *Cohomology of quotients in algebraic and symplectic geometry*, Princeton 1984
  (3) I. Dolgachev, *Introduction to geometric invariant theory*, 서울대 수학 연구소 강의록 25권
  (4) J. Le Potier, *Lectures on vector bundles*, Cambridge University Press, 1997
  (5) D. Gieseker, *Moduli of curves*, Tata lecture Note, 1982
- **Grading**: 수업참여도 20%+ 숙제 80%

대수기하 특강 – 2강

Chapter 0. Preliminaries.

Throughout this course, we assume that the base field k is an algebraically closed field of characteristic 0. In many places we will think of only $\mathbb{C}$.

(1) Sheaf

Let X be a topological space. The open sets in X form a category by inclusion $U \subset V$. A presheaf of abelian groups (resp. rings, modules, algebras) is a contravariant functor from the category of the open sets to the category of abelian groups (resp. rings, modules, algebras. In other words, to each open set U we can associate an abelian group $\mathcal{S}(U)$ and to each inclusion $U \subset V$ we can associate a (restriction) homomorphism $\rho_{VU} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that $\rho_{UU} = id_U$ and $\rho_{VU}\rho_{WV} = \rho_{WU}$ for $U \subset V \subset W$. Furthermore, a presheaf $\mathcal{S}$ is a sheaf if for each open cover of an open set $U = \cup U_i$ the following are satisfied:

(1) if $s_1, s_2 \in \mathcal{S}(U)$ satisfies $s_1|_{U_i} = s_2|_{U_i}$ for each i, then $s_1 = s_2$
(2) if we have $s_i \in \mathcal{S}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there is $s \in \mathcal{S}(U)$ such that $s|_{U_i} = s_i$.

Suppose X is equipped with a sheaf $\mathcal{O}_X$ of rings. A sheaf of $\mathcal{O}_X$-modules $\mathcal{S}$ is invertible (locally free) if for each $x \in X$ there is an open set U containing x such that $\mathcal{S}|_U$ is isomorphic to (a direct sum of) $\mathcal{O}|_U$.

(2) Affine variety

By weak Nullstellensatz, there is a one-to-one correspondence

$$k^n \leftrightarrow \{m \subset k[z_1, \cdots, z_n] \,:\, \text{maximal ideal}\}$$

given by $(a_1, \cdots, a_n) \to m = (z_1 - a_1, \cdots, z_n - a_n)$. For each ideal $I \subset k[z_1, \cdots, z_n]$, we get a subset

$$V(I) = \{(a_1, \cdots, a_n) \in k^n \,:\, f(a_1, \cdots, a_n) = 0 \text{ for all } f \in I\}.$$

By declaring that $V(I)$ is a closed set for each I – it is easy to check – we get a topology on k^n, called the Zariski topology. The sets $(k^n)_f = \{(a_1, \cdots, a_n) : f(a_1, \cdots, a_n) \neq 0\}$ for $f \in k[z_1, \cdots, z_n]$ are basic open set for the topology.

An Affine variety consists of 3 layers.

(1) a closed subset $X = V(I)$ of k^n
(2) induced Zariski topology : $X_f = X \cap (k^n)_f$ basic open sets (affine)
(3) sheaf of regular functions : $A(X) = \mathcal{O}(X) = k[z_1, \cdots, z_n]/\sqrt{I}$, $\mathcal{O}(X_f) = A(X)_{\overline{f}}$ localization.

set + topology + sheaf of rings = ringed space

There is a bijection

$$\{\text{affine varieties}\} \leftrightarrow \{\text{finitely generated integral domain}\}$$

given by $X \to \mathcal{O}(X)$.

(3) Varieties

A prevariety is a ringed space X which can be covered by finitely many open subsets which are isomorphic to affine varieties. A prevariety is a variety if the diagonal map

$$\Delta_X : X \to X \times X$$

has closed image. (Hausdorff axiom)

Example: $\mathbb{P}^n = \mathbb{C}^{n+1} - 0/\mathbb{C}^*$ is a variety. Let $z_0, \cdots, z_n$ be homogeneous coordinates for $\mathbb{P}^n$. Then the sets

$$U_i = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n \mid z_i \neq 0\}$$

for $i = 0, 1, \cdots, n$ give us an open cover, each element of which is isomorphic to k^n via $(z_0 : \cdots : z_n) \to (z_0/z_i, \cdots, z_n/z_i)$. $\mathbb{P}^n$ is certainly separated since the image of the diagonal map is given by $z_i z_j' = z_j z_i'$.

An open or a closed subset of a variety is a variety. A projective variety is a closed subvariety of $\mathbb{P}^n$. A quasi-projective variety is an open subset of a projective variety.

A variety X is irreducible if it is not a union of proper closed subsets. A variety X is complete (compact) if for any variety Y the projection $p_Y : X \times Y \to Y$ is closed.[1] If X is complete irreducible, then $\mathcal{O}(X) = k$.[2]

(1) Projective varieties are complete.
(2) A complete affine variety must be a finite set.
(3) A compactification of a variety X is a complete variety Y containing X as a dense open subset.

[1] If $(x_n, y_n) \in Z$, $y_n \to y$, then $x_{n_k} \to x$.

[2] If nonconstant, the image of $V(fy = 1) \subset X \times k \to k$ should be k which contains 0. Contradiction!

(4) Morphisms

A morphism of varieties $f : X \to Y$ is a continuous map which induces a homomorphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$, i.e. given a regular function $\phi \in \mathcal{O}_Y(U)$ on $U \subset Y$, the composition $\phi \circ f \in \mathcal{O}_X(f^{-1}(U))$ is regular.

A morphism $f : X \to Y$ is affine if for each affine open subset U of Y, $f^{-1}(U)$ is affine. f is finite if affine and $\mathcal{O}(f^{-1}(U))$ is integral over $\mathcal{O}(U)$. f is proper[3] if for any variety Z, the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed.

It is elementary to check the following:

(1) if X is a closed subset of Y, the inclusion $i : X \hookrightarrow Y$ is finite
(2) composition of two affine/finite/proper morphisms is affine/finite/proper
(3) if $f \circ g$ is proper, g is proper
(4) if $f \circ g$ is proper and g is surjective, then f is proper[4]
(5) X is complete if $X \to pt$ is proper. Inverse image of a complete variety by a proper morphism is complete
(6) a finite morphism is proper

Valuative criterion: Properness is difficult to prove by using its definition. Rather the valuative criterion is more useful. We need to use the language of schemes: For a commutative ring R, Spec$R$ denotes the set of prime ideals in R together with Zariski topology and the sheaf of rings R_f on Spec$R_f = \{\mathfrak{a} \in R : f \notin \mathfrak{a}\}$.

Let $R = k[[T]]$ be the ring of formal power series in T and K be its field of fractions, i.e. $K = R_{(T)}$. The inclusion $R \hookrightarrow K$ induces an inclusion Spec$K \to$ Spec$R$. A morphism $f : X \to Y$ is proper iff whenever we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}R & \longrightarrow & Y \end{array}$$

there is a morphism Spec$R \to X$ which makes the diagram commutative. For proof, see [Hartshorne] for instance.

A surjective morphism $f : X \to Y$ of irreducible varieties is *flat* if at each $x \in X$, the stalk $\mathcal{O}_{X,x} = \lim_{x \in U} \mathcal{O}(U)$ at x is a flat $\mathcal{O}_{Y,f(x)}$-module via the homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. If flat, the fibers do not vary discontinuously (e.g. dimension jump like $+ \to -$). A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is flat over Y if each stalk $\mathcal{F}_x$ is flat as a $\mathcal{O}_{Y,f(x)}$-module.

If f is projective, i.e. f factors through $X \to Y \times \mathbb{P}^n$ for some n, then f is flat iff the Hilbert polynomial of each fiber is independent of $y \in Y$.[5] See Hartshorne for a proof.

Let $X \to S$ and $Y \to S$ be two morphisms. Then the fibred product is defined as the unique variety $X \times_S Y$ such that for any $Z \to X$ and $Z \to Y$ .... there is a unique morphism $Z \to X \times_S Y$ ....

(5) Vector bundles

A *vector bundle* of rank r is a morphism $p : E \to X$ of algebraic varieties together with an open cover $\mathcal{U} = \{U_i\}$ and a set of isomorphisms

$$\beta_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$$

[3] Intuitively, it means that each fiber is compact.
[4] $(f \circ g)^{-1}(z) \to f^{-1}(z)$ surjects.
[5] a fiber is the fibred product $pt \times_Y X$.

such that the isomorphism $\beta_i \circ \beta_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ is given by a morphism $g_{ij} : U_i \cap U_j \to GL(r)$. A *line bundle* is a vector bundle of rank 1. A trivial bundle of rank r is $pr_X : I_r = X \times \mathbb{C}^r \to X$. The cocycle conditions $g_{ij}g_{jk} = g_{ik}$, $g_{ii} = 1$ are satisfied. The dual bundle of E is given by g_{ij}^{-1} and the determinant bundle is given by $det g_{ij}$. The direct sum and tensor product of two vector bundles are defined in the obvious fashion.

A homomorphism $h : E_1 \to E_2$ of vector bundles is a morphism such that $p_2 \circ h = p_1$ and h restricts to a linear map at each point. An isomorphism is a bijective homomorphism.

A section of a vector bundle is a morphism $s : X \to E$ such that $p \circ s = 1_X$. There is a bijection

$$\{\text{sections of } E\} \leftrightarrow Hom(I, E).$$

If $p : E \to X$ is a vector bundle and $Y \to X$ is a morphism, then the fiber product $E \times_X Y \to Y$ is the pull-back bundle.

Example: $\mathbb{P}^n \times \mathbb{C}^{n+1} \supset \{(x, v) : v \in x\} \to \mathbb{P}^n$ tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^n$. (The blow-up of $\mathbb{C}^{n+1}$ at 0 is $\mathcal{O}(-1)$ of $\mathbb{P}^n$.) The dual $\mathcal{O}(1)$ is the hyperplane bundle.

Claim: Let X be a complete variety and L be a line bundle. If $I_r \otimes L$ is trivial, then L is trivial.

Proof: $L^r = det(L \otimes I_r) \cong I$. A nonzero section s of L gives a section s^n of L^n which is nowhere vanishing. Hence s is nowhere vanishing. So, $L \cong I$.

대수기하 특강 – 4강

Chapter 1. The concept of moduli

Suppose we have a set A of objects (e.g. vector bundles, algebraic manifolds of given topological type) and equivalence relation $\sim$ (e.g. isomorphism).

Classification problem: Describe $A/\sim$ algebro-geometrically. [GIT or Stack] We need the concept of family in order to assign a topology on the moduli space.

§1. Families

Let S be a variety.

Examples:

(1) *Hypersurfaces of degree d in $\mathbb{P}^n$:* A hypersurface is the zero locus of a homogeneous polynomial in $z_0, \cdots, z_n$.

A=all hypersurfaces of degree d in $\mathbb{P}^n$, i.e. A is the projective space $\mathbb{P}^{N-1}$ where $N = (n+1)Hd = \binom{n+d}{d}$. (Given a homogeneous polynomial f, write $f = \sum a_{i_0 \cdots i_n} z_0^{i_0} \cdots z_n^{i_n}$. The ratio of $(a_{i_0 \cdots i_n})$ determine a hypersurface.

$\sim$= two hypersurfaces H and H' are equivalent if there is $g \in GL(n+1)$ such that H is mapped to H' by g.

A family of hypersurfaces parametrized by S is a pair (L, a) of a line bundle L over S and a set of sections $a = (a_{i_0 \cdots i_n})$ of L for $i_0 + \cdots + i_n = d$.

Two families (L, a) and (L', a') are isomorphic if there is an isomorphism $h : L \to L'$ which takes a to a'.

Two families are equivalent if there exists $g \in GL(n+1)$ such that (L, a) is isomorphic to (L', ga'). (The action of $GL(n+1)$ on $\mathbb{C}^n$ induces an action of $GL(n+1)$ on $\mathbb{C}^N$.)

Finding the moduli space for this equivalence, i.e. finding the quotient $\mathbb{P}^{N-1}//GL(n+1)$, was the major problem of classical invariant theory.

(2) *Family of complete varieties:* A= all complete varieties, $\sim$= isomorphism of varieties. A family of objects of A parametrized by S is a variety X and a proper *flat* morphism $f : X \to S$ whose fibers $X_s = f^{-1}(s)$ are objects in A.

If $S' \to S$ is a morphism, then we can define the *pull-back family* of $X \to S$ to S' as the fiber product $S' \times_S X$.

Two families $X \to S$ and $X' \to S$ are equivalent if there is an isomorphism $X \to X'$ over S.

(3) *Family of vector bundles:* X = fixed variety, A= vector bundles over X, $\sim$= isomorphism of vector bundles.

A family of vector bundles over X parametrized by S is a vector bundle E over $S \times X$. The restriction E_s of E to $X \cong s \times X$ is a vector bundle over X.

For any morphism $\phi : S' \to S$, the induced family is just the pull-back $(\phi \times 1_X)^* E$.

Two families of bundles E_1, E_2 over X parametrized by S are equivalent if $E_1 \cong E_2 \otimes p_S^* L$ for some line bundle L over S.

A *moduli problem* consists of *objects, families, equivalence relation of families* such that

(1) A family parametrized by a single point is a single object in A. The equivalence of two families parametrized by a point is the same as the equivalence of two objects in A.

(2) For any morphism $\phi : S' \to S$ and any family X parametrized by S, there is an induced family $\phi^* X$ parametrized by S'. Moreover, $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ and $1_S^* = identity$.
(3) The equivalence relation is compatible with the pull-back, i.e. $X \sim X'$ implies $\phi^* X \sim \phi^* X'$.

§2. Moduli spaces

(1) Fine moduli space:

It is necessary to make clear what we want from a solution. (Necessities make defitions.)

Suppose we are given a moduli problem. The goal is to describe $A/\sim$ algebro-geometrically.

Suppose we have a family X parametrized by S. Then each point $s \hookrightarrow S$ gives us an element of $A/\sim$ and thus we have a set-theoretic map $S \to A/\sim$, i.e. an element of $Hom_{sets}(S, A/\sim)$.

For a variety S, let $\mathcal{F}(S)$ be the set of all equivalence classes of families parametrized by S.

(Ex.: vector bundles over X. $\mathcal{F}(S)$ is the set of equivalence classes of vector bundles over $S \times X$.)

Then we have a map $\mathcal{F}(S) \to Hom_{sets}(S, A/\sim)$. What we would like to have is to find a variety structure M on A/S such that the map factors through $Hom_{var}(S, M)$. Furthermore, it would be best if the map is a bijection.

Let's vary S. Category theory is useful to keep track of the parameter space. Notice that a morphism $\phi : S' \to S$ induces a map $\phi^* : \mathcal{F}(S) \to \mathcal{F}(S')$ and we have $\phi^* \circ \psi^* = (\psi \circ \phi)^*$. Then $\mathcal{F}$ is a contravariant functor from the category of varieties to the category of sets.

Fix a variety M. Let $h_M(S) = Hom_{var}(S, M)$. Then h_M is a contravariant functor from the category of varieties to the category of sets.

Having a map $\mathcal{F}(S) \to h_M(S)$ for each S amounts to having a natural transformation $\mathcal{F} \to h_M$ of functors. So we make the following definition.

Definition: A fine moduli space for a moduli problem is a variety M together with an isomorphism of functors $\Phi : \mathcal{F} \to h_M$, i.e. M represents the functor $\mathcal{F}$.

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We learned that a fine moduli space is a variety which represents the moduli functor $S \to \mathcal{F}(S)$. I.e. there is an isomorphism of functors $\Phi : \mathcal{F} \to h_M$ where $h_M = Hom(-, M)$. It is unique up to isomorphism.

(2) Universal family and fine moduli space:

Let M be a fine moduli space. Consider $S = M$. Then $\mathcal{F}(M) \leftrightarrow Hom(M, M)$. Let U be the family parametrized by M corresponding to the identity morphism in $Hom(M, M)$. Then we call U the universal family for the moduli problem.

This is the reason why. Let X be any family parametrized by S and $\phi : S \to M$ be the corresponding morphism via $\mathcal{F}(S) \leftrightarrow Hom(S, M)$. Consider the pull-back $\phi^* U$ of the universal family U. Then we have $\phi^* U = X$ since they both correspond to ϕ.[6] In other words, for any family X parametrized by S, there is a unique morphism $\phi : S \to M$ such that $X = \phi^* U$. (universal property **)

Conversely, suppose U is a family parametrized by M with the above property. Then obviously, M is a fine moduli space given by $X \to \phi$.

<u>lemma</u> M is a fine moduli space for a moduli problem iff there is a family paprametrized by M with the property **.

(3) Coarse moduli space:

In many interesting cases there are no fine moduli spaces (e.g. moduli of curves, vector bundles, etc). So, we need to weaken the assumption. Here's an example.

<u>Example</u>: Moduli problem for (irreducible smooth complete) algebraic curves of genus 0. A=algebraic curves of genus 0, $\sim$=isomorphisms, a family parametrized by S is a proper flat morphism $X \to S$ whose fibers are genus 0 curves, two families $X \to S$ and $X' \to S$ are isomorphic if there is an isomorphism $X \to X'$ over S.

Suppose there is a fine moduli space M. Let $S = \mathbb{P}^1$ and consider two families $pr : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $Bl_{pt}\mathbb{P}^2 \to \mathbb{P}^1$. They are not isomorphic (though birational due to a -1-curve) but they both give the same map $\mathbb{P}^1 \to pt \to M$. Contradiction.

So, we need to be less ambitious i.e. we require a weaker condition on the natural transformation $\Phi : \mathcal{F} \to h_M$ than being an isomorphism of functors.

Notice that given any morphism $M \to N$, the composition of Φ and the obvious transformation $h_M \to h_N$ gives us a new functor $\Psi : \mathcal{F} \to h_N$. We require the following universal property.

<u>Definition</u>: A coarse moduli space for a given moduli problem is a variety M together with a natural transformation $\Phi : \mathcal{F} \to h_M$ such that

(1) as a set $M = A/\sim$

(2) given any variety N together with a natural transformation $\Psi : \mathcal{F} \to h_N$, there is a *unique* morphism $\phi : M \to N$ which makes the diagram $\phi^* \circ \Phi = \Psi$.

When the second condition is satisfied we say M *corepresents* the functor $\mathcal{F}$.

The coarse moduli space is unique.

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$$\begin{array}{ccc} \mathcal{F}(M) & \longrightarrow & h_M(M) \\ {\scriptstyle \phi^*}\downarrow & & \downarrow \\ \mathcal{F}(S) & \longrightarrow & h_M(S) \end{array} .$$

Start with 1 on the upper-right.

Proposition: If M_1, M_2 are coarse moduli spaces, then $M_1 \cong M_2$. (Proof: obvious.)

Relation of coarse moduli space with fine moduli space: A fine moduli space is a coarse moduli space (trivial to check). But a coarse moduli space is not necessarily a fine moduli space.

Proposition: A coarse moduli space (M, Φ) is a fine moduli space iff

(1) there is a family U parametrized by M such that for each $m \in M$, $U_m \in \Phi(pt)^{-1}(m)$

(2) for families X, X' parametrized by a variety S, the corresponding morphisms are equal $\nu_X = \nu_{X'} : S \to M$ iff $X \sim X'$.

Proof: (1) iff Φ surj. (2) iff Φ inj.

Though a fine moduli space does not exist in many interesting cases, a coarse moduli space exists and that is something we call often the "moduli space".

Remarks: We considered 3 moduli problems; hypersurfaces, complete varieties, and vector bundles. But there is no coarse moduli space for Hyp and VB. For the hypersurface problem, consider the family

$$x(x - \lambda y) = 0 \subset \mathbb{P}^1 \times \mathbb{C}$$

For $\lambda \neq 0$, the hypersurfaces are all equivalent but the fiber over 0 is not equivalent. (Jump phenomenon)

Similarly, for VB, can construct a family X_s such that $X_s \sim X_{s'}$ for $s, s' \neq 0$ but X_s is not similar to X_0. (Choose two line bundles L_0, L_1 of degree 0 and 1 over a Riemann surface. Choose a line in $Ext^1(L_1, L_0)$. This gives us a family over a line $\mathbb{C}$.)

We need to get rid of some hypersurfaces and vector bundles in order to get a separated moduli space.

§3. Moduli and Quotients.

In the previous lecture, we defined fine and coarse moduli spaces. The simplest example of a fine moduli space is a projective space (lines passing through the origin in a complex vector space) or Grassmannians (subspaces of a vector space).

In this lecture, we will learn how the problem of constructing a coarse moduli space is related to the problem of forming a quotient of a variety by a group action.

(1) Local universal property:

In general, given a moduli problem, it is difficult (impossible in many cases) to find a family with universal property but it is not so hard to find a family which satisfies the universal property locally.

Definition: We say a family X parametrized by a variety S has the local universal property if for any family X' parametrized by S' and $s \in S'$ there exists a neighborhood U of s such that $X'|_U \sim \phi^* X$ for some morphism $U \to S$. We say a variety has the local universal property if there is a family with the local universal property.

For example, let us consider the moduli problem End_n.

$A = \{(V, T) \mid \dim V = n, T : V \to V \text{ hom } \}$

$(V, T) \sim (V', T')$ iff there is an isom $h : V \to V'$ such that $T' = hTh^{-1}$.

A family of endomorphisms parametrized by S is a vector bundle E of rank n over S together with a homomorphism $T : E \to E$. Two families (E, T) and (E', T') are equivalent if there is an isomorphism of vector bundles $h : E \to E'$ such that $T' = hTh^{-1}$.

These define a moduli problem End_n. Basically, it is the classification problem of $n \times n$ matrices up to similarity. Let $\mathcal{F}(S)$ be the set of isomorphism classes of families of endomorphisms parametrized by S.

This moduli problem does not have even a coarse moduli space due to the jump phenomenon as we saw in the previous lecture. For instance, consider the morphism $\mathbb{C} \to M(2)$ given by $t \to B_t = (\lambda\ t \,||\, 0, \lambda)$. For $t \neq 0$, the matrix B_t is similar to B_1. Hence they are mapped to the same point in M and so is the matrix B_0.

However we can easily find a family with local universal property. For example, let $S = M(n)$, $F = I_n = S \times \mathbb{C}^n$, and define $T : F \to F$ by $(f, v) = (f, fv)$. Then we get a family.

Proposition: The family F has the local universal property.

Proof: Let (E, T) be any family parametrized by S' and $s \in S'$. Since vector bundles are locally trivial, there is an open subset U of S' containing s such that $E|_U \cong U \times \mathbb{C}^n$. Now by this isomorphism T gives us a morphism $\phi : U \to M(n)$ and certainly the family $(E|_U, T|_U)$ is the pull-back of the family F.

(2) Local universal property and coarse moduli space:

Now let us think about the problem of finding a coarse moduli space when we have a family with local universal property.

Suppose we have a local universal family F parametrized by S. Then we have the following proposition.

Proposition: (1)For any natural transformation $\Phi : \mathcal{F} \to h_M$, consider the morphism $\phi : S \to M$ given by the family F. Then ϕ is constant on equivalence classes, i.e. if $F_s \sim F_{s'}$, then $\phi(s) = \phi(s')$.

(2) Conversely, if $\phi : S \to M$ is any morphism which is constant on equivalence classes, then we have a natural transformation $\Phi : \mathcal{F} \to h_M$ such that ϕ is the morphism associated with the family F.

Proof: (1) Think of s as a morphism $pt \to S$. From the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\Phi(S)} & h_M(S) \\ {\scriptstyle s^*}\downarrow & & {\scriptstyle s^*}\downarrow \\ \mathcal{F}(pt) & \xrightarrow{\Phi(pt)} & h_M(pt) = M \end{array}$$

we have $\phi(s) = s^*(\phi) = s^*(\Phi(S)(F)) = \Phi(pt)(s^*(F)) = \Phi(pt)F_s$. Similarly, $\phi^*(s') = \Phi(pt)F_{s'}$. Since $F_s \sim F_{s'}$, we have $\phi(s) = \phi(s')$.

(2) For any family X' parametrized by S', we have to find a natural morphism $S' \to M$. Since S has the local universal propety, for each point $s \in S'$ there is a neighborhood U and a morphism $U \to S$ such that the pull-back of F by this morphism is $X'|_U$. Compose $U \to S$ with $S \to M$ to get a morphism $U \to M$. For another open set U' with a morphism $U' \to S$ such that the pull-back of F is $X'|_{U'}$ we do the same to get a morphism $U' \to M$. They should be identical on $U \cap U'$ since ϕ is constant on equivalence classes. Hence we get a morphism $S' \to M$. This gives us a natural transformation $\Phi : \mathcal{F} \to h_M$.

Suppose there is a family F parametrized by S. Then the problem of finding a coarse moduli space becomes the following.

Lemma: Suppose there is a family F parametrized by S. Then the coarse moduli space is the variety M together with a morphism $\phi : S \to M$ which is constant on equivalence classes, such that

(1) if $\psi : S \to N$ is any morphism constant on equivalence classes, there is a unique morphism $\gamma : M \to N$ such that $\gamma \circ \phi = \psi$.
(2) each fiber of ϕ consists of only one equivalence class.

(3) Categorical quotient:

In particular, suppose the equivalence classes are the orbits of a group action. The above result gives us motivates the following.

Definition: Let G be a group acting on a variety X. A categorical quotient of X by G is a variety Y together with a morphism $\phi : X \to Y$ which is constant on each orbit, such that for any variety Z and a morphism $\psi : X \to Z$, constant on orbits, there is a unique morphism $\gamma : Y \to Z$ such that $\gamma \circ \phi = \psi$.

We say Y is an orbit space if in addition each fiber of ϕ consists of only one orbit. Obviously, the categorical quotient is unique up to isomorphism.

An obvious consequence is the following.

Proposition: Suppose that there is a family X parametrized by S with the local universal property. Suppose the equivalence classes are the orbits, i.e. $X_s \sim X_t$ iff s, t lie in the same orbit. Then

(1) a coarse moduli space is a categorical quotient of S by G
(2) a categorical quotient is a coarse moduli space iff it is an orbit space.

Proof: Obvious.

Let us finish with an example of categorical quotient.

Consider the set of all $n \times n$ matrices $M(n)$ and the action of $GL(n)$ by conjugation. Since the characteristic polynomial is invariant under conjugation, we get a morphism $\phi : M(n) \to k^n$ by considering the coefficients of the characteristic polynomial.

Proposition: $\phi : M(n) \to k^n$ is a categorical quotient.

Proof: Suppose $\psi : M(n) \to Z$ is constant on each orbit.

Claim: if two matrices have the same characteristic polynomial then their images in Z are identical. (Consider the Jordan canonical form. Note $(\lambda, 1//0, \lambda) \sim (\lambda, t//0, \lambda)$ for $t \neq 0$ and thus it has the same image as the diagonal matrix.)

Hence ϕ factors through a map $\gamma : k^n \to Z$. It suffices to show that γ is a morphism. But this is obvious since γ is the composition of $k^n \to M(n)$ by $C_t = (0, 0, -t_3//1, 0, -t_2//0, 1, -t_1)$ with ψ.

대수기하 특강 – 7강
Chapter 2. Quotients

In the previous lecture, we learned how the problem of constructing a coarse moduli space is related to the problem of constructing a quotient. From now on, we will focus on the problem of constructing the quotients.

§1. Actions of algebraic groups

(1) Algebraic groups.

We first learn the definition of algebraic groups.

Definition: (i) an algebraic group G is a variety with a group structure such that the multiplication $\mu : G \times G \to G$ and the inverse $G \to G$ are morphisms.

(ii) an algebraic group G is affine if the variety G is an affine variety.

(iii) an algebraic group G is linear if it is a closed subgroup of $GL(n)$ for some n.

Remark: an algebraic group is affine iff linear.

Suppose G is affine. Let $\mathcal{O}(G)$ be the ring of regular functions on G. Then the multiplication gives the comultiplication $\mu^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ and the inverse gives the coinverse $\mathcal{O}(G) \to \mathcal{O}(G)$. (The axioms they satisfy are formulated as "Hopf algebra".) This is a way of giving a group structure on a variety G.

Example: (i) additive group $\mathbf{G}_a^n = \mathbb{C}^n$. We have $\mathcal{O}(\mathbb{C}^n) = k[z_1, \cdots, z_n]$. Consider the map $z_i \to z_i \otimes 1 + 1 \otimes z_i$ and $z_i \to -z_i$. These make $\mathbf{G}_a^n$ an abelian algebraic group.

(ii) multiplicative group $\mathbf{G}_m = \mathbb{C}^*$. We have $\mathcal{O}(\mathbb{C}^*) = k[z, z^{-1}]$. The multiplication is given by $z \to z \otimes z$ and the inverse is given by $z \to z^{-1}$. We call $\mathbf{G}_m^n = (\mathbb{C}^*)^n$ a torus.

(iii) general linear group $GL(n)$. It is the complement of the closed subvariety $det = 0$ in $\mathbb{C}^{n^2}$. Thus it is affine and we have $\mathcal{O}(G) = k[z_{ij}, det(z_{ij})^{-1}]$. The multiplication is given by $z_{ij} \to \sum_k z_{ik} \otimes z_{kj}$.

Definition: a homomorphism of algebraic groups G, G' is a morphism $\phi : G \to G'$ which is also a group homomorphism, i.e.

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ G' \times G' & \longrightarrow & G'. \end{array}$$

(2) Algebraic group actions.

Next, we think about algebraic group actions.

Definition (i) an action of an algebraic group G on a variety X is a morphism $\sigma : G \times X \to X$ such that the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mu \times 1} & G \times X \\ {\scriptstyle 1\times\sigma}\big\downarrow & & {\scriptstyle \sigma}\big\downarrow \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

commutes and the composition $X \to^{e\times 1} G \times X \to^{\sigma} X$ is the identity.

(ii) For $x \in X$, the stabilizer of x is $G_x = \{g \in G : gx = x\}$. The orbit of x is $Gx = \{gx : g \in G\}$.

(iii) a point $x \in X$ is invariant under G if $gx = x$ for every $g \in G$. A subset W of X is invariant if $gW \subset W$ for every $g \in G$.

(iv) given an action of G on X, Y, a morphism $\phi : X \to Y$ is equivariant (or G-morphism) if $\phi(gx) = g\phi(x)$. We say ϕ is invariant if ϕ is equivariant and G acts trivially on Y.

Suppose G and X are affine. An action is given by a homomorphism $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G) \times \mathcal{O}(X)$, called coaction. The diagram

$$\begin{array}{ccc} \mathcal{O}(G) \otimes \mathcal{O}(X) & \xleftarrow{\sigma^*} & \mathcal{O}(X) \\ {\scriptstyle 1\otimes\sigma^*}\downarrow & & {\scriptstyle \sigma^*}\downarrow \\ \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(X) & \xleftarrow{\mu^*\otimes 1} & \mathcal{O}(G) \otimes \mathcal{O}(X) \end{array}$$

commutes and $\mathcal{O}(X) \to^{\sigma^*} \mathcal{O}(G) \otimes \mathcal{O}(X) \to^{e\otimes 1} \mathcal{O}(X)$.

A function $f \in \mathcal{O}(X)$ is G-invariant if $\sigma^*(f) = 1 \otimes f$. (This means $f(gx) = f(x)$ for all $g \in G$.) We let $\mathcal{O}(X)^G$ be the subalgebra of G-invariant functions.

Example: Suppose $\mathbf{G}_m = \mathbb{C}^*$ acts on an affine variety X. Then we have a homomorphism $\sigma^* : \mathcal{O}(X) \to k[z, z^{-1}] \otimes \mathcal{O}(X)$. Let $\sigma^*(f) = \sum_{i\in\mathbb{Z}} z^i \otimes f_i$. The assignment $f \to f_i$ gives us a map $\mathcal{O}(X) \to \mathcal{O}(X)$. Since $\sigma^*(f_i) = z^i \otimes f_i$,[7] p_i is a projection. Let $\mathcal{O}(X)_i = p_i(\mathcal{O}(X))$. Then $\mathcal{O}(X) \oplus_i \mathcal{O}(X)_i$.[8] Hence, a $\mathbb{C}^*$ action on an affine variety X gives us a $\mathbb{Z}$-grading on $\mathcal{O}(X)$.

Conversely, suppose we are given a $\mathbb{Z}$-grading of $\mathcal{O}(X) = \oplus_i \mathcal{O}(X)_i$. Then define $\sigma^* : \mathcal{O}(X) \to k[z, z^{-1}] \otimes \mathcal{O}(X)$ by $f = \sum f_i \to \sum z^i \otimes f_i$.

Proposition: For an affine variety X, there is a bijection

$$\{\mathbb{C}^*\text{-actions on } X\} \leftrightarrow \{\mathbb{Z}\text{-grading on } \mathcal{O}(X)\}.$$

Example: any homomorphism $\phi : G \to GL(n)$ gives rise to an action of G on k^n by $g \cdot v = \phi(g)v$, matrix multiplication. Such a homomorphism is called a rational representation and such an action is called a linear action.

(3) Rational action.

In order to solve a geometric problem about group actions, we sometimes need to convert the problem into a purely algebraic problem. For this purpose, we need to extract some algebraic properties of algebraic group actions.

Lemma Let G be an algebraic group acting on a variety X. Let W be a finite dimensional subspace of $\mathcal{O}(X)$. Then we have (i) W is contained in a finite dimensional *invariant* subspace of $\mathcal{O}(X)$, (ii) if W is invariant, the action of G on W is given by a rational representation.

Proof: (i) Find a basis $f_1, \cdots, f_r$ of W. Let W' be the subspace spanned by f_i^g for all i and $g \in G$. Certainly W' is invariant and it suffices to show that W' is finite dimensional. Let $\sigma^*(f_i) = \sum \rho_{ij} \otimes f_{ij}$ and W'' be the finite dimensional subspace spanned by f_{ij}. Since $f_i^g = \sum \rho_{ij}(g) f_{ij}$, $W' \subset W''$ which implies that W' is finite dimensional.

(ii) $\sigma^*(f_i) = \sum \rho_{ij} \otimes f_j$, $\rho_{ij} \in \mathcal{O}(G)$ regular function on G. Hence we get a morphism $\rho = (\rho_{ij}) : G \to M(n)$. This has to factor through $GL(n)$ since every element of G is invertible.

Definition: Let G be an algebraic group, R be a k-algebra. A rational action of G on R is a map $R \times G \to R$ such that

(1) $f^{gg'} = (f^g)^{g'}$, $f^e = f$

(2) the map $f \to f^g$ is a k-algebra automorphism of R for all $g \in G$

[7] $\sum z^i \otimes (z^j \otimes f_{ij}) = \sum z^i \otimes z^i \otimes f_i$.

[8] $f \to \sum z^i \otimes f_i \to \sum 1 \otimes f_i \to \sum f_i = f$.

(3) every element of R is contained in a finite dimensional invariant subspace on which G acts by a rational representation.

대수기하 특강 – 8강

In the previous lecture, we learned about (1) algebraic groups (2) algebraic group actions (3) rational actions. Today, we will think about reductive groups.

(4) Reductive groups.

Most of the algebraic groups we shall deal with are reductive groups like torus, SL(n), GL(n), PGL(n). The definition is as follows.

Definition: a linear algebraic group is reductive (resp. semisimple) if a maximal solvable connected normal subgroup (radical) is a torus (resp. trivial).

Remarks: (1) A complex algebraic group is reductive iff it is the complexification of a compact Lie group i.e. G has a maximal compact subgroup K such that $Lie(K) \otimes_{\mathbb{R}} \mathbb{C} = Lie(G)$. A compact connected Lie group is the product of a torus and a semisimple Lie group modulo a finite group action.

(2) Any rational representation of a reductive group is completely reducible, i.e. it is a direct sum of irreducible representations. Weyl's theorem.

Definition (1) a linear algebraic group G is linearly reductive if for any linear action of G on k^n, and every invariant point $v \in k^n$, there is an invariant homogeneous polynomial f of degree 1 such that $f(v) \neq 0$.

(2) a linear algebraic group G is geometrically reductive if for any linear action of G on k^n, and every invariant point $v \in k^n$, there is an invariant homogeneous polynomial f of degree $d \geq 1$ such that $f(v) \neq 0$.

Remark: a reductive group is linearly reductive: Put $k^n = kv \oplus V$. Consider the projection onto kv. This gives us a homogeneous invariant polynomial of degree 1 which does not vanish on v. Of course, a linearly reductive group is geometrically reductive. Also, it was proved that every geometrically reductive group is reductive [Nagata and Miyata]. Hence, the three definitions are all equivalent.

A consequence of reductivity is the following lemma.

Lemma: Let G be a reductive group acting on an affine variety X. Let W_1, W_2 be disjoint closed invariant subsets of X. Then there is $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$, $f(W_2) = 1$.

Proof: By (HW1), there is $h \in \mathcal{O}(X)$ such that $h(W_1) = 0$, $h(W_2) = 1$. Consider the subspace spanned by h^g for $g \in G$ is finite dimensional. Choose a basis $h_1, \cdots, h_n$. This gives a morphism $\psi : X \to k^n$ which is equivariant. We have $\psi(W_1) = 0$ and $\psi(W_2)$ is a nonzero invariant vector. Since geometrically reductive, there is a homogeneous invariant function g on k^n such that $g(v) = 1$. Let $f = g \circ h$.

§2. Nagata's Theorem.

We are interested in constructing categorical quotients.

Let X be a G-space. An invariant morphism $X \to Z$ (i.e. constant on orbits) induces $\mathcal{O}(Z) \to \mathcal{O}(X)$ which factors through $\mathcal{O}(X)^G$. Suppose the categorical quotient Y of X by G is affine. Then, $\mathcal{O}(Y) = \mathcal{O}(X)^G$. In order to have an affine quotient, we need to know whether $\mathcal{O}(X)^G$ is finitely generated. (There are no nilpotents in $\mathcal{O}(X)^G$ since $\mathcal{O}(X)$ is already reduced.)

Question: Given a rational action of G on a finitely generated k-algebra R, is the invariant subalgebra R^G finitely generated?

In general, the answer to this question is No. But when G is a reductive group, it is Yes. This is the content of Nagata's theorem!

Recall that we assume $char(k) = 0$ all the time.

Theorem: Let G be a reductive group acting rationally on a finitely generated k-algebra R. Then R^G is finitely generated.

Remark: Popov proved the converse, i.e. if R^G is finitely generated for any rational action of G on a finitely generated k-algebra R, then G is reductive.

Simple case (Hilbert): Let G be a reductive group acting on k^n linearly. Let $A = \mathcal{O}(k^n) = k[z_1, \cdots, z_n]$, $A^G = \oplus_d A_d^G$. Since a linear representation is completely reducible, we have a G-invariant projection $r_d : A_d \to A_d^G$. So we have a unique linear map $r : A \to A^G$ which is an A^G-module homomorphism, i.e. $r(ab) = a\,r(b)$ for $a \in A^G, b \in A$.

Now let I be the ideal of A generated by homogeneous polynomials of positive degree in A^G. Hilbert Basis Theorem says I is generated by a finite set $f_1, \cdots, f_N$ where $f_i \in A_{d_i}^G$. For any $f \in A_d^G$, $f = \sum a_i f_i$ where $a_i \in A_{d-d_i}$. We have $f = r(f) = \sum r(a_i) f_i$. By induction on degree of f, f is a polynomial of f_i. Therefore, A^G is finitely generated.

General case: Let G be a reductive group acting on a finitely generated k-algebra rationally. Choose a set of generators $f_1, \cdots, f_n$ such that $Span(f_1, \cdots, f_n)$ is G-invariant. Let $f_i^g = \sum \alpha_{ij}(g) f_j$. Let $A = k[z_1, \cdots, z_n]$ and consider the action of G on A given by $z_i^g = \sum \alpha_{ij}(g) z_j$. Then we have a k-algebra homomorphism $A \to S = A/I$ which commutes with the G-action. So it suffices to prove the following.

Lemma: Let G be a reductive group acting rationally on a k-algebra A. Let I be an invariant ideal. Then we have

$$(A/I)^G = A^G/I \cap A^G.$$

Proof: $(\supset)$ is obvious. We prove $(\subset)$. Let h be an element of A whose image $\bar{h} \in A/I$ is a nonzero element in $(A/I)^G$. It suffices to find $f \in A^G$ such that $f - h \in I$.

Let $M = Span\{h^g | g \in G\}$, finite dimensional since the action is rational. Let $N = M \cap I$. Since $\bar{h} \neq 0$, $h \notin N$. But $h^g - h \in M \cap I = N$ for all $g \in G$. This implies that $\dim M = \dim N + 1$, i.e. $M = N \oplus k$. Let $l : M \to k$ be the map $ah + h' \to a$. Then l is G-invariant since $(ah+h')^g = ah^g + (h')^g = ah + a(h^g - h) + (h')^g \in ah + N$. Therefore $l \in (M^*)^G$.

Choose a basis of M such that $v_1 = h$, $v_2, \cdots, v_n \in N$. Then $(M^*) \cong k^n$ and $l = (1, 0, \cdots, 0)$. Since linearly reductive, there is a G-invariant linear function f on M^* such that $f(l) \neq 0$. But $(M^*)^* = M = Span(v_1, \cdots, v_n)$ and we may assume $f = h + a_2 v_2 + \cdots + a_n v_n \in M^G$. Hence we found $f \in M^G \subset A^G$ such that $f - h = a_2 v_2 + \cdots + a_n v_n \in N \subset I$. So we are done.

Complete the proof of Nagata's theorem.

§3. Affine Quotients.

We are now ready to construct the quotients. Let's start with affine varieties.

Let G be a reductive group acting on an affine variety X. Consider the invariant subalgebra $\mathcal{O}(X)^G$ of $\mathcal{O}(X)$. Then we know it is finitely generated with no nilpotents. Hence there is an affine variety Y such that $\mathcal{O}(Y) = \mathcal{O}(X)^G$. The inclusion $\mathcal{O}(Y) = \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ gives rise to a morphism of affine varieties $\phi : X \to Y$. (Warm-up HW 3.) This morphism satisfies the following properties.

Theorem:

(1) ϕ is G-invariant
(2) ϕ is surjective
(3) for each open subset U of Y, the pull-back $\phi^* : \mathcal{O}(U) \to \mathcal{O}(\phi^{-1}(U))$ is an isomorphism onto $\mathcal{O}(\phi^{-1}(U))^G$, i.e. $\mathcal{O}_Y \cong [\phi_*(\mathcal{O}_X)]^G$.
(4) if W is a closed invariant subset of X, then $\phi(W)$ is closed
(5) if W_1, W_2 are disjoint invariant closed subsets of X, then $\phi(W_1)\cap\phi(W_2) = \emptyset$.

Proof: (1) Suppose $\phi(gx) \neq \phi(x)$. Choose a regular function $f \in \mathcal{O}(Y)$ such that $f(\phi(gx)) = 0$, $f(\phi(x)) = 1$. Then $\phi^*(f) = f \circ \phi$ is not in $\mathcal{O}(X)^G$. Contradiction.

(2) Let $R = \mathcal{O}(X)$. We want to show that given a maximal ideal m of R^G, we can find a maximal ideal m' of R such that $m' \cap R^G = m$.

Choose a set of generators $f_1, \cdots, f_n$. Consider the ideal $\sum f_i R$ of R. Suppose $\sum f_i R \neq R$. Then we can choose a maximal ideal m' of R containing $\sum f_i R$. The intersection $m' \cap R^G$ is a maximal ideal (since it cannot contain 1) containing m and thus $m' \cap R^G = m$.

So it remains to show that given $f_1, \cdots, f_n \in R^G$ such that $\sum f_i R = R$, we have $\sum f_i R^G = R^G$: We use induction on n. Let $\bar{R} = R/f_1 R$. Then $\sum f_i R = R$ implies that $\sum_{i=2}^n \bar{f}_i \bar{R} = \bar{R}$. By induction hypothesis, $\sum_{i=2}^n \bar{f}_i \bar{R}^G = \bar{R}^G$. Hence

$$1 = \sum_{i=2}^n \bar{f}_i \bar{a}_i$$

for some $\bar{a}_i \in \bar{R}^G$.

Since $\bar{R}^G = R^G / f_1 R \cap R^G$ by the lemma we proved above, we can find $a_i \in R^G$ whose image in R is $\bar{a}_i$. Hence

$$1 - \sum_{i=2}^n f_i a_i = b_1 f_1$$

for some $b_1 \in R$. Let f be the left hand side of the equation. Then $f = f_1 b_1 \in R^G$. Thus $f_1(b_1^g - b_1) = f^g - f = 0$.

Let $J = \{h \in R : f_1 h = 0\}$, ideal in R. Then the image $\bar{b}_1$ of b_1 in R/J lies in $(R/J)^G = R^G / J \cap R^G$. Hence there is an element $a_1 \in R^G$ such that $b_1 - a_1 \in J$. This implies that

$$f = f_1 b_1 = f_1 a_1 + f_1(b_1 - a_1) = f_1 a_1$$

and $1 = \sum_{i=1}^n f_i a_i \in \sum f_i R^G$. The proof in the case where $n = 1$ is an exercise.

(3) Since $\mathcal{O}$ is a sheaf, it suffices to show $\mathcal{O}(U) \cong \mathcal{O}(\phi^{-1}(U))^G$ for $U = Y_f$ since they form a basis. But $\mathcal{O}(Y_f) = \mathcal{O}(Y)_f$ and $\mathcal{O}(\phi^{-1}(Y_f)) = \mathcal{O}(X)_f$. So it suffices to show that $(R^G)_f = (R_f)^G$ for $f \in R^G$. The natural homomorphism $R \to R_f$ induces $R^G \to (R_f)^G$ which factors through $(R^G)_f$ (since the image of f is invertible). It is easy to show that this is an isomorphism.

(5) We proved that there is $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$, $f(W_2) = 1$. Since $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)^G$ is an isomorphism, there is $g \in \mathcal{O}(Y)$ such that $f = g \circ \phi$. Hence, $g(\phi(W_1)) = 0$, $g(\phi(W_2)) = 1$. Thus $\overline{\phi(W_1)} \cap \overline{\phi(W_2)} = \emptyset$.

(4) Suppose $\overline{\phi(W)} - \phi(W) \neq \emptyset$. Choose a point y in the set. Consider the fiber $\phi^{-1}(y)$ over y. Then W and $\phi^{-1}(y)$ are disjoint invariant closed subsets of X. But $\overline{\phi(W)} \cap \phi(\phi^{-1}(y)) \neq \emptyset$. Contradiction.

대수기하 특강 – 10강

Finish the proof of the Theorem from last time.

We can generalize (5) slightly.

Lemma: Let U be an open subset of Y. If W_1, W_2 are disjoint invariant subsets of $\phi^{-1}(U)$, closed in $\phi^{-1}(U)$, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

Proof: Suppose $y \in \phi(W_1) \cap \phi(W_2)$. Consider the closed subsets $\phi^{-1}(y)$, $\bar{W}_1, \bar{W}_2$. Since their images in Y intersect, the closed sets $\phi^{-1}(y) \cap \bar{W}_1$ and $\bar{W}_2$ intersect. Hence,

$$W_1 \cap W_2 = \phi^{-1}(U) \cap \bar{W}_1 \cap \bar{W}_2 \supset \phi^{-1}(y) \cap \bar{W}_1 \cap \bar{W}_2 \neq \emptyset.$$

Contradiction.

Now, we prove that $\phi : X \to Y$ is a categorical quotient.

Corollary: For any open subset U of Y, (U, ϕ) is a categorical quotient of $\phi^{-1}(U)$.

Proof: Let $\psi : \phi^{-1}(U) \to Z$ be a G-invariant morphism. We want to show that there is a unique morphism $\chi : U \to Z$ such that $\chi \circ \phi = \psi$.

Simple case: Suppose Z is affine. Then ψ gives us a homomorphism $\mathcal{O}(Z) \to \mathcal{O}(\phi^{-1}(U))$. Since G-invariant it must factor through $\mathcal{O}(\phi^{-1}(U))^G = \mathcal{O}(U)$. The homomorphism $\mathcal{O}(Z) \to \mathcal{O}(U)$ gives rise to a morphism $\chi : U \to Z$. (Warm-up HW 3.) By construction, we have $\psi = \chi \circ \phi$.

General case: We first show that there is a unique map $\chi : U \to Z$. For this we need to show that for any $y \in U$, $\psi(\phi^{-1}(y))$ consists of a single point. Suppose $z_1, z_2 \in \psi(\phi^{-1}(y))$. Then $\psi^{-1}(z_1)$, $\psi^{-1}(z_2)$ are two disjoint invariant closed subsets of $\phi^{-1}(U)$. By the lemma above, $\phi(\psi^{-1}(z_1) \cap \phi(\psi^{-1}(z_2) = \emptyset$ but it contains y. Contradiction. Hence, we have a well-defined set-theoretic map $\chi : U \to Z$.

Next we claim that χ is continuous. Let V be an open set in Z. Note that each fiber $\psi^{-1}(z) = \cup_{\chi(y)=z} \phi^{-1}(y)$. Hence $\chi^{-1}(z) = \phi(\psi^{-1}(z))$ and thus

$$\chi^{-1}(V) = \phi(\psi^{-1}(V)) = U - \phi(\phi^{-1}(U) - \psi^{-1}(V))$$

since ϕ is surjective. By (4) in the above theorem, we see that $\chi^{-1}(V)$ is open.

Finally, we claim that χ is a morphism. Let V be an affine open subset of Z. Consider the open subsets $\chi^{-1}(V)$ and $\psi^{-1}(V) = \phi^{-1}(\chi^{-1}(V))$ of U and $\phi^{-1}(U)$ respectively. Then we are in the situation of the simple case. So we are done.

Let's see an example.

Example: Consider the action of $G = \mathbb{Z}_2 = \{1, -1\}$ on $\mathbb{C}^2$ by $(-1) \cdot (t_1, t_2) = (-t_1, -t_2)$. What is $\mathbb{C}^2/\mathbb{Z}_2$?

$\mathcal{O}(\mathbb{C}^2) = k[t_1, t_2]$ and the invariant subalgebra is

$$\mathcal{O}(\mathbb{C}^2)^G = k[t_1^2, t_1t_2, t_2^2] = k[x_1, x_2, x_3]/(x_1x_3 - x_2^2)$$

where $x_1 = t_1^2$, $x_2 = t_1t_2$, $x_3 = t_2^2$. Hence we have $\mathcal{O}(\mathbb{C}^2/\mathbb{Z}_2) = k[x_1, x_2, x_3]/(x_1x_3 - x_2^2)$ and the quotient is the hypersurface of $\mathbb{C}^3$ given by the equation $x_1x_3 = x_2^2$. The map $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}_2 \to \mathbb{C}^3$ is given by $(t_1, t_2, t_3) \to (t_1^2, t_1t_2, t_2^2)$.

대수기하 특강 – 11강

In the previous lecture, we proved the following.

Let X be an affine variety acted on by a reductive group G. Let Y be an affine variety satisfying $\mathcal{O}(Y) \cong \mathcal{O}(X)^G$ and $\phi : X \to Y$ be the morphism induced from the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$. Then

(1) ϕ is invariant
(2) ϕ is surjective
(3) for any open subset U of Y, the restriction $\phi : \phi^{-1}(U) \to U$ induces an isomorphism $\mathcal{O}(U) \cong \mathcal{O}(\phi^{-1}(U))^G$
(4) the image of an invariant closed subset W of X is closed in Y[9]
(5) if W_1, W_2 are disjoint invariant closed subsets of X, $\phi(W_1) \cap \phi(W_2) = \emptyset$.[10]

We also proved that $\phi : X \to Y$ is a categorical quotient. In general, when X is an arbitrary variety, we wish to construct the quotient by gluing the local affine quotients. The above results motivate the following defintion.

Definition: Let G be an algebraic group acting on a variety X. A good quotient of X by G is an affine morphism $\phi : X \to Y$ of varieties satisfying (1)-(5) above. A geometric quotient is a good quotient which is also an orbit space.

The proofs I gave last time also prove the following.

Proposition: Let $\phi : X \to Y$ is a good quotient of X by G. Then ϕ is a categorical quotient.

In general, a good quotient is a categorical quotient but not an orbit space.

Proposition: Let $\phi : X \to Y$ be a good quotient. Then

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset.$$

Proof: ($\Leftarrow$) If $\phi(x_1) \neq \phi(x_2)$, $\phi^{-1}(\phi(x_1))$ and $\phi^{-1}(\phi(x_2))$ are disjoint invariant closed subsets which contain $\overline{Gx_1}$ and $\overline{Gx_2}$ respectively.

($\Rightarrow$) The subsets $\overline{Gx_1}$ and $\overline{Gx_2}$ are closed invariant subsets. If they are disjoint, their images are disjoint, i.e. $\phi(x_1) \neq \phi(x_2)$.

Even if a good quotient is not an orbit space, its restriction to a suitable open subset may be an orbit space.

Proposition: Let $\phi : X \to Y$ be a good quotient. Let U be an open subset of Y. Suppose the action of G on $\phi^{-1}(U)$ is closed (i.e. the orbits are closed). Then U is an orbit space.

Proof: We must show that each fiber consists of only one orbit, i.e. $\phi(x_1) = \phi(x_2) \Rightarrow Gx_1 = Gx_2$. Suppose $Gx_1 \neq Gx_2$. Then Gx_1, Gx_2 are disjoint closed invariant subsets. Hence $\phi(Gx_1) \neq \phi(Gx_2)$.

[9] More generally, the image of an invariant subset of $\phi^{-1}(U)$ which is closed in $\phi^{-1}(U)$ is closed in U.

[10] More generally, if W_1, W_2 are disjoint invariant subsets of $\phi^{-1}(U)$ closed in $\phi^{-1}(U)$ then their images are disjoint.

In order to find an open subset U of Y such that the action of G on $\phi^{-1}(U)$ is closed, we need the following lemma.

Lemma: Let G be an algebraic group acting on a variety X. Then

(1) for any $x \in X$, Gx is an open subset of $\overline{Gx}$ and $\overline{Gx} - Gx$ is a union of orbits of dimension $< \dim Gx$

(2) for any $x \in X$, $\dim Gx = \dim G - \dim G_x$

(3) $\{x \in X : \dim Gx \geq n\}$ is open for any integer n.

Proof: (1) Consequence of Chevalley's theorem. See any book on algebraic groups (e.g. Borel's).

(2) The morphism $G \to Gx$ has fiber G_x.

(3) Consider the morphism $G \times X \to X \times X$ given by $(g, x) \to (x, gx)$. The fiber over (x, x) is $G_x \times \{x\}$. Since the dimension of fiber is an upper semi-continuous function, the function $x \to \dim Gx$ is lower semi-continuous.

Now, we can describe the subset X' of X such that the restriction of ϕ is a geometric quotient.

Proposition: Let X' be the subset of points $x \in X$ such that $\dim Gx$ is maximal and Gx is closed in X. Then there is an open subset Y' of Y such that $\phi^{-1}(Y') = X'$. Furthermore, the restriction $\phi : X' \to Y'$ is a geometric quotient for the action of G on X'.

Proof: Let X^{max} be the subset of points $x \in X$ such that $\dim Gx$ is maximal. We have to get rid of the orbits in X^{max}, which are not closed. Let $Y' = Y - \phi(X - X^{max})$. This is an open subset since X^{max} is open by the above lemma and $X - X^{max}$ is closed invariant. Obviously $X' \subset \phi^{-1}(Y')$.

Conversely, if $x \notin X^{max}$, $\phi(x) \notin Y'$. If Gx is not closed, for any $y \in \overline{Gx} - Gx$, $\dim Gy < \dim Gx$, $y \notin X^{max}$, $\phi(x) = \phi(y) \notin Y'$, $x \notin \phi^{-1}(Y')$.

The concept of good quotient (resp. geometric quotient) is local.

Proposition: If $\phi : X \to Y$ is a morphism and $\{U_i\}$ is an open covering of Y such that $\phi|_{\phi^{-1}(U_i)}$ is a good (resp. geometric) quotient for each i , then ϕ is a good (resp. geometric) quotient of X by G. Conversely, if $\phi : X \to Y$ is a good (geometric quotient) of X by G and U is open in Y, then $\phi : \phi^{-1}(U) \to U$ is a good (geometric) quotient.

The proof is obvious and we omit it.

We end this lecture with the following proposition.

Proposition: Let $\psi : X_1 \to X$ be an affine equivariant morphism of G-varieties. If X has a good quotient, say $\phi : X \to Y$ then X_1 has a good quotient $\phi' : X_1 \to Y_1$ and the induced morphism $\psi' : Y_1 \to Y$ is affine.

Proof: (Sketch) Find an affine open covering $\{V_i\}$ of Y. Since ϕ and ψ are affine, $\psi^{-1}(\phi^{-1}(V_i))$ is affine and thus we have a good quotient $\phi'_i : \psi^{-1}(\phi^{-1}(V_i)) \to V'_i$. Since a good quotient is a categorical quotient, we get a morphism $\psi'_i : V'_i \to V_i$. Over the intersection $V_i \cap V_j$, the varieties $(\psi'_j)^{-1}(V_i \cap V_j)$ and $(\psi'_i)^{-1}(V_i \cap V_j)$ are both good quotients of $\psi^{-1}(\phi^{-1}(V_i \cap V_j))$. Hence there is an isomorphism $\beta_{ij} : (\psi'_j)^{-1}(V_i \cap V_j) \to (\psi'_i)^{-1}(V_i \cap V_j)$. We may glue V'_i with V'_j using this isomorphism to get a variety Y_1 and a morphism $\psi' : Y_1 \to Y$. From the construction, it is obvious that ψ' is affine.

대수기하 특강 – 12강

§4. Projective Quotients.

Today, we think about the problem of constructing a good quotient of a projective variety X by a reductive group G. It is expected that the quotient is again projective.

Suppose $X \subset \mathbb{P}^n$ is a projective variety and $\hat{X}$ be the affine subvariety of $\mathbb{C}^{n+1}$ lying over X, i.e. the homogeneous polynomial equations of X define $\hat{X}$. Suppose a reductive group G acts on $\hat{X}$ via a homomorphism $G \to GL(n+1)$. We have an induced action of G on X. Our goal is to find the quotient of X by G.

Definition: Let X be a projective variety in $\mathbb{P}^n$. A linear action of an algebraic group G on X is an action of G via a homomorphism $G \to GL(n+1)$.[11]

Notice that the center of $GL(n+1)$ is a torus $\mathbb{C}^*$ and so the action of G commutes with the action of $\mathbb{C}^*$. Hence we have an action of $G \times \mathbb{C}^*$ on $\hat{X}$.

There are two ways to take the quotient of $\hat{X}$ by $G \times \mathbb{C}^*$. We may

(1) take the quotient of $\hat{X}$ by G and then by $\mathbb{C}^*$ or
(2) take the quotient of $\hat{X}$ by $\mathbb{C}^*$ and then by G.

It is obviously expected that we should get the same results "if there is justice on earth".

Let us make clear what we mean by "quotient by $\mathbb{C}^*$". We know the quotient of $\mathbb{C}^{n+1} - 0/\mathbb{C}^*$ is $\mathbb{P}^n$ and the quotient $\hat{X} - 0/\mathbb{C}^*$ is X. Notice that $\mathbb{P}^n = \mathrm{Proj}\mathcal{O}(\mathbb{C}^{n+1})$ and $X = \mathrm{Proj}\mathcal{O}(\hat{X})$.[12] In general, if $Z = \mathrm{Spec}(A)$ is an affine variety with an action of $\mathbb{C}^*$ which equips $\mathcal{O}(Z)$ with a $\mathbb{Z}_{\geq 0}$-grading, the variety $\mathrm{Proj}\mathcal{O}(Z)$ can be thought of as the quotient of $Z - 0$ by $\mathbb{C}^*$.

(1) Consider the first method: Let $R = \mathcal{O}(\hat{X})$. The quotient of $\hat{X}$ by G is $\hat{\phi} : \hat{X} = \mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$. Now the quotient of $\mathrm{Spec}(R^G)$ by $\mathbb{C}^*$ is just $\mathrm{Proj}(R^G)$.

(2) Next the second method: The quotient of $\hat{X}$ by $\mathbb{C}^*$ is $X = \mathrm{Proj}(R)$. Thus the result we get by the second method should be the quotient of X by G.

Consequently, we should have $X//G = \mathrm{Proj}(R^G)$ if $X = \mathrm{Proj}(R)$.[13]

What is the quotient map then? To answer this question, we think about the first method above. We have the quotient morphism $\hat{\phi} : \hat{X} \to \hat{X}//G$. In order to take the quotient of $\hat{X}//G$ by $\mathbb{C}^*$, we have to get rid of the vertex $v = \hat{\phi}(0)$ of the cone $\hat{X}//G = \mathrm{Spec}(R^G)$. Hence we have to remove $\hat{\phi}^{-1}(v)$ from $\hat{X}$. But we know $\hat{\phi}(x) \neq \hat{\phi}(0)$ iff $\overline{Gx} \cap \overline{G0} = \emptyset$ iff $\exists f \in R^G$ such that $f(x) \neq 0$ but $f(0) = 0$. By lifting f to $\mathcal{O}(\mathbb{C}^{n+1})$ and taking a homogeneous part, the last condition is equivalent to saying that $\exists f$ nonconstant homogeneous invariant polynomial satisfying $f(x) \neq 0$. Let $\hat{X}^{ss}$ be the set of such points. Then we have morphism $\hat{X}^{ss} \to \hat{X}//G - v$. Now

[11] In other words, X is an invariant subset of $\mathbb{P}^n$ by the action of G on $\mathbb{P}^n$ via the homomorphism $G \to GL(n+1)$.

[12] Recall that Proj of a graded ring $A = \oplus_{d \geq 0} A_d$ is the set of homogeneous prime ideals, not equal to $\oplus_{d>0} A_d$, with an affine open covering $\{Spec(A_{(f)}) \,|\, f \in A_d \text{ for some } d > 0\}$ where $A_{(f)}$ is the subring of elements of degree 0 in the localized ring A_f. If A is finitely generated, then $\mathrm{Proj}(A)$ is always projective. (If A is generated by $f_1, \cdots, f_r$ with $\deg(f_i) = d_i$, then let $d = \prod_i d_i$ and consider $S = \oplus r \geq 0 S_r$ with $S_r = A_{rd}$. Then S is generated by S_1 and thus $\mathrm{Proj}(S)$ is projective by [Hartshorne, II §5, Corollary 5.16]. By [Hartshorne, II §5, Exercise 5.13], we have $\mathrm{Proj}(A) \cong \mathrm{Proj}(S)$.)

[13] Compare this with the affine case: the quotient of $\mathrm{Spec}(R)$ by G is $\mathrm{Spec}(R^G)$.

we may take the quotients by $\mathbb{C}^*$ and we get a morphism $\phi : X^{ss} \to X//G$ where X^{ss} is the image of $\hat{X}^{ss}$ via the quotient map $\mathbb{C}^{n+1} - 0 \to \mathbb{P}^n$. We shall see that this is a good quotient. We proved last time that if we restrict ϕ to the set of points whose orbits are closed in X^{ss} with maximal dimension, then we get a geometric quotient. So we make the following definition.

Definition: Let X be a projective variety in $\mathbb{P}^n$ on which a reductive group G acts linearly.

(1) A point $x \in X$ is semi-stable if there is a nonconstant invariant homogeneous polynomial f such that $f(x) \neq 0$.
(2) A point $x \in X$ is stable if $\dim Gx = \dim G$ and there is a nonconstant invariant homogeneous polynomial f such that $f(x) \neq 0$ and the action of G on X_f is closed.

It is obvious from the definition that X^{ss} and X^s are open subsets of X.

We summarize the above discussions into the following theorem.

Theorem: Let X be a projective variety in $\mathbb{P}^n$ on which a reductive group G acts linearly. Then

(1) there is a good quotient $\phi : X^{ss} \to Y$ of X^{ss} by G and Y is projective.[14]
(2) there is an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and $\phi|_{X^s}$ is a geometric quotient.
(3) for $x_1, x_2 \in X^{ss}$, $\phi(x_1) = \phi(x_2)$ iff $\overline{Gx_1} \cap \overline{Gx_2} \cap X^{ss} \neq \emptyset$.
(4) a semi-stable point x is stable iff $\dim Gx = \dim G$ and Gx is closed in X^{ss}.

Proof: We constructed $\phi : X^{ss} \to X//G =: Y$. For (1), check that $Y = \mathrm{Proj}(R^G)$ is covered by affine open sets Y_f and $\phi^{-1}(Y_f) = X_f$. It is easy to see that $\phi : X_f \to Y_f$ is the affine quotient and thus a good quotient. Since good quotient is a local concept, we deduce that ϕ is a good quotient. The rest of the proof is easy and we omit it.

We learned how to construct the quotient of a projective variety by a linear action of a reductive group. The lesson is that we cannot take the quotient of the whole variety but we have to get rid of some bad points. (The homomorphism $R^G \hookrightarrow R$ induces a rational map $\mathrm{Proj}(R) \dashrightarrow \mathrm{Proj}(R^G)$ which is a morphism only on an open subset of prime ideals which does not contain $\oplus_{d>0} R^G_d$. This open set is precisely X^{ss}.)

But the concept of (semi)stability depends on the choice of the embedding $X \subset \mathbb{P}^n$, i.e. the choice of an ample line bundle on X. So the GIT quotient $X//G$ depends on the choice of an ample line bundle! But all the quotients are birationally equivalent and in many good cases they are related by "flips".

We end this lecture with some more terminologies. We say

(1) a point $x \in X$ is unstable if x is not semistable.
(2) a point x is strictly semistable if x is semistable but not stable.

Our terminologies are different from Mumford's [e.g. stable (us) =properly stable (Mumford)].

[14] Projectivity is an important fact. This is the reason why GIT is so useful in compactification problems.

대수기하 특강 – 13강

§5. Linearization.

In the previous lecture, we learned that in order to get the good quotient of a projective variety we have to get rid of unstable points and the notion of stability depends on the choice of an ample line bundle. Recall that a line bundle L is ample iff $\exists r \geq 1$ such that a basis of $H^0(L^r)$ gives rise to an embedding of X into $\mathbb{P}^n$ where $n = \dim H^0(L^r) - 1$.

We generalize the construction to arbitrary varieties with reductive group actions.

Let X be any variety on which an algebraic group acts. Let $p : L \to X$ be a line bundle over X.

Definition: A linearization of the action of G with respect to L is an action of G on L such that p is equivariant and the map $g : L_x \to L_{gx}$ is linear. A linearization is an isomorphism $\Phi : pr_2^*(L) \to \sigma^*(L)$ where $\sigma : G \times X \to X$ is the group action and pr_2 is the projection onto the second factor.

A linear action with respect to L is an action of G on X equipped with a linearization.

Given a linearization, we can think about the stability. Recall that in the projective case $X \subset \mathbb{P}^n$, a point $x \in X$ is semi-stable if there is a nonconstant homogeneous polynomial which does not vanish at x. But a homogeneous polynomial f of degree r is a section of the line bundle $\mathcal{O}_{\mathbb{P}^n}(r) = \mathcal{O}_{\mathbb{P}^n}(1)^r$. Let L be the restriction of the ample line bundle $\mathcal{O}(1)$ to X. Then a point $x \in X$ is semi-stable iff there is a section f of L^r such that $f(x) \neq 0$.[15] This motivates the following definition.

Definition: (1) A point $x \in X$ is semi-stable if for some $r \geq 1$ there is an invariant section $f \in H^0(L^r)^G$ of L^r such that $f(x) \neq 0$ and $X_f = \{y \in X \,:\, f(y) \neq 0\}$ is affine. Let $X^{ss}(L)$ denote the set of semi-stable points with respect to L.

(2) A point $x \in X$ is stable if $\dim Gx = \dim G$ and there is an invariant section f of L^r such that $f(x) \neq 0$, X_f affine and the action of G on X_f is closed. Let $X^s(L)$ be the set of stable points with respect to L.

The condition of X_f being affine is to enable us to take the affine quotient of X_f by G and then glue these to form the global quotient of X by G.

Remark: Obviously, this definition is compatible with the previous definition in the projective case since the hyperplane complement of a projective space is affine.

Lemma: A line bundle L over a variety X is ample iff for all $x \in X$, there is a section f of L^r for some $r \geq 1$ such that $f(x) \neq 0$ and X_f is affine. (For a proof, see [Hartshorne, II, §7, Proof of Theorem 7.6].)

By the definition of semi-stability, we see that $L|_{X^{ss}}$ is ample and X^{ss} is quasi-projective. Hence it seems reasonable to expect a quasi-projective quotient.

Theorem: Let X be a variety and L a line bundle over X. Suppose a linear action of G with respect to L is given. Then

(1) there is a good quotient $\phi : X^{ss}(L) \to Y$ of $X^{ss}(L)$ by G and Y is quasi-projective

(2) there is an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s(L)$ and the restriction $\phi|_{X^s(L)}$ is a geometric quotient

(3) for $x_1, x_2 \in X^{ss}(L)$, $\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{Gx_1} \cap \overline{Gx_1} \cap X^{ss}(L) \neq \emptyset$

[15]Notice that the restriction map $H^0(\mathbb{P}^n, \mathcal{O}(r)) \to H^0(X, \mathcal{O}(r))$ is surjective for sufficiently large r since $H^1(\mathbb{P}^n, \mathcal{I}_X \otimes \mathcal{O}(r)) = 0$ for large r by Serre's theorem and $f(x) \neq 0 \Leftrightarrow f^r(x) \neq 0$.

(4) a semi-stable point x is stable iff $\dim Gx = \dim G$ and Gx is closed in $X^{ss}(L)$.

Proof: (1) Choose an affine covering $\{X_{f_i}\}$ and take the affine quotients $\phi_i : X_{f_i} \to Y_i$. Glue these to get a variety Y and a morphism $\phi : X^{ss}(L) \to Y$. The ratio f_i/f_j is an invariant nowhere vanishing function on $X_{f_i} \cap X_{f_j}$ and hence a nowhere vanishing function on $Y_i \cap Y_j$. This gives us a line bundle which must be ample by the lemma above. (Exercise: Check the details!) The rest of the proof is also easy and we omit it.

We end this lecture with a few words about linearizations.

Let $Pic(X) = H^1(X, \mathcal{O}^*)$ be the set of isomorphism classes of line bundles on X. Then with tensor product as multiplication and the trivial bundle I or rank 1 as the identity, $Pic(X)$ becomes a group. If there is an action of an algebraic group G on X, we may consider the set $Pic^G(X)$ of isomorphism classes of the line bundles together with a linearization. Then tensor product and I with trivial action give a group structure to $Pic^G(X)$. The forgetful map $\alpha : Pic^G(X) \to Pic(X)$ is a group homomorphism obviously. The kernel of α is the set of linearizations on the trivial line bundle over X.

Proposition: Let G be an affine algebraic group acting on a variety X. The homomorphism α fits into an exact sequence

$$0 \to \mathrm{Hom}(G, \mathbb{C}^*) \to Pic^G(X) \to Pic(X) \to Pic(G)$$

and $Pic(G)$ is finite. Hence for any line bundle L over X there is an integer r such that L^r admits a linearization.

Since we are not going to use it, we don't prove it here.

In particular, if $G = SL(m)$, the forgetful map $Pic^G(X) \to Pic(X)$ is injective since $\mathrm{Hom}(SL(m), \mathbb{C}^*) = \{1\}$.[16]

Corollary: For each line bundle L, the action of $SL(m)$ on X has at most one linearization with respect to L.

Hence, our good quotient $X//G$ depends only on the choice of an ample bundle.

Remark: In applications, we shall deal with the action of $PGL(m)$ on $\mathbb{P}^n$ via a homomorphism $PGL(m) \to PGL(n+1)$. This action is not linear with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$. But we can linearize the action with respect to $\mathcal{O}_{\mathbb{P}^n}(n+1)$. Notice that (semi-)stability with respect to L is equivalent to the (semi-)stability with respect to L^r for any $r \geq 1$.

Another way to deal with this problem is to lift the homomorphism $PGL(m) \to PGL(n+1)$ to $SL(m) \to SL(n+1)$. This is possible since $SL(m)$ is a universal covering of $PGL(m)$. Thus we get a linear action of $SL(m)$ on $\mathbb{P}^n$.

[16] If $G = SL(m)$, $[G, G] = G$ and hence any homomorphism to an abelian group is trivial.

대수기하 특강 – 14강

§6. Slice theorem and descent lemma.

Two important tools in studying good quotients are the slice theorem which gives us the local structure of a quotient and the descent lemma which tells us when we can descend a vector bundle to a quotient. As before, the base field is an algebraically closed field of characteristic 0 *or just* $\mathbb{C}$. *Throughout this lecture, unless mentioned otherwise,* G *is a reductive group.*

A. Slice theorem.

Recall the following definition.

Definition: (1) A morphism $f : X \to Y$ of varieties of finite type is étale if it is smooth of relative dimension 0.[17]

(2) An equivariant morphism $f : X \to Y$ of affine G-varieties is strongly étale if the induced map $f' : X//G \to Y//G$ is étale and $(f, \phi_X) : X \to Y \times_{Y/G} X/G$ is an isomorphism.[18]

Now we can state the "amazing" slice theorem of Luna.

Theorem: Let X be a normal affine variety acted on by a reductive group G. If an orbit Gx is closed in X, there is a locally closed affine subvariety W, with $x \in W$, on which the stabilizer G_x acts, such that

(1) $U = GW = \{gw : g \in G, w \in W\}$ is open

(2) $G \times_{G_x} W \to U$ is strongly étale.

In case X is smooth at x, there is a strongly étale G_x-equivariant morphism from W to a neighborhood of 0 in $N_{Gx/X} = TX/T(Gx)$.

We omit the proof since it is quite technical.

In particular, if Gx is closed, a neighborhood of Gx is biholomorphic to $G \times_{G_x} W$. For many moduli problems, the normal space $N_{Gx/X}$ can be described by "deformation theory" and the slice theorem gives us a local description of the quotient: Given any good quotient $\phi : X \to Y$ and a point $y \in Y$, there is a unique closed orbit Gx in $\phi^{-1}(y)$. (Homework 4번.) Find the normal space N to Gx by deformation theory. Then there is a neighborhood of y in Y which is biholomorphic to $N//G_x$.

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[17] I.e. an etale morphism is a flat morphism with $\Omega_{X/Y} = 0$. Intuitively, this means that f is an (unramified) covering map.

[18] This implies that f is étale.

B. Descent lemma.

Next, we think about the descending problem: Given a good quotient $\phi : X \to Y$ and a vector bundle E over X, when can we descend E to Y? In other words, can we find a vector bundle F on Y such that $\phi^*F \cong E$?

Suppose $\phi : X \to Y$ is a good quotient by a reductive group G. If F is a vector bundle over Y, then its pull-back $\phi^*(F)$ is a vector bundle on X which is equipped with an action of G by $g(x, v) = (gx, v)$ for $x \in X$, $v \in F_{\phi(x)}$.

Let E be a G-vector bundle of rank r over X.[19]

Definition: We say a G-vector bundle E on X descends to Y if it is equivariantly isomorphic to the pull-back $\phi^*(F)$ of a vector bundle F on Y.

Lemma: E descends to Y iff for each point $y \in Y$ there is a neighborhood U of y and an equivariant isomorphism

$$E|_{\phi^{-1}(U)} \cong \phi^{-1}(U) \times \mathbb{C}^r$$

of vector bundles where G acts trivially on $\mathbb{C}^r$.

Proof: ($\Rightarrow$) Suppose $E \cong \phi^*F$. Take a neighborhood of y on which F is trivial.

($\Leftarrow$) We can find a covering $\{U_i\}$ of Y such that there is an equivariant isomorphism $f_i : E|_{\phi^{-1}(U_i)} \to \phi^{-1}(U_i) \times \mathbb{C}^r$. Consider the isomorphism $g_{ij} = f_j \circ f_i^{-1} : \phi^{-1}(U_i \cap U_j) \times \mathbb{C}^r \to \phi^{-1}(U_i \cap U_j) \times \mathbb{C}^r$. This is represented by a matrix of regular functions on $\phi^{-1}(U_i \cap U_j)$ which must be G-invariant since the action of G on $\mathbb{C}^r$ is trivial. As ϕ is a good quotient, the entries of the matrix are regular functions on $U_i \cap U_j$. Thus by gluing trivial bundles using these transition matrices, we get a vector bundle F. It is now obvious that $\phi^*F \cong E$.

We are now ready to prove the "descent lemma" due to Kempf.

Theorem: Let E be a G-vector bundle over X. Then E descends to Y iff for each point $x \in X$ with closed orbit, the stabilizer G_x acts trivially on the fiber E_x.

Proof: ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $x \in X$ such that Gx is closed in X. By the above lemma, it suffices to find an open neighborhood U of $\phi(x)$ and an equivariant isomorphism

$$s : \phi^{-1}(U) \times \mathbb{C}^r \to E|_{\phi^{-1}(U)}.$$

This is the same as finding r G-invariant sections

$$s_i : \mathcal{O}_{\phi^{-1}(U)} \to E|_{\phi^{-1}(U)}$$

that generate $E|_{\phi^{-1}(U)}$.

Let $u_1, \cdots, u_r$ be a basis of E_x. By the assumption, we have r sections $\sigma_i : \mathcal{O}_{Gx} \to E|_{Gx}$, given by $g \to gu_i$ as $Gx = G/G_x$. Of course, these sections are invariant and generate $E|_{Gx}$. So the question is whether for some U we can extend the sections $\sigma_i \in H^0(Gx, E|_{Gx})^G$ to sections $s_i \in H^0(\phi^{-1}(U), E|_{\phi^{-1}(U)})^G$ which generate $E|_{\phi^{-1}(U)}$.

Let V be an open affine neighborhood of $\phi(x)$. Because ϕ is a good quotient, $\phi^{-1}(V)$ is affine open containing Gx as a closed subset. Consider the restriction map

$$H^0(\phi^{-1}(V), E) \to H^0(Gx, E)$$

which must be surjective. Hence, we can find sections $s'_i \in H^0(\phi^{-1}(V), E)$ that extends σ_i. The problems are (1) s'_i may not be invariant and (2) they may not generate $E|_{\phi^{-1}(V)}$.

[19] This means that there is an action of G on E such that $p : E \to X$ is equivariant.

We deal with the first problem. Consider the action of G on $H^0(\phi^{-1}(V), E)$. We claim there is a homomorphism

$$R : H^0(\phi^{-1}(V), E) \to H^0(\phi^{-1}(V), E)^G$$

which is functorial with respect to restrictions. Suppose we proved the claim. Then, let $s_i'' = R(s_i') \in H^0(\phi^{-1}(V), E)^G$. Since σ_i is G-invariant, $s_i''|_{Gx} = \sigma_i$ by functoriality. So the first problem has been cleared.

[To prove the claim, it suffices to prove that $H^0(\phi^{-1}(V), E)$ is a union of finite dimensional invariant subspaces.[20] Certainly it is sufficient to show that for each $s \in H^0(\phi^{-1}(V), E)$ there is an invariant finite dimensional subspace which contains s.

Choose an open affine dense subset V_0 of $\phi^{-1}(V)$ on which E is trivial. Then we have an injection $H^0(\phi^{-1}(V), E) \to H^0(V_0, E)$. Consider the morphism

$$G \times V_0 \to E|_{V_0} \cong V_0 \times \mathbb{C}^r$$

defined by

$$(g, v_0) \to gs(g^{-1}v_0).$$

This gives rise to r regular functions $f_i : G \times V_0 \to \mathbb{C}$. But $\mathcal{O}(G \times V_0) \cong \mathcal{O}(G) \otimes \mathcal{O}(V_0)$. Write $f_i = \sum_j \xi_{ij} \otimes \nu_{ij}$ with $\xi_{ij} \in \mathcal{O}(V_0)$, $\nu_{ij} \in \mathcal{O}(G)$. Then $Gs|_{V_0}$ is contained in the subspace of $H^0(V_0, E) \cong \mathcal{O}(V_0)^r$ generated by ξ_{ij}. Since Gs is contained in the intersection of this subspace with $H^0(\phi^{-1}(V), E)$, we proved the claim.]

Now the second problem. Let W be the invariant closed subset of $\phi^{-1}(V)$ where s_i'' do not generate E. The closed subsets W and Gx are disjoint invariant and thus $\phi(W) \cap \{\phi(x)\} = \emptyset$. Put $U = V - \phi(W)$ and $s_i = s_i''|_{\phi^{-1}(U)}$. Then s_i generate E. So we are done.

[20]For each finite dimensional representation V of a reductive group G, we have a homomorphism $V \to V^G$ since completely reducible.

대수기하 특강 – 15강

Chapter 3. Hilbert-Mumford Criterion.

§1. A criterion for stability.

Let $X \subset \mathbb{P}^n$ be a projective variety acted on linearly by a reductive group G via a homomorphism $G \to GL(n+1)$. Recall that a point $x \in X$ is semi-stable iff there is a nonconstant invariant homogeneous polynomial f such that $f(x) \neq 0$. Also, a point $x \in X$ is stable iff $\dim Gx = \dim G$ and there is a nonconstant homogeneous polynomial f such that $f(x) \neq 0$ and Gx is closed in X^{ss}.

The problem is that the conditions are difficult to check! The Hilbert-Mumford criterion gives us a numerical method to determine stable points.

Let us first consider the simplest case: $G = \mathbb{C}^*$.

When $\mathbb{C}^*$ acts linearly on $\mathbb{C}^{n+1}$, we can find a basis of $\mathbb{C}^{n+1}$ such that G acts by

$$t \cdot (x_0, \cdots, x_n) = (t^{r_0} x_0, \cdots, t^{r_n} x_n)$$

where $r_0 \leq r_1 \leq \cdots \leq r_n$ is an increasing sequence of integers.[21]

Lemma: (a) A point $(x_0 : \cdots : x_n) \in X$ is semistable iff

$$\min\{r_i \,|\, x_i \neq 0\} \leq 0 \leq \max\{r_j \,|\, x_j \neq 0\}.$$

(b) A point $(x_0 : \cdots : x_n) \in X$ is stable iff

$$\min\{r_i \,|\, x_i \neq 0\} < 0 < \max\{r_j \,|\, x_j \neq 0\}.$$

Proof: (a) An invariant polynomial f is a linear combination of monomials of the form $z_0^{m_0} z_1^{m_1} \cdots z_n^{m_n}$ where

$$r_0 m_0 + r_1 m_1 + \cdots + r_n m_n = 0. \tag{1}$$

If $f(x_0, \cdots, x_n) \neq 0$, at least one of the monomials does not vanish at the point. Choose a nonvanishing monomial in f. Suppose $r_i > 0$ whenever $x_i \neq 0$. In order to satisfy the equation (1), we should have $m_i = 0$ whenever $x_i \neq 0$. But then the monomial vanishes at the point, contradicting our assumption. Hence there exists i such that $x_i \neq 0$ and $r_i \leq 0$. Similarly by reversing the inequalities we see that there exists j such that $x_j \neq 0$ and $r_j \geq 0$.

Conversely, if we have $r_i \leq 0$ for some $x_i \neq 0$ and $r_j \geq 0$ for some $x_j \neq 0$, then we can find a pair integers $(m_i, m_j) \neq (0, 0)$ such that $r_i m_i + r_j m_j = 0$. Thus $z_i^{m_i} z_j^{m_j}$ is a nonconstant invariant monomial which does not vanish at $(x_0, \cdots, x_n)$.

(b) The condition $\dim Gx = \dim G = 1$ is equivalent to saying that r_i for $x_i \neq 0$ are not all identical. The closure of the orbit minus the orbit consists of two points

- $\lim_{t \to 0} t(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$ where $y_i = x_i$ for $r_i = \min\{r_i \,|\, x_i \neq 0\}$ and $y_i = 0$ for $r_i \neq \min\{r_i \,|\, x_i \neq 0\}$
- $\lim_{t \to \infty} t(x_0 : \cdots : x_n) = (y_0' : \cdots : y_n')$ where $y_i' = x_i$ for $r_i = \max\{r_j \,|\, x_j \neq 0\}$ and $y_i' = 0$ for $r_i \neq \max\{r_j \,|\, x_j \neq 0\}$.

The orbit Gx is closed in X^{ss} iff the two points are not semistable iff $r_0 \neq 0$ and $r_n \neq 0$.

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[21] The coaction of $GL(n+1)$ on $\mathcal{O}(\mathbb{C}^{n+1}) = \mathbb{C}[z_0, \cdots, z_n]$ is given by $z_i \to \sum t_{ij} \otimes z_j$ and the coaction of $\mathbb{C}^*$ is given by $z_i \to \sum \lambda^*(t_{ij}) \otimes z_j$ where $\lambda : \mathbb{C}^* \to GL(n+1)$ is the linear action. Since the grading is preserved by the action, the subspace V of linear polynomials is an invariant subspace and we get a $\mathbb{Z}$ grading $V = \oplus V_i$ where $\mathbb{C}^*$ acts on V_i with weight i. (See the section on algebraic group action.) Choose a basis from each V_i and we get a basis $f_0, \cdots, f_n$ for V. We can find $g \in GL(n+1)$ such that $z_i \to f_i$. Then with this new basis, the action of $\mathbb{C}^*$ is diagonalized.

So in the case $G = \mathbb{C}^*$ we have an explicit numerical criterion for (semi)stability. Let us now think about the general case where G is any reductive group. For this purpose we make the following definitions.

<u>Definition</u>: (a) A 1-parameter subgroup (1-PS) of G is a non-trivial homomorphism $\lambda : \mathbb{C}^* \to G$.

(b) Let $x \in X$ and λ be a 1-PS of G. Let $r_0 \le \cdots \le r_n$ be the weights of the action of $\mathbb{C}^*$ by $\lambda : \mathbb{C}^* \to G \to GL(n+1)$. We define $\mu(x, \lambda) = -\min\{r_i \,|\, x_i \neq 0\}$.

There is another way of defining this: If we let $y = \lim_{t \to 0} t \cdot x$, then y is a fixed point by the 1-PS. Hence $\mathbb{C}^*$ acts on the fiber $\mathcal{O}_X(1)|_y$ of the ample line bundle. Then the weight of this action is exactly $-\mu(x, \lambda)$.[22]

If a point $x \in X$ is semistable, there is a G-invariant nonconstant homogeneous polynomial f which does not vanish at x. Since f is G-invariant, f is invariant with respect to the action of any 1-PS. Hence, x is semistable with respect to the 1-PS. Let X^{ss}_λ be the set of semistable points with respect to a 1-PS λ. Then we have $X^{ss} \subset \cap_\lambda X^{ss}_\lambda$ for any 1-PS λ. The above lemma tells us

$$x \in X^{ss} \Rightarrow \mu(x, \lambda) \ge 0 \text{ for any 1-PS } \lambda. \tag{2}$$

Suppose $x \in X^s$ and $\mu(x, \lambda) = 0$ for some 1-PS λ. Let $y = (y_0 : \cdots : y_n) = \lim_{t \to 0} t \cdot x$ where $y_i = x_i$ if $r_i = 0$ and $y_i = 0$ if $r_i \neq 0$. Then y is in $\overline{Gx} \cap X^{ss} = Gx$.[23] But since the 1-PS λ acts trivially on y, the stabilizer of y in G is not finite, and thus y cannot be stable. Hence x is not stable. Therefore we have

$$x \in X^s \Rightarrow \mu(x, \lambda) > 0 \text{ for any 1-PS } \lambda. \tag{3}$$

The Hilbert-Mumford criterion says the converses to (2) and (3) are also true!

<u>Theorem</u>: Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then

$$x \in X^{ss} \Leftrightarrow \mu(x, \lambda) \ge 0 \text{ for any 1-PS } \lambda$$
$$x \in X^s \Leftrightarrow \mu(x, \lambda) > 0 \text{ for any 1-PS } \lambda.$$

We skip the proof since (i) it is quite technical (will takes several lectures to complete) (ii) we don't have to know the proof to apply the theorem.

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[22] This definition makes sense for any proper G-variety with a linearization with respect to an ample line bundle. The Hilbert-Mumford criterion holds in this more general situation.

[23] The point $(y_0, \cdots, y_n) \in \mathbb{C}^{n+1}$ is the limit $\lim_{t \to 0} t \cdot (x_0, \cdots, x_n)$ in $\mathbb{C}^{n+1}$. If a nonconstant invariant homogeneous polynomial does not vanish at $(x_0, \cdots, x_n)$, then it does not vanish at $(y_0, \cdots, y_n)$. Therefore, $y = (y_0 : \cdots : y_n) \in \mathbb{P}^n$ is semistable with respect to the action of G.

대수기하 특강 – 16강

In the previous lecture, we learned about the Hilbert-Mumford criterion. We begin this lecture with the "reduction to torus" technique which simplifies the Hilbert-Mumford criterion considerably.

The following facts are well-known for a reductive group G.

- The image of any 1-PS is contained in a maximal torus of G
- Fix a maximal torus T. Then any maximal torus of G is conjugate to T.

On the other hand, it is easy to prove the followng (Exercise!).

- x is (semi)stable iff gx is (semi)stable for $g \in G$.
- $\mu(x, \lambda) = \mu(g^{-1}x, g^{-1}\lambda g)$.[24]

So we have

$$\begin{aligned} x \in X^{ss} \quad &\Leftrightarrow \mu(x,\lambda) \geq 0 \text{ for any 1-PS } \lambda \text{ of } G \\ &\Leftrightarrow \mu(x, g^{-1}\lambda g) \geq 0 \text{ for any 1-PS } \lambda \text{ of } T \text{ and } g \in G \\ &\Leftrightarrow \mu(gx, \lambda) \geq 0 \text{ for any 1-PS } \lambda \text{ of } T \text{ and } g \in G \end{aligned}$$

Similarly, $x \in X^s \Leftrightarrow \mu(gx, \lambda) > 0$ for any 1-PS λ of T and $g \in G$. Hence, to determine (semi)stability, it suffices to consider only the 1-PS of a maximal torus T. In particular, if $G = SL(m)$ it suffices to consider the 1-PS of the form

$$\lambda(t) = diag(t^{r_1}, \cdots, t^{r_m})$$

where $\sum r_i = 0$.

Let us now see some examples where we can apply the Hilbert-Mumford criterion explicitly. The following two examples serve as the test cases!

[24] For the latter, choose a basis $v_0, \cdots, v_n$ of $\mathbb{C}^{n+1}$ such that $\lambda(t) = diag(t^{r_0}, \cdots, t^{r_n})$. Then $g^{-1}v_0, \cdots, g^{-1}v_n$ is a basis of $\mathbb{C}^{n+1}$ for which $g^{-1}\lambda g$ is diagonalized as $diag(t^{r_0}, \cdots, t^{r_n})$. If $x = \sum x_i v_i = (x_0, \cdots, x_n)$ then $g^{-1}x = \sum x_i(g^{-1}v_i) = (x_0, \cdots, x_n)$. Now everything is completely identical except for the bases.

§2. Binary forms.

Our first example is the binary forms. Let $G = SL(2)$. Let V_n be the irreducible representation of G with $\dim V_n = n$.

Here is a way to describe the irreducible representation. The natural action of G on $\mathbb{C}^2$ induces an action of G on the polynomial ring $\mathbb{C}[z_1, z_2]$ by $(g \cdot f)(z_1, z_2) = f(g^{-1}(z_1, z_2))$. Since the action is linear, it preserves the grading. The subspace of degree n homogeneous polynomials is the irreducible representation V_{n+1}.

We consider the action of G on $\mathbb{P}^n = \mathbb{P}V_{n+1}$. A 1-PS of the maximal torus of G is of the form $\lambda_r(t) = diag(t^r, t^{-r})$ and any 1-PS of G is conjugate to λ_r for some integer r, i.e. $\exists g \in G$ such that $g^{-1}\lambda g = \lambda_r$.

Let $f = \sum a_i z_1^{n-i} z_2^i$. Then $\lambda_r(t) \cdot f = \sum t^{r(2i-n)} a_i z_1^{n-i} z_2^i$. Hence $\mu(f, \lambda_r) = r(n - 2i)$ where $i = \min\{j \,|\, a_j \neq 0\}$. So, $\mu(f, \lambda_r) < 0$ iff $i > n/2$ iff $a_j = 0$ whenever $i \leq n/2$ iff $(1:0)$ is a zero of f with multiplicity $> n/2$.

A binary form $f \in \mathbb{P}^n$ is unstable iff $\mu(f, \lambda) < 0$ for some 1-PS λ iff $\mu(g^{-1} \cdot f, \lambda_r) < 0$ for some $g \in G$ and $r \in \mathbb{Z}$ iff $g \cdot (1:0)$ is a zero of f with multiplicity $> n/2$ iff f has a zero in $\mathbb{P}^1$ of multiplicity $> n/2$. By switching $<$ and $\leq$, we see that $f \in \mathbb{P}^n$ is not stable iff f has a zero in $\mathbb{P}^1$ of multiplicity $\geq n/2$. So, we proved the following.

Proposition: A binary form of degree n is stable (semistable) iff no point of $\mathbb{P}^1$ occurs as a point of multiplicity $\geq n/2$ ($> n/2$). In particular, if n is odd, semistable points are all stable and the orbit space $(\mathbb{P}^n)/G$ is a projective variety.

§3. Ordered points in $\mathbb{P}^1$.

A closely related example is about ordered point in $\mathbb{P}^1$. Let $G = SL(2)$. The natural action of G on $\mathbb{C}^2$ gives us an action of G on $\mathbb{P}^1$. Let $X = (\mathbb{P}^1)^N$ and consider the diagonal action of G on X.[25]

We consider the Segre embedding of X into a projective space. Namely, the embedding is given by the ample bundle $L = \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. For $x = (x_0 : x_1) \in \mathbb{P}^1$, $\mu(x, \lambda_r)$ is r if $x_1 \neq 0$ and is $-r$ if $x_1 = 0$. For $x^{(j)} = (x_0^{(j)} : x_1^{(j)})$, $j = 1, 2, \cdots, N$, the coordinates of the N-tuple $(x^{(1)}, \cdots, x^{(N)})$ with respect to the Segre embedding are given by $x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_N}^{(N)}$ for $i_j = 0, 1$. The weight of the action of λ_r on $x_0^{(j)}$ is r and that on $x_1^{(j)}$ is $-r$. Thus,

$$\mu((x_1, \cdots, x_n), \lambda_r) = (N - 2q)r$$

where $q = \#\{j \,|\, x_1^{(j)} = 0\}$. Hence, $\mu < 0$ iff $q > N/2$ iff more than half of the points are $(1:0)$. Also, $\mu \leq 0$ iff $q \geq N/2$ iff at least half of the points are $(1:0)$. To get X^{ss} or X^s we have to get rid of the G-orbits of the above points. Therefore, we get

- The complement of X^{ss} consists of N-tuples $(x^{(1)}, \cdots, x^{(N)})$ which contains a point more than $N/2$ times.
- The complement of X^s consists of N-tuples $(x^{(1)}, \cdots, x^{(N)})$ which contains a point at least $N/2$ times.

In particular, semistable points are automatically stable when N is odd.

[25] This is related to the previous problem since $\mathbb{P}^n = (\mathbb{P}^1)^n / S_n$ where S_n is the symmetric group of n letters.

Proposition:

$$[(\mathbb{P}^1)^N]^s = \{\text{no points of } \mathbb{P}^1 \text{ occurs as a component of } x \geq N/2 \text{ times}\}$$

$$[(\mathbb{P}^1)^N]^{ss} = \{\text{no points of } \mathbb{P}^1 \text{ occurs as a component of } x > N/2 \text{ times}\}$$

In particular, if N is odd, the orbit space $[(\mathbb{P}^1)^N]^s/SL(2)$ has a structure of a projective variety.

§4. Sequences of linear subspaces.

The Hilbert-Mumford criterion says

$$X^{ss} = \{x \in X \mid \mu(gx, \lambda) \geq 0 \text{ for any 1-PS } \lambda \text{ of } T \text{ and any } g \in G\}.$$

Let $\mathbf{G}_{n,q}$ be the Grassmannian of q-dimensional subspaces of $\mathbb{P}^n$, i.e. $q+1$-dimensional subspaces of $\mathbb{C}^{n+1}$.

For a $q+1$-dimensional subspace $L \in \mathbf{G}_{n,q}$, choose a basis $(x_{j0}, x_{j1}, \cdots, x_{jn})$, $j = 0, 1, \cdots, q$. The Plücker coordinates are the maximal minors $p_{i_0,\cdots,i_q}$ of the $(q+1) \times (n+1)$ matrix (x_{ji}) and they give us an embedding of $\mathbf{G}_{n,q}$ into $\mathbb{P}^N$, $N = \binom{n+1}{q+1} - 1$.[26]

The group $G = SL(n+1)$ acts naturally on $\mathbf{G}_{n,q}$ by $(x_{ji})A$ for $A \in GL(n+1)$ and it is an elementary exercise that the maximal minors of $(x_{ji})A$ is a linear combination of the maximal minors of (x_{ji}). This implies that the action of $SL(n+1)$ on $\mathbf{G}_{n,q}$ is linear with respect to the Plücker embedding.

Let $X = (\mathbf{G}_{n,q})^m$ which parametrizes sequences of linear subspaces. The group $SL(n+1)$ acts on X diagonally and the Plücker embedding composed with the Segre map $X \hookrightarrow (\mathbb{P}^N)^m \hookrightarrow \mathbb{P}^M$ gives us a linearization.

To find the (semi)stable points, we compute $\mu(x, \lambda)$ for a 1-PS λ of the maximal torus of diagonal matrices in $SL(n+1)$, i.e. $\lambda(t) = diag(t^{r_0}, \cdots, t^{r_n})$, $\sum r_i = 0$, $r_0 \geq r_1 \geq \cdots \geq r_n$. Let $x = (L_1, \cdots, L_m) \in (\mathbf{G}_{n,q})^m$. Then we know

$$\mu(x, \lambda) = \sum \mu(L_i, \lambda).$$

Let $0 \subset V_0 \subset \cdots \subset V_n = \mathbb{C}^{n+1}$ be the filtration defined $V_i = \text{Span}\{e_0, \cdots, e_i\}$ where $e_0, e_1, \cdots, e_n$ is the basis of $\mathbb{C}^{n+1}$ which diagonalizes the action of λ. Let $L \in \mathbf{G}_{n,q}$ be a $q+1$-dimensional subspace of $\mathbb{C}^{n+1}$. Then $\exists \nu_0 < \nu_1 < \cdots < \nu_q$ such that $\dim(V_{\nu_j} \cap L) = j+1$ and $\dim(V_{\nu_j - 1} \cap L) = j$. Choose a vector from $V_{\nu_j} \cap L - V_{\nu_j - 1} \cap L$. Then the subspace L has a basis which are the rows of the matrix

$$\begin{matrix}
a_{00} & \cdots & a_{0\nu_0} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
a_{10} & \cdots & \cdots & a_{1\nu_1} & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{q0} & \cdots & \cdots & \cdots & \cdots & a_{q\nu_q} & 0 & \cdots & 0
\end{matrix}$$

such that $a_{j\nu_j} \neq 0$. Then the Plücker coordinate with minimal weight is $p_{\nu_0\nu_1\cdots\nu_q}$ with minimal weight $r_{\nu_0} + r_{\nu_1} + \cdots + r_{\nu_q}$. Hence,

$$\begin{aligned}
\mu(L, \lambda) &= -\textstyle\sum_{k=0}^{q} r_{\nu_k} = -\sum r_j (\dim(V_j \cap L) - \dim(V_{j-1} \cap L)) \\
&= -r_n(q+1) + \textstyle\sum_{j=0}^{n-1} (r_{j+1} - r_j) \dim(V_j \cap L)
\end{aligned}$$

For $x = (L_1, \cdots, L_m) \in X$, we have

$$\mu(x, \lambda) = \sum_{i=1}^{m} \mu(L_i, \lambda) = -mr_n(q+1) + \sum_i [\sum_{j=0}^{n-1} (r_{j+1} - r_j) \dim(V_j \cap L_i)].$$

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[26] A different choice of basis results in the matrix $C(x_{ji})$ for some $C \in GL(q+1)$. The Plücker coordinates are just multiplied by $\det(C)$.

This is a linear function of $r = (r_0, \cdots, r_n)$ and the region in $\mathbb{Q}^{n+1}$ for $\sum r_i = 0$ and $r_0 \geq r_1 \geq \cdots \geq r_n$ is a convex polyhedral cone[27] whose points are positive linear combinations of the extreme cases

$$r_0 = \cdots = r_p = n - p, \quad r_{p+1} = \cdots = r_n = -(p+1)$$

for $p = 0, 1, \cdots, n-1$. Therefore, $\mu(x, \lambda) \geq 0$ for any r iff

$$m(q+1)(p+1) - (n+1)\sum_{i=1}^{m} \dim L_i \cap V_p \geq 0$$

for $p = 0, 1, \cdots n-1$. Hence $\mu(gx, \lambda) \geq 0$ for any r and any $g \in G$ is the same as

$$m(q+1)\dim V - (n+1)\sum_{i=1}^{m} \dim L_i \cap V \geq 0$$

for any proper subspace V of $\mathbb{C}^{n+1}$.

Theorem: A sequence $(L_1, \cdots, L_m)$ of $q+1$-dimensional subspace of $\mathbb{C}^{n+1}$ is semistable iff for any proper subspace V of $\mathbb{C}^{n+1}$

$$(n+1)\sum_{i=1}^{m} \dim(L_i \cap V) \leq m(q+1)\dim V.$$

We get stability if we replace $\leq$ by $<$.

Corollary: Let $H_{p,r}$ be the Grassmannian of r-dimensional quotients of $\mathbb{C}^p$, i.e. $H_{p,r} \cong Gr(p-r, p)$. A point $y = (Q_1, \cdots, Q_m) \in (H_{p,r})^m$ is semistable iff for any proper subspace V of $\mathbb{C}^p$,

$$\rho(V) = \frac{1}{m \dim V}\sum_{i=1}^{m} \dim V_i - \frac{r}{p} \geq 0$$

where V_i is the image of V in Q_i. We get stability if we replace $\geq$ by $>$.

Here are some examples.

Example: (1) $n = 1, q = 0$. This is just the example of m ordered points in $\mathbb{P}^1$. The above theorem says, a point $(x_1, \cdots, x_m) \in X$ is semistable iff $2\#\{i : x_i = p\} \leq m$ for any $p \in \mathbb{P}^n$ iff no more than $m/2$ points may coincide. This coincides with our previous result.

(2) $n = 2, q = 0$ (ordered points in $\mathbb{P}^2$). By the theorem above, a sequence $(x_1, \cdots, x_m)$ is semistable iff

- for any point $p \in \mathbb{P}^2$, $\#\{i \,|\, x_i = p\} \leq m/3$
- for any line L in $\mathbb{P}^2$, $\#\{i \,|\, x_i \in L\} \leq 2m/3$.

We get stability by simply replacing $\leq$ by $<$.

(3) $n = 3, q = 1$ (lines in $\mathbb{P}^3$). Consider the inequality in the theorem for $(L_1, \cdots, L_m)$.

- If $\dim V = 1$, we get the condition
$$\#\{i \,|\, p \in L_i\} \leq m/2$$
for any point $p \in \mathbb{P}^3$, i.e. no more than $m/2$ lines intersect at one point.
- If $\dim V = 2$, we get the condition
$$2\#\{i \,|\, L_i = L\} + \#\{i \,|\, L_i \neq L, L \cap L_i \neq \emptyset\} \leq m$$
for any line L in $\mathbb{P}^3$, i.e. no more than $m/2$ lines coincide and no more than $m - 2t$ lines intersect a line L_j which is repeated t times.

[27] When $n = 2$, it is the region given by $y \leq x$, $x \geq -2y$, $z = -x - y$.

- If $\dim V = 3$, we get the condition
$$2\#\{i \mid L_i \subset W\} + \#\{i \mid L_i \not\subseteq W\} \leq 3m/2$$
for any plane W in $\mathbb{P}^3$, i.e. no more than $m/2$ lines are coplanar.

대수기하 특강 – 18강
김영훈

Chapter 4. Vector bundles over a curve

§1. Coherent sheaves over a curve.

Let X be a nonsingular irreducible projective curve of genus g. Let F be a vector bundle over X of rank r and degree d.[28] The topological type of a vector bundle over X is completely determined by rank and degree, i.e. if two vector bundles F_1 and F_2 have the same degree and rank, then $\exists$ continuous bijective bundle map $F_1 \to F_2$ over X.

Let $\mathcal{F}$ be a coherent sheaf over X. Then the i-th cohomology group $H^i(X,\mathcal{F}) = H^i(\mathcal{F})$ is the cohomology of the chain complex

$$0 \to I^0(X) \to I^1(X) \to I^2(X) \to \cdots$$

where $0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \cdots$ is an injective resolution of $\mathcal{F}$. In particular, $H^0(X,\mathcal{F}) = \mathcal{F}(X)$. Let $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$ and $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$.

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves on X, then we have the long exact sequence[29]

$$0 \to H^0(\mathcal{F}') \to H^0(\mathcal{F}) \to H^0(\mathcal{F}'') \to H^1(\mathcal{F}') \to H^1(\mathcal{F}) \to H^1(\mathcal{F}'') \to 0$$

Hence, we see that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$, i.e. the Euler characteristic χ is additive.

For a vector bundle F of rank r and degree d, the Riemann-Roch theorem says $\chi(F) = d - r(g-1)$. Let H be an ample line bundle with $\deg H = h > 0$.[30] Then the degree of $\mathcal{F}(m) := \mathcal{F} \otimes H^{\otimes m}$ is $d + rhm$ since $det(\mathcal{F}(m)) = \det(\mathcal{F}) \otimes H^{\otimes rm}$. Hence, the Hilbert polynomial

$$\chi(\mathcal{F}(m)) = d + rhm - r(g-1) = rhm + d - r(g-1).$$

If $\mathcal{F}$ is not a vector bundle, we have to be a bit careful about the concepts – rank and degree. The tensor product $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ makes sense and so does the Hilbert polynomial $\chi(\mathcal{F}(m))$ which we know must be of the form $a + bm$ because $\dim X = 1$. For a coherent sheaf $\mathcal{F}$, we define $\mathrm{rank}(\mathcal{F}) = b/h =: r$ and $\deg(\mathcal{F}) = \chi(\mathcal{F}) + r(g-1)$. Another way to define $\mathrm{rank}(\mathcal{F})$ is as follows: Let $T(\mathcal{F})$ be the torsion subsheaf of $\mathcal{F}$.[31] Then we have a short exact sequence $0 \to T(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}/T(\mathcal{F}) \to 0$ and the quotient $\mathcal{F}/T(\mathcal{F})$ is torsion-free. Since X is smooth, a torsion-free sheaf is locally free. The rank of $\mathcal{F}$ is the rank of the vector bundle $\mathcal{F}/T(\mathcal{F})$. We leave it as an exercise to check that the two definitions are equivalent.

In our case, the Serre duality is easy to describe. The cotangent bundle over X is a line bundle K of degree $2g-2$.[32] We call K the canonical line bundle and the sheaf of its sections is called the canonical sheaf. The obvious pairing $H^1(\mathcal{F}) \otimes H^0(\mathbf{Hom}(\mathcal{F},K)) \to H^1(K) \cong \mathbb{C}$ is non-degenerate, i.e. $H^1(\mathcal{F}) \cong H^0(\mathbf{Hom}(\mathcal{F},K))^*$ for any coherent sheaf $\mathcal{F}$ on X.

A short exact sequence of coherent sheaves $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ is called an extension of $\mathcal{F}$ by $\mathcal{G}$. Two extensions of $\mathcal{F}$ by $\mathcal{G}$ are isomorphic if there is an

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[28] $\deg(F) = \deg(det(F))$. The degree of a line bundle L is the number of zeros minus the number of poles of a meromorphic section of L. Or simply, $\deg(F)$ is the first Chern class of F.

[29] Warm-up Homework 5.

[30] It is easy to see that a line bundle L over X is ample iff $\deg L > 0$, because $H^1(L^m(-x-y)) \cong H^0(Hom(L^m(-x-y),K))^* = 0$ for $m\deg(L) - 2 > 2g-2$.

[31] This is supported over finitely many points.

[32] By definition, $\dim H^1(X,\mathcal{O}) = g = \dim H^0(X,K)$. Also, $res : H^1(X,K) \cong \mathbb{C}$. Hence by Riemann-Roch, $\deg K = 2g-2$.

isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle id}\downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Let $\mathbf{Ext}(\mathcal{F}, \mathcal{G})$ be the set of isomorphism classes of extensions of $\mathcal{F}$ by $\mathcal{G}$.

On the other hand, $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$ is define as the first cohomology of the complex $\mathrm{Hom}(\mathcal{F}, I^{\cdot})$ where $0 \to \mathcal{G} \to I^1 \to \cdots$ is an injective resolution.

Lemma: $\mathbf{Ext}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$.

Proof: From an extension $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$, we get the long exact sequence $0 \to Hom(\mathcal{F}, \mathcal{G}) \to Hom(\mathcal{F}, \mathcal{E}) \to Hom(\mathcal{F}, \mathcal{F}) \to Ext^1(\mathcal{F}, \mathcal{G}) \to \cdots$. The image of 1 in $Hom(\mathcal{F}, \mathcal{F})$ in $Ext^1(\mathcal{F}, \mathcal{G})$ is the associated element of the extension.

Conversely, given an element $\omega \in Ext^1(\mathcal{F}, \mathcal{G})$, find a representative $w \in Hom(\mathcal{F}, I^1)$. Let $\mathcal{E} = Ker[(w, d) : \mathcal{F} \oplus I^0 \to I^1)]$. The kernel of the composition $\mathcal{E} \hookrightarrow \mathcal{F} \oplus I^0 \to \mathcal{F}$ is precisely $\mathcal{G}$ and thus we get an extension $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$.

We leave it as an exercise to verify that this association is the inverse of the previous one.

Lemma: There is a short exact sequence

$$0 \to H^1(\mathbf{Hom}(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \to H^0(\mathbf{Ext}^1(\mathcal{F}, \mathcal{G})) \to \cdots .$$

In particular, if $\mathcal{F}$ is locally free, we have an isomorphism

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \cong H^1(\mathbf{Hom}(\mathcal{F}, \mathcal{G})).$$

Proof: This follows from a spectral sequence associated to the double complex $C^p(\mathcal{U}, \mathbf{Hom}(\mathcal{F}, I^q))$ used to define the Ext group. (There are two ways to compute.)

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김영훈

X is always a smooth projective curve of genus g.

Subsheaves are closely related to subbundles.

Proposition: Let F be a vector bundle over X and $\mathcal{F}$ be the sheaf of its sections. For any subsheaf $\mathcal{G}$ of $\mathcal{F}$, $\exists\,!$ subbundle H of F with sheaf of sections $\mathcal{H}$ such that $\mathcal{H}/\mathcal{G}$ is a torsion sheaf, i.e. supported on a finite set. In particular, $\mathrm{rank}(H) = \mathrm{rank}(\mathcal{G})$ and $\deg(H) \geq \deg(\mathcal{G})$.

Proof: Let T be the torsion subsheaf of $\mathcal{F}/\mathcal{G}$. Then $\mathcal{Q} = (\mathcal{F}/\mathcal{G})/T$ is torsion-free and hence locally free. The homomorphism $\mathcal{F} \to \mathcal{Q}$ is surjective by construction. The kernel of this homomorphism must be a subbundle H which contains $\mathcal{G}$ and $\mathcal{H}/\mathcal{G} \cong T$ is a torsion sheaf.

Conversely, if H is a subbundle of F such that $\mathcal{H}/\mathcal{G}$ is torsion, then the image of $\mathcal{H}$ in $\mathcal{F}/\mathcal{G}$ is torsion and thus contained in T. Hence, $\mathcal{H}$ lies in the kernel $\mathcal{K}$ of the homomorphism $\mathcal{F} \to \mathcal{Q}$. Since $\mathcal{H}$ and $\mathcal{K}$ are both subbundles of the same rank with $H \subset K$, they must be equal.

The last statement is obvious.

One way of giving a subsheaf is by giving a subspace V of $H^0(\mathcal{F})$ through the homomorphism $X \times V \to X \times H^0(\mathcal{F}) \to \mathcal{F}$. The subbundle we found above for the subsheaf generated by V is called the subbundle generically generated by V. In particular, any nonzero section s of a vector bundle F gives us a line subbundle. Since $F \otimes H^m$ has a nonzero section for $m >> 0$ and H ample, $F \otimes H^m$ has a subbundle of rank 1 and so does F.

Corollary: Every vector bundle over X has a subbundle of rank 1.

Our goal is to construct the moduli space of vector bundles of degree d and rank r over a smooth projective curve X of genus g. We first consider the case $g = 0$, i.e. $X = \mathbb{P}^1$.

We know the tautological line bundle

$$U = \{(x, v) \mid v \in x\} \subset \mathbb{P}^1 \times \mathbb{C}^2$$

over $\mathbb{P}^1$ is of degree -1. Let $\mathcal{O}(1)$ be the dual bundle of U. Then $\deg(\mathcal{O}(1)) = 1$.

Lemma: Any line bundle L over $\mathbb{P}^1$ of degree d is isomorphic to $\mathcal{O}(d)$. In particular, we have $Pic(\mathbb{P}^1) = \mathbb{Z}$.

Proof: Let $M = L \otimes \mathcal{O}(-d)$. Then M is of degree 0. By Riemann-Roch, $h^0(M) \geq h^0(M) - h^1(M) = 1$. Let s be a nonzero section of M. Since $\deg(M) = 0$, s is nowhere vanishing. Therefore, $M \cong \mathcal{O}$.

We can now prove a theorem of Grothendieck.

Theorem: Any vector bundle F over $\mathbb{P}^1$ is a direct sum of line bundles, i.e.

$$F \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for an increasing sequence of integers $a_1 \geq a_2 \geq \cdots \geq a_r$.

Proof: Let $a_1 = \max\{i \mid H^0(F \otimes \mathcal{O}(-i)) \neq 0\}$.[33] By definition, there is a section s of $F \otimes \mathcal{O}(-a_1)$ but there is no section of $F \otimes \mathcal{O}(-a_1 - 1)$. We use induction on the rank r of F. When $r = 1$, there is nothing to prove. We know there is a subbundle

[33] a_1 exists since $F \otimes \mathcal{O}(-i)$ has no nonzero section for large i by Serre's theorem.

of $F \otimes \mathcal{O}(-a_1)$ of rank 1 generically generated by s but this subbundle must the trivial line bundle $\mathcal{O}$.[34] Hence we get a short exact sequence of vector bundles

$$0 \to \mathcal{O}(a_1) \to F \to F' \to 0$$

where F' is a vector bundle of rank $r-1$. By induction hypothesis, $F' = \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$ for some integers $a_2 \geq \cdots \geq a_r$. From the exactness of $0 = H^0(F \otimes \mathcal{O}(-a_1-1)) \to H^0(F' \otimes \mathcal{O}(-a_1-1)) \to H^1(\mathcal{O}(-1)) = 0$, we deduce that $H^0(F' \otimes \mathcal{O}(-a_1-1)) = 0$ and thus $a_1 \geq a_2$.

We have $Ext^1(F', \mathcal{O}(a_1)) \cong H^1(Hom(F', \mathcal{O}(a_1))) = 0$ and hence the above extension splits. So we complete the proof of the theorem.

[34] Otherwise its degree is ≥ 1 and we get a contradiction to the maximality of a_1.

§2. Semistable bundles.

We will construct the moduli space of vector bundles as the good quotient of a projective variety. The (semi)stability we introduced in the previous chapter will give us the notion of (semi)stable bundles.

For any vector bundle F, the slope of F is $\mu(F) = \deg(F)/\text{rank}(F)$.

Definition: A vector bundle F over X is (semi)stable iff for any nonzero proper subbundle G of F, $\mu(G) < (\leq)\mu(F)$ iff for any nonzero proper quotient bundle Q of F, $\mu(F) < (\leq)\mu(Q)$.

We have the following basic facts.

Lemma: (1) Every line bundle is stable.

(2) If F is (semi)stable, then $F \otimes L$ is (semi)stable for any line bundle L.

(3) If F_1, F_2 are stable with $\mu(F_1) = \mu(F_2)$, then every nonzero homomorphism $h : F_1 \to F_2$ is an isomorphism.

(4) If F is (semi)stable, then so is F^*.

Proof: (1) clear. (2) For any subbundle G of $F \otimes L$, $G \otimes L^{-1}$ is a subbundle of F. Since $\deg G \otimes L^{-1} = \deg(G) - \text{rank}(G)\deg(L)$, $\mu(G \otimes L^{-1}) = \mu(G) - \deg(L) < \mu(F)$. Thus $\mu(G) < \mu(F \otimes L)$.

(3) Let $\mu = \mu(F_1) = \mu(F_2)$. Suppose $ker(h) \neq 0$ or $im(h) \neq F_2$. Let G_1 be the subbundle of F_1 generically generated by $ker(h)$ and G_2 be the subbundle of F_2 generically generated by $im(h)$. From the short exact sequence $0 \to ker(h) \to F_1 \to im(h) \to 0$, we see that $\deg(F_1) \leq \deg(G_1) + \deg(G_2)$ and $\text{rank}(F_1) = \text{rank}(G_1) + \text{rank}(G_2)$. By stability, we have $\mu(G_1) < \mu$ and $\mu(G_2) < \mu$. Thus $\mu(F_1) < \mu$. Contradiction.

(4) Let Q be a quotient bundle of F^*. Then Q^* is a subbundle of F and thus $\mu(Q^*) < \mu(F)$. Hence $\mu(Q) > \mu(F^*)$. Therefore, F^* is stable.

Corollary: Every stable bundle is simple, i.e. $Hom(F, F) = \mathbb{C} \cdot 1$.

Proof: Let F be a stable bundle and $h : F \to F$ be a homomorphism. Choose an eigenvalue λ of $h_x : F_x \to F_x$. Then $h - \lambda \cdot 1$ is not an isomorphism and thus $h - \lambda \cdot 1 = 0$, i.e. $h = \lambda \cdot 1$.

대수기하 특강 – 20강
김영훈

We intend to construct a family of vector bundles over X with local universal property whose equivalence classes are orbits with respect to an action of a reductive group.

Lemma: Let F be a semistable bundle over X of rank r and degree $d > r(2g-1)$. Then F is generated by its sections and $H^1(F) = 0$.

Proof: Suppose $H^1(F) \neq 0$. Then by Serre duality, $H^0(\mathbf{Hom}(F, K)) = \mathrm{Hom}(F, K) \neq 0$. Let $h : F \to K$ be a nonzero homomorphism. Let G be the subbundle generically generated by $ker(h)$. Then $\deg(G) \geq \deg ker(h) \geq \deg(F) - \deg(K) = d - 2g + 2$ and $\mathrm{rank}(G) = r - 1$. Since F is semistable, $d/r \geq (d - 2g + 2)/(r - 1)$ and hence $d \leq r(2g - 2)$. Therefore, if $d > r(2g - 2)$, then $H^1(F) = 0$. Similarly, if $d > r(2g - 1)$, $H^1(m_x F) = 0$ where m_x is the ideal sheaf of regular functions vanishing at x.

From the short exact sequence $0 \to m_x F \to F \to F_x \to 0$, we get an exact sequence $H^0(F) \to H^0(F_x) \to H^1(m_x F)$. The bundle F is generated by its sections if $H^1(m_x F) = 0$.

Corollary: A semistable bundle F is a quotient of $E := \mathcal{O}(-m)^{\oplus \chi}$ where $\chi = H^0(F(m))$ for $m >> 0$.

Proof: For sufficiently large m, we have $\deg F(m) > r(2g - 1)$ and thus a surjection $\mathcal{O}^{\oplus \chi} \to F(m)$ since $F(m)$ is generated by its sections. Tensoring $\mathcal{O}(-m)$ gives us the desired result.

Fix a very ample line bundle $\mathcal{O}_X(1)$ over X and a polynomial P of degree 1. The Hilbert scheme parametrizes all quotients of E with the Hilbert polynomial P. Consider the functor $\mathcal{H}ilb : (\mathcal{V}ar) \to (\mathcal{S}ets)$ defined by

$$\mathcal{H}ilb(S) = \begin{array}{l}\{\text{coherent quotient sheaves } \mathcal{G} \text{ of } q_S^*(E) \text{ where } q_S : S \times X \to X \\ \text{ such that } \mathcal{G} \text{ is flat over } S \text{ and the Hilbert polynomial of } \mathcal{G}(s) \\ \text{ for any } s \in S \text{ is } P\}.\end{array}$$

For any morphism $f : S' \to S$ and a quotient $\mathcal{G}$ of $q_S^* E$, the pull-back of $\mathcal{G}$ is the inverse image $(f \times 1_X)^* \mathcal{G}$. This makes $\mathcal{H}ilb$ a contravariant functor and thus we have a moduli problem.

Theorem (Grothendieck): The moduli functor $\mathcal{H}ilb$ is represented by a projective variety $Hilb^P(E)$.[35]

Lemma: There is an integer ν such that for any quotient $G = E/\mathcal{F}$ with Hilbert polynomial P and for any integer $k \geq 0$, we have

(1) $H^1(X, F(\nu + k)) = 0$
(2) the obvious map $H^0(X, \mathcal{O}_X(k)) \otimes H^0(X, \mathcal{F}(\nu)) \to H^0(X, \mathcal{F}(\nu + k))$ is surjective.

[35] It is often called the "Quot scheme".

Proof: We have a finite morphism $f : X \to \mathbb{P}^1$ such that $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(1)$.[36] Since the direct image f_*E is torsion-free,[37] f_*F is torsion-free and thus locally free. By a theorem of Grothendieck we proved in the previous lecture,

$$f_*F = \oplus_{r_j \neq 0}\mathcal{O}(j)^{r_j}.$$

Since f_*F is a subsheaf of f_*E which must be also a direct sum of line bundles, the set $\{j \,|\, r_j \neq 0\}$ is bounded above. On the other hand, since f is finite, we have $H^*(X, F(i)) \cong H^*(\mathbb{P}^1, f_*F(i))$ by Leray spectral sequence.[38] Hence, the degree of f_*F is just $\chi(F) - \mathrm{rank}(f_*F)$ and hence $\sum jr_j$ is fixed. This means that the set $\{j \,|\, r_j \neq 0\}$ is also bounded below. Hence there are only finitely elements.

For (1), just note that

$$H^1(X, F(\nu + k)) \cong H^1(\mathbb{P}^1, f_*(F(\nu + k))) \cong H^1(\mathbb{P}^1, (f_*F)(\nu + k)) = 0$$

for large enough $\nu + k$.[39] For (2), observe that $H^0(X, F(\nu)) \cong H^0(\mathbb{P}^1, f_*F(\nu))$ and $H^0(X, F(\nu + k)) \cong H^0(\mathbb{P}^1, f_*F(\nu + k))$. The homomorphism in (2) together with the natural homomorphism $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \to H^0(X, \mathcal{O}_X(k))$ gives us the map

$$H^0(\mathbb{P}^1, \mathcal{O}(k)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(\nu)) \to H^0(\mathbb{P}^1, \mathcal{O}(\nu + k))$$

which is just the product of degree k homogeneous polynomial with degree ν homogeneous polynomial. This is obviously surjective and thus the homomorphism in (2) is also surjective.

Proof of the theorem (Sketch): Let ν be a sufficiently large integer such that $H^1(X, E(i)) = 0$ for $i \geq \nu$ and ν satisfies the conditions of the above lemma. Then for any quotient $E/F = G$, the short exact sequence $0 \to F(\nu) \to E(\nu) \to G(\nu) \to 0$ gives us a short exact sequence $0 \to H^0(F(\nu)) \to H^0(E(\nu)) \to H^0(G(\nu)) \to 0$ since $H^1(F(\nu)) = 0$. Let $\mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$ be the Grassmannian of subspaces of codimension $P(\nu) = \dim H^0(G(\nu))$. Then given a quotient $E/F = G$, the subspace $H^0(F(\nu))$ of $H^0(E(\nu))$ gives us a point in $\mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$, called the *Hilbert point*. Given a family $\mathcal{F} \to S \times X$ of subsheaves, the direct image $p_*\mathcal{F}(\nu)$, where $p : S \times X \to S$ is the projection, is a subbundle of $H^0(E(\nu)) \times S$. Thus we get a morphism $S \to \mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$ and hence a natural transformation $\Phi : \mathcal{H}ilb \to \mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$.

Conversely, the Hilbert point determines the quotient $G = E/F$: Since $F(\nu)$ is generated by its sections,[40] $F(\nu)$ is the image of the composition

$$H^0(F(\nu)) \times X \to H^0(E(\nu)) \times X \to E(\nu)$$

and F is the image of $H^0(F(\nu)) \otimes \mathcal{O}(-\nu) \to E$.

The Hilbert scheme is a closed subvariety of $\mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$.[41] The universal subbundle $U \to \mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))$ of the trivial bundle $H^0(E(\nu)) \times X$ gives

[36] The very ample line bundle $\mathcal{O}_X(1)$ gives rise to an embedding $X \hookrightarrow \mathbb{P}^n$. Project X to $\mathbb{P}^{n-1}$ from a point $x \notin X \cup \mathbb{P}^{n-1}$. Project the image of X to $\mathbb{P}^{n-2}$ from a point not in $\mathbb{P}^{n-2}$. Continue this way till we get a morphism $X \to \mathbb{P}^1$.

[37] The natural homomorphism $\mathcal{O}_{\mathbb{P}^1} \to f_*\mathcal{O}_X$ is injective. Since f_*E is a locally free $f_*\mathcal{O}_X$-module, f_*E is torsion-free.

[38] $H^p(\mathbb{P}^1, R^qf_*F) \Rightarrow H^{p+q}(X, F)$. Note $R^qf_*F = 0$ for $q > 0$ since f is finite.

[39] For the second isomorphism we used the projection formula $f_*(F \otimes f^*\mathcal{O}(\nu + k)) = f_*F \otimes \mathcal{O}(\nu + k)$.

[40] $H^1(X, F(\nu) \otimes m_x) = 0$ from the exact sequence $0 \to F(\nu - 1) \to F(\nu) \otimes m_x \to T \to 0$ where T is a torsion sheaf.

[41] The second condition of the above lemma determines the closed subvariety. Namely, $Hilb^P(E)$ is the locus of subspaces Γ satisfying $\dim H^0(E(m))/\Gamma H^0(X, \mathcal{O}(m - \nu) = P(m)$ if $m \geq \nu$.

us a homomorphism $U \boxtimes \mathcal{O}_X(-\nu) \to \mathcal{O}_{\mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))} \boxtimes E$. The image of this homomorphism, restricted to the closed subvariety $Hilb^P(E) \times X$, is the universal family $\mathcal{F} \to Hilb^P(E) \times X$ of subsheaves. The quotient $\mathcal{U} = \mathcal{O}_{\mathbf{Grass}^{P(\nu)}(H^0(E(\nu)))} \boxtimes E/\mathcal{F}$ is the universal quotient sheaf.

Let $P(m) = d + rmh - r(g-1)$ and $E = \mathcal{O}(-\nu)^{\oplus p}$ where $p = P(\nu)$. Let R be the subset of $Hilb^P(E)$ whose points q satisfy the following

- $\mathcal{U}_q$ is locally free
- the canonical map $E(\nu) = \mathcal{O}^p \to \mathcal{U}_q(\nu)$ induces an isomorphism $H^0(\mathcal{O}^p) \to H^0(\mathcal{U}_q(\nu))$.

The group $G = PGL(p)$ acts on $E = \mathcal{O}(-\nu)^{\oplus p}$ in the obvious fashion and hence on the Hilbert scheme $Hilb^P(E)$. The open subvariety R is G-invariant.

<u>Theorem</u>:

(1) The restriction $\mathcal{U}|_{R\times X}$ is a vector bundle
(2) The family $\mathcal{U}|_{R\times X}$ has the local universal property for families of bundles of rank r and degree d
(3) $U_{q_1} \cong U_{q_2}$ iff q_1 and q_2 lie in the same orbit of the action of $PGL(p)$ on R
(4) For $q \in R$, the stabilizer of q in G is isomorphic to $Aut(U_q)/\mathbb{C}^* \cdot 1$.

<u>Proof</u> (Sketch): (1) obvious. (2) Given a family, we can find a neighborhood of a point where the family is obtained as the quotient of E.

(3), (4) If $U_{q_1} \cong U_{q_2}$, we get an isomorphism $H^0(\mathcal{O}^p) \cong H^0(U_{q_1}) \cong H^0(U_{q_2}) \cong H^0(\mathcal{O}^p)$ and thus an element of G. The converse is easy.

대수기하 특강 – 21강, 22강
최인송

Let X be a smooth algebraic curve of genus g. We want to classify the vector bundles over X of rank r and degree d. The following observation of the last lecture provides the starting point of doing this:

Lemma Any semistable bundle F of rank r and degree d is a quotient of a fixed trivial bundle $E = \mathcal{O}_X^p$ where $p = d - r(g-1) \gg 0$. □

Note that the degree and rank are fixed if we fix the Hilbert polynomial $P(m) = (d + rhm) - r(g-1) = p + rhm$, and conversely. (Here, $h = deg\mathcal{O}_X(1)$). Next step is to construct the Quot scheme which "bounds" the set of isomorphism classes of semistable bundles with Hilbert polynomial P. By Grothendieck's theorem, there is a projective variety $Hilb^P(E)$ inside a big Grassmannian, whose points correspond to the coherent quotient sheaves $\mathcal{G} = E/\mathcal{F}$ with Hilbert polynomial P. Also, there is a universal quotient sheaf $\mathcal{U}$ over $Hilb^P(E) \times X$ such that $\mathcal{U}_q = \mathcal{U}|_{\{q\}\times X} \cong \mathcal{G}$ for each $q = [\mathcal{G} = E/\mathcal{F}] \in Hilb^P(E)$. This variety $Hilb^P(E)$ represents the moduli functor $\mathcal{H}ilb$:

$$\Phi : \mathcal{H}ilb \to Mor(\ -\ , Hilb^P(E)).$$

Let R be the subset of $Hilb^P(E)$ whose points q satisfy
(i) $\mathcal{U}_q$ is locally free and
(ii) the quotient map $E \to \mathcal{U}_q$ induces an isomorphism $H^0(E) \cong H^0(\mathcal{U}_q)$.
Note that any semistable quotient bundle E/F of rank r and degree $d \gg 0$ satisfies these two conditions. Since these are open conditions, R is an open subset and the restriction $\mathcal{U}|_{R\times X}$ is a vector bundle[42]. The family $\mathcal{U}|_{R\times X}$ has the local universal property for families of bundles of rank r and degree d satisfying condition (ii).

Now we want to identify the isomorphic vector bundles. From the action of $G = PGL(p)$ on E, we have an induced action on $Hilb^P(E)$ given by

$$g \cdot (E/\mathcal{F}) = E/(g \cdot \mathcal{F}) \text{ for } g \in G.$$

It is clear that R is invariant under G-action. The followings are easily checked:

Lemma
(1) For $q_1, q_2 \in R$, $\mathcal{U}_{q_1}$ and $\mathcal{U}_{q_2}$ are isomorphic if and only if q_1 and q_2 lie in the same orbit.
(2) For $q \in R$, the stabilizer G_q is isomorphic to $Aut(\mathcal{U}_q)/\{\lambda I\}$.

Proof.
(1) If $\mathcal{U}_{q_1} \cong \mathcal{U}_{q_2}$, we get isomorphism $H^0(E) \cong H^0(\mathcal{U}_{q_1}) \cong H^0(\mathcal{U}_{q_2}) \cong H^0(E)$ and thus an element $g \in G$. The converse is obvious.
(2) In the same way, any $\phi \in Aut(\mathcal{U}_q)/\{\lambda I\}$ must come from a unique element $g \in G$. Hence, the homomorphism

$$\text{Stabilizer of } q \text{ in } G \to Aut(\mathcal{U}_q)/\{\lambda I\}$$

is bijective. □

In view of this lemma, we need to construct a quotient of R by G. In next section, we relate this problem to the one considered in Lecture 17 (sequences of linear subspaces).

Remark. There is also a direct approach: One can show that the (semi)stable bundles corresponds to the (semi)stable points $q \in Hilb^P(E)$ under G-action. Knowing this, one can construct the quotients $(Hilb^P(E))^{ss}//G$ and $(Hilb^P(E))^s/G$. In some sense this is more natural but requires detailed study of (semi)stability of coherent sheaves.

§4. Construction of Quotients.

Let $R^s(R^{ss})$ be the subset of R consisting of those q for which $\mathcal{U}_q$ is (semi)stable.

[42] $\mathcal{U}$ is flat over $Hilb^P(E)$.

Fix any $x \in X$ and define a map $\tau_x : R \to Gr^r(E)$ by

$$\tau_x(q) = (\mathcal{U}_q)_x,$$

where $Gr^r(E)$ is the Grassmannian of r-dimension quotient spaces in $E = \mathcal{O}_X^p$. This τ_x is a $PGL(p)$-morphism. Indeed, for any $[F = E/K] \in R$ and $g \in PGL(p)$,

$$\tau_x(g \cdot [F]) = (E/(g \cdot K)_x = g \cdot (E/K)_x = g \cdot \tau_x([F]).$$

Similarly, for any sequence $x_1, x_2, \cdots, x_N$ of points in X, the map

$$\tau : R \to (Gr^r(E))^N = Z$$

given by

$$\tau(q) = ((\mathcal{U}_q)_{x_1}, (\mathcal{U}_q)_{x_2}, \cdots, (\mathcal{U}_q)_{x_N})$$

is a $PGL(p)$-morphism.

Lemma There is a sequence of points of X for which the corresponding morphism $\tau : R \to Z$ is injective.

Proof. First note that if $q_1 \neq q_2$, then $\mathcal{U}_{q_1}$ and $\mathcal{U}_{q_2}$ are distinct as a quotient bundle of E. Hence there is point $x \in X$ at which $\mathcal{U}_{q_1}$ and $\mathcal{U}_{q_2}$ have distinct fibers. This shows that by adding points $x \in X$, we can make τ to separate any two distinct points $q_1, q_2 \in R$.

Now let D be the closed subvariety of R given by the inverse image of the diagonal Δ_Z of $Z \times Z$ under the morphism $\tau \times \tau : R \times R \to Z \times Z$. Consider a family of such D for arbitrary choice of the sequence $x_1, x_2, \cdots, x_N$ for arbitrary N. Note that any D in this family contains the diagonal $\Delta_R \subset R \times R$. By the Noetherian property, this family has a minimal element D_0. Above argument shows that D_0 coincides with Δ_R, which means that the corresponding τ is injective. $\square$

Moreover, we can require this injective morphism $\tau : R \to Z$ to satisfy the following additional properties:

Theorem (Theorem 5.6 in the book) For any fixed r, there is an integer d_0 such that for all $d > d_0$, there exists a sequence of points of X for which the corresponding morphism $\tau : R \to Z$ satisfies the followings.

(1) τ is an immersion, i.e., R is isomorphic to $\tau(R)$,
(2) $R^{ss} = \tau^{-1}(Z^{ss})$,
(3) $R^s = \tau^{-1}(Z^s)$,
(4) $\tau : R^{ss} \to Z^{ss}$ is proper.

Proof. Postponed to §6. $\square$

Note that by (2), R^{ss} is open in R and hence it is a quasi-projective variety. By (1), we identify R^{ss} with its image in Z^{ss}. Under the $PGL(p)$-action on the Zariski closure of R^{ss}, R^{ss} and R^s coincide with the set of semistable and stable points of $\overline{R^{ss}}$ respectively.[43] So there is a good quotient $M(r,d) = R^{ss}//G$ which is a projective variety. Also there is an open subset $M^s(r,d)$ of $M(r,d)$ which is a geometric quotient R^s/G.

Theorem There exists a coarse moduli space $M^s(r,d)$ for stable bundles of rank r and degree d over X. Also it has a natural compactification to a projective variety $M(r,d)$.

Proof. If we take $d > d_0 \geq r(2g-1)$, $R^s(R^{ss})$ has the local universal property for (semi)stable bundles of rank r and degree d. Hence the geometric quotient $M^s(r,d)$ is a coarse moduli space (cf: Proposition 2.13 in the book).

In general, by tensoring $\mathcal{O}_X(m)$ with $m \gg 0$, we can argue in the same way for arbitrary degree d. In particular, $M^s(r,d) \cong M^s(r, d+rmh)$ for any m. $\square$

Note that two points in R^{ss} collapse to the same point in $M(r,d)$ if and only if the closures of their orbits meet in R^{ss}. What does a point of $M(r,d)$ stand for in terms of bundles? To answer this, we need the following notion.

[43] For a closed invariant subvariety Y of X, $Y^{ss} = X^{ss} \cap Y$ and $Y^s = X^s \cap Y$.

Definition (Jordan-Hölder filtration)
For any semistable bundle F, there is a sequence of subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = F$$

such that for each i, F_i/F_{i-1} is stable and $\mu(F_i/F_{i-1}) = \mu(F)$.
Moreover the bundle

$$gr(F) = \bigoplus_{i=1}^{k} F_i/F_{i-1}$$

is determined by F up to isomorphism.

This filtration is obtained by the Jordan-Hölder theorem for the category $\mathcal{C}(\mu)$ of semistable vector bundles of fixed slope μ. Now we have

Theorem Two semistable bundles F and F' determine the same point of $M(r,d)$ if and only if $gr(F) \cong gr(F')$. Hence $M(r,d)$ parameterizes the "S-equivalent" classes of semistable bundles of rank r and degree d.

Proof.
($\Leftarrow$) It suffices to show that F and $gr(F)$ collapses to the same point of $M(r,d)$. In particular it suffices to argue this for F and $F_1 \oplus F/F_1$ where F_1 is a subbundle of F of slope $\mu = \mu(F)$. For the family $Ext^1(F/F_1, F_1)$, we have a map[44]

$$Ext^1(F/F_1, F_1) \to M(r,d).$$

Consider the restriction of this map on the straight line F_t passing through the origin ($= F_1 \oplus F/F_1$) and F in $Ext^1(F/F_1, F_1)$. Since $F_t \cong F$ for $t \neq 0$, this map must be constant. Note that the scalar multiplication by λ on $Ext^1(F/F_1, F_1)$ corresponds to the extension

$$0 \to F_1 \to F \stackrel{\lambda p}{\to} F/F_1 \to 0.$$

($\Rightarrow$) To prove that two non-isomorphic polystable bundles (direct sums of stable bundles of the same slope) define distinct points, it suffices to show that any polystable bundle F has a closed orbit. In the closure $\overline{O(F)}$, take F_∞ which has a closed orbit. Above argument shows that F_∞ must be polystable. Now from the existence of a sequence of orbit points of F converging to F_∞, it is easily seen that $F \cong F_\infty$[45]. $\square$

Remarks.
(1) One can show that $M(r,d)$ is normal and irreducible, and that $M^s(r,d)$ is smooth.
(2) If $M^s(r,d) \neq \emptyset$, then

$$\dim M(r,d) = \dim M^s(r,d) = r^2(g-1) + 1.$$

Also, $M^s(r,d) = \emptyset$ if either ($g = 0$ and $r \geq 2$) or ($g = 1$ and $(r,d) \neq 1$). Otherwise, $M^s(r,d) \neq \emptyset$.
(3) When r and d are coprime, $M(r,d) = M^s(r,d)$ and the moduli space is a smooth projective variety.

§5. Existence of a fine moduli space.

The goal of this section is to prove the following

Theorem If r and d are coprime, then there is a bundle V over $M^s(r,d) \times X$ which gives a fine moduli space for stable bundles of rank r and degree d over X (with respect to the equivalence relation on families).

We start from

---

[44]From the local universal property of $M(r,d)$, the map is defined at least on a neighborhood of the origin of $Ext^1(F/F_1, F_1)$.

[45]For $F = \oplus F_i^{r_i}$ with each F_i stable, $\dim Hom(F_i, F_\infty) \geq r_i$. This implies that F_∞ has at least r_i copies of F_i as its direct summands.

Lemma Let T be a smooth variety. Let E be a vector bundle over $T \times X$ such that E_t is stable bundle over $\{t\} \times X$ for each $t \in T$. For another vector bundle F over $T \times X$,

$$E_t \cong F_t \; for \; all \; t \in T \; if \; and \; only \; if \; F \cong E \otimes (p_T)^* L$$

for some line bundle L over T and the projection map $p_T : T \times X \to T$. (In this case, E and F are called to be equivalent).

Proof.
($\Leftarrow$) Clear since $((p_T)^* L)_t$ is trivial over X for each $t \in T$.
($\Rightarrow$) Since E_t is stable for each $t \in T$, we have
(i) each nonzero homomorphism $E_t \to F_t$ is an isomorphism and
(ii) $H^0(Hom(E_t, F_t)) \cong \mathbb{C}$ (stable $\Rightarrow$ simple).
Hence $L := (P_T)_*(Hom(E, F))$ is a line bundle over T with fiber
$L_t = H^0(Hom(E_t, F_t))$ at $t \in T$[46]. From this we get a homomorphism[47]
$\phi : E \otimes (p_T)^* L \to F$ defined by

$$\phi_t : E_t \otimes H^0(Hom(E_t, F_t)) \to F_t,$$

which should be an isomorphism by (i) again. □

In view of this lemma, we get

Corollary If there exists a bundle V over $M^s(r, d) \times X$ such that, for all $q \in M^s(r, d)$, V_q is the stable bundle corresponding to q, then $M^s(r, d)$ is a fine moduli space for the stable bundles of rank r and degree d over X with respect to the equivalence relation on families (cf: Proposition 1.8 in the book).

Now recall the construction of $M^s(r, d)$. For $p = \dim E$, $GL(p)$ acts on $\mathcal{U} \to Hilb^P(E) \times X$, which restricts to $\mathcal{U}' := \mathcal{U}|_{R^s \times X} \to R^s \times X$. Here, the matrices $\{\lambda I\}$ in $GL(p)$ acts trivially on R^s and the moduli space $M^s(r, d)$ is obtained by the geometric quotient $R^s//PGL(p)$.

Hence the natural approach to get a bundle V in the above corollary is to try for something like $\mathcal{U}'//PGL(p)$. But the matrix λI acts as scalar multiplication by λ on $\mathcal{U}_q$, hence $PGL(p)$ does not act on $\mathcal{U}'$. Our strategy is to construct a line bundle L over R^s such that
(a) the action of $GL(p)$ on R^s lifts to an action on L ;
(b) λI acts on L by a scalar multiplication by λ.

Once such L obtained, we put $\hat{\mathcal{U}} := \mathcal{U}' \otimes (p_{R^s})^* L^{-1}$. Now λI acts on $\hat{\mathcal{U}}$ trivially and we get a $PGL(p)$-vector bundle $\hat{\mathcal{U}}$ over $R^s \times X$. Note that $\hat{\mathcal{U}}$ is equivalent to $\mathcal{U}'$. Hence by taking the geometric quotient, we get the wanted vector bundle $V = \hat{\mathcal{U}}//PGL(p)$ over $(R^s \times X)//PGL(p) = M^s(r, d) \times X$.

The last step can be justified by the decent lemma due to Kempf (lecture 14). To apply the descent lemma, we need to show that for each point $q \in R^s$, the stabilizer $PGL(p)_q$ acts trivially on $\hat{\mathcal{U}}_q$. But as was seen before, $PGL(p)_q \cong Aut(\hat{\mathcal{U}}_q)/\lambda I$, which is trivial since $\hat{\mathcal{U}}_q$ is stable.

Finally we prove

Lemma If $(r, d) = 1$, then there exists a line bundle L over R^s satisfying the above properties (a) and (b).

Proof. Fix a line bundle J over X with degree 1 and consider the bundle

$$E_m = U' \otimes (P_X)^* J^m$$

[46]Generally for a vector bundle F over $T \times X$, F is flat over T iff $\chi(F_t)$ is constant. In addition if $h^0(F_t)$ (and $h^1(F_t)$) remains constant, then $p_* F$ is locally free of rank $h^0(F_t)$.

[47]ϕ is obtained by the composition map
$E \otimes (p_T)^* L = E \otimes (p_T)^*(p_T)_* Hom(E, F) \to E \otimes Hom(E, F) \to F.$

over $R^s \times X$ of degree $d + rm$. For sufficiently large m, we have

$$H^1((E_m)_q) = H^1(U'_q \otimes J^m) = 0$$

for all $q \in R^s$. Hence

$$h^0((E_m)_q) = \chi((E_m)_q) = deg((E_m)_q) + r(1-g) = d + r(m-g+1).$$

Thus $(p_{R^s})_* E_m$ corresponds to a vector bundle, say F_m, over R^s of rank $d + r(m - g + 1)$. The $GL(p)$-action on $U' \to R^s \times X$ induces the $GL(p)$-action on $F_m \to R^s$, where λI acts by scalar multiplication by λ as before. Note that λI acts by scalar multiplication by $\lambda^{d+r(m-g+1)}$ on the determinant line bundle $detF_m$.

Since $(r, d) = 1$, we have

$$(d + r(m-g+1)\ ,\ d + r(m+1-g+1)) = 1$$

so there exists integers a and b such that

$$a\ (d + r(m-g+1))\ +\ b\ (d + r(m+1-g+1))\ =\ 1.$$

Now the line bundle

$$L = (detF_m)^a \otimes (detF_{m+1})^b$$

has the required properties (a) and (b). □

Remark. It is known that there is no fine moduli space if $(r, d) \neq 1$ ([Ramanan]).

§6. Proof of Theorem 5.6

First we prove the following:

Proposition Let F be a vector bundle over X generically generated by its global sections.

(1) Let λ be the number of distinct points x of X at which $H^0(F)$ does not generate F_x. Then $\lambda \leq deg(F)$.

(2) $h^0(F) \leq degF + rkF$.

Proof. In case F is a line bundle, we have an exact sequence

$$0 \to \mathcal{O}_X \to F \to T \to 0,$$

where T is a torsion sheaf. Thus

$$\lambda \leq \#\text{ of points of } SuppT \leq h^0(T) = degT = degF.$$

Also,

$$h^0(F) \leq h^0(\mathcal{O}_X) + h^0(T) = 1 + degF.$$

In general, choose a global section s of F and argue for F/L inductively, where L is the trivial line bundle defined by s. □

Now we prove Theorem 5.6.

Proof of Theorem 5.6 (1). Omitted. To prove this, it is necessary to study the differential properties of R.

Proof of Theorem 5.6 (2) and (3). We need several steps to prove these. First we show

Lemma There is d_1 such that, for $d > d_1$, every semistable bundle F of rank r and degree d satisfies the followings:

(1) Any subbundle G of F with slope $\mu(G) = d/r$ is also generated by global sections and $H^1(G) = 0$.

(2) For any subbundle G of F with $\mu(G) < d/r$, if it is generically generated by global sections, then

$$\frac{h^0(G)}{rkG} < \frac{h^0(F)}{rkF}\ .$$

Proof. (1) Suppose $d > r(2g-1)$. For $r' = rankG$ and $d' = degG$, note that $d' = (r'/r)d > r'(2g-1)$. Now since G is semistable, the wanted result follows from

what we proved in Lecture 20.
(2) Recall the Riemann-Roch formula: $h^0 - h^1 = d - r(g-1)$. If $H^1(G) = 0$, then

$$\frac{h^0(G)}{r'} = \mu' + 1 - g < \mu + 1 - g = \frac{h^0(F)}{r}$$

and we are done.

Now suppose $H^1(G) \neq 0$. By Serre duality, we have a map $h : G \to K_X$ for the canonical line bundle K_X. The subbundle generically generated by $Ker(h)$ has rank $r'' = r' - 1$ and degree d'',

$$d'' \geq deg(Ker(h)) = degG - deg(Im(h)) \geq degG - degK = d' - (2g-2).$$

From the semistability of F, we get[48]

$$r(d' - (2g-2)) \leq (r'-1)d.$$

From the above proposition, we know $h^0(G) \leq d' + r'$ and to prove (2), it suffices to show

$$d'/r' + 1 < d/r + 1 - g.$$

From above inequality, this is true if

$$d > r(2g-2) + r(r-1)g. \quad \square$$

Now consider the map

$$\tau(q) = ((\mathcal{U}_q)_{x_1}, (\mathcal{U}_q)_{x_2}, \cdots, (\mathcal{U}_q)_{x_N}).$$

For any subspace V of $H^0(\mathcal{O}_X^p)$, we let V_j denote the image of V in $(\mathcal{U}_q)_{x_j}$. Recall that $\tau(q)$ is a (semi)stable point if and only if for every nonzero proper subspace V of $H^0(\mathcal{O}_X^p) \cong H^0(\mathcal{U}_q)$,

$$\rho(V) = \frac{1}{N \ \dim V} \sum_{j=1}^{N} \dim V_j - \frac{r}{p} > 0 \ (\geq 0).$$

To show (2) and (3) of Theorem 5.6, we claim:
(A) $q \in R^{ss} \Rightarrow \rho(V) \geq 0$ for all V, hence $\tau(R^{ss}) \subset Z^{ss}$,
(B) $q \in R^s \Rightarrow \rho(V) > 0$ for all V, hence $\tau(R^s) \subset Z^s$,
(C) $q \in R^{ss} \backslash R^s \Rightarrow \rho(V) = 0$ for some V, hence $\tau(R^{ss}\backslash R^s) \subset Z^{ss}\backslash Z^s$.
(D) $\tau(R \backslash R^{ss}) \subset Z \backslash Z^{ss}$.

For (C), if $q \in R^{ss}\backslash R^s$, then there is a proper subbundle G of $\mathcal{U}_q$ such that $\mu(G) = d/r$. By (1) of the above lemma, $H^1(G) = 0$ and $H^0(G)$ generate G. Hence

$$\begin{aligned} \rho(H^0(G)) &= \frac{rkG}{h^0(G)} - \frac{r}{p} \\ &= \frac{rkG}{deg(G) + rkG(1-g)} - \frac{r}{p} \\ &= \frac{r}{d + r(1-g)} - \frac{r}{p} \\ &= 0. \end{aligned}$$

For (D), let $q \in R \backslash R^{ss}$. Choose a subbundle G of $\mathcal{U}_q$ of smallest possible rank such that $\mu(G) > d/r$, which is obviously stable. The same argument as above

[48] In case $r' = 1$, it can be seen that G is an invertible sheaf generically generate K_X and so $d' \leq 2g - 2$ and the inequality is still valid.

shows

$$\begin{aligned}\rho(H^0(G)) &= \frac{rkG}{h^0(G)} - \frac{r}{p} \\ &= \frac{rkG}{deg(G) + rkG(1-g)} - \frac{r}{p} \\ &> \frac{r}{d + r(1-g)} - \frac{r}{p} \\ &= 0.\end{aligned}$$

For (A) and (B), suppose $q \in R^{ss}$ and let V be any nonzero proper subspace of $H^0(\mathcal{U}_q)$. We write G for the subbundle of $\mathcal{U}_q$ generically generated by V, and so $V \subset H^0(G)$. We put

$$\sigma(V) = \frac{rkG}{\dim V} - \frac{r}{p}.$$

If $\mu(G) = d/r$, it follows from the above formula that

$$\sigma(V) \geq \rho(H^0(G)) = 0,$$

equality occurring if and only if $V = H^0(G)$[49]. In particular, when $V = H^0(G)$, (A) and (B) holds vacuously. So we can suppose that $V \neq H^0(G)$, and therefore $\sigma(V) > 0$.

On the other hand, if $\mu(G) < d/r$, then by (2) of the above lemma,

$$\sigma(V) \geq \frac{rkG}{h^0(G)} - \frac{r}{p} > \frac{r}{p} - \frac{r}{p} = 0.$$

So in any case $\sigma(V) > 0$ and hence (since $p\ \dim V \cdot \sigma(V)$ is an integer) $\sigma(V) \geq 1/p^2$. Now

$$\sigma(V) - \rho(V) = \frac{1}{N\ \dim V} \sum_{j=1}^{N} (rkG - \dim V_j).$$

If V generates the fibers of G at all x_j, then $\rho(V) = \sigma(V) > 0$ as required. In general, let λ be the number of points of X for which V does not generate G_x. Then

$$\sigma(V) - \rho(V) \leq \frac{\lambda \cdot rkG}{N\ \dim V} \leq \frac{\lambda}{N},$$

hence $N\rho(V) \geq N\sigma(V) - \lambda \geq N/p^2 - \lambda$.

Therefore, $\rho(V) > 0$ if $N > \lambda p^2$. By (1) of the above Proposition, we have $\lambda \leq degG$ and $degG \leq (rkG/r)d \leq d$. So it suffices to choose $N > dp^2$. □

Proof of (4) of Theorem 5.6. Put $Q = Hilb^P(E)$. First we show

Lemma There are integers $d_2, N_2(d)$ such that whenever $d > d_2$ and $N \geq N_2(d)$, there is a closed set Φ in $Q \times Z$ containing the graph of τ and

$$\Phi \cap (Q \times Z^{ss}) = \text{ graph of } \tau|_{R^{ss}}.$$

Note that $\tau : R^{ss} \to Z^{ss}$ has a factorization

$$R^{ss} \to \text{ graph of } \tau|_{R^{ss}} \subset Q \times Z^{ss} \xrightarrow{proj} Z^{ss},$$

the first being an isomorphism. By the above lemma, the graph of $\tau|_{R^{ss}}$ is closed in $Q \times Z^{ss}$, and since Q is projective, the last projection is proper. The result follows at once. □

Sketch of proof of the lemma. A detailed proof is given in the book. First we construct a closed set Φ in $Q \times Z$ such that

$$\Phi \cap (R \times Z) = \text{ graph of } \tau.$$

[49]Since G is semistable, we always have $\frac{rkG}{\dim V} \geq \frac{rkG}{h^0(G)} = \frac{r}{p}$.

To do this, we need to think an open subset

$$Q_x = \{q \in Q : \mathcal{U}_q \text{ is locally free at } x\}$$

for $x \in X$ and extend the morphism τ to $\tau'_x : Q_x \to Gr^r(E)$. First define

$$\Phi_x = (\text{graph of } \tau'_x|_{Q_x}) \bigcup ((Q \backslash Q_x) \times Gr^r(E))$$

and then put $\Phi = \bigcap_{j=1}^N \Phi_x$. It is easily seen that $\Phi \cap (R \times Z)$ coincides with the graph of τ.

Next step is to show that if we take sufficiently large $d_2, N_2(d)$, then

$$\Phi \cap (Q \times Z^{ss}) = \text{ graph of } \tau|_{R^{ss}},$$

as stated in the lemma. Inclusion ($\supset$) is already proven in above (A). Also, For $q \in R\backslash R^{ss}$, $\tau(q)$ is unstable by above (D).

The most difficult thing is to show that if $(q, y) \in \Phi$ with $q \in Q\backslash R$, then y is unstable. Here, $\mathcal{U}_q$ may not be locally free and the isomorphism $H^0(\mathcal{O}_X^p) \cong H^0(\mathcal{U}_q)$ is not always guaranteed. The proof in the book provides a way how to take care this problem. $\square$

대수기하 특강 – 23강
최인송 정리

Let X be a smooth algebraic curve of genus g, $M = M_X(r,d)$ the moduli space of semistable vector bundles over X of rank r and degree d. Our aim is to prove the following:

<u>Main Theorem</u> If E is a stable vector bundle over X, then M is smooth at $[E]$ and $T_{[E]}M = Ext^1(E,E) \cong H^1(X, End(E))$. □

As a corollary, we can see that if M^s is not empty, then

$$\dim M = h^1(X, End(E)) = -\chi(End(E)) + h^0(End(E)) = r^2(g-1)+1.$$

We start from considering the scheme-theoretic conditions for the smoothness (reference: Le Potier, Chapter 8). Let Y be a variety and $a \in Y$. By definition, Y is smooth at a if $\dim(T_aY) = \dim \mathcal{O}_{Y,a}$.

- Tangent space

Let T denote the tangent space $T_aY = (\mathfrak{m}/\mathfrak{m}^2)^*$ for the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_{Y,a}$. A non-reduced scheme $D = Spec(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ supported at one point is called the *dual number*. It is easy to check that

$$T_aY = (\mathfrak{m}/\mathfrak{m}^2)^* = Mor_a(D,Y).$$

Also note that

$$T_aY = (\mathfrak{m}/\mathfrak{m}^2)^* = Spec(Sym(\mathfrak{m}/\mathfrak{m}^2)) = Spec(\oplus_{i\geq 0} Sym^i(\mathfrak{m}/\mathfrak{m}^2)).$$

- Tangent cone

Let $C = CY$ denote the tangent cone $Spec(\bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1})$ of Y with vertex o. Roughly speaking, it is cut out by the minimal degree terms of the polynomials in the ideal of Y, while the tangent space T is cut out by the linear terms. The corresponding maximal ideal at o is given by $\bigoplus_{i>0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

For example, Let Y be the plane curve cut out by $y^2 - x^3 = 0$. Then $T_oY \cong \mathbb{C}^2$, while the tangent cone CY at the origin is the double line ($y^2 = 0$).

Since $Sym^i(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow (\mathfrak{m}^i/\mathfrak{m}^{i+1})$ for each i, $Sym(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow \bigoplus_{i\geq 0}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ and so $C \subset T$.

<u>Lemma</u> $\dim_a Y = dim_o C$

Proof. Consider the map $n \mapsto \dim_{\mathbb{C}}(R/\mathfrak{m}^n)$, where $R = \mathcal{O}_{Y,a}$. For $n \gg 0$, this is a polynomial map, called the Hilbert-Samuel polynomial, whose degree is $\dim R = \dim_a Y$ (Hartshorne, V, Ex. 3.4). In the same way, $\dim_o C$ is equal to the degree of the polynomial map

$$n \mapsto \dim_{\mathbb{C}}(\bigoplus_{i\geq 0}(\mathfrak{m}^i/\mathfrak{m}^{i+1})/\bigoplus_{i\geq n}(\mathfrak{m}^i/\mathfrak{m}^{i+1})),$$

but this coincides with the map

$$n \mapsto \dim_{\mathbb{C}}(\bigoplus_{i=0}^{n-1}(\mathfrak{m}^i/\mathfrak{m}^{i+1})) = \dim(R/\mathfrak{m}^n). \ \square$$

<u>Proposition</u> Y is smooth at a if and only if $C = T$.

Proof. By the above lemma, $\dim_a Y = \dim T$ if and only if $\dim_o C = \dim T$. Since C is a closed subset of the irreducible variety T, the equality holds if and only if $C = T$. □

To catch the "very local" behavior, we consider the following notion corresponding to the Taylor expansions in analysis.

Definition The k-th infinitesimal neighborhood of a in Y is defined by $Y_k = Spec(\mathcal{O}_{Y,a}/\mathfrak{m}^{k+1})$.

Theorem Y is smooth at a if and only if for any surjection $\tilde{A} \to A$ of Artinian local rings over $\mathbb{C}$, any morphism $Spec A \to Y$ supported at a lifts to $Spec\tilde{A}$.
(An Artinian local ring is a Noetherian local ring with unique prime ideal $\mathfrak{m}$ such that $\mathfrak{m}^k = 0$ for some n. A typical example of Artinian local ring is: $\mathbb{C}[t_1, t_2, \cdots, t_d]_o/\mathfrak{n}^k$, $\mathfrak{n}$ its maximal ideal. Reference: Atiyah and Macdonald, Chapter 8.)

Proof. The details can be found in, e.g., Le Potier, pp.123-125. Here we just prove the following what we need later: Let T_k denote the k-th infinitesimal neighborhood of the tangent space $T = T_aY$ at the origin. If the natural embedding $T_1 = Y_1 \hookrightarrow Y$ lifts to $T_k \to Y$, then Y is smooth at a.

First notice that $T_k = Spec(\mathbb{C}[t_1, \cdots, t_d]_o/\mathfrak{n}^{k+1})$, where $d = \dim_a Y$. By the assumption, we have a map $T_k \to Y$. Taking the tangent cones, we have $C(T_k) = T_n \to CY \hookrightarrow T$. Thus obtained map $T_k \to T$ is given by the surjection $\mathbb{C}[t_1, \cdots, t_d] \twoheadrightarrow \mathbb{C}[t_1, \cdots, t_d]_o/\mathfrak{n}^{k+1}$, which implies that $T_k \to T$ is an embedding. Hence we see that $T_k \to C = CY$ is also an embedding for each $k \geq 1$.

From this, we get

$$\dim \mathcal{O}_{T,o}/\mathfrak{n}^{k+1} = \dim \mathcal{O}_{T_k,o} \leq \dim \mathcal{O}_{C,o}/\tilde{\mathfrak{n}}^{k+1},$$

where $\tilde{\mathfrak{n}}$ denotes the maximal ideal of $\mathcal{O}_{C,o}$. Since both sides correspond to the values evaluated at $(k+1)$ of the Hilbert-Samuel polynomial maps of (T, o) and (C, o) respectively, we conclude that $\dim T \leq \dim C$. This shows that $T = C$ and thus Y is smooth at a. $\square$

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Now we prove the Main Theorem. The moduli space M is given by the GIT quotient $\phi : R^{ss} \to M = R^{ss}//PGL(p)$. For $q \in R^s$ lying over a stable bundle $[E]$, the stabilizer group $Stab(q) = Aut(E)/\lambda I$ is trivial. Hence by Luna's slice theorem, there is a locally closed subvariety S of R^{ss}, passing through q, such that $\phi|_S : S \to M$ is étale. This implies that $\hat{\mathcal{O}}_{S,Q} \cong \hat{\mathcal{O}}_{M,[E]}$ (Hartshorne, III Ex. 10.4).

Claim 1: For any Artinian local ring A,

$$Mor_q(SpecA, S) \cong Mor_{[E]}(SpecA, M).$$

Proof. We use the completion of local rings (reference: Atiyah and Macdonald, Chap.10). Note that since A is an Artinian local ring, $A = \hat{A}$.

$$\begin{aligned} SpecA \to (S,q) \quad &\Leftrightarrow \quad \mathcal{O}_{S,q} \to A \;\Leftrightarrow\; \hat{\mathcal{O}}_{S,q} \to \hat{A} = A \\ &\Leftrightarrow \quad \hat{\mathcal{O}}_{M,[E]} \to A \;\Leftrightarrow\; \mathcal{O}_{M,[E]} \to A \\ &\Leftrightarrow \quad SpecA \to (M,[E]). \;\square \end{aligned}$$

Definition A *deformation of* E *with base* $SpecA$ is a coherent sheaf $\mathcal{E}$ over $SpecA \times X$, flat over $SpecA$, whose central fiber at $Spec(A/\mathfrak{m}) \times X$ is isomorphic to E.

In other words, $\mathcal{E}$ is a coherent sheaf of $(\mathcal{O}_X \otimes_{\mathbb{C}} A)$-modules, flat over A, such that $\mathcal{E}/\mathfrak{m}\mathcal{E} \cong E$. (By Nakayama's lemma, $\mathcal{E}$ is locally free). We let $Def^E(SpecA)$ denote the set of isomorphism classes of deformations of E with base $SpecA$.

Claim 2: $Def^E(SpecA) = Mor_{[E]}(SpecA, M)$.

Proof. From above Claim 1, $SpecA \to (M,[E]) \Leftrightarrow SpecA \to (S,q)$, where $S \subset R$. From the existence of the universal bundle on R, we see that $Mor_q(SpecA, S) \cong Def^E(SpecA)$. $\square$

Claim 3: M is smooth at $[E]$ if and only if for any surjection $\tilde{A} \twoheadrightarrow A$ of Artinian local rings, any deformation of E over $SpecA$ lifts to that over $Spec\tilde{A}$, i.e.,

$$Def^E(Spec\tilde{A}) \twoheadrightarrow Def^E(SpecA).$$

Proof. By Claim 2, we may prove instead that M is smooth at $[E]$ iff $Mor_{[E]}(Spec\tilde{A}, M) \twoheadrightarrow Mor_{[E]}(SpecA, M)$. But this is just the smoothness criterion we have already proven. $\square$

Claim 4: Suppose that E is a stable bundle. Then the above deformation lifting property holds.

Proof. Recall that in the proof of the smoothness criterion, it was enough to consider the embedding $T_1 \hookrightarrow T_k$, which amounts to considering the surjection $\mathbb{C}[t_1, \cdots, t_d]_o/\mathfrak{n}^{k+1} \twoheadrightarrow \mathbb{C}[t_1, \cdots, t_d]_o/\mathfrak{n}^2$. This map can be decomposed into a sequence of maps of the following type: $\tilde{A} \to A = \tilde{A}/I$, where I is 1-dimensional, $I = \mathbb{C} \cdot \nu$. Also for the maximal ideals $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$ of A and $\tilde{A}$ respectively, we have vector space decompositions $\tilde{A} = \mathbb{C} \cdot 1 \oplus \mathfrak{m} \oplus \mathbb{C} \cdot \nu$ and $\tilde{\mathfrak{m}} = \mathfrak{m} \oplus \mathbb{C} \cdot \nu$, where $\nu^2 = \nu\mathfrak{m} = 0$.

Now choose a basis $\mu_1, \mu_2, \cdots, \mu_n$ of $\mathfrak{m}$. Then $\mu_1, \mu_2, \cdots, \mu_n, \nu$ give a basis of $\tilde{\mathfrak{m}}$. We denote the multiplications in A and $\tilde{A}$ by $\cdot$ and $*$ respectively. They are related as follows:

$$(\sum a_i\mu_i) * (\sum a'_i\mu_i) = (\sum a_i\mu_i) \cdot (\sum a'_i\mu_i) + \sum_{i,j} b_{ij}a_i a'_j \nu,$$

where $\sum b_{ij}a_i a'_j$ is a symmetric bilinear form.

For a given deformation $\mathcal{E}$ of E over $SpecA$, we can find an open cover $\{U_\alpha\}$ of X such that $E|_{U_\alpha}$ is a free $\mathcal{O}_X(U_\alpha)$-module and $\psi_\alpha : \mathcal{E}|_{U_\alpha} \cong E|_{U_\alpha} \otimes_{\mathbb{C}} A$ a free $(\mathcal{O}_X \otimes_{\mathbb{C}} A)$-module. (Nakayama's lemma)

On $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $D_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ is an endomorphism of $E|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} A$, which is a Čech 1-cocycle. Put $D_{\alpha\beta} = I_E + \sum_{i=1}^n D^i_{\alpha\beta}\mu_i$. The cocycle condition is given by

$$(I_E + \sum_{i=1}^n D^i_{\alpha\beta}\mu_i) \cdot (I_E + \sum_{i=1}^n D^i_{\beta\gamma}\mu_i) = (I_E + \sum_{i=1}^n D^i_{\alpha\gamma}\mu_i).$$

If an extension $\tilde{\mathcal{E}}$ with base $Spec\tilde{A}$ exists, then

$$\tilde{D}_{\alpha\beta} = I_E + \sum_{i=1}^n D^i_{\alpha\beta}\mu_i + G_{\alpha\beta}\nu,$$

where $G_{\alpha\beta} \in H^0(U_{\alpha\beta}, End(E))$. The cocycle condition is given by $\tilde{D}_{\alpha\beta} * \tilde{D}_{\beta\gamma} = \tilde{D}_{\alpha\gamma}$. From the above cocycle condition for $D_{\alpha\beta}$, this holds iff

$$\sum_{i,j} b_{ij} D^i_{\alpha\beta} D^j_{\beta\gamma} = -G_{\alpha\beta} - G_{\beta\gamma} - G_{\gamma\alpha} = -\partial\{G_{\alpha\beta}\}.$$

We conclude that there is an extension $\tilde{\mathcal{E}}$ with base $Spec\tilde{A}$ iff the 2-cocycle in the left-hand side is a coboundary of some 1-cycle $\{-G_{\alpha\beta}\}$.

Since $\dim X = 1$, $H^2(X, End(E)) = 0$ and so any 2-cocycle is a coboundary. This completes the proof that M is smooth at $[E]$. $\square$

Now we turn to the tangent space description of M. As already indicated, $T_{[E]} = Mor_{[E]}(D, M)$ for the dual number $D = Spec(\mathbb{C}[\varepsilon]/(\varepsilon^2))$. By Claim 2, this coincides with $Def^E(D)$. As we have seen in the above proof, a deformation of E with base D yields a 1-cocycle $\{D_{\alpha\beta}\}$. Since the trivial deformation corresponds to the coboundaries, we have injection $Def^E(D) \hookrightarrow H^1(X, EndE)$. Again the vanishing $H^2(X, EndE) = 0$ shows that there is no obstruction for the deformation, proving that

$$T_{[E]}M = H^1(X, EndE) = Ext^1(E, E).$$

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- 80점이상 A^+, 50점이상 A^0, 30점이상 A^-, 30점미만 B, 안내거나 베끼면 F.

(1) Let $P1$ and $P2$ be two moduli problems with the sets of objects A_1 and A_2 respectively. We can define the product moduli problem Pr by considering the product of the sets of objects $A_1 \times A_2$ and by letting $\mathcal{F}_{Pr}(S) = \mathcal{F}_{P1}(S) \times \mathcal{F}_{P2}(S)$ where $\mathcal{F}_*(S)$ denotes the set of isomorphism classes of the families parametrized by S for the moduli problem $*$. Show that if M_1 and M_2 are the coarse moduli spaces for $P1$ and $P2$ respectively, then $M_1 \times M_2$ is the coarse moduli space for Pr.

(2) Prove the "reduction in stages" principle: Let G be an algebraic group acting on a variety X. Let H be a normal subgroup of G and $K = G/H$. Suppose the categorical quotients $Y = X//H$ and $Z = Y//K$ exist. Prove that Z is the categorial quotient of X by G.

(3) Let $\mathbb{C}^*$ act on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by
$$\lambda \cdot (t_1, \cdots, t_4) = (\lambda t_1, \lambda t_2, \lambda^{-1} t_3, \lambda^{-1} t_4).$$
  (a) Find the quotient $\mathbb{C}^4//\mathbb{C}^*$.
  (b) Find the quotient $\mathbb{P}(\mathbb{C}^4)//\mathbb{C}^*$.

(4) Let $\phi : X^{ss} \to X//G$ be a good quotient of a variety X. Show that in each fiber of ϕ there is a unique *closed* orbit.

(5) Prove the "quantization commutes with reduction" principle: Suppose X is a projective variety with very ample line bundle $\mathcal{O}_X(1)$ which is acted on linearly by a reductive group G. Let Y be the good quotient of X^{ss} by G and $\mathcal{O}_Y(1)$ be the induced ample line bundle. Then
$$H^0(Y, \mathcal{O}_Y(k)) = H^0(X, \mathcal{O}_X(k))^G$$
for sufficiently large k.

(6) Suppose a reductive group G acts on varieties X and Y linearly (with respect to the line bundles L_1, L_2 respectively). Consider the diagonal linear action of G on $X \times Y$ (with respect to $pr_1^* L_1 \otimes pr_2^* L_2$). Show that for each one parameter subgroup λ of G we have
$$\mu((x, y), \lambda) = \mu(x, \lambda) + \mu(y, \lambda).$$

(7) Let X be a variety with a linear action of a reductive group G. The action of G on L induces a linear action of G on L^r for any integer r. Prove that for any integer $r \geq 1$, a point $x \in X$ is (semi)stable with respect to L iff x is (semi)stable with respect to L^r.

(8) Let G be a reductive group acting on varieties X and Y and let $\phi : X \to Y$ be a finite equivariant morphism. Suppose a good quotient $Y//G$ exists. Prove that a good quotient $X//G$ of X by G exists and the induced morphism $X//G \to Y//G$ is finite.

(9) Let U, W be finite dimensional vector spaces. Let $Grass_P(U \otimes W)$ be the Grassmannian of P dimensional subspaces of the vector space $U \otimes W$. Consider the $SL(U)$ action on $Grass_P(U \otimes W)$ induced from the natural $SL(U)$ action on U. Prove that a point $L \in Grass_P(U \otimes W)$ is semistable with respect to the Plücker embedding iff
$$(\dim L)(\dim U') - (\dim U)(\dim U' \otimes W \cap L) \geq 0$$
for any proper nonzero subspace U' of U. Prove that we get stability if we replace $\leq$ by $<$.

(10) Let Z, W be projective varieties and $\phi : Z \to W$ be a projective morphism with relatively ample bundle $\mathcal{O}_Z(1)$ over Z. Suppose a reductive group G acts on Z, W linearly with respect to ample line bundles $\mathcal{O}_Z(1), \mathcal{O}_W(1)$ respectively. Suppose ϕ is a G-equivariant morphism. Now consider the line bundle $L = \phi^*(\mathcal{O}_W(a)) \otimes \mathcal{O}_Z(1)$ and the induced action of G on L. Prove that for sufficiently large a we have

(a) $\phi^{-1}(W^s) \subset Z^s$

(b) $\phi(Z^{ss}) \subset W^{ss}$

where the (semi)stability for points in Z (resp. W) is with respect to L (resp. $\mathcal{O}_W(1)$).

Comments and Hints

(1) See chapter 1, section 2.
(2) See chapter 2, section 4.
(3) See chapter 3, section 3 for (a), chapter 3, section 4 for (b).
(4) See chapter 3, section 3.
(5) See chapter 3, section 4. Also, see II 2.5, Ex. 5.14 in Hartshorne's book.
(6) See chapter 4, section 2.
(7) See chapter 3, section 5.
(8) See chapter 3, section 4. This problem is from the paper, "On the moduli of vector bundles on an algebraic surface" by D. Gieseker, Annals of Math, 1977, pages 45–60.
(9) See chapter 4, section 2. This problem is from the paper, "Moduli of representations of the fundamental group of a smooth projective variety", by C. Simpson, Publ. IHES, 1994, pages 47-129.
(10) See chapter 4. This is from the paper "Quotient spaces modulo reductive algebraic groups", by C. Seshadri, Annals of Math, vol. 95, 1972, pages 511–556.