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THE KIRWAN MAP FOR SINGULAR SYMPLECTIC QUOTIENTS

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Abstract

Let M be a Hamiltonian K-space with proper moment map μ. The symplectic quotient $X = \mu^{-1}(0)/K$ is a singular stratified space with a symplectic structure on the strata. In this paper we generalise the Kirwan map, which maps the K equivariant cohomology of $\mu^{-1}(0)$ to the middle perversity intersection cohomology of X, to this symplectic setting.

The key technical result which allows us to do this is a decomposition theorem exhibiting the intersection cohomology of a blowup of X as a direct sum of terms including the intersection cohomology of X.

Introduction

Intersection cohomology, introduced in [**4**] and [**5**], has proven to be a very useful invariant for singular spaces. However, it is generally difficult to compute, largely because it is not very functorial. In this paper we demonstrate the existence of a map, which we dub the Kirwan map, from the equivariant cohomology of the zero set Z of the moment map of a Hamiltonian action of a compact connected Lie group K on a symplectic manifold M to the intersection cohomology of the symplectic reduction Z/K. (Throughout this paper all cohomology groups will be taken with rational coefficients unless otherwise stated.) This generalises the construction of such a map for geometric invariant theory quotients in [**14**].

In the companion paper [**11**] we use this map to study and compute the intersection cohomology of quotients by weakly-balanced actions. Similar techniques are used in [**10**] to study quotients by circle actions. The key point in the analyses is that the Kirwan map is surjective in these cases. In the algebraic setting the Kirwan map is always surjective, see [**13**, 3.10] and [**22**] for a correction. However this seems hard to prove in the symplectic setting, again because of the absence of the Decomposition Theorem, and presently we are confined to dealing with the special cases treated in [**11**] and [**10**].

Our construction of the Kirwan map is closely modelled on that in [**14**]. The idea is as follows. When 0 is a regular value of the moment map K acts on Z with only finite stabilisers. Hence, since we are using rational coefficients, there is a natural isomorphism $H^*_K(Z) \cong H^*(Z/K)$. Further Z/K is an orbifold and so $H^*(Z/K) \cong IH^*(Z/K)$. The Kirwan map is defined to be the composition of these isomorphisms. What if 0 is not a regular value? Now Z need not be a manifold and K may act with infinite stabilisers so that, in general, $H^*_K(Z)$ and $IH^*(Z/K)$ are no longer isomorphic. The approach we take is to use a resolution of the singularities of the reduction i.e. a symplectic manifold $\widehat{M}$ equipped with a Hamiltonian K action such that 0 is a regular value of the moment map and there

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is an equivariant map $\widehat{M} \to M$ which maps the zero set $\widehat{Z}$ onto Z. There is then an induced map $\widehat{Z}/K \to Z/K$ between the reductions which partially resolves the singularities of Z/K. (We use the term 'partially' to mean that orbifold singularities may remain.) Meinrenken and Sjamaar give a procedure for constructing such a partial desingularisation in [**18**]. Given this, we obtain a map

$$H_K^*(Z) \to H_K^*(\widehat{Z}) \cong IH^*(\widehat{Z}/K) \tag{0.1}$$

by composing the pullback with the Kirwan map already constructed for the regular reduction of $\widehat{M}$. In the algebro-geometric setting of [**14**] the Decomposition Theorem of [**1**, §6] tells us that $IH^*(Z/K)$ is a direct summand of $IH^*(\widehat{Z}/K)$. Composing (0.1) with the projection we obtain the Kirwan map

$$H_K^*(Z) \to H_K^*(\widehat{Z}) \cong IH^*(\widehat{Z}/K) \to IH^*(Z/K).$$

However, we cannot directly apply the Decomposition Theorem for symplectic quotients, instead we have to construct a projection

$$IH^*(\widehat{Z}/K) \to IH^*(Z/K)$$

by hand. This involves a careful analysis of the steps involved in the construction of the partial desingularisation and the inductive application of

Theorem 1 (cf.Theorem 5). *Suppose X is a compact stratified space with even dimensional strata and that B is a closed submanifold of codimension $2m$, a neighbourhood U_B of which is homeomorphic to a fibre bundle*

$$P \times_G C \to B$$

where the fibre C is an affine complex variety cut out by homogeneous equations on which the structure group G of the principal bundle P acts unitarily. Let $\pi : \widetilde{X} \to X$ be the blow up of X along B formed by replacing U_B with a neighbourhood modelled on the fibre bundle $P \times_G \widetilde{C}$ where $\widetilde{C} = Bl(C, 0)$, the blow up of C at the origin. Then there is a non-canonical direct sum decomposition

$$IH^*(\widetilde{X}) \cong IH^*(X) \oplus \bigoplus_{j\in\mathbb{Z}} H^{*-(m+j)}(B; \mathcal{L}^j)$$

of the intersection cohomology $IH^*(\widetilde{X})$ of $\widetilde{X}$ where $\mathcal{L}^j$ is a locally constant sheaf on B with stalk

$$\mathcal{L}_x^j \cong \begin{cases} IH^{m+j-2}(\widetilde{C}) & j < 0 \\ IH^{m+j}(\widetilde{C}) & j \geq 0. \end{cases}$$

To get a feel for this theorem, suppose X is a manifold and $P \times_G C \to B$ is (up to isomorphism) the normal bundle to the submanifold B. It follows that $C \cong \mathbb{C}^m$ so $IH^*(\widetilde{C}) \cong H^*(\widetilde{C}) \cong H^*(\mathbb{P}^{m-1})$ and the $\mathcal{L}^j$ are zero or have one dimensional stalks. In fact the $\mathcal{L}^j$ are all constant sheaves in this case and we recover

Theorem (McDuff [**16**]). *Suppose $M \subset X$ is a (real) codimension $2m$ submanifold of a manifold X and that the structure group of the normal bundle to M in X reduces to the unitary group. Then the cohomology of the blowup $\tilde{X}$ of X along M fits into a short exact sequence*

$$0 \to H^*(X) \to H^*(\tilde{X}) \to F^* \to 0$$

where F^* *is a free module over* $H^*(M)$ *with one generator* $a_i \in F^{2i}$ *for* $i = 1, 2, \dots, m-1$.

Another special case is when B is reduced to a point. Then Theorem 1 follows from the Decomposition Theorem applied to a blowup and a patching argument.

The structure of the paper is as follows. We begin with some preliminaries about intersection cohomology and perverse sheaves in §1. The main technical result of the paper, Theorem 1, is proved in §2. In §2.1 we use the fact that the inclusion of the exceptional divisor is normally nonsingular to construct a Gysin map

$$IH^*(\widetilde{X}) \to IH^{*+2}(\widetilde{X}).$$

In fact this map arises at the level of sheaves, more precisely, in the derived category of constructible sheaves. In §2.2 we apply the Decomposition Theorem locally and then use a theorem of Deligne's to prove that this Gysin map induces a decomposition of $IH^*(\widetilde{X})$ as in Theorem 1. Again this decomposition arises from one in the derived category.

§3 shows how Theorem 1 can be applied to prove the existence of a Kirwan map for singular symplectic quotients. In §3.1 we briefly recall the results of [**20**] and [**18**] on the structure of singular symplectic quotients, in particular the stratification by orbit and infinitesimal orbit types. The partial desingularisation of a singular reduction Z/K constructed in [**18**] is outlined in §3.2. Since this involves a finite sequence of blow ups we can inductively apply Theorem 1 to obtain a decomposition of the cohomology of the partial desingularisation with $IH^*(Z/K)$ as a direct summand.

In fact life is slightly more complicated since to partially desingularise a singular reduction we may need to blow up along symplectic orbifolds with a normal structure which has exceptional fibres at the orbifold points. In §3.3 we prove the small extensions of Theorem 1 which we require to deal with this. Finally in §3.4 the projection map associated to this decomposition is used to define the Kirwan map. The approach is analogous to that in [**14**] but has the minor technical advantage that the Kirwan map is shown to arise from a morphism in the derived category.

1. *Preliminaries*

We review some technical material on derived categories of sheaves on stratified spaces in order to introduce notation and provide references for any reader who is unfamiliar with the area.

1.1. *Perverse sheaves and intersection cohomology*

Suppose X is a connected compact topologically stratified space of dimension $2n$ with even dimensional strata $\{S_\alpha\}$. Furthermore, suppose the unique open stratum is orientable. Let $\mathbf{D}(X)$ be the bounded derived category of sheaves of rational vector spaces on X. Let the *constructible derived category* $\mathbf{D}^S(X)$ of X be the full subcategory of (cohomologically) constructible complexes (with respect to the given stratification). This is a triangulated category with a shift functor which we denote by $\mathcal{E} \mapsto \mathcal{E}[1]$ — see [**6**, Ch. 7, 1.6.1]. In §3 we will also use the bounded below version of this construction which we denote $\mathbf{D}^S_+(X)$.

The constructible derived category can be built up stratum by stratum. More precisely, suppose $Y \subset X$ is an open union of strata. Suppose Z is a stratum not in Y but such that if $Z \subset \overline{S_\alpha}$ then $S_\alpha \subset Y$. Let

$$Y \overset{\imath}{\hookrightarrow} Y \cup Z \overset{\jmath}{\hookleftarrow} Z$$

be the inclusions. These give rise to so-called glueing data i.e. six triangulated functors

$$\mathbf{D}^S(Y) \underset{\imath^*}{\overset{R\imath_*\ ,\ \imath_!}{\rightleftarrows}} \mathbf{D}^S(Y \cup Z) \underset{\jmath_*}{\overset{\jmath^*\ ,\ \jmath^!}{\rightleftarrows}} \mathbf{D}^S(Z)$$

obeying certain relations (see [**6**, Ch. 5, 3.9.1]) which describe how $\mathbf{D}^S(Y \cup Z)$ is built up from $\mathbf{D}^S(Y)$ and $\mathbf{D}^S(Z)$. In particular for any $\mathcal{E} \in \mathbf{D}^S(Y \cup Z)$ there are distinguished triangles

$$\imath_!\imath^*\mathcal{E} \to \mathcal{E} \to \jmath_*\jmath^*\mathcal{E} \quad \text{and} \quad \jmath_*\jmath^!\mathcal{E} \to \mathcal{E} \to R\imath_*\imath^*\mathcal{E}.$$

Since objects of $\mathbf{D}^S(X)$ are complexes of sheaves the triangulated category $\mathbf{D}^S(X)$ has a natural bounded **t**-structure given by truncation functors $\tau_{\leq r}$ and $\tau_{\geq r}$ — see [**6**, Ch. 5] for an introduction to triangulated categories and **t**-structures. We denote the associated cohomology sheaves by $\mathcal{H}^*(-)$ i.e.

$$\mathcal{H}^r(\mathcal{E}) = \tau_{\geq r}\tau_{\leq r}\mathcal{E}.$$

The heart of this **t**-structure is the Abelian category of sheaves constructible with respect to $\{S_\alpha\}$. We will say that the shift $\mathcal{F}[-r]$ of an object $\mathcal{F}$ in the heart of this **t**-structure is a *sheaf in dimension* r.

For any locally compact space X there is a Poincaré–Verdier duality functor $D : \mathrm{D}(X)^{\mathrm{op}} \to \mathrm{D}(X)$ and a natural transformation $\chi : 1 \to D^2$. One of the most important properties of the constructible derived category is that D restricts to a functor

$$\mathbf{D}^S(X)^{\mathrm{op}} \to \mathbf{D}^S(X)$$

such that χ becomes an isomorphism, and in particular D becomes an equivalence (see [**6**, Ch. 7, 1.6.2]).

Suppose X is a manifold with the trivial stratification with only one stratum. Then the heart of the natural **t**-structure is the category of locally constant sheaves on X. This is preserved by duality in the following sense; if $\mathcal{E}$ is a locally constant sheaf then $D\mathcal{E}[-2n]$ is also locally constant with the dual stalks. If the stratification of X is not trivial then this no longer holds; duality does not preserve constructible sheaves. To see this consider the effect of dualising on glueing data. There are canonical isomorphisms of functors

$$DR\imath_* \cong \imath_!D\ ,\ D\imath^* \cong \imath^*D\ ,\ D\jmath_* \cong \jmath_*D \text{ and } D\jmath^* \cong \jmath^!D.$$

Both $\imath_!$ and $\jmath^*$ are **t**-exact in the natural **t**-structure i.e they commute with the truncations and so preserve constructible sheaves. However, neither $R\imath_*$ nor $\jmath^!$ are **t**-exact and it follows that the Poincaré–Verdier dual D cannot be either. This observation leads to the notion of perverse sheaves.

On a stratified space these are the 'correct' generalisation of locally constant sheaves. They arise as the heart of the *perverse* **t**-structure on $\mathbf{D}^S(X)$.

Theorem 2 [**6**, Ch. 7, 1.2.1 and §1.7]. *The pair of full subcategories*

$$\begin{aligned} {}^p\mathbf{D}^S_{\leq 0}(X) &= \{\mathcal{A}^\bullet \mid H^i(\jmath_\alpha^*\mathcal{A}^\bullet) = 0 \textit{ for } i > \operatorname{codim}(S_\alpha)/2\} \\ \textit{and } {}^p\mathbf{D}^S_{\geq 0}(X) &= \{\mathcal{A}^\bullet \mid H^i(\jmath_\alpha^!\mathcal{A}^\bullet) = 0 \textit{ for } i < \operatorname{codim}(S_\alpha)/2\} \end{aligned}$$

defines a bounded $\mathbf{t}$*-structure on* $\mathbf{D}^S(X)$ *with truncation functors*

$${}^p\tau_{\leq 0}\colon \mathbf{D}^S(X) \to {}^p\mathbf{D}^S_{\leq 0}(X) \textit{ and } {}^p\tau_{\geq 0}\colon \mathbf{D}^S(X) \to {}^p\mathbf{D}^S_{\geq 0}(X)$$

respectively right and left adjoint to the inclusions. The heart ${}^p\mathbf{D}^S_{\leq 0}(X) \cap {}^p\mathbf{D}^S_{\geq 0}(X)$ *is the Abelian category* $\mathbf{Perv}(X)$ *of perverse sheaves. The Poincaré–Verdier duality functor commutes with the perverse truncations and so induces an equivalence of categories*

$$\mathbf{Perv}(X)^{\mathrm{op}} \longrightarrow \mathbf{Perv}(X) : \mathcal{P} \longmapsto D\mathcal{P}[-\dim X].$$

Remark 1. There are several extant indexing conventions for perverse sheaves. A common one is to shift by $[\dim X/2]$ so that a perverse sheaf on a manifold is a locally constant sheaf *in dimension* $-\dim X/2$. With this definition the Poincaré–Verdier dual takes perverse sheaves to perverse sheaves (rather than to shifted perverse sheaves as with our definition).

We can explicitly construct the perverse truncation functors ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$. On the open strata they are the natural truncations $\tau_{\leq 0}$ and $\tau_{\geq 0}$ — perverse sheaves on a manifold are locally constant sheaves. Consider, as above, the addition of a stratum Z of codimension $2m$ to an open union of strata Y. Suppose we have already constructed the perverse truncation ${}^p\tau_{\leq 0}$ on $\mathbf{D}^S(Y)$. Given $\mathcal{E} \in \mathbf{D}^S(Y \cup Z)$ we can (uniquely up to isomorphism) construct a diagram

$$\begin{tikzcd}
 & & \jmath_*\tau_{\geq m+1}\jmath^*\mathcal{E} \arrow[dr] & & \\
 & \mathcal{F} \arrow[ur] \arrow[dr] & & \mathcal{G} \arrow[dr, bend left] & \\
 & & \mathcal{E} \arrow[ur] \arrow[drr, bend right] & & \\
{}^p\tau_{\leq 0}\mathcal{E} \arrow[uur, bend left] \arrow[urr, bend right] & & & & R\imath_*{}^p\tau_{\geq 1}\imath^*\mathcal{E}
\end{tikzcd}$$

whose straight(ish) lines are distinguished triangles. This defines the perverse truncation ${}^p\tau_{\leq 0}$ on $\mathbf{D}^S(Y \cup Z)$. Inductively we can construct the perverse truncation below functor — see [**1**, p48] for details. Note that applying $\imath^*$ we have

$$\imath^{*p}\tau_{\leq 0}\mathcal{E} \cong \imath^*\mathcal{F} \cong {}^p\tau_{\leq 0}\imath^*\mathcal{E}. \tag{1.1}$$

The perverse truncation above functor is constructed analogously from the diagram

$$
\begin{tikzcd}
\imath_! {}^p\tau_{\leq -1}\imath^*\mathcal{E} \arrow[rrd, bend left=15] \arrow[rdd, bend right=20] & & & & {}^p\tau_{\geq 0}\mathcal{E} \\
& & \mathcal{E} \arrow[rru, bend left=15] \arrow[rd] & & \\
& \mathcal{G}' \arrow[ru] \arrow[rd] & & \mathcal{F}' \arrow[ruu, bend right=20] & \\
& & \jmath_*\tau_{\leq m+1}\jmath^!\mathcal{E} \arrow[ru] & &
\end{tikzcd}
$$

It follows that

$$\imath^{*p}\tau_{\geq 0}\mathcal{E} \cong \imath^*\mathcal{F}' \cong {}^p\tau_{\geq 0}\imath^*\mathcal{E}. \tag{1.2}$$

The *perverse cohomology sheaves*

$${}^p\mathcal{H}^r(\mathcal{A}) = {}^p\tau_{\leq r}{}^p\tau_{\geq r}\mathcal{A},$$

where ${}^p\tau_{\leq r}\mathcal{A} = ({}^p\tau_{\leq 0}(\mathcal{A}[r]))[-r]$ are the cohomology sheaves associated to the perverse **t**-structure.

The simplest perverse sheaves are described by

Lemma 1. *A perverse sheaf $\mathcal{E}$ supported on a closed stratum S is the extension by zero of a locally constant sheaf on S in dimension* $\operatorname{codim} S/2$.

Proof. It follows easily from the explicit constructions of the perverse truncations that when $\mathcal{E}$ is supported on a stratum S we have

$${}^p\tau_{\geq 0}\mathcal{E} \cong \tau_{\geq \operatorname{codim} S/2}\mathcal{E} \text{ and } {}^p\tau_{\leq 0}\mathcal{E} \cong \tau_{\leq \operatorname{codim} S/2}\mathcal{E}.$$

So $\mathcal{E} \cong {}^p\mathcal{H}^0(\mathcal{E}) \cong \mathcal{H}^{\operatorname{codim} S/2}(\mathcal{E})$. Since $\mathcal{E}$ is in the constructible derived category the latter is a locally constant sheaf on S in dimension $\operatorname{codim} S/2$. □

A more interesting perverse sheaf, playing an analogous role to that of the constant sheaf in the category of locally constant sheaves on a manifold, is the intersection cohomology complex $\mathcal{IC}^{\bullet}(X)$.

Theorem 3 [**5**, §3]. *Up to isomorphism in the derived category there is a unique perverse sheaf $\mathcal{IC}^{\bullet}(X)$ satisfying the conditions*

$$
\begin{aligned}
H^i(\jmath_\alpha^*\mathcal{IC}^{\bullet}(X)) = 0 \ \ i \geq \operatorname{codim}(S_\alpha)/2 \\
H^i(\jmath_\alpha^!\mathcal{IC}^{\bullet}(X)) = 0 \ \ i \leq \operatorname{codim}(S_\alpha)/2
\end{aligned}
$$

for all strata of strictly positive codimension, and which is isomorphic to the constant sheaf with rational coefficients $\mathbb{Q}_U$ on the unique open stratum $U \subset X$.

The hypercohomology of $\mathcal{IC}^{\bullet}(X)$ is the intersection cohomology of X, which we denote $IH^*(X)$. In general intersection cohomology is not a ring. However it is naturally a module over the cohomology ring $H^*(X)$ and so we can make sense of multiplication of an intersection cohomology class by a cohomology class.

Given an orientation $\mathbb{Q}_U \cong D\mathbb{Q}_U[-2n]$ of the nonsingular part U of X there is a unique extension to an isomorphism

$$\mathcal{IC}^\bullet(X) \xrightarrow{\phi} D\mathcal{IC}^\bullet(X)[-2n]$$

of perverse sheaves. This induces the generalised Poincaré duality isomorphisms

$$IH^i(X) \cong IH_{2n-i}(X)$$

where IH_* denotes the intersection homology groups.

1.2. *The Equivariant Case*

If a compact Lie group G acts continuously on X then all of the above constructions can be made equivariantly. In the situations we consider the group will preserve the stratification, but this is not necessary for the constructions — see [**2**, p.29]. There is a bounded constructible equivariant derived category $\mathbf{D}^S_G(X)$. Equivariant maps induce the standard functors of sheaf theory. $\mathbf{D}^S_G(X)$ has a natural $\mathbf{t}$-structure whose heart is the Abelian category of constructible G-equivariant sheaves and a perverse $\mathbf{t}$-structure whose heart is the Abelian category $\mathbf{Perv}_G(X)$ of equivariant perverse sheaves. (An important and subtle point is that $\mathbf{D}^S_G(X)$ is not, in general, the derived category of the heart of the natural $\mathbf{t}$-structure i.e. of the constructible G-equivariant sheaves — see [**2**, 2.5.4].) There is an equivariant Verdier duality functor $D : \mathbf{D}^S_G(X)^{\mathrm{op}} \to \mathbf{D}^S_G(X)$ which is an equivalence and preserves the equivariant perverse sheaves (up to a shift $[-\dim X]$).

The forgetful functor $\mathrm{For} : \mathbf{D}^S_G(X) \to \mathbf{D}^S(X)$ commutes with the standard sheaf theory functors, both the natural and perverse $\mathbf{t}$-structures and also with Verdier duality (see [**2**, Theorems 3.4.1, 3.5.2 and sections 2.2 and 5.1]). The constant sheaf $\mathbb{Q}_X$ and the intersection cohomology complex $\mathcal{IC}^\bullet(X)$ have natural lifts to equivariant objects, respectively $\mathbb{Q}_{X,G}$ and $\mathcal{IC}^\bullet_G(X)$, along the forgetful functor.

Suppose $\{S'_\beta\}$ is a stratification of X/G whose inverse image is a refinement of the stratification $\{S_\alpha\}$ of X. Then by [**2**, §6] the quotient map $X \to X/G$ induces a pushforward functor

$$Q_* : \mathbf{D}^S_G(X) \to \mathbf{D}^S_+(X/G). \tag{1.3}$$

to the *bounded below* constructible derived category of X/G. As a rational vector space the equivariant cohomology $H^*_G(X;\mathcal{E})$ of an object $\mathcal{E} \in \mathbf{D}^S_G(X)$ is defined to be the hypercohomology of $H^*(X; Q_*\mathcal{E})$. This description ignores the fact that $H^*_G(X;\mathcal{E})$ also has the structure of an H^*_G-module.

The construction of the equivariant derived category is rather subtle. However, if G is finite, there are significant simplifications. ln this case a G-equivariant sheaf is nothing but a sheaf together with a G-action compatible with that on X. Such objects form an Abelian category and the constructible equivariant derived category can be described as the full subcategory of constructible objects in the derived category of this Abelian category. In particular, since $\mathbb{Q}_X$ and $\mathcal{IC}^\bullet(X)$ are topological invariants, they carry natural G-actions which define the equivariant lifts $\mathbb{Q}_{X,G}$ and $\mathcal{IC}^\bullet_G(X)$.

2. *Blowups and Decompositions*

2.1. *The Gysin morphism*

Suppose B is a maximal depth stratum in X with $\operatorname{codim} B = 2m$. For simplicity assume that there are no other strata of codimension $2m$, otherwise we deal with them separately. Further, suppose a neighbourhood U_B of B is homeomorphic to a fibre bundle

$$P \times_G C \to B$$

where the fibre C is an affine complex variety cut out by homogeneous equations and P is a principal G-bundle, where G acts unitarily on C. Let $\pi : \widetilde{X} \to X$ be the 'blow up' of X along B formed by replacing U_B with a neighbourhood modelled on the fibre bundle $P \times_G \widetilde{C}$ where $\widetilde{C} = Bl(C, 0)$, the blow up of C at the origin.

The inclusion $\epsilon : E = \pi^{-1}B \hookrightarrow \widetilde{X}$ of the 'exceptional divisor' is normally nonsingular of codimension 2. An orientation of the normal bundle to E corresponds to an isomorphism $\epsilon^*\mathbb{Q}_{\widetilde{X}}[-2] \cong \epsilon^!\mathbb{Q}_{\widetilde{X}}$ where $\mathbb{Q}_{\widetilde{X}}$ is the constant sheaf with rational coefficients on $\widetilde{X}$. Composing with the units of standard adjunctions leads to a morphism

$$\mathbb{Q}_{\widetilde{X}}[-2] \to \epsilon_*\epsilon^*\mathbb{Q}_{\widetilde{X}}[-2] \cong \epsilon_*\epsilon^!\mathbb{Q}_{\widetilde{X}} \to \mathbb{Q}_{\widetilde{X}} \tag{2.1}$$

and hence to a natural transformation $[-2] \to \mathrm{id}$ given by

$$\mathcal{E}[-2] \cong \mathbb{Q}_{\widetilde{X}}[-2] \otimes \mathcal{E} \to \mathbb{Q}_{\widetilde{X}} \otimes \mathcal{E} \cong \mathcal{E}$$

for $\mathcal{E} \in \mathbf{D}^S(\widetilde{X})$.

Remark 2. Since the rational cohomology $H^i(\widetilde{X})$ is isomorphic to the morphisms from $\mathbb{Q}_{\widetilde{X}}$ to $\mathbb{Q}_{\widetilde{X}}[i])$ in the derived category, this natural transformation corresponds to a 2-dimensional cohomology class. This class defines an intersection cohomology class which is the Poincaré dual of the intersection homology class represented by E (it is easy to check that normal nonsingularity implies the satisfaction of the allowability conditions for a cycle in intersection homology).

Let β be the result of applying the natural transformation to $\mathcal{IC}^\bullet(\widetilde{X})$ i.e. we have

$$\beta : \mathcal{IC}^\bullet(\widetilde{X})[-2] \to \mathcal{IC}^\bullet(\widetilde{X}) \tag{2.2}$$

Applying $R\pi_*$ we obtain a Gysin morphism

$$\gamma : R\pi_*\mathcal{IC}^\bullet(\widetilde{X})[-2] \to R\pi_*\mathcal{IC}^\bullet(\widetilde{X}). \tag{2.3}$$

It follows from the definition that γ factors through the restriction to B so that the restriction of this morphism to any open set which does not meet B is zero.

Remark 3. In the sequel we will abuse notation by using γ to denote not only the morphism in $\mathbf{D}^S(X)$ but also its various restrictions and induced maps on hypercohomology and on cohomology sheaves with respect to both the natural and perverse $\mathbf{t}$-structures.

PROPOSITION 1. *If we restrict to B there is a distinguished triangle*

$$R\pi_*\mathcal{IC}^\bullet(E)[-2] \xrightarrow{\gamma} R\pi_*\mathcal{IC}^\bullet(E) \to \jmath^* R\imath_*\mathcal{IC}^\bullet(X \setminus B) \tag{2.4}$$

where $\jmath : B \hookrightarrow X$ and $\imath : X \setminus B \hookrightarrow X$ are the inclusions. Localising further to a point $x \in B$ and taking cohomology we obtain the long exact Gysin sequence

$$\ldots \to IH^{*-2}(\widetilde{C}) \xrightarrow{\gamma} IH^*(\widetilde{C}) \to IH^*(C \setminus \{0\}) \to \ldots \tag{2.5}$$

The composition $\gamma^r : IH^{m-1-r}(\widetilde{C}) \to IH^{m-1+r}(\widetilde{C})$, where $2m = \operatorname{codim}_{\mathbb{R}} B$, is an isomorphism. In particular $\gamma : IH^{*-2}(\widetilde{C}) \to IH^*(\widetilde{C})$ is injective for $* \leq m$ and surjective for $* \geq m$.

Proof. There is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\epsilon"] \arrow[d, "\pi"] & \widetilde{X} \arrow[d, "\pi"] & \widetilde{X} \setminus E \arrow[l, "\imath"'] \arrow[d, equal] \\ B \arrow[r, "\jmath"'] & X & X \setminus B. \arrow[l, "\imath"] \end{tikzcd} \tag{2.6}$$

Identifying $\widetilde{X} \setminus E$ with $X \setminus B$ we have a distinguished triangle

$$R\epsilon_*\epsilon^!\mathcal{IC}^\bullet(\widetilde{X}) \to \mathcal{IC}^\bullet(\widetilde{X}) \to R\imath_*\mathcal{IC}^\bullet(X \setminus B)$$

Applying ϵ^* we obtain a second distinguished triangle

$$\epsilon^!\mathcal{IC}^\bullet(\widetilde{X}) \to \epsilon^*\mathcal{IC}^\bullet(\widetilde{X}) \to \epsilon^* R\imath_*\mathcal{IC}^\bullet(X \setminus B). \tag{2.7}$$

Using the orientation of the normal bundle to E and [**5**, §5.4], we have canonical isomorphisms

$$\epsilon^*\mathcal{IC}^\bullet(\widetilde{X}) \cong \mathcal{IC}^\bullet(E) \quad \text{and} \quad \epsilon^!\mathcal{IC}^\bullet(\widetilde{X}) \cong \mathcal{IC}^\bullet(E)[-2]. \tag{2.8}$$

Note that these differ from those in [**5**] by a shift since we are using a different indexing scheme (see Remark 1). Using these identities we can identify the first map in the distinguished triangle (2.7) with $\epsilon^*\beta$ where β is defined in (2.2). It follows from (2.6) that $R\pi_*\epsilon^* \cong \jmath^* R\pi_*$ (see, for instance, [**5**, §1.13]) so that after applying $R\pi_*$ the first map in the triangle is $R\pi_*\epsilon^*\beta \cong \jmath^* R\pi_*\beta \cong \jmath^*\gamma$ and we obtain (2.4) as required.

Localising to a point $x \in B$ and taking cohomology we obtain the Gysin sequence (2.5). Note that $\widetilde{C}$ is isomorphic to the tautological line bundle $\mathcal{L}$ on $\mathbb{P}(C \setminus \{0\})$ which is anti-ample. There is a stratification preserving retraction

$$\widetilde{C} \to \mathbb{P}(C \setminus \{0\}).$$

so that $IH^*(\widetilde{C}) \cong IH^*(\mathbb{P}(C \setminus \{0\}))$. It follows from (2.5) that we can identify γ with multiplication by the first Chern class of $\mathcal{L}$. Furthermore, by [**1**, §6], there is a Hard Lefschetz Theorem for the intersection cohomology groups of $\mathbb{P}(C \setminus \{0\})$ which says that multiplication by a power of the first Chern class of the ample line bundle $\mathcal{L}^{-1}$ induces isomorphisms

$$IH^{m-1-i}(\mathbb{P}(C \setminus \{0\})) \to IH^{m-1+i}(\mathbb{P}(C \setminus \{0\}))$$

for $i > 0$. Since $c_1(\mathcal{L}) = -c_1(\mathcal{L}^{-1})$ we are done. □

2.2. *Decomposing Intersection Cohomology*

In this section we show how the Gysin morphism γ can be used to decompose $R\pi_*\mathcal{IC}^{\bullet}(\widetilde{X})$ with $\mathcal{IC}^{\bullet}(X)$ as a direct summand. The key ingredients in our proof are Proposition 1 and

Theorem 4 (Deligne [**3**], see also [**21**]). *Let $\mathbf{A}$ be a triangulated category with a bounded $\mathbf{t}$-structure and heart $\mathbf{A}_0$. Let $H^0 : \mathbf{A} \to \mathbf{A}_0$ denote the associated cohomology functor and $H^i(\mathcal{A}) = H^0(\mathcal{A}[i])[-i]$.*

Suppose $\mathcal{A} \in \mathbf{A}$ and $\phi : \mathcal{A}[-1] \to \mathcal{A}[1]$ is such that

$$H^0(\phi^k) : H^{-k}(\mathcal{A})[-k] \to H^k(\mathcal{A})[k]$$

is an isomorphism for $k \geq 0$. Then there exists an isomorphism

$$\mathcal{A} \cong \bigoplus_{k\in\mathbb{Z}} H^{-k}(\mathcal{A}).$$

Remark 4. The isomorphism is not canonical; there is no unique choice essentially because the cone on a morphism in a triangulated category is unique up to isomorphism but *not up to unique isomorphism.*

We apply this with $\mathbf{A} = \mathbf{D}^S(X)$ equipped with the perverse $\mathbf{t}$-structure so that $\mathbf{A}_0 = \mathbf{Perv}(X)$. We take the object $\mathcal{A} = R\pi_*\mathcal{IC}^{\bullet}(\widetilde{X})$ and the morphism $\phi = \gamma[1]$. We need to show that ${}^p\mathcal{H}^0(\gamma^k[k])$ is an isomorphism for $k \geq 0$. To do this we must identify the perverse cohomology sheaves of $\mathcal{A}$.

Lemma 2. *For $i \neq 0$ the perverse sheaf ${}^p\mathcal{H}^i(\mathcal{A})$ in dimension i is supported on B and is the extension by zero of a locally constant sheaf on B in dimension $m+i$.*

Proof. Note that the restriction of $\mathcal{A}$ to $X \setminus B$ is isomorphic to the perverse sheaf $\mathcal{IC}^{\bullet}(X \setminus B)$ so that ${}^p\mathcal{H}^i(\mathcal{A})$ is supported on B for $i \neq 0$. The result follows from Lemma 1. $\square$

Now we study how $\mathcal{A}$ is built up by successive extensions by its perverse cohomology sheaves. For each j there is a distinguished triangle

$${}^p\tau_{\leq j-1}\mathcal{A} \to {}^p\tau_{\leq j}\mathcal{A} \to {}^p\mathcal{H}^j(\mathcal{A}).$$

When $j \neq 0$ we can use the above lemma and the fact that $\mathcal{H}^k({}^p\tau_{\leq j}\mathcal{A}) = 0$ for $k > m+j$ to see that there is a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \\ \longrightarrow \mathcal{H}^{m+j-1}({}^p\tau_{\leq j-1}\mathcal{A}) \longrightarrow \mathcal{H}^{m+j-1}({}^p\tau_{\leq j}\mathcal{A}) \longrightarrow 0 \longrightarrow \\ \longrightarrow 0 \longrightarrow \mathcal{H}^{m+j}({}^p\tau_{\leq j}\mathcal{A}) \longrightarrow {}^p\mathcal{H}^j(\mathcal{A}) \longrightarrow \\ \longrightarrow 0 \end{array} \tag{2.9}$$

We deduce that for $j \neq 0$ we have

$$\mathcal{H}^k({}^p\tau_{\leq j}\mathcal{A}) \cong \begin{cases} \mathcal{H}^k({}^p\tau_{\leq j-1}\mathcal{A}) & k < m+j \\ {}^p\mathcal{H}^j(\mathcal{A}) & k = m+j \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

When $j = 0$ the situation is more complicated. We only know that $\mathcal{H}^k({}^p\tau_{\leq -1}\mathcal{A}) = 0$ for $k \geq m$, so we have a long exact sequence

$$\begin{aligned} &\mathcal{H}^{m-2}({}^p\mathcal{H}^0(\mathcal{A})) \longrightarrow \\ &\longrightarrow \mathcal{H}^{m-1}({}^p\tau_{\leq -1}\mathcal{A}) \longrightarrow \mathcal{H}^{m-1}({}^p\tau_{\leq 0}\mathcal{A}) \longrightarrow \mathcal{H}^{m-1}({}^p\mathcal{H}^0(\mathcal{A})) \longrightarrow \\ &\longrightarrow 0 \longrightarrow \mathcal{H}^{m}({}^p\tau_{\leq 0}\mathcal{A}) \longrightarrow \mathcal{H}^{m}({}^p\mathcal{H}^0(\mathcal{A})) \longrightarrow \\ &\longrightarrow 0 \end{aligned}$$

In particular it follows from this and (2.10) with $k = m$ and $j = 1, 2, \ldots$ that

$$\begin{aligned} \mathcal{H}^m({}^p\mathcal{H}^0(\mathcal{A})) &\cong \mathcal{H}^m({}^p\tau_{\leq 0}\mathcal{A}) \\ &\cong \mathcal{H}^m({}^p\tau_{\leq 1}\mathcal{A}) \\ &\cong \ldots \\ &\cong \mathcal{H}^m(\mathcal{A}). \end{aligned} \tag{2.11}$$

Proposition 2. *There is a direct sum decomposition*

$${}^p\mathcal{H}^0(\mathcal{A}) \cong \mathcal{IC}^\bullet(X) \oplus \mathcal{H}^m(\mathcal{A}).$$

Proof. For ease of notation write $\mathcal{P}$ for the perverse sheaf ${}^p\mathcal{H}^0(\mathcal{A})$. We know that $\widetilde{X}$ is a compact stratified space of dimension $2n$ with only even dimensional strata. Furthermore a choice of orientation of the unique open stratum of X yields an orientation of the unique open stratum of $\widetilde{X}$ and thence a Poincaré–Verdier duality isomorphism

$$\widetilde{\phi} : \mathcal{IC}^\bullet(\widetilde{X}) \to D(\mathcal{IC}^\bullet(\widetilde{X}))[-2n].$$

Since π is proper we obtain an isomorphism $\mathcal{A} \to D\mathcal{A}[-2n]$ by pushing forward and, by applying ${}^p\mathcal{H}^0(-)$ and using Theorem 2, an isomorphism

$$\psi : \mathcal{P} \to D\mathcal{P}[-2n].$$

Recall that $\jmath : B \hookrightarrow X$ is the inclusion of the unique stratum of codimension $2m$ and that all other strata have strictly lower codimension. Since $\mathcal{P}$ is perverse we can check that $\jmath_*\tau_{\leq m}\jmath^!\mathcal{P}$ is a locally constant sheaf in dimension m supported on B, and so is itself perverse. Furthermore adjunction and truncation yield a natural morphism $\rho : \jmath_*\tau_{\leq m}\jmath^!\mathcal{P} \to \mathcal{P}$ of perverse sheaves. We claim that the composition $D\rho[-2n] \circ \psi \circ \rho$, which is a morphism of locally constant sheaves (in dimension m) on B, is an isomorphism. It is sufficient to show that

$$\mathcal{H}^m(\jmath^*(D\rho[-2n] \circ \psi \circ \rho))$$

is an isomorphism. Unwinding the definition this is given by the composition

$$\mathcal{H}^m(\jmath^!\mathcal{P}) \to \mathcal{H}^m(\jmath^*\mathcal{P}) \to \mathcal{H}^m(\jmath^* D\mathcal{P}[-2n]). \tag{2.12}$$

where the first map arises from the first morphism in the distinguished triangle

$$\jmath^!\mathcal{P} \to \jmath^*\mathcal{P} \to \jmath^* R\imath_*\imath^*\mathcal{P}.$$

and the second from the isomorphism $\jmath^*\psi$.

Taking cohomology of the above distinguished triangle and localising to $x \in B$ we obtain a long exact sequence

$$\ldots \to \mathcal{H}^m_x(\jmath^!\mathcal{P}) \to IH^m(\widetilde{C}) \to IH^m(C \setminus \{0\}) \to \ldots$$

where we have used the fact that $\imath^*\mathcal{P} \cong \mathcal{IC}^\bullet(X \setminus B)$ and identified $\mathcal{H}^m_x(\mathcal{P}) \cong IH^m(\widetilde{C})$ using (2.11). It follows from Proposition 1 that the second map is zero so that the first is a surjection. Hence (2.12) is the composition of a surjection and an isomorphism and so is itself a surjection. However since $\jmath^!\mathcal{P}$ is dual to $\jmath^* D\mathcal{P}[-2n]$ the stalks have the same dimension and so it is an isomorphism. In particular the first map in (2.12) is an isomorphism. Since

$$\mathcal{H}^m(\jmath^!\mathcal{P}) \cong \mathcal{H}^m(\jmath_*\tau_{\le m}\jmath^!\mathcal{P}) \cong \jmath_*\tau_{\le m}\jmath^!\mathcal{P} \quad \text{and} \quad \mathcal{H}^m(\jmath^*\mathcal{P}) \cong \mathcal{H}^m(\mathcal{P})$$

it follows that $\jmath_*\tau_{\le m}\jmath^!\mathcal{P} \cong \mathcal{H}^m(\mathcal{P})$.

By the above ρ is a split injection so there is a split short exact sequence

$$0 \to \jmath_*\tau_{\le m}\jmath^!\mathcal{P} \to \mathcal{P} \to \mathcal{Q} \to 0 \tag{2.13}$$

of perverse sheaves. Considering the associated long exact sequence of cohomology sheaves we see that $\mathcal{H}^k(\mathcal{Q}) \cong 0$ for $k \ge m$. By applying $\jmath^!$ to (2.13) we obtain a distinguished triangle

$$\tau_{\le m}\jmath^!\mathcal{P} \to \jmath^!\mathcal{P} \to \jmath^!\mathcal{Q}.$$

It follows that $\jmath^!\mathcal{Q} \cong \tau_{\ge m+1}\jmath^!\mathcal{P}$ so $\mathcal{H}^k(\jmath^!\mathcal{Q}) \cong 0$ for $k \le m$. Hence by Theorem 3 we have $\mathcal{Q} \cong \mathcal{IC}^\bullet(X)$ as required. □

We can now complete our identification of the perverse cohomology sheaves.

PROPOSITION 3. *For $j \neq 0$ there is an injection of locally constant sheaves in dimension $m + j$ supported on B*

$${}^p\mathcal{H}^j(\mathcal{A}) \to \mathcal{H}^{m+j}(\mathcal{A}).$$

Furthermore the image of the perverse cohomology sheaf ${}^p\mathcal{H}^j(\mathcal{A})$ is the image of the sheaf map

$$\gamma : \mathcal{H}^{m+j-2}(\mathcal{A})[-2] \to \mathcal{H}^{m+j}(\mathcal{A}).$$

Proof. There are natural maps ${}^p\mathcal{H}^j(\mathcal{A}) \leftarrow {}^p\tau_{\le j}\mathcal{A} \to \mathcal{A}$. Applying $\mathcal{H}^{m+j}$ and using Lemma 2 and the long exact sequence (2.9) we obtain a map of sheaves

$${}^p\mathcal{H}^j(\mathcal{A}) \cong \mathcal{H}^{m+j}({}^p\mathcal{H}^j(\mathcal{A})) \cong \mathcal{H}^{m+j}({}^p\tau_{\le j}\mathcal{A}) \to \mathcal{H}^{m+j}(\mathcal{A}).$$

For $j > 0$ it follows from (2.10) that this is an isomorphism. Note that the Gysin map

$$\gamma : \mathcal{H}^{m+j-2}(\mathcal{A})[-2] \to \mathcal{H}^{m+j}(\mathcal{A})$$

is surjective for $j \geq 0$ by Proposition 1. Hence we have the result for $j > 0$.

Now suppose $j < 0$. Using the fact that $\imath^{*p}\tau_{\leq k} \cong {}^p\tau_{\leq k}\imath^*$ for any k and that $\imath^*\mathcal{A} \cong \mathcal{IC}^\bullet(X \setminus B)$ is a perverse sheaf we see there is a diagram

$$\begin{tikzcd}
{}^p\tau_{\leq -1}\mathcal{A} \arrow[r] \arrow[d] & {}^p\tau_{\leq 0}\mathcal{A} \arrow[r] \arrow[d] & {}^p\mathcal{H}^0(\mathcal{A}) \arrow[d] \\
R\imath_*\imath^{*p}\tau_{\leq -1}\mathcal{A} \arrow[r] \arrow[d, equal] & R\imath_*\imath^{*p}\tau_{\leq 0}\mathcal{A} \arrow[r] \arrow[d, equal] & R\imath_*\imath^{*p}\mathcal{H}^0(\mathcal{A}) \arrow[d, equal] \\
0 \arrow[r] & R\imath_*\mathcal{IC}^\bullet(X \setminus B) \arrow[r, equal] & R\imath_*\mathcal{IC}^\bullet(X \setminus B)
\end{tikzcd}$$

whose rows are distinguished triangles. When $j < 0$ it follows from (2.10) that

$$\mathcal{H}^{m+j}({}^p\tau_{\leq -1}\mathcal{A}) \cong {}^p\mathcal{H}^j(\mathcal{A}) \text{ and } \mathcal{H}^{m+j}({}^p\tau_{\leq 0}\mathcal{A}) \cong \mathcal{H}^{m+j}(\mathcal{A})$$

and by Proposition 2 we have

$$\mathcal{H}^{m+j}({}^p\mathcal{H}^0(\mathcal{A})) \cong \mathcal{H}^{m+j}(\mathcal{IC}^\bullet(X)).$$

Hence taking cohomology we have a diagram

$$\begin{tikzcd}
{} \arrow[r] & {}^p\mathcal{H}^j(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{H}^{m+j}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{H}^{m+j}(\mathcal{IC}^\bullet(X)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & \mathcal{H}^{m+j}(R\imath_*\mathcal{IC}^\bullet(X \setminus B)) \arrow[r, equal] & \mathcal{H}^{m+j}(R\imath_*\mathcal{IC}^\bullet(X \setminus B)) \arrow[r] & {}
\end{tikzcd}$$

with long exact sequences for rows. Since C has dimension $2m$ it is a standard property of intersection cohomology (following directly from the definition in Theorem 3) that the final vertical arrow is an isomorphism. Proposition 1 says that the middle vertical arrow is surjective (for $j < 0$). Hence the upper long exact sequence breaks up into short exact sequences and ${}^p\mathcal{H}^j(\mathcal{A})$ is identified with the kernel of

$$\mathcal{H}^{m+j}(\mathcal{A}) \to \mathcal{H}^{m+j}(R\imath_*\mathcal{IC}^\bullet(X \setminus B)).$$

By comparing with the Gysin sequence (2.5) we can identify ${}^p\mathcal{H}^j(\mathcal{A})$ with the image of the Gysin map as required. □

PROPOSITION 4. *The Gysin morphism γ induces isomorphisms*

$$\gamma^k[k] : {}^p\mathcal{H}^{-k}(\mathcal{A})[-k] \to {}^p\mathcal{H}^k(\mathcal{A})[k]$$

for $k > 0$.

Proof. It follows from Proposition 3 that for $k > 0$ there is a commutative diagram

$$\begin{tikzcd}
{}^p\mathcal{H}^{-k}(\mathcal{A})[-k] \arrow[r, equal] \arrow[d, "\gamma^k[k]"'] & \mathcal{H}^{m-k}({}^p\tau_{\leq -k}\mathcal{A})[-k] \arrow[r] \arrow[d, "\gamma^k[k]"'] & \mathcal{H}^{m-k}(\mathcal{A})[-k] \arrow[d, "\gamma^k[k]"] \\
{}^p\mathcal{H}^k(\mathcal{A})[k] \arrow[r, equal] & \mathcal{H}^{m+k}({}^p\tau_{\leq k}\mathcal{A})[k] \arrow[r, equal] & \mathcal{H}^{m+k}(\mathcal{A})[k]
\end{tikzcd}$$

and that the top row is an injection with image the same as that of $\gamma[-k]$ in $\mathcal{H}^{m-k}(\mathcal{A})[-k]$. The result follows from Proposition 1. □

THEOREM 5. *There is a direct sum decomposition*

$$IH^*(\widetilde{X}) \cong IH^*(X) \oplus \bigoplus_{j\in\mathbb{Z}} H^*(B;\mathcal{L}^j)$$

where $\mathcal{L}^j$ is the image of $\gamma : \mathcal{H}^{m+j-2}(\mathcal{A}) \to \mathcal{H}^{m+j}(\mathcal{A})$. In particular $\mathcal{L}^j$ is a locally constant sheaf on B in dimension $m+j$ with stalk

$$\mathcal{L}^j_x \cong \begin{cases} IH^{m+j-2}(\widetilde{C}) & j<0 \\ IH^{m+j}(\widetilde{C}) & j \geq 0. \end{cases}$$

Proof. By Theorem 4 and Proposition 4 we have a direct sum decomposition

$$R\pi_*\mathcal{IC}^\bullet(\widetilde{X}) \cong \mathcal{A} \cong \bigoplus_{j\in\mathbb{Z}} {}^p\mathcal{H}^j(\mathcal{A}).$$

In Lemma 2 we showed that $\mathcal{IC}^\bullet(X)$ is a direct summand of ${}^p\mathcal{H}^0(\mathcal{A})$ with complementary summand $\mathcal{H}^m(\mathcal{A})$.

Proposition 3 identifies ${}^p\mathcal{H}^j(\mathcal{A})$ with the image of the Gysin map for $j \neq 0$. Finally note that the Gysin map

$$\gamma : \mathcal{H}^{m-2}(\mathcal{A}) \to \mathcal{H}^m(\mathcal{A})$$

is surjective by Proposition 1 and so has image the remaining summand $\mathcal{H}^m(\mathcal{A})$. We can now use Proposition 1 to identify the stalks of the $\mathcal{L}^j$ as required. □

REMARK 5. Suppose a finite group F acts on X fixing the points of B and acting unitarily on the fibres C of the model $P\times_G C$ of a neighbourhood of B. Then the blowup construction can be made equivariantly so that F acts compatibly on $\widetilde{X}$. Thus we can ask whether $R\pi_*\mathcal{IC}^\bullet_F(\widetilde{X})$ can be decomposed in $\mathbf{D}^S_F(X)$ with $\mathcal{IC}^\bullet_F(X)$ as a summand?

Mimicking the earlier construction, the F-equivariant inclusion of the exceptional divisor yields an equivariant Gysin morphism

$$\gamma_F : R\pi_*\mathcal{IC}^\bullet_F(\widetilde{X})[-2] \to R\pi_*\mathcal{IC}^\bullet_F(\widetilde{X}).$$

All we need do is check whether powers of γ_F induce isomorphisms between perverse cohomology sheaves and whether the analogue of Proposition 2 holds (again a matter of checking whether a map of perverse sheaves is an isomorphism). Since F is finite, we know that $\mathbf{D}^S_F(X)$ is equivalent to the full subcategory of constructible objects in the derived category of F-equivariant sheaves. It follows that a morphism α in $\mathbf{D}^S_F(X)$ is an isomorphism if, and only if, its image $\mathrm{For}(\alpha)$ in $\mathbf{D}^S(X)$ under the forgetful functor is an isomorphism. The forgetful functor commutes with the natural and perverse **t**-structures and with Verdier duality. Also it follows from the naturality of the Gysin morphism and [**2**, Theorem 3.4.1] that $\mathrm{For}(\gamma_F) = \gamma$. Hence we deduce from the above results for $\mathbf{D}^S(X)$ that we have the required isomorphisms in $\mathbf{D}^S_F(X)$. In short, we can lift the decomposition along the forgetful functor.

3. *The Kirwan map*

One approach to the study of the intersection cohomology of a singular quotient space is to relate it to the equivariant cohomology of the original space. For geometric invariant theory quotients Kirwan showed in [**14**] how to construct a map (which we call the Kirwan map) exhibiting the intersection cohomology of the quotient space as a quotient of the equivariant cohomology of the original space. We follow a similar programme for symplectic quotients in this section, but using Theorem 5 instead of the more powerful Decomposition Theorem of [**1**, §6] which does not necessarily apply here.

We do not yet know how to prove the map we construct is surjective so we do not obtain the intersection cohomology as a quotient of the equivariant cohomology. The companion paper [**11**] discusses a class of examples for which we can show surjectivity and thence compute the intersection cohomology.

3.1. *Singular symplectic quotients*

Suppose (M,ω) is a Hamiltonian K-space for some compact connected Lie group K. By this we mean that K acts on M preserving the symplectic form ω in such a way that there is an equivariant moment map $\mu : M \to \mathfrak{k}^*$ satisfying

$$\langle d\mu, a\rangle = \imath_{a_M}\omega$$

where $a \in \mathfrak{k}$ and a_M is the vector field on M arising from the infinitesimal action of a. We will assume that μ is proper, and in this case call M a *proper Hamiltonian K-space.*

If 0 is a regular value of μ then the topological space $M_0 = \mu^{-1}(0)/K$ can naturally be given the structure of a compact symplectic orbifold. When 0 is not a regular value Lerman and Sjamaar show in [**20**] that M_0 is a stratified symplectic space. To avoid unnecessary symmetry we will always assume that there is at least one point of $\mu^{-1}(0)$ with zero dimensional stabiliser. We briefly describe the stratification.

Let $Z = \mu^{-1}(0)$. Suppose H is a compact subgroup of K with Lie algebra $\mathfrak{h}$. Let Z_H be the subset of points of Z whose stabiliser is precisely H and $Z_{(H)}$ be the subset whose stabiliser is conjugate to H. The reduced space M_0 is topologically stratified by non-empty connected components of the $Z_{(H)}/K$, each of which is a symplectic manifold, with a symplectic structure induced from ω. This stratification makes M_0 into a compact topological stratified space with even dimensional strata, and in particular we can define $\mathbf{D}^S(M_0)$ and the intersection cohomology complex $\mathcal{IC}^\bullet(M_0) \in \mathbf{D}^S(M_0)$.

We also need to consider a coarser decomposition (it is not a topological stratification) of M_0 into locally closed symplectic orbifolds. This arises by considering infinitesimal stabilisers; we define

$$Z_{\mathfrak{h}} = \{z \in Z : \operatorname{Lie}\operatorname{Stab} z = \mathfrak{h}\} \quad \text{and} \quad Z_{(\mathfrak{h})} = \{z \in Z : \operatorname{Lie}\operatorname{Stab} z \in (\mathfrak{h})\}$$

where $(\mathfrak{h})$ is the set of subalgebras of $\mathfrak{k}$ conjugate to $\mathfrak{h}$. The reduced space M_0 is decomposed into the non-empty connected components of the $Z_{(\mathfrak{h})}/K$, each of which is a symplectic orbifold formed from the union of finitely many strata of the orbit type stratification. These pieces are rational homology manifolds. Just as for a stratification, the pieces form a poset and we will talk about the depth of a piece etc.

We give a local description of M_0 near one of the pieces of this decomposition. To do so we need to introduce a choice of compatible K-invariant almost complex structure J and metric g on M.

Let $P_{\mathfrak{h}}$ be a connected component of $Z_{(\mathfrak{h})}/K$. Set $H_0 = \exp(\mathfrak{h})$. Suppose $\mathcal{O}$ is an orbit in $P_{\mathfrak{h}}$. Choose $p \in \mathcal{O}$ so that the stabiliser of p is H with $\operatorname{Lie} H = \mathfrak{h}$ i.e. the connected component of the identity of H is H_0. Define

$$V = \left(T_p\mathcal{O} \oplus JT_p\mathcal{O}\right)^{\perp} \leq T_pM$$

where $\perp$ denotes the orthogonal complement with respect to g, and set

$$W = (V_{\mathfrak{h}})^{\perp} \leq V \tag{3.1}$$

where $V_{\mathfrak{h}}$ is the subspace invariant under the infinitesimal $\mathfrak{h}$ action. Note that V and W are, up to isomorphism, independent of the point p and the orbit $\mathcal{O}$.

There is a Hermitian structure on both $V_{\mathfrak{h}}$ and W, induced from the triple (J, g, ω), with respect to which they become Hamiltonian H-spaces, with H acting via the unitary group. The symplectic reduction can be identified with the geometric invariant theory quotient by the linear action of the complexified group (see [**12**] and [**19**]).

A neighbourhood of the point $[p] \in M_0$ represented by $p \in \mathcal{O}$ is modelled by the geometric invariant theory quotient

$$(V_{\mathfrak{h}} \oplus W) /\!\!/ H^{\mathbb{C}} \cong \left[(V_{\mathfrak{h}} \oplus W) /\!\!/ H_0^{\mathbb{C}}\right] /\pi_0 H \cong \left[V_{\mathfrak{h}} \oplus \left(W /\!\!/ H_0^{\mathbb{C}}\right)\right] /\pi_0 H,$$

see [**18**][§3]. This description can be thought of as a chart

$$\begin{array}{ccc} V_{\mathfrak{h}} \oplus \left(W/\!\!/ H_0^{\mathbb{C}}\right) & \longrightarrow & \left[V_{\mathfrak{h}} \oplus \left(W/\!\!/ H_0^{\mathbb{C}}\right)\right] /\pi_0 H \\ \downarrow & & \downarrow \\ V_{\mathfrak{h}} & \longrightarrow & V_{\mathfrak{h}}/\pi_0 H \end{array} \tag{3.2}$$

for an orbibundle with fibre $W/\!\!/ H_0^{\mathbb{C}}$ over a neighbourhood of $[p] \in P_{\mathfrak{h}}$ homeomorphic to $V_{\mathfrak{h}}/\pi_0 H$. These charts patch together to give

PROPOSITION 5. *A neighbourhood of $P_{\mathfrak{h}}$ in M_0 is modelled on a neighbourhood of the zero section in an orbibundle with general fibre $W/\!\!/ H_0^{\mathbb{C}}$ and whose structure group acts on the fibre via the unitary group $U(W)$.*

Proof. This follows from the description of the local model in [**18**] §3, in particular Theorem 3.4. (Our situation is slightly simpler since we insist that M is a manifold and not an orbifold as in [**18**].) □

3.2. *Partial desingularisation*

Just as for geometric invariant theory quotients (see [**13**]) we can partially desingularise a singular symplectic quotient (but we cannot necessarily remove orbifold singularities). As above let $P_{\mathfrak{h}}$ be a piece of the decomposition by infinitesimal orbit types and suppose further that $P_{\mathfrak{h}}$ is of maximal depth and hence closed. In [**18**] Meinrenken and Sjamaar show that one can blow up along $P_{\mathfrak{h}}$ by replacing the fibre $W/\!\!/ H_0^{\mathbb{C}}$ of the local model by its blowup at the origin. This decreases the depth locally (there may be other disjoint pieces of equal depth) and so, by induction, we can successively blow up until we have a symplectic orbifold.

These blowups arise as reductions of symplectic blowups of a neighbourhood U of the zero set Z of the moment map. More precisely, we can choose a neighbourhood U of Z in M with an equivariant retraction $r : U \to Z$ and an invariant symplectic submanifold Y such that $Y \cap Z = Z_{(\mathfrak{h})}$. We can symplectically blow U up along Y

to obtain a proper Hamiltonian K-space $\widetilde{U}$ with an equivariant map $\widetilde{U} \to U$. Let $\widetilde{Z}$ be the zero set of the moment map on $\widetilde{U}$. Let π_K be the composition

$$\widetilde{Z} \hookrightarrow \widetilde{U} \to U \xrightarrow{r} Z$$

and $\pi : \widetilde{M_0} \to M_0$ the induced map on quotients. Then π induces a homeomorphism $\pi^{-1}(M_0 \backslash P_{\mathfrak{h}}) \to M_0 \backslash P_{\mathfrak{h}}$. Let $E = \pi^{-1}(P_{\mathfrak{h}})$. A neighbourhood of E in $\widetilde{M_0}$ is modelled by the total space of an orbibundle over $P_{\mathfrak{h}}$ with fibre

$$Bl(W /\!\!/ H_0^{\mathbb{C}}, 0) \cong Bl(W, 0) /\!\!/ H_0^{\mathbb{C}}.$$

We write $\widetilde{W}$ for $Bl(W, 0)$. Alternatively we can think of a neighbourhood of E in $\widetilde{M_0}$ as a 2-dimensional vector orbibundle over E with charts of the form

$$\begin{array}{ccc} V_{\mathfrak{h}} \oplus \left(\widetilde{W} /\!\!/ H_0^{\mathbb{C}}\right) & \longrightarrow & \left[V_{\mathfrak{h}} \oplus \left(\widetilde{W} /\!\!/ H_0^{\mathbb{C}}\right)\right] / \pi_0 H \\ \downarrow & & \downarrow \\ V_{\mathfrak{h}} \oplus \left(\mathbb{P}W /\!\!/ H_0^{\mathbb{C}}\right) & \longrightarrow & \left[V_{\mathfrak{h}} \oplus \left(\mathbb{P}W /\!\!/ H_0^{\mathbb{C}}\right)\right] / \pi_0 H \end{array} \tag{3.3}$$

induced by the vector bundle $\widetilde{W} /\!\!/ H_0^{\mathbb{C}} \to \mathbb{P}W /\!\!/ H_0^{\mathbb{C}}$. See [**18**] for details.

3.3. *The decomposition*

In this section we explain how to decompose $R\pi_* \mathcal{IC}^\bullet(\widetilde{M_0})$ with $\mathcal{IC}^\bullet(M_0)$ as a summand. As in Theorem 5 the object $R\pi_* \mathcal{IC}^\bullet(\widetilde{M_0})$ will decompose as a direct sum of its cohomology sheaves with respect to a non-standard **t**-structure. Recall that the objects in $\mathbf{D}^S(M_0)$ are cohomologically constructible with respect to the orbit type stratification of M_0. However the **t**-structure on $\mathbf{D}^S(M_0)$ we will use is associated to the coarser decomposition by *infinitesimal* orbit types. Let $\{P_\alpha\}$ be the set of pieces of this decomposition and

$$\jmath_\alpha : P_\alpha \hookrightarrow M_0 \setminus (\overline{P_\alpha} \setminus P_\alpha)$$

be the associated closed inclusions.

LEMMA 3. *The subcategories*

$${}^p\mathbf{D}^S_{\leq 0}(M_0) = \{\mathcal{A}^\bullet \mid H^i(\jmath_\alpha^* \mathcal{A}^\bullet) = 0 \text{ for } i > \operatorname{codim}(P_\alpha)/2 \quad \forall \alpha\}$$
$${}^p\mathbf{D}^S_{\leq 0}(M_0) = \{\mathcal{A}^\bullet \mid H^i(\jmath_\alpha^! \mathcal{A}^\bullet) = 0 \text{ for } i < \operatorname{codim}(P_\alpha)/2 \quad \forall \alpha\}$$

define a bounded **t***-structure on* $\mathbf{D}^S(M_0)$ *with truncation functors*

$${}^p\tau_{\leq 0} : \mathbf{D}^S(M_0) \to {}^p\mathbf{D}^S_{\leq 0}(M_0) \text{ and } {}^p\tau_{\geq 0} : \mathbf{D}^S(M_0) \to {}^p\mathbf{D}^S_{\geq 0}(M_0)$$

respectively right and left adjoint to the inclusions.

Proof. The **t**-structure can be constructed inductively using the glueing data associated to the inclusion of a stratum just as in §1.1. □

We will refer to this as the perverse **t**-structure on $\mathbf{D}^S(M_0)$. We show that $\mathcal{IC}^\bullet(M_0)$, even though it may not be constructible with respect to the infinitesimal orbit type stratification, lies in the heart of this perverse **t**-structure. This is a local matter so we work in our local model.

For ease of notation write L for $V_{\mathfrak{h}} \oplus \left(W /\!\!/ H_0^{\mathbb{C}}\right)$ and F for $\pi_0 H$. The quotient of L

by the finite group F is a model for a neighbourhood in M_0 of a point in $P_{\mathfrak{h}}$. Hence L/F is stratified by the restriction to this neighbourhood of the stratification of M_0 by orbit types and we use this stratification to define the constructible derived categories $\mathbf{D}^S(L/F)$ and $\mathbf{D}^S_+(L/F)$.

The inverse image of this stratification under $L \to L/F$ is a refinement of the product stratification of L arising from the stratification of $W/\!\!/H_0^{\mathbb{C}}$ by orbit types of $H_0^{\mathbb{C}}$. We use this product stratification to define the constructibility conditions for $\mathbf{D}^S_F(L)$. By §1.2 the quotient map induces a functor

$$Q_* : \mathbf{D}^S_F(L) \to \mathbf{D}^S_+(L/F).$$

We also have perverse $\mathbf{t}$-structures: $\mathbf{D}^S_+(L/F)$ inherits a perverse $\mathbf{t}$-structure associated to the decomposition by infinitesimal orbit types and $\mathbf{D}^S_F(L)$ has a perverse $\mathbf{t}$-structure associated to the inverse image decomposition. The latter can also be described as the product decomposition of L arising from the decomposition of $W/\!\!/H_0^{\mathbb{C}}$ by infinitesimal orbit types of $H_0^{\mathbb{C}}$.

It follows from the statement and proof of [**2**, Theorem 8.7.1] that

(1) Q_* preserves the natural $\mathbf{t}$-structures i.e. $\tau_{\leq 0} Q_* \cong Q_* \tau_{\leq 0}$ etc,
(2) Q_* preserves the perverse $\mathbf{t}$-structures i.e. ${}^p\tau_{\leq 0} Q_* \cong Q_* {}^p\tau_{\leq 0}$ etc,
(3) $Q_* \mathcal{IC}^\bullet_F(L) \cong \mathcal{IC}^\bullet(L/F)$.

In particular it follows from 1) that Q_* lifts to a functor (which we also denote Q_*) to the *bounded* derived category $\mathbf{D}^S(L/F)$.

By definition $\mathcal{IC}^\bullet_F(L)$ is in the heart of the perverse $\mathbf{t}$-structure on $\mathbf{D}^S_F(L)$ if, and only if, $\mathcal{IC}^\bullet(L)$ is in the heart of the perverse $\mathbf{t}$-structure on $\mathbf{D}^S(L)$. In turn this holds if, and only if, $\mathcal{IC}^\bullet(W/\!\!/H_0^{\mathbb{C}})$ is in the heart of the perverse $\mathbf{t}$-structure on $\mathbf{D}^S(W/\!\!/H_0^{\mathbb{C}})$ associated to the decomposition by infinitesimal orbit types. This can be checked by using the standard cone calculation to verify the conditions at the vertex and by induction on the depth of the decomposition elsewhere. Thus it now follows from 2) and 3) that $\mathcal{IC}^\bullet(M_0)$ is in the heart of the perverse $\mathbf{t}$-structure on $\mathbf{D}^S(M_0)$ as required.

Let $\widetilde{L}$ be the blowup

$$V_{\mathfrak{h}} \oplus \left(\widetilde{W}/\!\!/H_0^{\mathbb{C}} \right)$$

of L along $V_{\mathfrak{h}}$. Arguing in a similar fashion to above we have a functor

$$\widetilde{Q}_* : \mathbf{D}^S_F(\widetilde{L}) \to \mathbf{D}^S(\widetilde{L}/F)$$

with $Q_* \mathcal{IC}^\bullet_F(\widetilde{L}) \cong \mathcal{IC}^\bullet(\widetilde{L}/F)$. Furthermore, by [**2**, 6.12] the commutative square of spaces

$$\begin{tikzcd} \widetilde{L} \arrow[r] \arrow[d, "\pi_F"'] & \widetilde{L}/F \arrow[d, "\pi"] \\ L \arrow[r] & L/F \end{tikzcd}$$

gives rise to a commutative square of functors

$$\begin{tikzcd} \mathbf{D}^S_F(\widetilde{L}) \arrow[r, "\widetilde{Q}_*"] \arrow[d, "R\pi_{F*}"'] & \mathbf{D}^S(\widetilde{L}/F) \arrow[d, "R\pi_*"] \\ \mathbf{D}^S_F(L) \arrow[r, "Q_*"'] & \mathbf{D}^S(L/F). \end{tikzcd}$$

In particular $Q_* R\pi_{F*}\mathcal{IC}^\bullet_F(\widetilde{L}) \cong R\pi_* \widetilde{Q}_* \mathcal{IC}^\bullet_F(\widetilde{L}) \cong R\pi_* \mathcal{IC}^\bullet(\widetilde{L}/F)$.

Lemma 4. *The inclusion $E \hookrightarrow \widetilde{M}_0$ induces a Gysin morphism*

$$\gamma : R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0)[-2] \to R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0).$$

If we restrict to the neighbourhood L/F then $\gamma = Q_*\gamma_F$ where γ_F is the equivariant Gysin morphism arising from the F-equivariant, codimension 2, normally nonsingular inclusion

$$V_{\mathfrak{h}} \oplus \left(\mathbb{P}W /\!\!/ H_0^{\mathbb{C}}\right) \hookrightarrow V_{\mathfrak{h}} \oplus \left(\widetilde{W} /\!\!/ H_0^{\mathbb{C}}\right) = \widetilde{L}.$$

Proof. Since the fibres of the normal orbibundle to E in $\widetilde{M}_0$ are finite group quotients of $\mathbb{R}^2$ an orientation of the orbibundle induces the required Gysin morphism just as in §2.1. That $\gamma = Q_*\gamma_F$ follows because on a chart orientations of the orbibundle correspond to orientations of the normal bundle to the inclusion

$$V_{\mathfrak{h}} \oplus \left(\mathbb{P}W /\!\!/ H_0^{\mathbb{C}}\right) \hookrightarrow V_{\mathfrak{h}} \oplus \left(\widetilde{W} /\!\!/ H_0^{\mathbb{C}}\right) = \widetilde{L}.$$

Proposition 6. *There is a (non-canonical) direct sum decomposition*

$$IH^*(\widetilde{M}_0) \cong IH^*(M_0) \oplus \bigoplus_{i\in\mathbb{Z}} H^*(S;\mathcal{L}^i)$$

where $\mathcal{L}^i$ is the sheaf in dimension $m+i$ supported on S given by the image of

$$\gamma : \mathcal{H}^{m+i-2}(R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0)) \to \mathcal{H}^{m+i}(R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0)).$$

Proof. We follow the same steps as in the proof of Theorem 5. First, we must show that ${}^p\mathcal{H}^i(\gamma^i[i])$ is an isomorphism for $i = l, 2, \ldots$ where

$$\gamma : R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0)[-2] \to R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0)$$

is the Gysin morphism and the perverse cohomology sheaves are computed with respect to the $\mathbf{t}$-structure on $\mathbf{D}^S(M_0)$ coming from the stratification by infinitesimal orbit types. This is a local matter. It is trivial for any open subset of M_0 not meeting S so we need only consider a neighbourhood of a point in S homeomorphic to L/G.

By Remark 5 we know that

$$\gamma_F : R\pi_*\mathcal{IC}^\bullet_F(\widetilde{L})[-2] \to R\pi_*\mathcal{IC}^\bullet_F(\widetilde{L})$$

induces isomorphisms ${}^p\mathcal{H}^i(\gamma_F^i[i])$ for $i = l, 2, \ldots$. Thus, by the fact that Q_* preserves the $\mathbf{t}$-structures and by Lemma 4,

$$\begin{aligned} Q_*{}^p\mathcal{H}^i(\gamma_F^i[i]) &= {}^p\mathcal{H}^i(Q_*\gamma_F^i[i]) \\ &= {}^p\mathcal{H}^i(\gamma^i[i]) \end{aligned}$$

is an isomorphism as required.

Second, we need to decompose ${}^p\mathcal{H}^0(R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0))$ with $\mathcal{IC}^\bullet(M_0)$ as a summand. Once again the argument (in Proposition 2) rests on establishing an isomorphism of perverse sheaves and so can be checked locally. By Remark 5 a corresponding isomorphism will hold in $\mathbf{D}^S_F(L)$ and can be pushed forward using Q_* to an isomorphism in $\mathbf{D}^S(L/F)$. Since $Q_*\mathcal{IC}^\bullet_F(L) \cong \mathcal{IC}^\bullet(L/F)$ and

$$Q_*{}^p\mathcal{H}^0(R\pi_{F*}\mathcal{IC}^\bullet_F(\widetilde{L})) \cong {}^p\mathcal{H}^0(Q_*R\pi_{F*}\mathcal{IC}^\bullet_F(\widetilde{L})) \cong {}^p\mathcal{H}^0(R\pi_*\mathcal{IC}^\bullet(\widetilde{L}/F))$$

we are done. □

3.4. *The definition of the Kirwan map*

THEOREM 6. *Let $R_* : \mathbf{D}^S_K(Z) \to \mathbf{D}^S_+(M_0)$ be the pushforward functor induced by the quotient map $r : Z \to Z/K = M_0$. Then we can define (non-canonically) a morphism*

$$\kappa : R_*\mathbb{Q}_Z \to \mathcal{IC}^\bullet(M_0)$$

where $\mathbb{Q}_Z$ is the natural lift of the constant sheaf to an equivariant object, which extends the natural isomorphism between the two over the nonsingular part of M_0. This induces a Kirwan map

$$H^*_K(Z) \to IH^*(M_0).$$

Proof. We prove this inductively on the depth of the stratification by infinitesimal orbit types. The base case is when K acts with only finite stabilisers. [**2**, Theorem 9.1 (ii)] tells us that there is an isomorphism $R_*\mathbb{Q}_Z \cong \mathbb{Q}_{M_0}$. The proof is based on Luna's slice theorem for geometric quotients but this can be replaced by the local normal form theorem in [**20**]. Since M_0 is in this case a rational homology manifold we also have an isomorphism $\mathbb{Q}_{M_0} \cong \mathcal{IC}^\bullet(M_0)$. Composing the two yields the desired map $R_*\mathbb{Q}_Z \to \mathcal{IC}^\bullet(M_0)$.

Now suppose we are in the general case. By performing blow ups we can obtain a proper Hamiltonian K-space $\widetilde{U}$ such that the infinitesimal orbit type decomposition of the reduction $\widetilde{Z}/K = \widetilde{M}_0$ has strictly lesser depth than that of $Z/K = M_0$. There is an equivariant map $\pi_K : \widetilde{Z} \to Z$. Adjunction gives a morphism $\mathbb{Q}_Z \to R\pi_{K*}\pi_K^*\mathbb{Q}_Z \cong R\pi_{K*}\mathbb{Q}_{\widetilde{Z}}$. Since

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\pi_K"] \arrow[d] & Z \arrow[d] \\ \widetilde{M}_0 \arrow[r, "\pi"'] & M_0 \end{tikzcd}$$

is commutative [**2**, 6.12] tells us that

$$R_*\mathbb{Q}_Z \to R_*R\pi_{K*}\mathbb{Q}_{\widetilde{Z}} \cong R\pi_*R_*\mathbb{Q}_{\widetilde{Z}}.$$

By induction on the depth of the infinitesimal orbit type decomposition we may assume that $\widetilde{\kappa} : R_*\mathbb{Q}_{\widetilde{Z}} \to \mathcal{IC}^\bullet(\widetilde{M}_0)$ has already been constructed. Furthermore, by Proposition 6, there is a projection map $R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0) \to \mathcal{IC}^\bullet(M_0)$. Hence we can define κ to be the composition

$$R_*\mathbb{Q}_Z \to R\pi_*R_*\mathbb{Q}_{\widetilde{Z}} \to R\pi_*\mathcal{IC}^\bullet(\widetilde{M}_0) \to \mathcal{IC}^\bullet(M_0).$$

REMARK 6. There are choices involved in the definition of the Kirwan map. Whenever we apply Proposition 6 we have to choose the isomorphism giving the decomposition. For every blowup in the partial desingularisation we must make such a choice.

Since it arises from a morphism in the derived category the Kirwan map is clearly natural under restriction i.e. for any open $U \subset M_0$ there is a commutative square

$$\begin{tikzcd} H_K^*(Z) \arrow[r] \arrow[d] & H_K^*(r^{-1}U) \arrow[d] \\ IH^*(M_0) \arrow[r] & IH^*(U) \end{tikzcd}$$

whose vertical maps are Kirwan maps *where we make compatible choices of the decompositions required to define them.*

In addition, both $H_K^*(Z)$ and $IH^*(M_0)$ are graded $H^*(M_0)$-modules. From the point of view of the derived category we see this by noting that

$$\begin{aligned} H^i(M_0) &\cong \mathrm{Hom}(\mathbb{Q}_{M_0}, \mathbb{Q}_{M_0}[i]), \\ H_K^i(Z) &\cong \mathrm{Hom}(\mathbb{Q}_{M_0}, R_*\mathbb{Q}_Z[i]), \\ \text{and } IH^i(M_0) &\cong \mathrm{Hom}(\mathbb{Q}_{M_0}, \mathcal{IC}^\bullet(M_0)[i]). \end{aligned}$$

The module structure arises from composition of morphisms. Since composition on the left and right commute we see that the Kirwan map is a map of graded $H^*(M_0)$-modules. In particular it is surjective onto the image of the map $H^*(M_0) \to IH^*(M_0)$ and uniquely defined on the image of the map $H^*(M_0) \to H_K^*(Z)$.

References

1. A. Beilinson, J. Bernstein, and P. Deligne. 'Faisceaux pervers', *Astérisque* 100 Proc. C.I.R.M. conférence: Analyse et topologie sur les espaces singuliers (1982).

2. J. Bernstein, and V. Lunts. *Equivariant sheaves and functors*, Lecture Notes in Mathematics 1578 (Springer–Verlag, Berlin, 1994).

3. P. Deligne. 'Décompositions dans la categorie dérivée', *Motives*, Proc. Sympos. Pure Math. 55 (AMS, Providence, RI, 1994) pp.115–128.

4. M. Goresky and R. MacPherson. 'Intersection homology theory', *Topology* 19 (1980) pp.135–162.

5. M. Goresky and R. MacPherson. 'Intersection homology theory II', *Invent. Math.* 71 (1983) pp.77–129.

6. S.I. Gelfand and Yu.I. Manin. *Algebra V*, Encyclopedia of Mathematics 38 (eds A. Kostrikin and I. Shafarevich, Springer–Verlag, 1994).

7. V. Guillemin and S. Sternberg. 'A normal form for the moment map', *Differential geometric methods in mathematical physics* (ed S. Sternberg, Reidel Publishing company, Dordrecht, 1984).

8. M. Kashiwara and P. Schapira. *Sheaves on manifolds*, Grundlehren der mathematischen wissenschaften 292 (Springer–Verlag, 1990).

9. Y.-H. Kiem. 'Intersection cohomology of quotients of nonsingular varieties', *Invent. Math.* (1) 155 (2004) pp.163–202.

10. Y.-H. Kiem and J. Woolf. 'Intersection cohomology of symplectic quotients by circle actions', (accepted by the LMS 2004).

11. Y.-H. Kiem and J. Woolf. 'The cosupport axiom, equivariant cohomology and the intersection cohomology of certain symplectic quotients', preprint, 2004.

12. F. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes 34 (Princeton University Press, 1984).

13. F. Kirwan. 'Partial desingularisations of quotients of nonsingular varieties and their Betti numbers', *Annals of Mathematics* 122 (1985) pp.41–85.

14. F. Kirwan. 'Rational intersection homology of quotient varieties', *Invent. Math.* 86 (1986) pp.471–505.

15. E. Lerman and S. Tolman. 'Intersection cohomology of S^1 symplectic quotients and small resolutions', *Duke Math. J.* (1) 103 (2000) pp.79–99.

16. D. McDuff. 'Examples of simply-connected symplectic non-Kählerian manifolds', *J. Diff. Geom.* (1) 20 (1984) pp.267–277.

17. C.M. Marle. 'Modéle d'action Hamiltonienne d'un groupe de Lie sur une variété symplectique', *Rendiconti del Seminario Matematico* 43 (1985) pp.227–251.

18. E. Meinrenken and R. Sjamaar. 'Singular reduction and quantisation', *Topology* (4) 38 (1999) pp.699–762.
19. L. Ness. 'A stratification of the null cone via the moment map', *Am. J. Math.* (6) 106 (1984) pp.1281–1330.
20. R. Sjamaar and E. Lerman. 'Stratified symplectic spaces and reduction', *Annals of Maths* 134 (1991) pp.375–422.
21. M. Van den Bergh. 'A remark on a theorem by Deligne', preprint, arXiv:math.CT/0212197, 2002.
22. J. Woolf. 'The decomposition theorem and the Kirwan map', *Journal of Pure and Applied Algebra* 182 (2003) pp.317–328.

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