# INTRODUCTION TO EQUIVARIANT COHOMOLOGY THEORY

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## 1. Definitions and Basic Properties

1.1. **Lie group.** Let $G$ be a Lie group (i.e. a manifold equipped with differentiable group operations mult : $G \times G \to G$, inv : $G \to G$, id $\in G$ satisfying the usual group axioms). We shall be concerned only with linear groups (i.e. a subgroup of $GL(n) = GL(n, \mathbb{C})$ for some $n$) such as the unitary group $U(n)$, the special unitary group $SU(n)$. A connected compact Lie group $G$ is called a *torus* if it is abelian. Explicitly, they are products $U(1)^n$ of the circle group $U(1) = \{e^{i\theta} \,|\, \theta \in \mathbb{R}\} = S^1$. Since a complex linear (reductive) group is homotopy equivalent[1] to its maximal compact subgroup, it suffices to consider only compact groups. For instance, the equivariant cohomology for $SL(n)$ ($GL(n)$, $\mathbb{C}^*$, resp.) is the same as the equivariant cohomology for $SU(n)$ ($U(n)$, $S^1$, resp.).

1.2. **Classifying space.** Suppose a compact Lie group $G$ acts on a topological space $X$ continuously. We say the group action is *free* if the stabilizer group $G_x = \{g \in G \,|\, gx = x\}$ of every point $x \in X$ is the trivial subgroup. A topological space $X$ is called *contractible* if there is a homotopy equivalence with a point (i.e. $\exists h : X \times [0,1] \to X$ such that $h(x,0) = x_0$, $h(x,1) = x$ for $x \in X$).

**Theorem 1.** For each compact Lie group $G$, there exists a *contractible* topological space $EG$ on which $G$ acts *freely*.

**Proof:** Since $G \subset GL(n)$ for some $n$, it suffices to construct a contractible free space for $G = GL(n)$. Let $A_m$ be the space of $n \times m$ matrices with complex entries of rank $n$ for $m > n$. Of course, $G$ acts freely on $A_m$ but $A_m$ is not contractible. We claim the limit $\lim_{m\to\infty} A_m$ is contractible and this is the desired space. The set $M^{n\times m}$ of all $n \times m$ matrices is contractible and the complement $M^{n\times m} - A_m$ is a union of locally closed submanifolds whose codimensions grow to infinity. Use Gysin sequence[2] and Whitehead's theorem[3] to conclude that the limit is contracitble. □

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[1]Two spaces $X$ and $Y$ are *homotopy equivalent* if $\exists f : X \to Y, g : Y \to X$ such that $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are *homotopic* (=continuously deformable) to the identity maps. Ordinary cohomology groups are invariant under homotopy equivalence.

[2]Gysin sequence: Let $F \subset M$ be a submanifold of codimension $c$ whose normal bundle is oriented. Then we have an exact sequence

$$\cdots \to H^{i-c}(F) \to H^i(M) \to H^i(M-F) \to H^{i+1-c}(F) \to H^{i+1}(M) \to H^{i+1}(M-F) \to \cdots .$$

[3]Whitehead's theorem: If $f : X \to Y$ is continuous map of reasonably nice spaces such that $f^* : H^i(Y) \to H^i(X)$ is an isomorphism for all $i$ (with integer coefficients), then $f$ is a homotopy equivalence.

**Definition.** The *classifying space* of a group $G$ is the quotient space $BG = EG/G$ of $EG$. It is unique up to homotopy equivalence.[4] Since the action of $G$ on $EG$ is free, each fiber of the quotient map $\pi : EG \to BG$ is an orbit $Gx$, homeomorphic to $G$.

**Examples.**

(1) $ES^1 = S^\infty$, $BS^1 = S^\infty/S^1 = \lim_{n\to\infty} S^{2n+1}/S^1 = \mathbb{CP}^\infty$. (Exercise: Prove that $S^\infty$ is contractible. Prove that $B(\mathbb{Z}_2) = S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$.) It is well known that $H^*(\mathbb{CP}^n;\mathbb{R}) = \mathbb{R}[t]/(t^{n+1})$. Hence, $H^*(BS^1) = \mathbb{R}[t]$ by taking inverse limit.[5]

(2) For $G = G_1 \times G_2$, $EG = EG_1 \times EG_2$ and $BG = BG_1 \times BG_2$. In particular, $B(S^1 \times \cdots \times S^1) = \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$ and $H^*(B(S^1 \times \cdots \times S^1)) = H^*(BS^1 \times \cdots \times BS^1) \cong H^*(BS^1) \otimes \cdots \otimes H^*(BS^1) \cong \mathbb{R}[t_1, \cdots, t_n]$.

(3) Suppose $H$ is a subgroup of $G$. Then $BH \cong EG/H$ and we have a map $BH \to BG$ whose fibers are $G/H$.

## 1.3. Equivariant cohomology.

Suppose a topological group $G$ acts on a topological space $X$ continuously. When the group action is not free, the quotient $X/G$ may be terribly ugly. The idea of Borel is to replace $X$ by a free $G$-space $X^\dagger$ which is homotopically equivalent to $X$. The obvious choice is $X^\dagger = X \times EG$ on which $G$ acts diagonally $g \cdot (x, e) = (gx, ge)$.

**Definition.** The *homotopy quotient* of a $G$-space $X$ is $X_G = X \times_G EG = (X \times EG)/G$. The *equivariant cohomology* of $X$ is $H^*_G(X) = H^*(X_G)$. We shall use real coefficients $\mathbb{R}$ for convenience.

**Borel's diagram:**

$$\begin{array}{ccccc} EG & \xleftarrow{\;X\;} & X \times EG & \xrightarrow{\;EG\;} & X \\ \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle Gx} \\ BG & \xleftarrow[\;X\;]{} & X \times_G EG & \xrightarrow[\;BG_x\;]{} & X/G \end{array}$$

## 1.4. Basic properties.

Almost all basic properties follow directly from the definition.

(1) Suppose $G$ acts freely on $X$. Then the fibers of $X_G = (X \times EG)/G \to X/G$ are contractible $EG$, and hence the map $X_G \to X/G$ is a homotopy equivalence. In particular, $H^*_G(X) = H^*(X_G) \cong H^*(X/G)$.

(2) Suppose a normal subgroup $K$ of $G$ acts freely on $X$. Then $S = G/K$ acts on $Y = X/K$ ($\cong X \times_K EK \cong X \times_K EG$). We have $H^*_G(X) \cong H^*_S(Y)$.
(Proof: $EG \times ES$ is a contractible space acted on freely by $G$ via the quotient homomorphism $G \to S$. Then the obvious map $X_G = X \times_G (EG \times ES) \to X \times_G ES = Y \times_S ES = Y_S$ is a homotopy equivalence with contractible fiber $EG$.)

[4] We will prove this later.

[5] For convenience, we will use real coefficients for cohomology groups in this lecture course but there is no privilege for $\mathbb{R}$ in equivariant cohomology theory. Most authors prefer $\mathbb{Q}$ coefficients.

(3) Suppose $\mathsf{G}$ acts trivially on $\mathsf{X}$. Then $\mathsf{X}_\mathsf{G} = \mathsf{X} \times_\mathsf{G} \mathsf{EG} = \mathsf{X} \times \mathsf{BG}$ and hence $\mathsf{H}^*_\mathsf{G}(\mathsf{X}) \cong \mathsf{H}^*(\mathsf{X}) \otimes \mathsf{H}^*(\mathsf{BG})$. In particular, $\mathsf{H}^*_\mathsf{G}(\mathsf{X})$ is a *free* $\mathsf{H}^*(\mathsf{BG})$-module. (The module structure comes from the map $\mathsf{X}_\mathsf{G} \to \mathsf{BG}$.)

(4) Suppose $\mathsf{K}$ is a subgroup of $\mathsf{G}$. Let $\mathsf{Y}$ be a $\mathsf{K}$-space. Then $\mathsf{X} = \mathsf{G} \times_\mathsf{K} \mathsf{Y} = (\mathsf{G} \times \mathsf{Y})/\mathsf{K}$ admits an obvious action of $\mathsf{G}$. We have $\mathsf{H}^*_\mathsf{G}(\mathsf{X}) = \mathsf{H}^*_\mathsf{K}(\mathsf{Y})$ because $\mathsf{X}_\mathsf{G} = (\mathsf{G} \times_\mathsf{K} \mathsf{Y}) \times_\mathsf{G} \mathsf{EG} = \mathsf{Y} \times_\mathsf{K} \mathsf{EG} = \mathsf{Y} \times_\mathsf{K} \mathsf{EK} = \mathsf{Y}_\mathsf{K}$.

## 2. Cartan model and spectral sequence

Let $G$ be a compact Lie group acting on a *smooth compact* manifold $M$ differentiably. Let $\mathfrak{G}$ be the Lie algebra of $G$, i.e. the tangent space of $G$ at the identity. There is a natural action of $G$ on $\mathfrak{G}$ by conjugation $g\cdot v = \lim_{t\to 0} g\gamma(t)g^{-1}$ where $\frac{d}{dt}|_{t=0}\gamma(t) = v$. Let $S(\mathfrak{G}^*) = \oplus_{j\geq 0} S^j(\mathfrak{G}^*)$ be the polynomial algebra generated by the dual space $\mathfrak{G}^*$ of $\mathfrak{G}$, i.e. the set of polynomial functions on the vector space $\mathfrak{G}$. Of course, there is an induced $G$-action on $S(\mathfrak{G}^*)$.

The differential forms on $M$ form a complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots .$$

A theorem of de Rham says that the cohomology $H^*(M)$ with real coefficients is isomorphic to the cohomology of $(\Omega(M), d)$. When there is an action of $G$ on $M$, we get an induced action on $\Omega(M)$. The main theorem of this section is the following due to H. Cartan.

**Theorem 2.** Let $\Omega^i_G(M) = \oplus_j [S^j(\mathfrak{G}^*)\otimes \Omega^{i-2j}(M)]^G$ where $[-]^G$ denotes the $G$-invariant subspace.[6] Let $d_G : \Omega^i_G(M) \to \Omega^{i+1}_G(M)$ be defined as follows: For a basis $\{\xi_a\}$ of $\mathfrak{G}$ and its dual basis $\{f_a\}$, $d_G(F\otimes\sigma) = F\otimes d\sigma - \sum_a f_a F\otimes \iota_{\xi_a}\sigma$.[7] Then the equivariant cohomology $H^*_G(M)$ of $M$ is the cohomology of the complex

$$0 \longrightarrow \Omega^0_G(M) \xrightarrow{d_G} \Omega^1_G(M) \xrightarrow{d_G} \Omega^2_G(M) \xrightarrow{d_G} \cdots .$$

**Proof (sketch):** The de Rham complex of the quotient $M_G = (M\times EG)/G$ is embedded into the de Rham complex of $M\times EG$ as $G$-invariant subspaces. So it suffices to find the de Rham complex for $M\times EG$ and take the invariant (horizontal) part. For $M$, we have the ordinary de Rham complex $\Omega(M)$ but what is the de Rham complex for the infinite dimensional contractible space $EG$? Recall that $EG = \lim_{m\to\infty} A_m$ and hence $\Omega(EG) = \lim_{\leftarrow} \Omega(A_M)$. Then one finds that $\Omega(EG)$ can be chopped down to a more economical complex, namely the Koszul complex $W := S(\mathfrak{G}^*)\otimes \wedge(\mathfrak{G}^*)$ of $S(\mathfrak{G}^*)$-modules. So we obtain

$$H^*_G(M) = H^*(M\times_G EG) \cong H^*([\Omega(M)\otimes W]_{hor}).$$

Then by pure algebra, we obtain an isomorphism

$$H^*([\Omega(M)\otimes W]_{hor}) \cong H^*(\Omega_G(M)).$$

□

**Corollary (homogeneous spaces).** $H^*_G(G/K) \cong H^*(BK) \cong S(\mathfrak{K}^*)^K$ is the space of $K$-invariant polynomial functions on $\mathfrak{K}$. A nonzero element of $\mathfrak{K}^*$ corresponds to a degree 2 class, i.e. $H^{2j}(BK) \cong S^j(\mathfrak{K}^*)^K$ and $H^{odd}(BK) = 0$.

**Examples.** (1) If $G$ is the $n$-dimensional torus $(S^1)^n$, then $\mathfrak{G} \cong \mathbb{R}^n$ and $G$ acts trivially on $\mathfrak{G}$. Hence $H^*_G(pt) = H^*(BG) = S(\mathbb{R}^n) = \mathbb{R}[t_1,\cdots,t_n]$ as expected.

(2) For $G = U(n)$, $\mathfrak{G}$ is the space of skew-hermitian $n\times n$ complex matrices. For $A\in\mathfrak{G}$, we consider the coefficients $c_i(A)$ of the characteristic polynomial

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[6] We think of an element in $\Omega_G(M)$ as a differential form valued polynomial function defined on $\mathfrak{G}^*$. For $f\in\mathfrak{G}^*$, $\deg f = 2$.

[7] $\iota_{\xi_a}$ denotes the interior product by the vector field on $M$ generated by $\xi_a$, i.e. $(\xi_a)_x = \frac{d}{dt}|_{t=0} e^{\xi_a t}x$.

$\det(\lambda - A) = \sum(-1)^i c_i(A)\lambda^{n-i}$. In particular, $c_1(A) = \mathrm{tr}(A)$ and $c_n(A) = \det(A)$. It is well known that $S(\mathfrak{G}^*)^G = \mathbb{R}[c_1, c_2, \cdots, c_n]$ and hence

$$H^*_G(pt) = H^*(BG) = \mathbb{R}[c_1, c_2, \cdots, c_n].$$

The $c_i$ is called the $i$-th Chern class.

(3) For $G = O(n)$, $\mathfrak{G}$ is the space of skew-symmetric $n \times n$ real matrices. Since $\det(\lambda - A) = \det(\lambda - A^t) = \det(\lambda + A)$, we have the vanishing of odd degree coefficients $c_i(A) = 0$ for $i$ odd. The even coefficients $p_i(A) = c_{2i}(A)$ generate the invariant subring $S(\mathfrak{G}^*)^G$. The $p_i$ is called the $i$-th Pontryagin class.

(4) For $G = SO(2n)$, we have an additional class, called the Pfaffian. A skew-symmetric matrix gives us a skew-symmetric bilinear form which can be put into a standard form. The determinant turns out to be the square of an invariant polynomial $\mathsf{Pfaff}$.

An indispensable tool for equivariant cohomology theory is *spectral sequence*. Note that $\Omega_G(M) = [S(\mathfrak{G}^*) \otimes \Omega(M)]^G$ is bigraded and the differential[8] $d_G$ is the sum of its vertical part $d := 1 \otimes d$ and horizontal part $\delta := -\sum f_a \otimes \iota_{\xi_a}$.

A *double complex* is a bigraded vector space $C = \oplus_{p,q \geq 0} C^{p,q}$ with operators $d : C^{p,q} \to C^{p,q+1}$, $\delta : C^{p,q} \to C^{p+1,q}$ satisfying $d^2 = 0$, $\delta^2 = 0$ and $d\delta + \delta d = 0$. The *total complex* of $C$ is defined by $C^n = \oplus_{p+q=n} C^{p,q}$ with differential $d + \delta : C^n \to C^{n+1}$. The goal is to calculate the cohomology of the total complex by approximation.

Let $E_0^{p,q} = C^{p,q}$. Inductively, we can construct bigraded complexes $E_r$ for $r \geq 0$ together with differential $\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $E_{r+1}^{p,q}$ is the cohomology of $(E_r, \delta_r)$ at $(p, q)$-th place. Furthermore, in the limit $E_\infty$, the sum $\oplus_{p+q=n} E_\infty^{p,q}$ is isomorphic to the $n$-th cohomology of the total complex of $C$. We say the spectral sequence *collapses* at $E_r$ if $E_r \cong E_\infty$. This happens when $\delta_r$ vanishes and so do all higher differentials.

For practical use, it suffices to know only the first two terms $E_1$ and $E_2$.[9] For $E_1$, you ignore $\delta$ and take the cohomology at each place with only the vertial differential $d$. Then the $E_2$ term is given by taking cohomology of $E_1$ with the horizontal differential $\delta$.

For the Cartan model, the $E_1$ term is

$$E_1^{p,q} = [S^p(\mathfrak{G}^*) \otimes H^{q-p}(M)]^G.$$

If $G$ is connected, then $G$ acts trivially on $H^*(M)$ because a continuous deformation of cycles doesn't change the (co)homology classes. Hence, if $G$ is connected, we have

$$E_1^{p,q} = S^p(\mathfrak{G}^*)^G \otimes H^{q-p}(M) = H^{2p}(BG) \otimes H^{q-p}(M)$$

or $E_1^{*,*} \cong H^*(BG) \otimes H^*(M)$. For many friendly spaces, the spectral sequence degenerates at $E_1$ and so we obtain an isomorphism of $S(\mathfrak{G}^*)^G$-modules

$$H^*_G(M) \cong H^*(BG) \otimes H^*(M)$$

for connected $G$. We make this into a definition.

**Definition.** A manifold $M$ on which $G$ acts is called *equivariantly formal* if the spectral sequence collapses at $E_1$, i.e. $H^*_G(M) \cong [S(\mathfrak{G}^*) \otimes H^*(M)]^G$.

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[8]A differetial means nothing but a sequence of homomorphisms such that the composition of any two consecutive maps is zero.

[9]If you can't finish the calculation by $E_2$, it is often too difficult to continue.

Here are some criteria for equivariant formality.

**Theorem 3.**

(1) If $G$ is connected and $H^{odd}(M) = 0$, then $M$ is equivariantly formal.[10]
(2) Every compact symplectic manifold on which $G$ acts in a Hamiltonian fashion[11] is equivariantly formal. In partucular, any linear action on a smooth projective variety is equivariantly formal.

The first item is completely obvious but the proof of the second item requires the equivariant Morse theory (a.k.a. Atiyah-Bott-Kirwan theory). The point is that if you live with only compact symplectic or algebraic manifolds, equivariant formality is almost always there.

**Example.** Suppose a connected group $G$ acts on the complex projective space $\mathbb{P}^n$. Since all the odd cohomology groups for $\mathbb{P}^n$ vanish, it is equivariantly formal and hence we have an isomorphism of $S(\mathfrak{G}^*)^G$-modules

$$H^*_G(\mathbb{P}^n) \cong H^*(BG) \otimes H^*(\mathbb{P}^n).$$

More generally, we have the same isomorphism for arbitrary smooth projective varieties (with linear actions), such as Grassmannians and toric varieties.[12]

**Warning.** The above isomorphisms are *not* isomorphisms of rings!!

When $G$ is not connected, the $E_1$ term of the spectral sequence is the invariant subspace

$$[H^*(BG_0) \otimes H^*(M)]^{\pi_0(G)}$$

where $G_0$ is the identity component of $G$ and $\pi_0(G) = G/G_0$.

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[10]Obvioiusly, $\delta_r$ for $r \geq 1$ vanish.

[11]Whatever it is, this is a very weak requirement.

[12]Another way to obtain such an isomorphism is to use the *Leray spectral sequence*: When there is a fiber bundle $\pi : P \to B$ with fiber $F$ (i.e. $B$ is covered by open sets $U$ such that $\pi^{-1}(U) \cong U \times F$ and $\pi|_{\pi^{-1}(U)}$ is the projection to $U$) and $\pi_1(B) = 1$, there is a spectral sequences whose $E_2$-term is $E_2^{p,q} = H^p(B) \otimes H^q(F)$. If $F$ is a compact Kähler manifold, the Leray spectral sequence collapses at $E_2$ by Deligne's criterion.

## 3. Classifying maps and characteristic classes

3.1. **Classifying maps.** A *principal* $G$*-bundle* on a topological space $Y$ is a continuous map $\pi : X \to Y$ such that there is an open cover $\{U_i\}$ of $Y$ such that $\varphi_i : \pi^{-1}(U_i) \cong U_i \times G$ and over $U_i \cap U_j$, $(\varphi_j \circ \varphi_i^{-1})(y, g) = (y, a_{ij}(y)g)$ for continuous functions $a_{ij} : U_i \cap U_j \to G$. Such $\varphi_i$ are called *local trivialization* and $a_{ij}$ are called *transition functions*. The obvious (right) actions of $G$ on $U_i \times G$ glue to give us a free $G$-action on $X$ and we have $X/G = Y$.

Similarly, we define a complex vector bundle on $Y$ of rank $n$ as a continuous map $\pi : X \to Y$ such that there is an open cover $\{U_i\}$ of $Y$ such that $\varphi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$ and over $U_i \cap U_j$, $(\varphi_j \circ \varphi_i^{-1})(y, v) = (y, a_{ij}(y)v)$ for continuous functions $a_{ij} : U_i \cap U_j \to GL(n)$. Note that a complex vector bundle of rank $n$ and a principal $GL(n)$-bundle are given by the same data $\{a_{ij}\}$.

**Examples.** (1) When a compact Lie group $G$ acts on a manifold $X$ freely, the quotient space $X/G$ is also a manifold and the map $X \to X/G$ is a principal $G$-bundle (fibers are all $G$).

(2) Let $\pi : X \to Y$ be a principal $G$-bundle and let $\rho : G \to GL(n)$ be a representation. Then $X \times_G \mathbb{C}^n \to Y$ is a vector bundle of rank $n$ on $Y$. Conversely, given a vector bundle, the frame bundle (i.e. the set of all bases of fibers) gives us a principal bundle. In other words, $\{a_{ij}\}$ determine the principal $GL(n)$-bundle.

Topological classification of principal $G$-bundles is attained by means of the classifying space $BG$.

**Theorem 4.** Suppose $X$ is a free $G$-space. Then any section $s : X/G \to X_G = (X \times EG)/G$ (i.e. its composition with $X_G \to X/G$ is identity on $X/G$) uniquely determines a $G$-equivariant map[13] $h : X \to EG$ which makes the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & EG \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X/G & \xrightarrow[f]{} & BG. \end{array}$$

Any two sections are homotopic and the homotopy class of $(f, h)$ is independent of the section $s$.

**Proof (sketch):** An element of $X_G$ is an orbit $\{(gx, ge) \mid g \in G\} \subset Gx \times EG \subset X \times EG$ which is the graph of an equivariant map $Gx \to EG$. Hence

$$\sqcup_{Gx \in X/G} s(Gx) \subset X \times EG$$

determines an equivariant map $h : X \to EG$ given by

$$s(G \cdot x) = G \cdot (x, h(x)) \qquad \text{for all } x \in X.$$

Since the fibers of $X_G \to X/G$ is contractible $EG$, a section $s$ always exists and any two sections are homotopic.[14] □

**Corollary.** Let $E_1$ and $E_2$ be two contractible $G$-spaces. Then there exist $G$-equivariant maps $\phi : E_1 \to E_2$ and $\psi : E_2 \to E_1$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are

[13] A map $f : X \to Y$ of $G$-spaces is *equivariant* if $f(g \cdot x) = g \cdot f(x)$.

[14] An excellent reference for basics on principal bundles is Steenrod's classic book on fiber bundles.

homotopic to identity maps. In particular, $H^*(X \times_G E_1) \cong H^*(X \times_G E_2)$ for any $G$-space $X$ and $BG$ is unique up to homotopy equivalence.

3.2. **Characteristic classes.** By Whitehead's theorem, the homotopy class of a continuous map $f : Y \to BG$ is determined by the induced map $f^* : H^*(BG) \to H^*(Y)$.

(1) $G = U(n)$ (or $GL(n)$): A principal $G$-bundle is the same thing as a complex vector bundle of rank $n$ by taking orthonomal frames. From previous lecture, $H^*(BG)$ is freely generated by invariant polynomials $c_1 = \mathrm{tr}, c_2, c_3, \cdots, c_n = \det$. Hence, principal $G$-bundles (or vector bundles) on $Y$ are determined by the images of $c_1, \cdots, c_n$ in $H^*(Y)$. The image of $c_i$ is called the $i$-th *Chern class* of the principal bundle.
(2) $G = O(n)$: This case corresponds to real vector bundles of rank $n$. Principal $G$-bundles are determined by the images of $p_1, p_2, \cdots$ in $H^*(Y)$. We call them the *Pontryagin classes* of the principal bundle.
(3) $G = SO(2n)$: This case corresponds to real oriented vector bundles of rank $2n$. We have in addition the pullback of Pfaffian. This is called the *Euler class* $e$. By simple linear algebra, we have $c_n = e$ via $H^*(BSO(2n)) \to H^*(BU(n))$. The top Chern class is the Euler class!

3.3. **Whitney sum formula.** The total Chern class of a complex vector bundle $E \to Y$ of rank $n$ is defined as

$$c(E) = 1 + c_1(E) + \cdots + c_n(E).$$

Whitney sum formula says if $E = E_1 \oplus E_2$ then $c(E) = c(E_1) \cdot c(E_2)$, i.e. $c_k(E) = \sum_{i+j=k} c_i(E_1)c_j(E_2)$. In particular, if $E = L_1 \oplus \cdots \oplus L_n$ for $L_i$ line bundles (=vector bundles of rank 1), then $c(E) = \prod(1 + c_1(L_i))$.

**Example.** Let us calculate the Chern classes of the tangent bundle of complex projective space $\mathbb{P}^n$. By the famous Euler sequence

$$0 \to \underline{\mathbb{C}} \to L^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

where $\underline{\mathbb{C}} = \mathbb{P}^n \times \mathbb{C}$ is the trivial line bundle and $L \to \mathbb{P}^n$ is the line bundle whose fiber over a point $\xi$ in $\mathbb{P}^n$ is the dual space of the line in $\mathbb{C}^{n+1}$ represented by $\xi$. (Recall that $\mathbb{P}^n$ is the space of lines through $0$ in $\mathbb{C}^{n+1}$.) By Whitney's formula, we obtain

$$c(T_{\mathbb{P}^n}) = (1 + c_1(L))^{n+1} = \sum_k \binom{n+1}{k} \alpha^k$$

where $\alpha = c_1(L) \in H^2(\mathbb{P}^n)$ is the generator of $H^*(\mathbb{P}^n)$. In particular, $c_1(T_{\mathbb{P}^n}) = (n+1)\alpha$ and $e(T_{\mathbb{P}^n}) = c_n(T_{\mathbb{P}^n}) = (n+1)\alpha^n$. Note $\alpha^{n+1} = 0$.

3.4. **Equivariant characteristic classes.** Let $G$ and $K$ be compact Lie groups and $K \times G$ act on a manifold $P$. Suppose $K = K \times \{1_G\}$ acts freely on $P$. Then we get an induced action of $G$ on the quotient $X = P/K$ and the quotient map $P \to X$ is a $G$-equivariant principal $K$-bundle. So we obtain a principal $K$-bundle

$$P \times_G EG = P_G \to X_G = X \times_G EG$$

and its characteristic classes in $H^*_G(X) = H^*(X_G)$ are called the *equivariant characteristic classes* of the principal $K$-bundle $P \to X$. We will be only interested in the $K = U(n)$ case, i.e. complex vector bundles.

Let $\pi : E \to X$ be a complex vector bundle of rank $n$ which is $G$-equivariant. Then $E_G \to X_G$ is a vector bundle of rank $n$ and thus we obtain the *equivariant Chern classes*

$$c_i^G(E) := c_i(E_G) \in H^*(X_G) = H_G^*(X).$$

The *equivariant Euler class* is the top equivariant Chern class $e^G(E) = c_n^G(E)$.

**Example (point case).** The equivariant Chern classes are not trivial even if the bundle $E$ is trivial (with nontrivial group action). For example, consider a vector bundle $E \to pt$ over a point on which a group $G$ acts. This is nothing but a vector space $\mathbb{C}^n$ together with a homomorphism $G \to U(n) = K$. This homomorphism induces

$$\mathbb{R}[c_1, \cdots, c_n] \cong S(\mathfrak{K}^*)^K \to S(\mathfrak{G}^*)^G \cong H_G^*(pt).$$

The equivariant Chern classes of $E$ are the images of $c_j$ in $H_G^*(pt)$. If the $G$-action is not trivial, then the equivariant Chern classes are not trivial.

Furthermore, if $G$ is a torus, then any $\mathbb{C}$-representation $E$ of $G$ is the direct sum of 1-dimensional representations with weights $\alpha_j \in \mathfrak{G}^*$. The equivariant Chern classes are given by $\prod_j (1 + \alpha_j)$. More concretely, if $G = U(1)$ and $G$ acts on $E \cong \mathbb{C}^n$ as $e^{i\theta} \cdot (v_1, \cdots, v_n) = (e^{im_1\theta} v_1, \cdots, e^{im_n\theta} v_n)$, then the total equivariant Chern class is $\prod_{j=1}^n (1 + m_j t)$. In particular, the equivariant Euler class is nonzero iff $m_j \neq 0$ for all $j$.

3.5. **Equivariant integration.** For compact manifold $M$ acted on by a compact group $G$, we have a homomorphism

$$\int_M : H_G^*(M) \to H_G^*(pt) = H^*(BG)$$

by integrating the differential form part for any $d_G$-closed element in $\Omega_G(M) = [S(\mathfrak{G}^*) \otimes \Omega(M)]^G$. We call this homomorphism *equivariant integration.*

3.6. **Reduction to torus.** Let $G$ be a compact connected Lie group and $T$ be a maximal torus in $G$. For example, when $G = U(n)$ (resp. $SU(n)$), $T \cong U(1)^n$ (resp. $U(1)^{n-1}$). In many respects, tori are easier than arbitrary compact Lie groups. For a $G$-manifold $M$, we have a convenient isomorphism

$$H_G^*(M) \cong H_T^*(M)^W \subset H_T^*(M)$$

where $W = N(T)/T$ is the Weyl group. For instance, $W = S_n$ (resp. $S_{n-1}$ for $G = U(n)$ (resp. $SU(n)$).

The above isomorphism follows from the Leray spectral sequence associated to the map $M_T = M \times_T EG \to M \times_G EG = M_G$ whose fiber is $G/T$. The spectral sequence collapses at $E_2$ and thus $H_T^*(M) \cong H_G^*(M) \otimes H^*(G/T)$. Then take $W$-invariant subspace.

When $M$ is a point, we obtain

$$H^*(BG) \cong S(\mathfrak{G}^*)^G \cong S(\mathfrak{T}^*)^W \cong H^*(BT)^W \subset H^*(BT).$$

## 4. Localization theorem

In this section, $G$ is a torus $U(1)^r$.

4.1. **Cohomology with compact support.** Let $M$ be a manifold and $X$ be a closed submanifold. The relative cohomology $H^*(M, X)$ also has a de Rham theoretic interpretation: Let $\Omega(M-X)_c \subset \Omega(M-X)$ be the subcomplex of differential forms with compact support. We call the cohomology of this subcomplex *cohomology with compact support* and denote it by $H^*(M-X)_c$. Then we have an isomorphism

$$H^*(M, X) \cong H^*(M-X)_c.$$

and a long exact sequence

$$\cdots \to H^k(M-X)_c \to H^k(M) \to H^k(X) \to H^{k+1}(M-X)_c \to \cdots$$

When there is an action of $G$, we have an obvious analogue for the equivariant cohomology (by approximation by finite dimensional manifolds)

$$\cdots \to H^k_G(M-X)_c \to H^k_G(M) \to H^k_G(X) \to H^{k+1}_G(M-X)_c \to \cdots .$$

4.2. **Support of $S(\mathfrak{G}^*)$-module.** Recall that $S(\mathfrak{G}^*)$ is isomorphic to the polynomial ring $R = \mathbb{R}[t_1, \cdots, t_r] = H^*(BG)$ in $r$ variables and every $G$-equivariant cohomology is an $S(\mathfrak{G}^*)$-module. Suppose $A$ is a finitely generated $R$-module and let $I_A$ be the annihilator ideal $I_A = \{f \in R \,|\, fA = 0\}$. Then the *support* of $A$ is defined as

$$\text{supp}A = \{x \in \mathbb{C}^n \,|\, f(x) = 0 \text{ for all } f \in I_A\}$$

Suppose $A \to B \to C$ is an exact sequence of $R$-modules. Then $I_B \supset I_A \cap I_C$ and hence $\text{supp}B \subset \text{supp}A \cup \text{supp}C$. (If not, $\exists x \in \text{supp}B$ such that $f(x) \neq 0$, $g(x) \neq 0$ for some $f \in I_A$, $g \in I_C$. But then $fg \in I_A \cap I_C \subset I_B$ and $fg$ does not vanish at $x$. Contradiction.)

4.3. **Localization theorem.** Let $M$ be a compact $G$-manifold. Our goal is to prove

**Theorem 5.** Let $M^G$ be the $G$-fixed point set and $\imath : M^G \to M$ be the inclusion. If $M$ is equivariantly formal $H^*_G(M) \cong H^*(M) \otimes_{\mathbb{R}} R$, then $\imath^* : H^*_G(M) \to H^*_G(M^G)$ is injective.

**Lemma.** Let $K$ be a closed subgroup of $G$ and $\phi : M \to G/K$ be a $G$-equivariant map. Then $\text{supp}H^*_G(M)$ and $\text{supp}H^*_G(M)_c$ are contained in $\mathfrak{K}^{\mathbb{C}} := \mathfrak{K} \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{C}^n = \mathfrak{G}^{\mathbb{C}}$.

**Proof:** From $M \xrightarrow{\phi} G/K \to pt$, we get homomorphisms of rings

$$R = H^*_G(pt) \to H^*_G(G/K) = S(\mathfrak{K}^*) \to H^*_G(M).$$

Hence the $R$-module structure on $H^*_G(M)$ comes from the $S(\mathfrak{K}^*)$-module structure and hence $I_{H^*_G(M)}$ contains all polynomials vanishing on $\mathfrak{K}$. □

For $x \in M$ and any $G$-invariant tubular neighborhood $U$,[15] we have the obvious projection $U \to Gx = G/K$, where $K = G_x$ is the stabilizer. By the lemma above, we see that the support of $H^*_G(U)$ is contained in $\mathfrak{K}^{\mathbb{C}} \subset \mathbb{C}^n$.

[15]Invariant tubular nbd always exists for compact $G$ by slice theorem.

Suppose $M$ is compact. Then there are only finitely many closed subgroups which may be the stabilizer group of a point (Mostow's theorem). Let $\{K_i \mid i = 1, 2, \cdots, l\}$ be the list of stabilizers.

**Lemma.** The kernel of $\iota^* : H^*_G(M) \to H^*_G(M^G)$ has support in the set $\cup_i \mathfrak{K}_i^{\mathbb{C}}$.

**Proof:** By the long exact sequence

$$H^k_G(M - M^G)_c \to H^k_G(M) \to H^k_G(M^G) \to H^{k+1}_G(M - M^G)_c,$$

it suffices to prove that the support of $H^*_G(M - M^G)_c$ is contained in $\cup_i \mathfrak{K}_i^{\mathbb{C}}$.

Delete a small tubular neighborhood $U$ of $M^G$ so that $M - U$ is compact. Then we can cover $M - \bar{U}$ by invariant tubular neighborhoods $U_j$ of finitely many orbits $Gx_j$ with stabilizer group $G_{x_j} = K_i$ for some $i$. By the Mayer-Vietrois sequence

$$H^{k-1}_G(V_1 \cap V_2) \to H^k_G(V_1 \cup V_2) \to H^k_G(V_1) \oplus H^k_G(V_2)$$

and induction, we see that $H^*_G(M - \bar{U}) \cong H^*_G(M - M^G)$ has support in $\cup_i \mathfrak{K}_i^{\mathbb{C}}$. □

When $M$ is equivariantly formal, $H^*_G(M) \cong R \otimes H^*(M)$ and so $H^*_G(M)$ is a free $R$-module. On the other hand, since $G$ acts trivially on $M^G$, $H^*_G(M^G)$ is also a free $R$-module. Now the support of the kernel of $\iota^*$ is contained in $\cup_i \mathfrak{K}_i^{\mathbb{C}}$ and thus the kernel is a torsion submodule of $H^*_G(M)$. Since $H^*_G(M)$ is free, the torsion submodule has to be zero, i.e. $\iota^*$ is injective. □

**Example.** Suppose $G = U(1)$ acts on $M = \mathbb{P}^n$ with distinct weights $m_0, \cdots m_n$, i.e. $e^{i\theta} \cdot (x_0 : x_1 : \cdots : x_n) = (e^{im_0\theta}x_0 : \cdots : e^{im_n\theta}x_n)$. The fixed point set $M^G$ consists of $(n+1)$ points, $p_0, \cdots, p_n$ where $p_i = (0 : \cdots : 0 : 1 : 0 \cdots : 0)$ with 1 at $i$-th place. As a $R = \mathbb{R}[t]$-module, $H^*_G(M) = R \otimes H^*(\mathbb{P}^n)$ is a free $R$-module of rank $n+1$. Let $\xi = c_1(L)$ be the generator of $H^*(\mathbb{P}^n)$. Its restriction to the point $p_i$ is $-m_i t$ because the restriction of $L$ to $p_i$ has weight $-m_i$. By the localization theorem, $\prod_i(\xi + m_i t)$ is a relation in $H^*_G(\mathbb{P}^n)$ and this is the only relation because $R[\xi]/(\prod(\xi + m_i t))$ is already a free module of rank $n+1$. In summary, we obtained an isomorphism of rings

$$H^*_{U(1)}(\mathbb{P}^n) = \mathbb{R}[t, \xi]/(\prod_{i=0}^{n}(\xi + m_i t)).$$

4.4. **Integration formula.** As an application of the localization theorem, we obtain a useful integration formula, which basically says all integrals are *concentrated at the fixed point set.*

Let $\mathcal{F}$ be the set of fixed point components of the $G$-action on $M$. Each $F \in \mathcal{F}$ is a submanifold and suppose the normal bundle $N_F$ to $F$ is oriented of even rank. In this case, we have the *Gysin map*[16] $\iota_* : H^{k-c}_G(F) \to H^k_G(M)$ where $c$ is the real codimension of $F$. When composed with the restriction $\iota^* : H^k_G(M) \to H^k_G(F)$, it gives us multiplication by the equivariant Euler class of the normal bundle

$$\iota^* \circ \iota_* = e(N_F) : H^{k-c}_G(F) \to H^k_G(F).$$

Recall that we have the equivariant integration $\int_M : H^*_G(M) \to H^*_G(pt) = H^*(BG)$ for compact $M$.

[16] This is obtained by multiplying a differential form $\alpha$ supported in a small neighborhood of $F$ such that $\int_M \eta \wedge \alpha = \int_F \eta$ for all closed differential form $\eta$ on $F$.

**Theorem 6.** For $\xi \in H^*_G(M)$, we have
$$\int_M \xi = \sum_{F\in\mathcal{F}} \int_F \frac{\xi|_F}{e(N_F)}.$$

**Proof:** For $\xi \in H^*_G(M)$, since $\iota^*\iota_*$ is multiplication of $e(N_F)$ for each $F \in \mathcal{F}$,
$$\iota^*[\sum_{F\in\mathcal{F}} \iota_* \frac{\xi|_F}{e(N_F)} - \xi] = \sum_{F\in\mathcal{F}} \iota^*\iota_*\xi|_F/e(N_F) - \iota^*\xi = \xi|_F - \xi|_F = 0.$$
Because $\iota^*$ is injective, we find
$$\xi = \sum_{F\in\mathcal{F}} \iota_* \frac{\xi|_F}{e(N_F)}$$
and therefore
$$\int_M \xi = \sum_{F\in\mathcal{F}} \int_M \iota_* \frac{\xi|_F}{e(N_F)} = \sum_{F\in\mathcal{F}} \int_F \frac{\xi|_F}{e(N_F)}$$
as desired. □

Theorem 6 is the most important tool for Gromov-Witten theory and geometry of symplectic quotients.

**Wisdom of the area.** For many problems with equivariant cohomology, two things are quite useful:

(1) reduction to torus ($H^*_G(M) = H^*_T(M)^W$, Martin's trick,...),
(2) reduction to fixed point set (localization theorem).

Good luck!!

## References

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