Contraction Coefficients for Noisy Quantum Channels

Mary Beth Ruskai
ruskai@member.ams.org

joint work with F. Hiai

Daejeon February, 2016

Recall Relative Entropy

$\Phi : M_d \mapsto M_{d'}$ Quantum channel , i.e.,

completely positiv , trace preserving (CPT) map

Relative entropy $H(\rho, \gamma) = \mathrm{Tr}\left(\rho \log \rho - \rho \log \gamma\right)$

$\rho, \gamma \geq 0 \qquad \mathrm{Tr}\, \rho = \mathrm{Tr}\, \gamma = 1$

Basic property: Contracts under action of quantum channel, i.e.,

$H\big[\Phi(\rho), \Phi(\gamma)\big] \leq H(\rho, \gamma)$

Maximal contraction – measure of noisiness of channel Φ

$$\eta(\Phi) = \sup_{\rho \neq \gamma} \frac{H\big[\Phi(\rho), \Phi(\gamma)\big]}{H(\rho, \gamma)}$$

Many other quantities contract – compare contraction properties

Basic Notation for Operators on M_d

$\widehat{\Phi}$ denote adjoint wrt Hilbert-Schmidt inner prod $\langle P, Q \rangle = \mathrm{Tr}\, P^* Q$

i.e., $\quad \mathrm{Tr}\, [\Phi(P)]^* Q = \mathrm{Tr}\, P^* \widehat{\Phi}(Q)$

Density matrices $\mathcal{D} \equiv \{P \in M_d : P \geq 0, \mathrm{Tr}\, P = 1\}$

Tangent space $\{A = M_d : A = A^*, \; \mathrm{Tr}\, A = 0\}$

Def. Left and Right mult are linear operators on M_d

$$L_P(X) = PX \qquad \text{and} \qquad R_Q(X) = XQ$$

a) L_P and R_Q commute $L_P[R_Q(X)] = PXQ = R_Q[L_P(X)]$

b) $P = P^* \;\Rightarrow\; L_P, R_P$ self-adjoint wrt H-S inner prod

c) $P, Q > 0$ pos def $\;\Rightarrow\; L_P, R_P$ pos def, e.g $\mathrm{Tr}\, X^* P X \geq 0$

d) $P, Q > 0 \;\Rightarrow\; (L_P)^{-1} = L_{P^{-1}}, \quad (R_Q)^{-1} = R_{Q^{-1}}$

e) $f(L_P) = L_{f(P)}$, etc. $\quad \log R_Q = R_{\log Q}$ when $Q > 0$

Generalized Relative Entropy

Petz (1986) defined "quasi-entropy"

a.k.a. "generalized realtive entropy", "f-divergence"

$$\mathcal{G} = \{g : (0, \infty) \mapsto \mathbf{R} \mid \text{operator convex, } g(1) = 0\}$$

$$H_g(K, P, Q) \equiv \operatorname{Tr} \sqrt{Q} K^* \, g\big(L_P R_Q^{-1}\big)(K\sqrt{Q})$$

Thm: $H_g(K, P, Q)$ jointly convex in P, Q

Thm: $H_g\big[K, \Phi(P), \Phi(Q)\big] \leq H_g(\widehat{\Phi}(K), P, Q)$

$$g(x) \in \mathcal{G} \Leftrightarrow \widetilde{g}(x) = x\, g(x^{-1}) \in \mathcal{G} \qquad \widetilde{\widetilde{g}} = g$$

$$H_{\widetilde{g}}(K, P, Q) = H_g(K^*, Q, P)$$

$g(x) = x \log x \qquad H_g(I, P, Q) = \operatorname{Tr} P\,(\log P - \log Q)$

$\widetilde{g}(x) = -\log x \qquad \widetilde{H}_g(I, P, Q) = \operatorname{Tr} Q(\log Q - \log P)$

Recover WYD Entropy

$$g_t(x) = \begin{cases} \frac{1}{t(1-t)}(x - x^t) & t \neq 1 \\ x \log x & t = 1 \end{cases}. \qquad t \in (0,2]$$

$$\widetilde{g}_t(x) = x\, g_t(x^{-1}) = \begin{cases} \frac{1}{t(1-t)}(1 - x^t) & t \neq 0 \\ -\log x & t = 0 \end{cases} \qquad t \in [-1,1)$$

$$J_t(K,P,Q) \equiv \mathrm{Tr}\,\sqrt{Q}K^*\, g_t\big(L_P R_Q^{-1}\big)(K\sqrt{Q}) \qquad t \in [-1,2]$$

$$= \frac{1}{t(1-t)}\big(\mathrm{Tr}\,K^*PK - \mathrm{Tr}\,K^*P^tKQ^{1-t}\big)$$

$$J_1(K,P,Q) = \mathrm{Tr}\,KK^*P\log P - \mathrm{Tr}\,K^*PK\log Q$$

$$J_0(K,P,Q) = \mathrm{Tr}\,K^*K\,Q\log Q - \mathrm{Tr}\,KQK^*\log P$$

Recover both WYD entropy with linear term and $H(P,Q)$, $K = I$

Riemannian metrics or Fisher information

Henceforth consider only $K = I$ and write $H_g(P, Q)$

$$-\frac{\partial^2}{\partial a \partial b} H_g(P + aA, P + bB, I)\Big|_{a=b=0} = \mathrm{Tr}\, A\, \Omega_P^\kappa(B) = \langle A, \Omega_P^\kappa(B)\rangle$$

for $\mathrm{Tr}\, A = \mathrm{Tr}\, B = 0$ in LHS, get pos quad form which is RHS with

$$\Omega_P^\kappa(X) \equiv R_P^{-1} \kappa\big(L_P R_P^{-1}\big)\, X = L_P^{-1} \kappa\big(R_P L_P^{-1}\big)$$

$$\mathcal{K} = \{\kappa : (0, \infty) \mapsto (0, \infty) \mid \kappa \text{ op convex }, \kappa\big(x^{-1}\big) = x\kappa(x)\}$$

with $\kappa(x) = \frac{g(x) + x g(x^{-1})}{(1-x)^2} = \frac{g_{\mathrm{sym}}(x)}{(1-x)^2} \in \mathcal{K}$ relate H_g and M_P^κ

$\langle A, \Omega_P^\kappa(B)\rangle \equiv M_P^\kappa(A, B)$ is Riemmanian metric

Ω_P non-commutative multiplication by P^{-1}

$\langle A, \Omega_P^\kappa(A)\rangle$ quantum analogue of Fisher information

Examples

	$g(x)$	$\widetilde{g}(x)$	$\kappa(x)$	$\Omega_\rho(X)$
RelEnt	$x \log x$	$-\log x$	$\frac{\log x}{x-1}$	$\int_0^\infty \frac{1}{\rho+uI} X \frac{1}{\rho+uI}\, du$
WY	$4(1-\sqrt{x}\,)$	$4(x-\sqrt{x}\,)$	$\frac{4}{(1+\sqrt{x}\,)^2}$	$\frac{4}{\left(\sqrt{L_P}+\sqrt{R_P}\right)^2}(X)$
	$x^{-1/2}-x^{1/2}$		$x^{-1/2}$	$\rho^{-1/2} X \rho^{-1/2}$
max	$\frac{1}{2}(x^{-1}-1)$	$\frac{1}{2}(x^2-x)$	$\frac{1+x}{2x}$	$\frac{1}{2}\left(X\rho^{-1}+\rho^{-1}X\right)$
min			$\frac{2}{1+x}$	$\frac{2}{L_\rho+R_\rho}(X)$

Summarize

$$\mathcal{G} = \{g : (0,\infty) \mapsto \mathbf{R} \mid \text{operator convex}, \ g(1) = 0\}$$

$$\mathcal{G}_{\text{sym}} = \{g : (0,\infty) \mapsto \mathbf{R} \mid \text{op convex}, \ g(1) = 0, \ xg(x^{-1}) = g(x)\}$$

$$g \in \mathcal{G} \Rightarrow \widetilde{g}(x) = xg(x^{-1}) \in \mathcal{G} \Rightarrow g_{\text{sym}} = g(x) + \widetilde{g}(x) \in \mathcal{G}_{\text{sym}}$$

There is a 1-1 correspondence between

a) sym rel entropy $H_{g_{\text{sym}}}(P,Q) = H_{g_{\text{sym}}}(Q,P), \quad g_{\text{sym}} \in \mathcal{G}_{\text{sym}}$

b) op convex $g \in \mathcal{G}_{\text{sym}}$ with $g''(1) = 2$

c) op convex $\kappa : (0,\infty) \mapsto (0,\infty)$ with $\kappa(x^{-1}) = x\kappa(x), \kappa(1) = 1$

d) monotone Riemannian metrics

Any $g \in \mathcal{G}$ gives mono metric with $\kappa(x) = \frac{g(x)+xg(x^{-1})}{(x-1)^2} = \frac{g_{\text{sym}}(x)}{(x-1)^2}$

Geodesic distance

Define geodesic distance for each $\kappa \in \mathcal{K}$

$$D_\kappa(P, Q) \equiv \inf_{\xi(t)} \int_0^1 \sqrt{\mathrm{Tr}\, \xi'(t)\, \Omega_{\xi(t)} \xi'(t)} dt$$

where $\xi(t)$ smooth path in $\mathcal{D}$ with $\xi(0) = P,\ \xi(1) = Q$

Remark: Restriction to $\mathrm{Tr}\, \xi(t) = 1$ here typically gives larger distance than geodesic with $\xi(t)$ path over all pos matrices.

Trace distance $\|P - Q\|_1 = \mathrm{Tr}\, |P - Q|$ contracts under CPT maps

Contraction Theorems

All of above decrease under quantum channels, i.e.,

for any CPT map Φ and for all $P, Q \in \mathcal{D}$ and $\mathrm{Tr}\, A = 0$

Thm: $H_g[\Phi(P), \Phi(Q)] \leq H_g(P, Q) \qquad \forall\, g \in \mathcal{G}$

Thm: $\mathrm{Tr}\, \Phi(A)\, \Omega^{\kappa}_{\Phi(P)} \Phi(A) \leq \mathrm{Tr}\, A\, \Omega^{\kappa}_{P}(A) \qquad \forall\, \kappa \in \mathcal{K}$

Thm: $D_\kappa[\Phi(P), \Phi(Q)] \leq D_\kappa(P, Q) \qquad \forall\, k \in \mathcal{K}$

Thm: $\|\Phi(P) - \Phi(Q)\|_1 \leq \|P - Q|\|_1$ Only need Φ pos

$\kappa(x) = \frac{g(x) + x g(x^{-1})}{(1-x)^2} = \frac{g_{\mathrm{sym}}(x)}{(1-x)^2}$ op conv and $\kappa(x^{-1}) = x\kappa(x)$

Definition of Contraction Coefficients

$$\eta_g(\Phi)^{\mathrm{RelEnt}} \equiv \sup_{P \neq Q \in \mathcal{D}} \frac{H_g[\Phi(P), \Phi(Q)]}{H_g(P, Q)}$$

$$\eta_\kappa(\Phi)^{\mathrm{Riem}} \equiv \sup_{P \in \mathcal{D}} \sup_{\mathrm{Tr} A = 0} \frac{\langle \Phi(A)\, \Omega^k_{\Phi(P)} \Phi(A) \rangle}{\langle A\, \Omega^k_P(A) \rangle}$$

$$\eta_\kappa(\Phi)^{\mathrm{geod}} \equiv \sup_{P \neq Q \in \mathcal{D}} \frac{D_\kappa[\Phi(P), \Phi(Q)]}{D_\kappa(P, Q)}$$

$$\eta^{\mathrm{Tr}}(\Phi) \equiv \sup_{P \neq Q \in \mathcal{D}} \frac{\|\Phi(P) - \Phi(Q)\|_1}{\|P - Q\|_1}$$

Remark Classical: $\eta^{\mathrm{geod}}(\Phi) = \eta^{\mathrm{Riem}}(\Phi) = \eta^{\mathrm{RelEnt}}(\Phi) \leq \eta^{\mathrm{Tr}}(\Phi)$

Def. $H_g(p, q)$ for any $g : (0, \infty) \mapsto \mathbf{R}$ convex with $g(1) = 0$.

Only one Fisher information.

General Results

Recall $\kappa(x) = \frac{g(x)+xg(x^{-1})}{(1-x)^2} = \frac{g_{\rm sym}(x)}{(1-x)^2}$

Thm: $\eta_\kappa(\Phi)^{\rm geod} = \eta_\kappa(\Phi)^{\rm Riem} \leq \eta_{g_{\rm sym}}^{\rm RelEnt}(\Phi) \leq \eta_g^{\rm RelEnt}(\Phi) \leq 1$

$=$ follows from Hiai-Petz $\lim_{\epsilon \searrow 0} \frac{D_\kappa(\rho, \rho + \epsilon A)}{\epsilon} = \sqrt{\langle A, \Omega_\rho^\kappa(A)\rangle}$

other proofs straightforward

Thm: $\eta_\kappa^{\rm Riem}(\Phi) \geq \sqrt{\eta^{\rm Tr}(\Phi)}$

Conj: $\eta_\kappa^{\rm Riem}(\Phi) \leq \eta^{\rm Tr}(\Phi) \quad \forall\ \kappa$ False in general

Thm: $\eta_\kappa^{\rm Riem}(\Phi) \leq \eta^{\rm Tr}(\Phi)$ for $\kappa(x) = x^{-1/2}$

Integral Representations

$$g(x) = g'(1)(x-1) + c(x-1)^2 + \int_0^\infty \frac{(x-1)^2}{x+s}\, d\mu(s)$$

$$H_g(\rho,\gamma) = c\,\mathrm{Tr}\,(\rho-\gamma)^2\gamma^{-1} + \int_0^\infty \mathrm{Tr}\,(\rho-\gamma)\frac{1}{L_\rho + sR_\gamma}(\rho-\gamma)\, d\mu(s)$$

$$\kappa(x) = \int_0^1 \left(\frac{1}{x+s} + \frac{1}{sx+1}\right)\frac{1+s}{2}\, dm(s), \qquad \int_0^1 dm(s) = 1$$

$$\mathrm{Tr}\, A\,\Omega_P^\kappa(A) = \int_0^\infty \mathrm{Tr}\, A\left(\frac{1}{L_P + sR_P} + \frac{1}{R_P + sL_P}\right)A\,\tfrac{1+s}{2}\, dm(s)$$

$\Rightarrow$ Often suffices to prove bounds for $\mathrm{Tr}\, A\frac{1}{L_\rho + sR_\gamma}(A)$

Reformulation of $\eta_\kappa(\Phi)^{\mathrm{Riem}}$ as eigenvalue problem

Use HS inner product $\langle X, Y\rangle = \mathrm{Tr}\, X^* Y$ and $\widehat{\Phi}$ denote adjoint

Eigenvalue problem $\quad \widehat{\Phi} \circ \Omega^\kappa_{\Phi(P)} \circ \Phi(X) = \lambda \Omega^\kappa_P(X)$

$$\langle X, \widehat{\Phi} \circ \Omega^\kappa_{\Phi(P)} \circ \Phi(X)\rangle \leq \langle X, \Omega^\kappa_P(X)\rangle \quad \Rightarrow \quad \lambda \leq 1$$

$X = P = (\Omega^\kappa_P)^{-1}(I)$ e-vec to largest e-val 1

By max-min principle $\quad \lambda_2(\Phi, P) = \sup_{\mathrm{Tr}\, A=0} \dfrac{\langle \Phi(A)\, \Omega^\kappa_{\Phi(P)} \Phi(A)\rangle}{\langle A\, \Omega^\kappa_P(A)\rangle}$

$$\eta_\kappa^{\mathrm{Riem}}(\Phi) = \sup_{P \in \mathcal{D}} \lambda_2(\Phi, P) \quad = \sup_{P \in \mathcal{D}} \sup_{\mathrm{Tr}\, A=0} \frac{\langle \Phi(A)\, \Omega^\kappa_{\Phi(P)} \Phi(A)\rangle}{\langle A\, \Omega^\kappa_P(A)\rangle}$$

Remark: $(\Omega^\kappa_P)^{-1}$ Non-Comm. mult by P BUT $\neq \Omega^\kappa_{P^{-1}}$

$$(\Omega^\kappa_P)^{-1} = \Omega^{1/\kappa(x^{-1})}_{P^{-1}} \qquad \kappa(x) \in \mathcal{K} \;\Leftrightarrow\; 1/\kappa(x^{-1}) \in \mathcal{K}$$

Eigenvalue (cont.)

Equiv. Prob $\left[(\Omega_P^\kappa)^{-1} \circ \widehat{\Phi} \circ \Omega_{\Phi(P)}^\kappa\right] \Phi(X) = \lambda_2(\Phi, P) X$

Recall Υ positivity and trace preserving implies $\|\Upsilon(Y)\|_1 \leq \|Y\|_1$

Apply $\Upsilon_P^\kappa \equiv (\Omega_P^\kappa)^{-1} \circ \widehat{\Phi} \circ \Omega_{\Phi(P)}^\kappa$ get $\lambda_2(\Phi, P)\|X\|_1 \leq \|\Phi(X)\|_1$

$$\eta_\kappa^{\rm Riem}(\Phi) = \sup_{P \in \mathcal{D}} \lambda_2(\Phi, P) \leq \eta^{\rm Tr}(\Phi)$$

But Υ_P^κ is positive $\forall\ \Phi$, $\forall\ P$ only for $\kappa(x) = x^{-1/2} = 1/\kappa(x^{-1})$

$\kappa(x) = x^{-1/2}$ important in mixing times of Markov chains

Thm: $\eta_{x^{-1/2}}^{\rm Riem}(\Phi) \leq \eta^{\rm Tr}(\Phi) \quad \forall\ \Phi$

Conj: $\eta_\kappa^{\rm Riem}(\Phi) \leq \eta^{\rm Tr}(\Phi)$ False for CQ qubit channel

Aside on partial order for $\kappa \in \mathcal{K}$ and CP of Ω_P^κ

Def: $\kappa_1 \preccurlyeq \kappa_2$ if $\kappa_1(e^t)/\kappa_2(e^t)$ has positive Fourier transform

Equiv cond: matrix with els $\frac{\kappa_1(\lambda_j/\lambda_k)}{\kappa_2(\lambda_j/\lambda_k}$ is pos semi-def

Positive and C.P. equiv for Ω_P^κ because

$L_P R_P^{-1}(X)$ is Schur product $x_{jk} \mapsto \lambda_j \lambda_k^{-1} x_{jk}$ in e-vec basis

Thm: a) Ω_P^κ is C.P. $\Leftrightarrow$ $\kappa(x) \preccurlyeq x^{-1/2}$

b) $\big(\Omega_P^\kappa\big)^{-1}$ is C.P. $\Leftrightarrow$ $x^{-1/2} \preccurlyeq \kappa(x)$

$\Upsilon_P^\kappa \equiv (\Omega_P^\kappa)^{-1} \circ \widehat{\Phi} \circ \Omega_{\Phi(P)}^\kappa$ positive $\forall\, \Phi, P$ only for $\kappa(x) = x^{-1/2}$

Hiai, Kosaki, Petz & Ruskai:
analyze partial order for many families using these conds

Summary

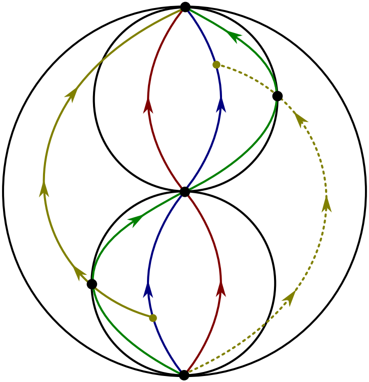

Figure: Diagram of families in $\mathcal{K}$ parameterized to increase in $\preccurlyeq$ order with the lower ball for $\mathcal{K}^+$ and the upper $\mathcal{K}^-$. The red curve describes the Heinz family k_α^{H} $(0, \frac{1}{2})$ and $\widetilde{k}_\alpha^{\mathrm{H}}$ $(\frac{1}{2}, 1)$; the blue curve the binomial family $k_{-\alpha}^{\mathrm{B}}$ $(-1, 1)$; the green curve the power difference family $k_{-\alpha}^{\mathrm{PD}}$ $(-2, 1)$. The left brown curve k_t^{WYD} with $t \in [\frac{1}{2}, 2]$ and the right dotted brown curve the dual $\widetilde{k}_t^{\mathrm{WYD}}$. Note crossings at $\frac{4}{1+\sqrt{x})^2}$ and $\frac{\log x}{x-1}$

Special Results for log

$$\Omega_P(A) = \int_0^\infty \frac{1}{P+uI} A \frac{1}{P+uI} du$$

$$\big(\Omega_P\big)^{-1}(X) = \int_0^1 P^t X P^{1-t} dt$$

Follows from $\frac{d}{dt} P^t X P^{1-t} = P^t \log\big(L_P R_P^{-1}\big) P^{1-t}$

Thm: $\eta_\kappa^{\mathrm{Riem}}(\Phi) = \eta_{g_{\mathrm{sym}}}^{\mathrm{RelEnt}}(\Phi) \le \eta_g^{\mathrm{RelEnt}}(\Phi) = \eta_{\widetilde{g}}^{\mathrm{RelEnt}}(\Phi)$

$$\kappa(x) = \frac{\log x}{x-1} \qquad \widehat{\kappa}(x) = 1/\kappa(x^{-1}) = \frac{x-1}{x \log x}$$

$$g_{\mathrm{sym}}(x) = (x-1)\log x \qquad g(x) = x \log x \qquad \widetilde{g}(x) = -\log x$$

Qubit channels

Bloch sphere representation $\rho = \frac{1}{2}\left[I + \mathbf{w}\cdot\sigma\right] = \frac{1}{2}\left[I + \sum_k w_k \sigma_k\right]$

linear TP map $\Phi : \left[I + \mathbf{w}\cdot\sigma\right] \mapsto I + \sum_k (\tau_k + \alpha_k w_k)\sigma_k$

$$\text{rep in Pauli basis} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_1 & \alpha_1 & 0 & 0 \\ \tau_2 & 0 & \alpha_2 & 0 \\ \tau_3 & 0 & 0 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tau & T \end{pmatrix}$$

- Unital $\tau = 0$ positivity preserving iff $|\alpha_k| \leq 1 \ \forall \ k$

  CP cond $(\alpha_1 \pm \alpha_2)^2 \leq (1 \pm \alpha_3)^2$ but not needed

- non-unital CQ $\Phi : \left[I + \mathbf{w}\cdot\sigma\right] \mapsto I + \alpha\sigma_1 + \tau w_3 \sigma_3$

  CP and positivity conditions coincide $\alpha^2 + \tau^2 \leq 1$

Unital qubit results

$$\begin{aligned}
\eta_\kappa(\Phi)^{\rm geod} &= \eta_\kappa(\Phi)^{\rm Riem} = \eta_{g_{\rm sym}}^{\rm RelEnt}(\Phi) = \eta_g^{\rm RelEnt}(\Phi) \\
&= \|T\|_\infty^2 = \max_j \alpha_j^2 \qquad \forall\, \kappa, g \\
&\leq \eta^{\rm Tr}(\Phi) = \|T\|_\infty = \max_j |\alpha_j|
\end{aligned}$$

Same as classical case.

Proofs convert action of Φ to linear op T on $\mathbf{R}_3$

and exploits fact that for $\Gamma_{\mathbf{w}}$ pos lin op on $\mathbf{R_3}$

$$\sup_{\mathbf{y}\in\mathbf{R}^3} \frac{\langle \mathrm{T}\mathbf{y}, \Gamma_{\mathrm{T}\mathbf{w}}^{-1} \mathrm{T}\mathbf{y}\rangle}{\langle \mathbf{y}, \Gamma_{\mathbf{w}}^{-1}\mathbf{y}\rangle} = \sup_{\mathbf{y}\in\mathbf{R}^3} \frac{\langle \mathrm{T}^*\mathbf{y}, \Gamma_{\mathbf{w}} \mathrm{T}^*\mathbf{y}\rangle}{\langle \mathbf{y}, \Gamma_{\mathrm{T}\mathbf{w}}\mathbf{y}\rangle}.$$

Non-unital CQ Qubit Results

$$\left[\Phi : I + \mathbf{w}\cdot\sigma\right] \mapsto I + \alpha w_1 \sigma_1 + \tau \sigma_3$$

$$\eta^{\mathrm{Riem}}_{\kappa_s}(\Phi) = \frac{\alpha^2}{1 - \left(\frac{1-s}{1+s}\right)^2 \tau^2}, \qquad 0 \leq s \leq 1$$

$\kappa_s(x) \equiv \frac{1+s}{2}\left(\frac{1}{x+s} + \frac{1}{1+sx}\right), \qquad 0 \leq s \leq 1$ extreme points of $\mathcal{K}$

$\Rightarrow \ \eta^{\mathrm{Riem}}_{\kappa}$ depends non-trivially on κ

Thm: $\exists$ range of s, α, τ such that second inequality is strict in

$$\eta^{\mathrm{RelEnt}}_{g_s}(\Phi_{\alpha,\tau}) \geq \eta^{\mathrm{RelEnt}}_{(g_s)_{\mathrm{sym}}}(\Phi_{\alpha,\tau}) > \eta^{\mathrm{Riem}}_{\kappa_s}(\Phi_{\alpha,\tau})$$

Non-unital CQ Qubit Example (cont.)

$$\Phi : \big[I + \mathbf{w}\cdot\sigma\big] \mapsto I + \alpha\sigma_1 + \tau w_3\sigma_3 \qquad \alpha^2 + \tau^2 \leq 1$$

$$\begin{aligned}
\eta^{\mathrm{Tr}}(\Phi) &= \alpha, \\
\eta^{\mathrm{Riem}}_{\max}(\Phi) &= \frac{\alpha^2}{1-\tau^2}, \\
\eta^{\mathrm{Riem}}_{\widehat{\mathrm{WY}}}(\Phi) \equiv \eta^{\mathrm{Riem}}_{(1+\sqrt{x})^2/4x}(\Phi) &\geq \alpha^2\,\frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)}, \\
\eta^{\mathrm{Riem}}_{x^{-1/2}}(\Phi) &\geq \frac{\alpha^2}{\sqrt{1-\tau^2}}, \\
\eta^{\mathrm{Riem}}_{(\log x)/(x-1)}(\Phi) &\geq \frac{\alpha^2}{2\tau}\log\tfrac{1+\tau}{1-\tau}, \\
\eta^{\mathrm{Riem}}_{\mathrm{WY}}(\Phi) \equiv \eta^{\mathrm{Riem}}_{4/(1+\sqrt{x})^2}(\Phi) &= \frac{2\alpha^2}{1+\sqrt{1-\tau^2}}, \\
\eta^{\mathrm{Riem}}_{\min}(\Phi) &= \alpha^2.
\end{aligned}$$

Implications of CQ Results

Recall C.P. condition $\alpha^2 + \tau^2 \leq 1$

Consistent: $\frac{\alpha^2}{\sqrt{1-\tau^2}} \leq \eta^{\rm Riem}_{\kappa(x)=x^{-1/2}}(\Phi) \leq \eta^{\rm Tr}(\Phi) = \alpha$

Conj: (Ruskai) $\quad \eta^{\rm Riem}_{\kappa}(\Phi) \leq \eta^{\rm Tr}(\Phi) \qquad \forall\ k \in \mathcal{K}$ False

$$\alpha > 1 - \tau^2 \quad \Rightarrow \quad \eta^{\rm Riem}_{\max}(\Phi) = \frac{\alpha^2}{1-\tau^2} > \alpha = \eta^{\rm Tr}(\Phi)$$

Conj: (Kastoryano-Temme) $\quad \eta^{\rm Riem}_{\kappa}(\Phi) \leq \eta^{\rm Riem}_{\kappa(x)=x^{-1/2}}(\Phi)\ \forall\ k \in \mathcal{K}$

False For $\alpha^2 + \tau^2 = 1$, first ineq is tight so

$$\eta^{\rm Riem}_{x^{-1/2}}(\Phi) \geq \frac{\alpha^2}{\sqrt{1-\tau^2}} \quad \Rightarrow \quad \eta^{\rm Riem}_{x^{-1/2}}(\Phi) = \frac{\alpha^2}{\sqrt{1-\tau^2}} = \alpha$$

$$\eta^{\rm Riem}_{x^{-1/2}}(\Phi) = \alpha < 1 = \eta^{\rm Riem}_{\max}(\Phi) = \frac{\alpha^2}{1-\tau^2}$$

$$\eta^{\rm Riem}_{x^{-1/2}}(\Phi) = \alpha < \tfrac{1}{2}(1+\alpha) = \alpha^2 \, \frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)} \leq \eta^{\rm Riem}_{\widehat{\rm WY}}(\Phi)$$

Remark on WYD relative entropy

$$H_{\mathrm{WYD}(t)}(P,Q) = \tfrac{1}{t(1-t)}\Big(\mathrm{Tr}\, P - \mathrm{Tr}\, P^t Q^{1-t}\Big) = \tfrac{1}{t(1-t)}\Big(1 - \mathrm{Tr}\, P^t Q^{1-t}\Big)$$

$$\eta^{\mathrm{RelEnt}}_{\mathrm{WYD}(t)}(\Phi) = \sup_{P,Q\in\mathcal{D}} \frac{1 - \mathrm{Tr}\, \Phi(P)^t \Phi(Q)^{1-t}}{1 - \mathrm{Tr}\, P^t Q^{1-t}}$$

Question: Does contract depend on K in $\sup_{P,Q} \dfrac{H_g\big[K, \Phi(P), \Phi(Q)\big]}{H_g(\widehat{\Phi}(K), P, Q)}$

$$\eta^{\mathrm{RelEnt}}_{\mathrm{WYD}(t)}(\Phi, K) \equiv \sup_{P\neq Q\in\mathcal{D}} \frac{\mathrm{Tr}\, K^*\Phi(P)K - \mathrm{Tr}\, K^*[\Phi(P)]^t K[\Phi(Q)]^{1-t}}{\mathrm{Tr}\, \widehat{\Phi}(K)^* P \widehat{\Phi}(K) - \mathrm{Tr}\, \widehat{\Phi}(K)^* P^t \widehat{\Phi}(K) Q^{1-t}}$$

Ex: tensor prod $\Phi(\rho_{AB}) = \mathrm{Tr}_B\, \rho_{AB} = \rho_A, \qquad \widehat{\Phi}(K_A) = K_A \otimes I_B$

References

D. Petz (1986) and (1996) and others given in refs below

A. Lesniewski and M. B. Ruskai, "Monotone Riemannian metrics and relative entropy on noncommutative probability spaces" *J. Math. Phys.* **40** (1999), 5702–5724.

F. Hiai and D. Petz "Riemannian metrics on positive definite matrices related to means" *Lin. Alg. Appl.* **430**, 3105-30 (2009).

F. Hiai, M. Kosaski, D. Petz and M.B. Ruskai, "Operator means associated with completely positive maps" *Lin. Alg. Appl.* **439**, 174991 (2013) (arXiv:1212.1337)

F. Hiai and M.B. Ruskai, "Contraction coefficients for noisy quantum channels" *J. Math. Phys.* **57** 015211 (2016)