Quantum divergences and reversibility

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Plan

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Usual notations

- $\mathcal{H}$ is a finite-dimensional Hilbert space.
- $\mathcal{B}(\mathcal{H})$ is the algebra of linear operators on $\mathcal{H}$.
- $\mathcal{B}(\mathcal{H})_+ := \{A \in \mathcal{B}(\mathcal{H}) : A \ge 0\}$.
- $\mathcal{B}(\mathcal{H})_{++} := \{A \in \mathcal{B}(\mathcal{H}) : A > 0$, i.e., positive invertible$\}$.
- For $A \in \mathcal{B}(\mathcal{H})_+$, A^{-1} is the generalized inverse of A, and A^0 is the support projection of A.
- Tr is the usual trace functional on $\mathcal{B}(\mathcal{H})$.
- The Hilbert-Schmidt inner product is

$$\langle X, Y\rangle := \mathrm{Tr}\, X^*Y, \qquad X, Y \in \mathcal{B}(\mathcal{H}).$$

- The left and the right multiplications of $A \in \mathcal{B}(\mathcal{H})$ are

$$L_A X := AX, \quad R_A X := XR, \qquad X \in \mathcal{B}(\mathcal{H}).$$

  If $A, B \in \mathcal{B}(\mathcal{H})_+$, then L_A and R_B are positive operators on $\mathcal{B}(\mathcal{H})$ with $L_A R_B = R_B L_A$.

Operator convex function

- Assume that $f : (0, \infty) \to \mathbb{R}$ is an operator convex function, i.e.,

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B), \qquad 0 \le \lambda \le 1$$

  for every $A, B \in \mathcal{B}(\mathcal{H})_{++}$ (of any $\mathcal{H}$).

- Set

$$f(+0) := \lim_{x \searrow 0} f(x), \qquad f'(\infty) := \lim_{x \to \infty} f'(x) = \lim_{x \to \infty} \frac{f(x)}{x}$$

  in $(-\infty, +\infty]$.

- The transpose of f is $\widetilde{f}(x) := x f(x^{-1})$, $x > 0$, which is operator convex on $(0, \infty)$ again.

Integral expression

The operator convex function f on $(0, \infty)$ admits the unique integral expression

$$f(x) = f(1) + f'(1)(x-1) + c(x-1)^2 + \int_{[0,\infty)} \frac{(x-1)^2}{x+s}\, d\mu(s), \qquad x \in (0,\infty)$$

with $c \geq 0$ and a positive measure μ on $[0, \infty)$ satisfying $\int_{[0,\infty)} (1+s)^{-1}\, d\mu(s) < +\infty$.

Standard f-divergences H_f

Definition

For $A, B \in \mathcal{B}(\mathcal{H})_{++}$ with spectral decompositions $A = \sum_{a\in\mathrm{Sp}(A)} aP_a$ and $B = \sum_{b\in\mathrm{Sp}(B)} bQ_b$, the (standard) f-divergence is

$$H_f(A,B) := \langle B^{1/2}, f(L_A R_{B^{-1}})B^{1/2}\rangle = \mathrm{Tr}\, B^{1/2} f(L_A R_{B^{-1}})(B^{1/2})$$

with
$$f(L_A R_{B^{-1}}) = \sum_{a\in\mathrm{Sp}(A)} \sum_{b\in\mathrm{Sp}(B)} f(ab^{-1}) L_{P_a} R_{Q_b},$$

which can be extended to general $A, B \in \mathcal{B}(\mathcal{H})_+$ as

$$H_f(A,B) := \lim_{\varepsilon\searrow 0} H_f(A+\varepsilon I, B+\varepsilon I).$$

The f-divergences are special cases of Petz's quasi-entropy.[1]

[1] D. Petz, Quasi-entropies for finite quantum systems, *Rep. Math. Phys.* **23** (1986), 57–65.

Properties of H_f

For every $A, B \in \mathcal{B}(\mathcal{H})_+$,

- Explicit formula

$$H_f(A,B) = \sum_{a>0}\sum_{b>0} bf(ab^{-1})\mathrm{Tr}\, P_a Q_b + f(+0)\mathrm{Tr}\,(I - A^0)B + f'(\infty)\mathrm{Tr}\, A(I - B^0).$$

- Transpose $H_{\tilde{f}}(A,B) = H_f(B,A)$.
- Continuity
  - If $f(+0) < +\infty$ or $B^0 \not\leq A^0$ or $A^0 = I$, then
$$H_f(A,B) = \lim_{0<L\to 0} H_f(A, B+L).$$
  - If $f'(\infty) < +\infty$ or $A^0 \not\leq B^0$ or $B^0 = I$, then
$$H_f(A,B) = \lim_{0<K\to 0} H_f(A+K, B).$$
  - If $f(+0) < +\infty$ and $f'(\infty) < +\infty$, then
$$H_f(A,B) = \lim_{0<K,L\to 0} H_f(A+K, B+L).$$

Major properties of H_f (for details, see [2] [3])

Joint convexity

For every $A_i, B_i \in \mathcal{B}(\mathcal{H})_+$ and $\lambda_i \ge 0$, $1 \le i \le k$,

$$H_f\Bigl(\sum_{i=1}^k \lambda_i A_i, \sum_{i=1}^k \lambda_i B_i\Bigr) \le \sum_{i=1}^k \lambda_i H_f(A_i, B_i).$$

Monotonicity or Data-processing inequality

Assume that $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a trace-preserving linear map such that the adjoint $\widehat{\Phi} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ satisfies $\widehat{\Phi}(Y^*Y) \ge \widehat{\Phi}(Y^*)\widehat{\Phi}(Y)$, $Y \in \mathcal{B}(\mathcal{K})$. For every $A, B \in \mathcal{B}(\mathcal{H})_+$,

$$H_f(\Phi(A), \Phi(B)) \le H_f(A, B).$$

[2] A. Lesniewski and M.B. Ruskai, Monotone Riemannian metrics and relative entropy on noncommutative probability spaces, *J. Math. Phys.* **40** (1999), 5702–5724.

[3] F.H., M. Mosonyi, D. Petz and C. Bény, Quantum f-divergences and error correction, *Rev. Math. Phys.* **23** (2011), 691–747.

Non-standard f-divergences H_f^* [4] [5]

Definition

For $A, B \in \mathcal{B}(\mathcal{H})_{++}$ define

$$H_f^*(A,B) := \operatorname{Tr} Bf(B^{-1/2}AB^{-1/2}) = \operatorname{Tr} B^{1/2}f(B^{-1/2}AB^{-1/2})B^{1/2},$$

which can be extended to general $A, B \in \mathcal{B}(\mathcal{H})_+$ as

$$H_f^*(A,B) := \lim_{\varepsilon \searrow 0} H_f^*(A+\varepsilon I, B+\varepsilon I).$$

[4] D. Petz and M.B. Ruskai, Contraction of generalized relative entropy under stochastic mappings on matrices, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** (1998), 83–89.

[5] K. Matsumoto, A new quantum version of f-divergence, Preprint, 2014; arXiv:1311.4722.

Properties of H_f^*

- $H_f^*(A,B)$ is jointly convex in $A,B \in \mathcal{B}(\mathcal{H})_+$. More strongly, $(A,B) \in \mathcal{B}(\mathcal{H})_{++} \times \mathcal{B}(\mathcal{H})_{++} \mapsto B^{1/2} f(B^{-1/2}AB^{-1/2})B^{1/2}$ is jointly operator convex.[6]
- Transpose $\quad H_{\tilde{f}}^*(A,B) = H_f^*(B,A)$.
- Continuity
  - If $f(+0) < +\infty$ and $B^0 = I$, then
  $$H_f^*(A,B) = \lim_{0<K\to 0} H_f^*(A+K,B).$$
  - If $f'(\infty) < +\infty$ and $A^0 = I$, then
  $$H_f^*(A,B) = \lim_{0<L\to 0} H_f^*(A,B+L).$$
  - If $f(+0) < +\infty$ and $f'(\infty) < +\infty$, then
  $$H_f^*(A,B) = \lim_{0<K,L\to 0} H_f^*(A+K,B+L).$$

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[6] E. Effros and F. Hansen, Non-commutative perspectives, *Ann. Funct. Anal.* **5** (2014), 74–79; arXiv:1309.7701.

Properties of H_f^* (cont.)

Monotonicity

For any trace-preserving positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and $A, B \in \mathcal{B}(\mathcal{H})_+$,

$$H_f^*(\Phi(A), \Phi(B)) \le H_f^*(A, B).$$

[The proof is essentially in [7]]

[7] F.H. and D. Petz, The proper formula for relative entropy and its asymptotics in quantum probability, *Comm. Math. Phys.* **143** (1991), 99–114.

Inequality

For every $A, B \in \mathcal{B}(\mathcal{H})_+$,

$$H_f(A,B) \le H_f^*(A,B).$$

[The proof is based on Matsumoto's construction of minimal reverse test.[5]]

Strict inequality

If $f(+0) < +\infty$ and the representing measure μ for f satisfies $|\mathrm{supp}\,\mu| \ge 2(\dim \mathcal{H})^2$, then

$$H_f(A,B) < H_f^*(A,B)$$

for every non-commuting $A, B \in \mathcal{B}(\mathcal{H})_+$ with $A^0 \le B^0$.

[The proof is based on the reversibility theorem below.]

Examples

(1) Quadratic function When $f(x) = x^2$,

$$H_{x^2}(A, B) = \mathrm{Tr}\, A^2 B^{-1} = H^*_{x^2}(A, B).$$

Hence, when $f(x) = ax^2 + bx + c$ with $a \geq 0$,
$H_f(A, B) = H^*_f(A, B)$ for all $A, B \in \mathcal{B}(\mathcal{H})_+$.

(2) Log function When $f(x) = x \log x$ and $\widetilde{f}(x) = -\log x$,

$$H_{x \log x}(A, B) = S(A\|B) := \mathrm{Tr}\, A(\log A - \log B)$$

is the Umegaki relative entropy, and $H_{-\log x}(A, B) = S(B\|A)$.
For $A^0 \leq B^0$,

$$H^*_{x \log x}(A, B) = S_{\mathrm{BS}}(A\|B) := \mathrm{Tr}\, A \log(A^{1/2} B^{-1} A^{1/2})$$

is the Belavkin-Staszewski relative entropy.

(3) Power functions For $\alpha \in (0,1) \cup (1,2]$ consider

$$f^{(\alpha)}(x) := \frac{x - x^\alpha}{\alpha(1-\alpha)}, \qquad \widetilde{f}^{(\alpha)}(x) = \frac{1 - x^{1-\alpha}}{\alpha(1-\alpha)}.$$

Note that $f^{(\alpha)}(x) \to x \log x$ and $\widetilde{f}^{(\alpha)}(x) \to -\log x$ as $\alpha \to 1$, and that $f^{(\alpha)}$ and $\widetilde{f}^{(\alpha)}$ cover all of operator convex power functions x^α ($-1 \le \alpha < 0$, $1 < \alpha \le 2$) and $-x^\alpha$ ($0 < \alpha < 1$). We write

$$H_{f^{(\alpha)}}(A,B) = \frac{\mathrm{Tr}\, A - \mathrm{Tr}\, A^\alpha B^{1-\alpha}}{\alpha(1-\alpha)},$$

$$H^*_{f^{(\alpha)}}(A,B) = \frac{\mathrm{Tr}\, A - \mathrm{Tr}\, B^{1/2}(B^{-1/2}AB^{-1/2})^\alpha B^{1/2}}{\alpha(1-\alpha)}.$$

When $0 < \alpha < 1$, we rewrite

$$H^*_{f^{(\alpha)}}(A,B) = \frac{\mathrm{Tr}\, A - \mathrm{Tr}\, B \,\#_\alpha\, A}{\alpha(1-\alpha)}.$$

Fact

From the strict inequality between H_f and H_f^*, for $A, B \in \mathcal{B}(\mathcal{H})_+$ with $A^0 \le B^0$ and $AB \ne BA$,

(1) $S(A\|B) < S_{\mathrm{BS}}(A\|B)$,

(2) $\mathrm{Tr}\, A \#_\alpha B < \mathrm{Tr}\, A^{1-\alpha} B^\alpha$ for $\alpha \in (0, 1)$,

(3) $\mathrm{Tr}\, A^\alpha B^{1-\alpha} < \mathrm{Tr}\, B^{1/2}(B^{-1/2} A B^{-1/2})^\alpha B^{1/2}$ for $\alpha \in (1, 2)$.

Note that $S(A\|B) \le S_{\mathrm{BS}}(A\|B)$ was first proved in [7],
$\mathrm{Tr}\, A \#_\alpha B \le \mathrm{Tr}\, A^{1-\alpha} B^\alpha$ is a consequence of the log-majorization [8],
and (3) seems new.

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[8] T. Ando and F.H., Log majorization and complementary Golden-Thompson type inequalities, *Linear Algebra Appl.* **197/198** (1994), 113–131.

Brief history

Reversibility problem Let D be a quantum divergence and $A, B \in \mathcal{B}(\mathcal{H})_+$. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a stochastic map (typically, a TPCP map). If $D(\Phi(A), \Phi(B)) = D(A, B) < +\infty$, then is there a reverse stochastic map $\Psi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that $\Psi(\Phi(A)) = A$ and $\Psi(\Phi(B)) = B$?

The problem treats the case of no contraction of D for given A, B, which is considered as complementary to the contraction coefficient problem [9] treating the maximal contraction

$$\eta_D(\Phi) := \sup_{\rho \neq \gamma} \frac{D(\Phi(\rho), \Phi(\gamma))}{D(\rho, \gamma)}$$

over invertible density operators ρ, γ in $\mathcal{B}(\mathcal{H})$.

[9] M.B. Ruskai's talk in the workshop.

Previous studies

- Petz (1986)[10]: When $\mathcal{N}$ is a subalgebra of a von Neumann algebra $\mathcal{M}$ and φ, ψ are faithful normal states, $\mathcal{N}$ is sufficient for φ, ψ iff $S(\psi|_{\mathcal{N}} \| \varphi|_{\mathcal{N}}) = S(\psi\|\varphi)$.
- Hayden, Jozsa, Petz and Winter (2004)[11]:
  A structural characterization of equality case of strong subadditivity, equivalently, equality case of relative entropy in a tripartite system.

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[10] D. Petz, Sufficient subalgebras and the relative entropy of states of a von Neumann algebra, *Comm. Math. Phys.* **105** (1986), 123–131.

[11] P. Hayden, R. Jozsa, D. Petz and A. Winter, Structure of states which satisfy strong subadditivity of quantum entropy with equality, *Comm. Math. Phys.* **246** (2004), 359–374.

Previous studies (cont.)

- Jenčová and Petz (2006)[12]: When $\alpha : \mathcal{N} \to \mathcal{M}$ is a unital 2-positive map between von Neumann algebras and $\mathcal{S}$ is a set of normal states with $\varphi \in \mathcal{S}$ such that both φ and $\varphi \circ \alpha$ are faithful, α is reversible for $\mathcal{S}$ iff $P(\psi \circ \alpha, \varphi \circ \alpha) = P(\psi, \varphi)$ for all $\psi \in \mathcal{S}$ iff $S(\psi \circ \alpha \| \varphi \circ \alpha) = S(\psi \| \varphi)$ for all $\psi \in \mathcal{S}$.
- Jenčová and Ruskai (2010)[13] A characterization for equality case in the joint convexity of quasi-entropies; joint convexity is a special case of the monotonicity under partial traces.

[12] A. Jenčová and D. Petz, Sufficiency in quantum statistical inference, *Comm. Math. Phys.* **263** (2006), 259–276.

[13] A. Jenčová and M.B. Ruskai, A unified treatment of convexity of relative entropy and related trace functions, with conditions for equality, *Rev. Math. Phys.* **22** (2010), 1099–1121.

Reversibility via H_f

We assume:

- $f(+0) < +\infty$, so f extends to an operator convex function on $[0, \infty)$.
- $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is trace-preserving and 2-positive.
- $A, B \in \mathcal{B}(\mathcal{H})_+$ with $B^0 = I$.

Set a unital 2-positive map $\Phi_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ as

$$\Phi_B(X) := \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2}, \qquad X \in \mathcal{B}(\mathcal{H}),$$

whose adjoint $\widehat{\Phi}_B : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is

$$\widehat{\Phi}_B(Y) = B^{1/2}\widehat{\Phi}(\Phi(B)^{-1/2}Y\Phi(B)^{-1/2})B^{1/2}, \qquad Y \in \mathcal{B}(\mathcal{K}).$$

Note that $\widehat{\Phi}_B(\Phi(B)) = B$.

Combining our previous paper (2011)[3] and Jenčová (2012)[14],

Theorem

The following conditions are equivalent:

(i) Φ is reversible for A, B.

(ii) $\widehat{\Phi}_B(\Phi(A)) = A$.

(iii) $H_f(\Phi(A), \Phi(B)) = H_f(A, B)$ for some f whose representing measure μ satisfies
$|\mathrm{supp}\,\mu| \ge |\mathrm{Sp}(L_A R_{B^{-1}}) \cup \mathrm{Sp}(L_{\Phi(A)} R_{\Phi(B)^{-1}})|$.

(iv) $H_f(\Phi(A), \Phi(B)) = H_f(A, B)$ for all f.

(v) $\widehat{\Phi}(\Phi(B)^{-1/2}\Phi(A)\Phi(B)^{-1/2}) = B^{-1/2}AB^{-1/2}$.

(vi) $B^{-1/2}AB^{-1/2} \in \ker(\mathrm{id} - \widehat{\Phi} \circ \Phi_B)$, the fixed-point subalgebra of $\widehat{\Phi} \circ \Phi_B$.

[14] A. Jenčová, Reversibility conditions for quantum operations, *Rev. Math. Phys.* **24** (2012), 1250016, 26 pp.

Theorem (cont.)

(vii) if $\rho := A/\mathrm{Tr}\, A$ and $\gamma := B/\mathrm{Tr}\, B$, then

$$\langle \Phi(\rho-\gamma), \Omega^{\kappa}_{\Phi(\gamma)}(\Phi(\rho-\gamma))\rangle = \langle \rho-\gamma, \Omega^{\kappa}_{\gamma}(\rho-\gamma)\rangle$$

for some monotone Riemannian metric $\langle\cdot\,, \Omega^{\kappa}_{\gamma}(\cdot)\rangle$ such that the representing measure ν of κ satisfies $|\mathrm{supp}\,\nu| \ge |\mathrm{Sp}(L_\gamma R_{\gamma^{-1}}) \cup \mathrm{Sp}(L_{\Phi(\gamma)} R_{\Phi(\gamma)^{-1}})|$.

Note Monotone metrics on $\{\gamma \in \mathcal{B}(\mathcal{H})_{++} : \mathrm{Tr}\,\gamma = 1\}$ are associated with operator decreasing functions $\kappa : (0,\infty) \to (0,\infty)$ such that $x\kappa(x) = \kappa(x^{-1})$, $x > 0$, and $\kappa(x) = 1$, admitting the integral expression

$$\kappa(x) = c + \int_{[0,\infty)} \frac{1}{x+s}\, d\nu(s), \qquad x \in (0,\infty),$$

with a unique positive measure ν on $[0,\infty)$.

Reversibility via H_f^*

- The multiplicative domain of Φ_B is

$$\begin{aligned}\mathcal{M}_{\Phi_B} &:= \{X \in \mathcal{B}(\mathcal{H}) : \Phi_B(XY) = \Phi_B(X)\Phi_B(Y), \\ &\qquad \Phi_B(YX) = \Phi_B(Y)\Phi_B(X),\ Y \in \mathcal{B}(\mathcal{H})\} \\ &= \{X \in \mathcal{B}(\mathcal{H}) : \Phi_B(X^*X) = \Phi_B(X)^*\Phi_B(X), \\ &\qquad \Phi_B(XX^*) = \Phi_B(X)^*\Phi_B(X)^*\}.\end{aligned}$$

- Associated with an operator monotone function $h \ge 0$ on $(0,\infty)$ with $h(1) = 1$, the operator mean σ_h (in the Kubo-Ando sense) is

$$A\,\sigma_h\,B := A^{1/2}h(A^{-1/2}BA^{-1/2})A^{1/2}, \quad A, B \in \mathcal{B}(\mathcal{H})_{++}.$$

We say that σ_h is non-linear if h is non-linear.

Note $\Phi(A\,\sigma\,B) \le \Phi(A)\,\sigma\,\Phi(B)$ holds for any operator mean σ and any general positive map Φ (essentially due to Ando, 1979).

The next theorem gives equivalent conditions for equality case of H_f^* under Φ. The implication (a) ⇒ (g) was shown by Matsumoto [5].

Theorem

The following conditions are equivalent:

(a) $H_f^*(\Phi(A),\Phi(B)) = H_f^*(A,B)$ for some non-linear f.

(b) $H_f^*(\Phi(A),\Phi(B)) = H_f^*(A,B)$ for all f.

(c) $\mathrm{Tr}\,\Phi(A)^2\Phi(B)^{-1} = \mathrm{Tr}\,A^2B^{-1}$.

(d) $\Phi(A)\,\sigma\,\Phi(B) = \Phi(A\,\sigma\,B)$ for some non-linear operator mean σ.

(e) $\Phi(A)\,\sigma\,\Phi(B) = \Phi(A\,\sigma\,B)$ for all operator means σ.

(f) $\Phi(A)\Phi(B)^{-1}\Phi(A) = \Phi(AB^{-1}A)$.

(g) $B^{-1/2}AB^{-1/2} \in \mathcal{M}_{\Phi_B}$.

Corollary

If $\Phi(A)$ and $\Phi(B)$ commute, then the conditions of the previous theorem are equivalent to the reversibility of Φ for A, B.

Corollary

If $\Phi(B)$ commutes with $\Phi(A)$ for all $A \in \mathcal{B}(\mathcal{H})$ (in particular, if Φ is a quantum-classical channel), then

$$\ker(\mathrm{id} - \widehat{\Phi} \circ \Phi_B) = \mathcal{M}_{\Phi_B}.$$

Note In particular, if Φ is a unital channel (trace-preserving) and $B = I$ (so $\Phi(B) = I$), then $\ker(\mathrm{id} - \widehat{\Phi} \circ \Phi) = \mathcal{M}_\Phi$, which is contained in [15].

[15] M.-D. Choi, N. Johnston and D.W. Kribs, The multiplicative domain in quantum error correction, *J. Phys. A: Math. Theor.* **42** (2009), 245303, 15 pp.

Qubit case

- If f is not of the form $ax^2 + bx + c$ with $a \geq 0$, then

$$H_f(A, B) < H_f^*(A, B)$$

  for every non-commuting $A, B \in \mathcal{B}(\mathbb{C}^2)_+$ with $B > 0$.
- If Φ is a unital qubit channel, then the conditions of the previous theorem are equivalent to the reversibility of Φ for any $A, B \in \mathcal{B}(\mathbb{C}^2)_+$ with $B > 0$.
  But, this is not true for some unital qutrit channel.

Sandwiched Rényi divergences and α-z-Rényi divergences

Definition For $\alpha, z > 0$ with $\alpha \neq 1$,

- The (old) Rényi divergence is

$$D_\alpha(A\|B) := \frac{1}{\alpha - 1} \log \frac{\operatorname{Tr} A^\alpha B^{1-\alpha}}{\operatorname{Tr} A}.$$

- The (new) sandwiched Rényi divergence[16] is

$$D_\alpha^*(A\|B) := \frac{1}{\alpha - 1} \log \frac{\operatorname{Tr} (B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}})^\alpha}{\operatorname{Tr} A}.$$

[16] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr and M. Tomamichel, On quantum Rényi entropies: A new generalization and some properties, *J. Math. Phys.* **54** (2013), 122203.

- The α-z-Rényi divergence[17] [18] is

$$D_{\alpha,z}(A\|B) := \frac{1}{\alpha-1}\log\frac{\mathrm{Tr}\,(B^{\frac{1-\alpha}{2z}}A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{2z}})^z}{\mathrm{Tr}\,A}.$$

Note $D_\alpha = D_{\alpha,1}$, $D_\alpha^* = D_{\alpha,\alpha}$ and

$$D_{1/2}^*(A\|B) = -2\log F(A,B) \text{ where } F(A,B) := \mathrm{Tr}\,|A^{1/2}B^{1/2}|,$$

$$\lim_{\alpha\to1} D_\alpha^*(A\|B) = S(A\|B) := \mathrm{Tr}\,A(\log A - \log B),$$

$$\lim_{\alpha\to\infty} D_\alpha^*(A\|B) = D_{\max}(A\|B) := \inf\{\gamma : A \le e^\gamma B\}.$$

[17] V. Jaksic, Y. Ogata, Y. Pautrat and C.-A. Pillet, Entropic fluctuations in quantum statistical mechanics. An Introduction, in: Quantum Theory from Small to Large Scales, August 2010, in: Lecture Notes of the Les Houches Summer School, vol. 95, Oxford University Press, 2012.

[18] K.M.R. Audenaert and N. Datta, α-z-Rényi relative entropies, *J. Math. Phys.* **56** (2015), 022202.

Monotonicity or Data-processing inequality

In each of the following cases, $\mathrm{Tr}\,(B^{\frac{1-\alpha}{2z}} A^{\frac{\alpha}{z}} B^{\frac{1-\alpha}{2z}})^z$ is jointly concave for $\alpha < 1$ and jointly convex for $\alpha > 1$, and the monotonicity

$$D_{\alpha,z}(\Phi(A)\|\Phi(B)) \le D_{\alpha,z}(A\|B)$$

holds for any TPCP map Φ.

Moreover, the monotonicity of D_α^* for $\alpha \ge 1$ is known for any trace-preserving positive map Φ (due to Beigi).

Monotonicity or Data-processing inequality (cont.)

- $0 < \alpha \le 1$ and $z \ge \max\{\alpha, 1-\alpha\}$ (F.H., 2013); hence $1/2 \le \alpha \le 1$ if $z = \alpha$,
- $1 \le \alpha \le 2$ and $z = 1$ (Ando, 1979),
- $\alpha \ge 1$ and $z = \alpha$ (Müller-Lennert et al.[16], Frank and Lieb[a], Beigi[b], Wilde, Winter and Yang[c])
- $1 \le \alpha \le 2$ and $z = \alpha/2$ (Carlen, Frank and Lieb[d]).

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[a] R.L. Frank and E.H. Lieb, Monotonicity of a relative Rényi entropy, *J. Math. Phys.* **54** (2013), 122201.

[b] S. Beigi, Sandwiched Rényi divergence satisfies data processing inequality, *J. Math. Phys.* **54** (2013), 122202.

[c] M.M. Wilde, A. Winter and D. Yang, Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy, *Comm. Math. Phys.* **331** (2014), 593–622.

[d] E.A. Carlen, R.L. Frank and E.H. Lieb, Some operator and trace function convexity theorems, *Linear Algebra Appl.* **490** (2016), 174–185.

Reversibility via α-z-Rényi divergences

Theorem

Assume: $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is 2-positive bistochastic map and

$$D_{\alpha,z}(\Phi(A)\|\Phi(B)) = D_{\alpha,z}(A\|B).$$

Then Φ is reversible for A, B if each of the following is satisfied:

- $\alpha \le z \le 1$, $\Phi(B) = B$ and $A^0 \le B^0$ (without 2-positivity),
- $\alpha < z \le 1$, $\Phi(B) = B$ and $B^0 \le A^0$,
- $0 < 1 - \alpha \le z \le 1$, $\Phi(A) = A$ and $B^0 \le A^0$ (without 2-positivity),
- $0 < 1 - \alpha < z \le 1$, $\Phi(A) = A$ and $A^0 \le B^0$,
- $\alpha \ge z \ge \max\{1, \alpha/2\}$, $\Phi(B) = B$ and $A^0 \le B^0$ (without 2-positivity),
- $\alpha > 1$, $z \ge 1$, $z > \alpha - 1$, $\Phi(A) = A$ and $B^0 = I$.

Corollary

Assume: $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is 2-positive bistochastic map and

$$D_\alpha^*(\Phi(A)\|\Phi(B)) = D_\alpha^*(A\|B).$$

Then Φ is reversible for A, B if each of the following is satisfied:

- $\Phi(B) = B$ and $A^0 \le B^0$ (for arbitrary $\alpha \in (0, \infty) \setminus \{1\}$ without 2-positivity),
- $1/2 \le \alpha < 1$, $\Phi(A) = A$ and $B^0 \le A^0$ (without 2-positivity),
- $1/2 < \alpha < 1$, $\Phi(A) = A$ and $A^0 \le B^0$.

Notes

- If $D_2^*(\Phi(A)\|\Phi(B)) = D_2^*(A\|B)$, then Φ is reversible for A, B in the general setting.
- When $\alpha \in (0, 2) \setminus \{1\}$, if $D_\alpha(\Phi(A)\|\Phi(B)) = D_\alpha(A\|B)$, then Φ is reversible for A, B (the case of f-divergence for $f(x) = x^\alpha$).
- When $\alpha = z = 1/2$ (the case of fidelity), $F(\Phi(A), \Phi(B)) = F(A, B)$ implies the reversibility of Φ for A, B when $\Phi(B) = B$ and $A^0 \le B^0$ (or $\Phi(A) = A$ and $B^0 \le A^0$).
- The reversibility via $D_{\max}$ does not hold even when $\Phi(B) = B$ and $A^0 = B^0$.

Thank you for your attention.