MAQIT Conference
Dajeon, Korea
February 15-19, 2014

Inequalities for quantum channels

Frank Hansen
Institute for Excellence in Higher Education
Tohoku University

Regular operator functions

Let $F\colon \mathcal{D} \to B(\mathcal{H})$ be a mapping of k operator variables defined in a domain $\mathcal{D}$ in $B(\mathcal{H}) \times \cdots \times B(\mathcal{H})$. We say that F is regular if

(i) The domain $\mathcal{D}$ is invariant under unitary transformations of $\mathcal{H}$ and

$$F(u^* x_1 u, \dots, u^* x_k u) = u^* F(x_1, \dots, x_k) u$$

for every $x = (x_1, \dots, x_k) \in \mathcal{D}$ and every unitary u on $\mathcal{H}$

(ii) For a projection p and a k-tuple $(px_1p, \dots, px_kp) \in \mathcal{D}$ we have

$$F(px_1p, \dots, px_kp) = pF(px_1p, \dots, px_kp)p$$

(iii) With suitable interpretations we have

$$F\left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_k & 0 \\ 0 & y_k \end{pmatrix}\right) = \begin{pmatrix} F(x_1, \dots, x_k) & 0 \\ 0 & F(y_1, \dots, y_k) \end{pmatrix}$$

Jensen's sub-homogeneous inequality

Consider the domain

$$\mathcal{D}^k(\mathcal{H}) = \{(A_1, \dots, A_k) \mid A_1, \dots, A_k \geq 0\}$$

of k-tuples of positive semi-definite operators acting on an infinite dimensional Hilbert space $\mathcal{H}$.

Theorem

Let $F\colon \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})_{sa}$ be a convex regular map with $F(0, \dots, 0) \leq 0$, and let $C\colon \mathcal{H} \to \mathcal{K}$ be a linear contraction. Then the inequality

$$F(C^*A_1C, \dots, C^*A_kC) \leq C^*F(A_1, \dots, A_k)C$$

holds for k-tuples $(A_1, \dots, A_k)$ in $\mathcal{D}^k(\mathcal{K})$.

Jensen's inequality for regular operator maps

Theorem

Let $F\colon \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})_{sa}$ *be a convex regular map, and let*

$$C_1, \ldots, C_n\colon \mathcal{H} \to \mathcal{K}$$

be linear maps of $\mathcal{H}$ *into a Hilbert space* $\mathcal{K}$ *such that*

$$C_1^* C_1 + \cdots + C_n^* C_n = 1_{\mathcal{H}}.$$

Then the inequality

$$F\Big(\sum_{i=1}^n C_i^* A_{i1} C_i, \ldots, \sum_{i=1}^n C_i^* A_{ik} C_i\Big) \le \sum_{i=1}^n C_i^* F(A_{i1}, \ldots, A_{ik}) C_i$$

holds for k*-tuples* $(A_{i1}, \ldots, A_{ik})$ *in* $\mathcal{D}^k(\mathcal{K})$*, where* $i = 1, \ldots, n$.

Corollary

Let for finite dimensional spaces $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ *be a completely positive unital linear map, and let* F *be a convex regular map. Then*

$$F\big(\Phi(A_1),\ldots,\Phi(A_k)\big) \le \Phi\big(F(A_1,\ldots,A_k)\big)$$

for $A_1,\ldots,A_k \in \mathcal{D}_k(\mathcal{H})$.

Proof: By Choi's decomposition theorem there exist operators $C_1,\ldots,C_n$ in $B(\mathcal{K},\mathcal{H})$ with $C_1^*C_1+\cdots+C_n^*C_n = 1_{\mathcal{K}}$ such that

$$\Phi(A) = \sum_{i=1}^{n} C_i^* A C_i \qquad \text{for} \qquad A \in B(\mathcal{H}).$$

The statement now follows by the preceding theorem by choosing

$$(A_{i1},\ldots,A_{ik}) = (A_1,\ldots,A_k)$$

for $i = 1,\ldots,n$.

The perspective of a regular map

We consider for $k = 1, 2, \dots$ the convex domain

$$\mathcal{D}_+^k(\mathcal{H}) = \{(A_1, \dots, A_k) \mid A_1, \dots, A_k > 0\}$$

of positive definite and invertible operators acting on $\mathcal{H}$.

Let $F\colon \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ be a regular map. The perspective map

$$\mathcal{P}_F\colon \mathcal{D}_+^{k+1}(\mathcal{H}) \to B(\mathcal{H})$$

is defined by setting

$$\mathcal{P}_F(A_1, \dots, A_k, B) = B^{1/2} F(B^{-1/2} A_1 B^{-1/2}, \dots, B^{-1/2} A_k B^{-1/2}) B^{1/2},$$

and it is a positively homogeneous regular map.

Theorem

The perspective of a convex regular map is convex

The perspective filtered through a CP map

Theorem

Let $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map ($\mathcal{H}, \mathcal{K}$ finite dimensional), and let $F\colon \mathcal{D}_+^k \to B(\mathcal{H})$ be a convex regular map. Then

$$\mathcal{P}_F\big(\Phi(A_1), \dots, \Phi(A_{k+1})\big) \le \Phi\big(\mathcal{P}_F(A_1, \dots, A_{k+1})\big),$$

for $(A_1, \dots, A_{k+1})$ in $\mathcal{D}_+^k(\mathcal{H})$, where $\mathcal{P}_F$ is the perspective of F.

Proof: To a fixed positive definite $B \in B(\mathcal{H})$ we set

$$\Psi(X) = \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2}$$

and notice that $\Psi\colon B(\mathcal{H}) \to B(\mathcal{K})$ is a unital linear map. We realise, by the definition of complete positivity, that also Ψ is completely positive.

Since F is convex we thus obtain:

$$F\big(\Psi(B^{-1/2}A_1B^{-1/2}),\ldots,\Psi(B^{-1/2}A_kB^{-1/2})\big) \le \Psi\big(F(B^{-1/2}A_1B^{-1/2},\ldots,B^{-1/2}A_kB^{-1/2})\big).$$

Inserting Ψ we obtain the inequality

$$F\big(\Phi(B)^{-1/2}\Phi(A_1)\Phi(B)^{-1/2},\ldots,\Phi(B)^{-1/2}\Phi(A_k)\Phi(B)^{-1/2}\big)$$
$$\le \Phi(B)^{-1/2}\Phi\big(B^{1/2}F(B^{-1/2}A_1B^{-1/2},\ldots,B^{-1/2}A_kB^{-1/2})B^{1/2}\big)\Phi(B)^{-1/2}.$$

We multiply from the left and right with $\Phi(B)^{1/2}$ and obtain

$$\begin{aligned}\mathcal{P}_F\big(\Phi(A_1),\ldots,\Phi(A_k),\Phi(B)\big) &= \Phi(B)^{1/2}F\big(\Phi(B)^{-1/2}\Phi(A_1)\Phi(B)^{-1/2},\\ &\qquad \ldots,\Phi(B)^{-1/2}\Phi(A_k)\Phi(B)^{-1/2}\big)\Phi(B)^{1/2}\\ &\le \Phi\big(B^{1/2}F(B^{-1/2}A_1B^{-1/2},\ldots,B^{-1/2}A_kB^{-1/2})B^{1/2}\big)\\ &= \Phi\big(\mathcal{P}_F(A_1,\ldots,A_k,B)\big),\end{aligned}$$

which is the assertion.

Applications to partial traces

Corollary

Consider a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite dimensions, and let $F\colon \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ be a convex regular map. Then

$$\mathcal{P}_F(\mathrm{Tr}_2 A_1, \ldots, \mathrm{Tr}_2 A_{k+1}) \leq \mathrm{Tr}_2 \mathcal{P}_F(A_1, \ldots, A_{k+1})$$

for operators $(A_1, \ldots, A_{k+1})$ in $\mathcal{D}_+^k(\mathcal{H})$, where $\mathcal{P}_F$ is the perspective of F and $\mathrm{Tr}_2 B$ is the partial trace of B on $\mathcal{H}_1$.

Setting $k = 1$, the inequality takes the form

$$\mathcal{P}_f(\mathrm{Tr}_2 A, \mathrm{Tr}_2 B) \leq \mathrm{Tr}_2 \mathcal{P}_f(A, B),$$

where $f\colon (0, \infty) \to \mathbf{R}$ is an operator convex function.

Homogeneous and convex regular maps

Consider a regular map

$$F\colon \mathcal{D}_+^{k+1}(\mathcal{H}) \to B(\mathcal{H})$$

for some $k = 1, 2 \ldots.$

Proposition

If F is convex and positively homogeneous, then F is the perspective of its restriction $G\colon \mathcal{D}_+^k \to B(\mathcal{H})$ given by

$$G(A_1, \ldots, A_k) = F(A_1, \ldots, A_k, 1).$$

The perspective $\mathcal{P}_G$ of a convex regular map $G\colon \mathcal{D}_+^k \to B(\mathcal{H})$ is thus the unique extension of G to a positively homogeneous convex regular map $F\colon \mathcal{D}_+^{k+1} \to B(\mathcal{H})$.

Some operator means of several variables

The Karcher mean of k positive definite invertible operators $A_1, \ldots, A_k$ is defined as the unique positive definite solution $\Lambda_k(A_1, \ldots, A_k)$ to the equation

$$\sum_{i=1}^{k} \log\left(X^{1/2} A_i X^{1/2}\right) = 0.$$

The harmonic mean $H_k(A_1, \ldots, A_k)$ is defined by setting

$$H_k(A_1, \ldots, A_k) = \frac{k}{A_1^{-1} + \cdots + A_k^{-1}}$$

for k positive definite invertible operators.

The Karcher mean and the harmonic mean are concave and positively homogeneous regular operator maps.

Applications to operator means of several variables

The Karcher mean Λ_k and the harmonic mean H_k of k operator variables satisfy the inequalities

$$\begin{aligned} \mathrm{Tr}_2 \Lambda_k(A_1, \ldots, A_k) &\leq \Lambda_k(\mathrm{Tr}_2 A_1, \ldots, \mathrm{Tr}_2 A_k) \\ \mathrm{Tr}_2 H_k(A_1, \ldots, A_k) &\leq H_k(\mathrm{Tr}_2 A_1, \ldots, \mathrm{Tr}_2 A_k) \end{aligned}$$

for $A_1, \ldots, A_k \in \mathcal{D}_+^k(\mathcal{H})$ on a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces of finite dimensions.

For $k = 2$ we obtain the inequality of Ando

$$(A\#B)_1 \leq A_1 \# B_1,$$

where A_1 and B_1 are the partial traces of A and B on $\mathcal{H}_1$ and

$$A\#B = B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1/2}B^{1/2}$$

is the (canonical) geometric mean of two variables.

Lieb-Ruskai's convexity theorem

Lieb and Ruskai proved convexity of the map

$$L(A, K) = K^* A^{-1} K,$$

where A is positive definite. If also K is positive definite then

$$KA^{-1}K = K^{1/2}\big(K^{-1/2}AK^{-1/2}\big)^{-1}K^{1/2}$$

is the perspective of the operator convex function $t \to t^{-1}$.

For operators on a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ we have

$$K_1^* A_1^{-1} K_1 \leq (K^* A^{-1} K)_1$$

for positive definite A and arbitrary K.

For K positive definite this follows from the above theory. The general case is proved by using convexity and Choi's decomposition.

Alternative proof of Lieb-Ruskai's convexity theorem

There is a way to consider Lieb-Ruskai's convexity theorem which points to generalisations to more than two operators.

The geometric mean G_1 of one positive definite operator is trivially given by $G_1(A) = A$. It is a concave regular map and its inverse

$$A \to G_1(A)^{-1} = A^{-1}$$

is thus a convex regular map. The perspective

$$\mathcal{P}_{G_1^{-1}}(A, B) = B^{1/2} G_1(B^{-1/2} A B^{-1/2})^{-1} B^{1/2} = BA^{-1}B = L(A, B)$$

is therefore a convex regular map, and it is increasing when filtered through a partial trace.

A similar construction may be carried out for any number of operator variables.

Generalisations to $n+1$ operator variables

Any geometric mean G_n studied in the literature is a positive, concave, and regular map. The inverse

$$G_n(A_1, \cdots, A_n)^{-1} = G_n(A^{-1}, \ldots, A_n^{-1})$$

is therefore convex and regular. The perspective

$$\begin{aligned}
&\mathcal{P}_{G_n^{-1}}(A_1, \ldots, A_n, C) \\
&= C^{1/2} G_n\big(C^{-1/2} A_1 C^{-1/2}, \ldots, C^{-1/2} A_n C^{-1/2}\big)^{-1} C^{1/2} \\
&= C^{1/2} G_n\big(C^{1/2} A_1^{-1} C^{1/2}, \ldots, C^{1/2} A_n^{-1} C^{1/2}\big) C^{1/2} \\
&= C G_n\big(A_1^{-1}, \ldots, A_n^{-1}\big) C = C G_n(A_1, \ldots, A_n)^{-1} C \\
&= L(A_1, \ldots, A_n, C),
\end{aligned}$$

where we used self-duality and congruence invariance of the geometric mean.

The map $L(A_1, \ldots, A_n, C)$ of $n+1$ operator variables

The operator map

$$L(A, \ldots, A_n, C) = CG_n(A_1, \ldots, A_n)^{-1}C,$$

defined in positive definite and invertible operators is thus positively homogeneous, concave, and regular.

If in addition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a bipartite system of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite dimensions then

$$L(\mathrm{Tr}_2 A_1, \ldots, \mathrm{Tr}_2 A_n, \mathrm{Tr}_2 C) \leq \mathrm{Tr}_2 L(A, \ldots, A_n, C).$$

Notice that if $A_1, \ldots, A_n$ commute then

$$L(A_1, \ldots, A_n, C) = CA_1^{-1/n} \cdots A_n^{-1/n} C$$

and in particular $L(A, \ldots, A, C) = CA^{-1}C$.

The map $L(A, B, C)$ of $2 + 1$ operator variables

The geometric mean of two variables is the unique extension of the function $(t, s) \to t^{1/2} s^{1/2}$ to a positively homogeneous, regular and concave operator map. Therefore,

$$L(A, B, C) = CB^{-1/2}\left(B^{1/2}A^{-1}B^{1/2}\right)^{1/2}B^{-1/2}C$$

is the only sensible extension of Lieb-Ruskai's map to $2 + 1$ positive definite and invertible operators with $L(A, B, C) = L(B, A, B)$.

We may also use the weighted geometric mean,

$$G_2(\alpha; A, B) = B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{\alpha}B^{1/2} \qquad 0 \leq \alpha \leq 1,$$

and obtain convexity of the map

$$L(\alpha; A, B, C) = CB^{-1/2}\left(B^{1/2}A^{-1}B^{/12}\right)^{\alpha}B^{-1/2}C.$$

It reduces to $L(\alpha; A, B, C) = CA^{-\alpha}B^{-(1-\alpha)}C$ for commuting A and B.

Another approach to the extended Lieb-Ruskai map

Let A be positive definite and invertible. It is a the well-known fact that a block matrix of the form

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

is positive semi-definite if and only if $B \geq C^* A^{-1} C$.

For $n = 2$ we put

$$L(A, B, C) = C^* G_2(A, B)^{-1} C,$$

where C now is arbitrary and A, B are positive definite and invertible.

Take $\lambda \in [0, 1]$, arbitrary operators C_1, C_2 and positive definite and invertible operators A_1, A_2 and B_1, B_2 and set

$$\begin{aligned} C &= \lambda C_1 + (1 - \lambda) C_2 \\ T &= \lambda C_1^* G_2(A_1, B_1)^{-1} C_1 + (1 - \lambda) C_2^* G_2(A_2, B_2)^{-1} C_2. \end{aligned}$$

Convexity alone without using perspectives

We obtain the equality

$$X = \begin{pmatrix} \lambda G_2(A_1, B_1) + (1-\lambda) G_2(A_2, B_2) & C \\ C^* & T \end{pmatrix}$$

$$= \lambda \begin{pmatrix} G_2(A_1, B_1) & C_1 \\ C_1^* & C_1^* G_2(A_1, B_1)^{-1} C_1 \end{pmatrix}$$

$$+(1-\lambda) \begin{pmatrix} G_2(A_2, B_2) & C_2 \\ C_2^* & C_2^* G_2(A_2, B_2)^{-1} C_2 \end{pmatrix}$$

and observe that the two last block matrices by construction are positive semi-definite.

The block matrix X is thus positive semi-definite; therefore

$$T \geq C^* \bigl(\lambda G_2(A_1, B_1) + (1-\lambda) G_2(A_2, B_2) \bigr)^{-1} C.$$

We thus obtain

$$
\begin{aligned}
&\lambda L(A_1, B_1, C_1) + (1-\lambda)L(A_2, B_2, C_2)\\
&= \lambda C_1^* G_2(A_1, B_1)^{-1} C_1 + (1-\lambda) C_2^* G_2(A_2, B_2)^{-1} C_2 = T\\
&\geq C^* \bigl(\lambda G_2(A_1, B_1) + (1-\lambda) G_2(A_2, B_2)\bigr)^{-1} C\\
&\geq C^* G_2(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2)^{-1} C\\
&= L(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda B_2), \lambda C_1 + (1-\lambda C_2),
\end{aligned}
$$

where we in the last inequality used concavity of the geometric mean and operator convexity of the inverse function.

This construction only uses concavity of G_2. However, if we want $L(A, B, C)$ to be positively homogenous and

$$
L(t, s, 1) = t^{1/2} s^{1/2} \qquad \text{for positive numbers,}
$$

then G_2 is the only regular solution.

Corollary

This way of reasoning extends to any number of variables.

Let G_n be any geometric mean of n positive semi-definite and invertible operators. The operator function

$$L(A_1, \ldots, A_n, C) = C^* G_n(A_1, \ldots, A_n)^{-1} C$$

is convex in arbitrary C and positive definite and invertible $A_1, \ldots, A_n$.

If in addition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a bipartite system of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite dimensions then

$$L(\mathrm{Tr}_2 A_1, \ldots, \mathrm{Tr}_2 A_n, \mathrm{Tr}_2 C) \leq \mathrm{Tr}_2 L(A, \ldots, A_n, C),$$

for arbitrary C and positive definite $A_1, \ldots, A_n$.

If $A_1, \ldots, A_n$ commute then

$$L(A_1, \ldots, A_n, C) = C^* A_1^{1/n} \cdots A_n^{1/n} C.$$