Operator theoretical approach to quantum error correction

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The Hong Kong Polytechnic University

February 15, 2016

Mathematical Aspects in Current Quantum Information Theory
Daejeon, Korea

Based on a joint work with
Zejun Huang (Hunan U) and Shiyu Shi (HK PolyU)

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Outline

- Quantum Error Correction - A very very brief Review
- Bit-flip Quantum Channel
- The $[n, k, d]$ Code
- Fully Correlated Quantum Channel
- Summary

Pauli matrices

- The Pauli matrices, also known as the spin matrices, and defined by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- Notice that for two computational basis states $|0\rangle$ and $|1\rangle$,

$$\begin{array}{rclrclrcl} X|0\rangle & = & |1\rangle & Y|0\rangle & = & i|1\rangle & Z|0\rangle & = & |0\rangle \\ X|1\rangle & = & |0\rangle & Y|1\rangle & = & -i|0\rangle & Z|1\rangle & = & -|1\rangle \end{array}$$

- In general, for any pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$\begin{array}{rclcl} X|\psi\rangle & = & X(\alpha|0\rangle + \beta|1\rangle) & = & \alpha|1\rangle + \beta|0\rangle \\ Y|\psi\rangle & = & Y(\alpha|0\rangle + \beta|1\rangle) & = & i\alpha|1\rangle - i\beta|0\rangle \\ Z|\psi\rangle & = & Z(\alpha|0\rangle + \beta|1\rangle) & = & \alpha|0\rangle - \beta|1\rangle \end{array}$$

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Quantum error correction

A quantum channel $\mathcal{E} : M_n \to M_n$ is a completely positive, trace preserving linear map of the form

$$\mathcal{E} : \rho \mapsto \sum_{j=1}^{r} F_j \rho F_j^\dagger \quad \text{with} \quad \sum_j F_j^\dagger F_j = I.$$ [Choi, LAA 10:285-290 (1975)]

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A Textbook Example: Bit-flip channel [Nakahara, Ohmi, CRC press, 2008]

Suppose in a noisy 3-qubit quantum channel, each qubit flips independent with a probability $p << 1$. Further, assume that at most one of qubits can be flipped. Mathematically, the three-qubit bit-flip channel $\mathcal{E} : M_8 \to M_8$ is defined by

$$\mathcal{E}(\rho) = \sum_{j=1}^{4} F_j \rho F_j^\dagger,$$

with error operators

$$\begin{aligned}
F_1 &= \sqrt{p_1}\, I \otimes I \otimes I, \\
F_2 &= \sqrt{p_2}\, X \otimes I \otimes I, \\
F_3 &= \sqrt{p_3}\, I \otimes X \otimes I, \\
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\end{aligned}$$

where $\sum_{j=1}^{4} p_j = 1$.

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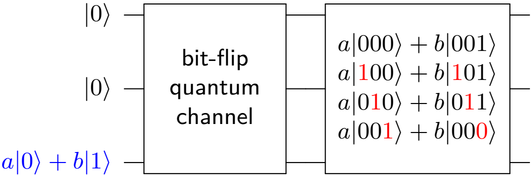
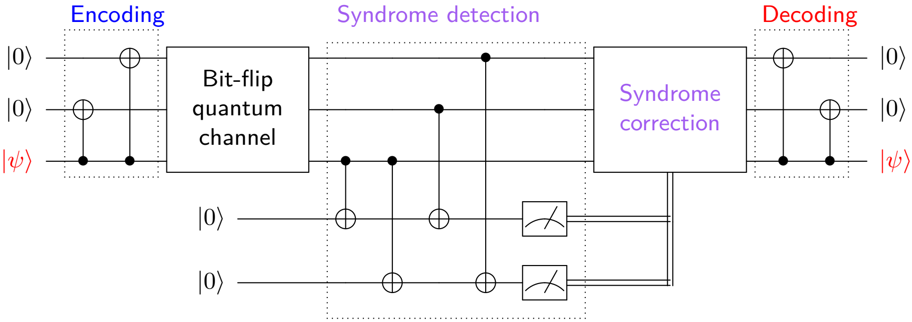


Quantum error correction

Assume that at most one of qubits can be flipped: $|0\rangle \leftrightarrow |1\rangle$



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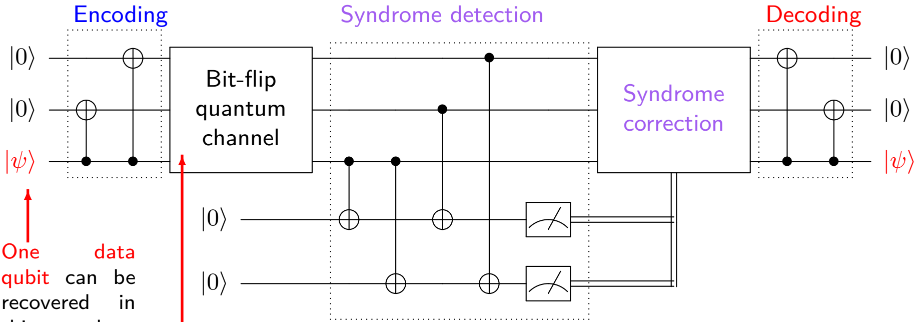

One data qubit can be recovered in this three qubit system.

$$F_1 = \sqrt{p_1}\, I \otimes I \otimes I$$
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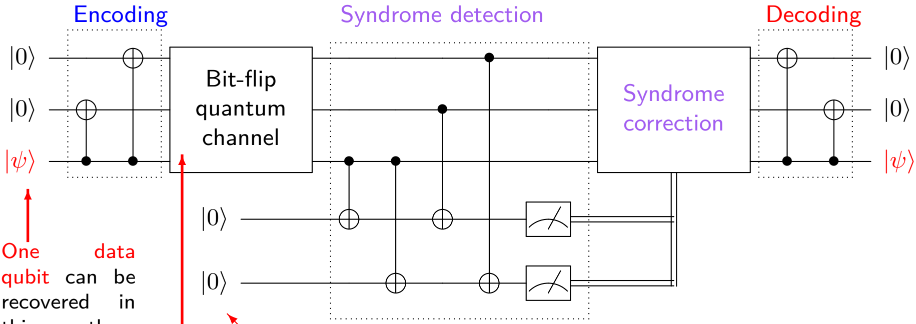

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Two more ancilla qubits are used in syndrome detection.

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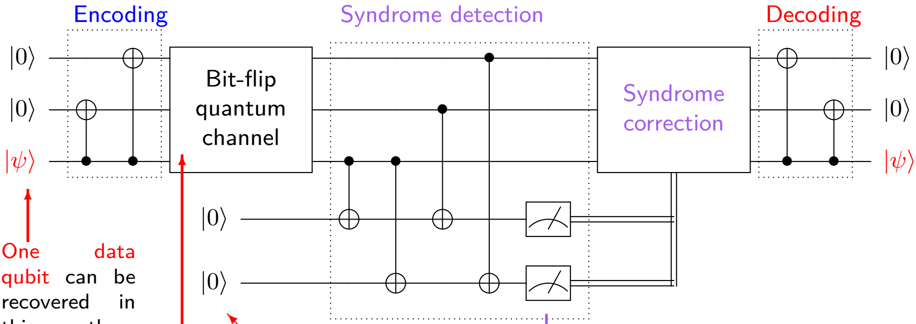

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Possible Outcome:
$|00\rangle$
$|11\rangle$
$|10\rangle$
$|01\rangle$

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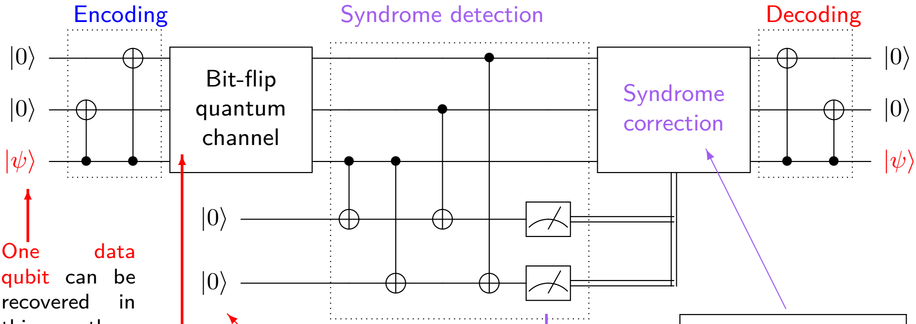

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Possible Outcome:
$|00\rangle$
$|11\rangle$
$|10\rangle$
$|01\rangle$

$\lvert 00\rangle$:	$I \otimes I \otimes I$
$\lvert 01\rangle$:	$X \otimes I \otimes I$
$\lvert 10\rangle$:	$I \otimes X \otimes I$
$\lvert 11\rangle$:	$I \otimes I \otimes X$

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Quantum error correction

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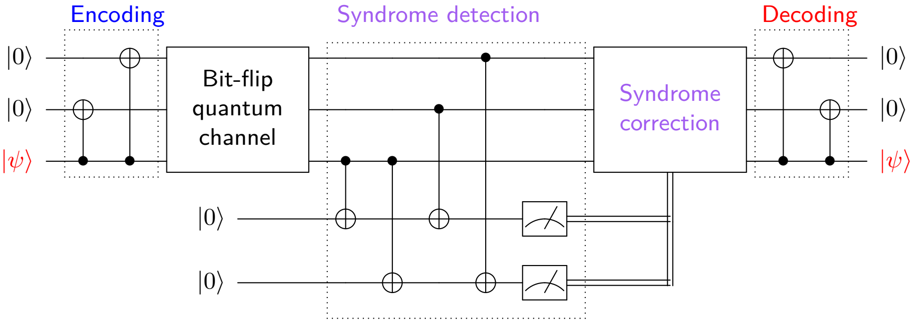

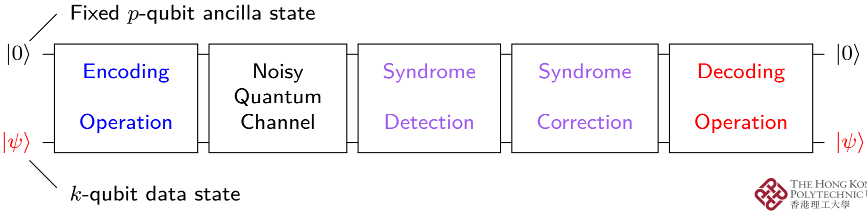

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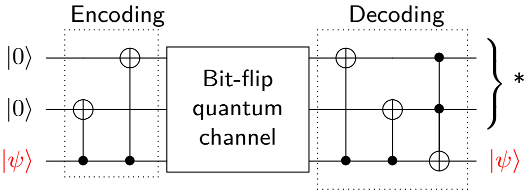
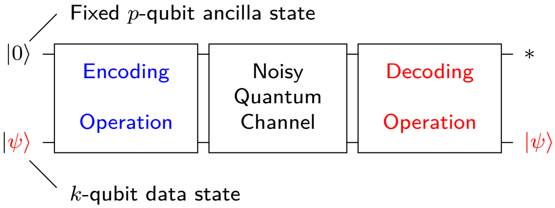
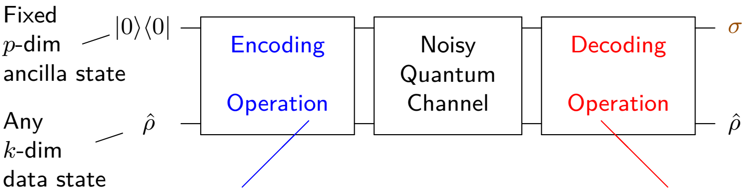


[Nakahara, Tomita arXiv:1101.0413 (2011)]



Quantum Error Correction Code (QECC)



$$|0\rangle\langle 0| \otimes \hat{\rho} \to U(|0\rangle\langle 0| \otimes \hat{\rho})U^\dagger \to \mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \hat{\rho}\right)U^\dagger\right) \to R^\dagger \mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \hat{\rho}\right)U^\dagger\right) R$$

$$= \sigma \otimes \hat{\rho}$$

In the original Knill-Laflamme result, a recovery channel is needed.

$$R^\dagger(\mathcal{E}(\rho))R = \rho \qquad \forall P_{\mathbf{V}}\rho P_{\mathbf{V}} = \rho$$

An orthogonal projection of $\mathbb{C}^n$ to a k dimensional subspace $\mathbf{V}$

Quantum Error Correction Code [Knill and Laflamme, PRA 55:900-911 (1997)]

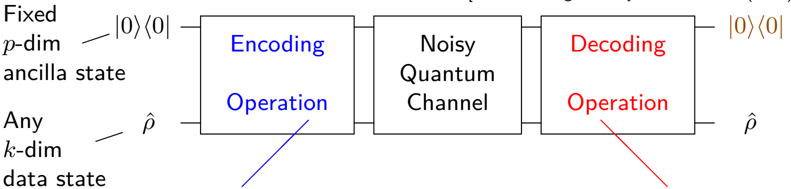
A subspace $\mathbf{V}$ of $\mathbb{C}^n$ is a QECC for $\mathcal{E}$ if and only if

$$P_{\mathbf{V}} F_i^\dagger F_j P_{\mathbf{V}} = \lambda_{ij} P_{\mathbf{V}} \quad \text{for all} \quad 1 \le i, j \le r.$$

[Li, Nakahara, Poon, S., Tomita, QIC 12:149-158 (2012)]

Decoherence Free Subspace (DFS)

[Duan, Guo, PRL 79:1953 (1997)]
[Zanardi, Rasetti, PRL, 79:3306 (1997)]
[Lidar, Chuang, Whaley, PRL, 81:2594 (1998)]



$$|0\rangle\langle 0| \otimes \hat{\rho} \to U(|0\rangle\langle 0| \otimes \hat{\rho})U^\dagger \to \mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \hat{\rho}\right)U^\dagger\right) \to U^\dagger \mathcal{E}\left(U\left(|0\rangle\langle 0| \otimes \hat{\rho}\right)U^\dagger\right)U$$

$$= |0\rangle\langle 0| \otimes \hat{\rho}$$

$$\mathcal{E}(\rho) = \rho \qquad \forall P_{\mathbf{V}} \rho P_{\mathbf{V}} = \rho$$

An orthogonal projection of $\mathbb{C}^n$ to a k-dimensional subspace $\mathbf{V}$

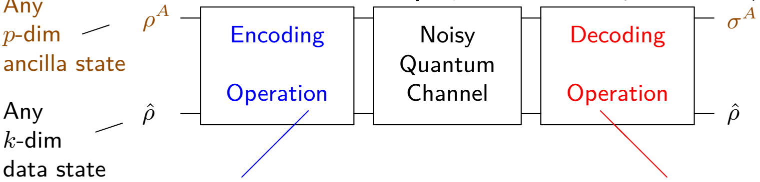
Decoherence Free Subspace [Kribs, Laflamme, Poulin, Lesosky, QIC 6:383-399 (2006)]

A subspace $\mathbf{V}$ of $\mathbb{C}^n$ is a DFS for $\mathcal{E}$ if and only if

$$F_j P_{\mathbf{V}} = \lambda_j P_{\mathbf{V}} \quad \text{for all} \quad 1 \le j \le r.$$

Noiseless Subsystem (NS)

[Knill, Laflamme, Viola, PRL 84:2525 (2000)]
[Zanardi, PRA 63:12301 (2001)]
[Kempe, Bacon, Lidar, Whaley, PRA 63:42307 (2001)]



$$\rho^A \otimes \hat{\rho} \to U(\rho^A \otimes \hat{\rho})U^\dagger \to \mathcal{E}\left(U\left(\rho^A \otimes \hat{\rho}\right)U^\dagger\right) \to U^\dagger \mathcal{E}\left(U\left(\rho^A \otimes \hat{\rho}\right)U^\dagger\right)U = \sigma^A \otimes \hat{\rho}$$

$\updownarrow$

$$\mathcal{E}(\rho^A \otimes \hat{\rho}) = \sigma^A \otimes \hat{\rho}$$

Noiseless Subsystem [Kribs, Laflamme, Poulin, Lesosky, QIC 6:383-399 (2006)]

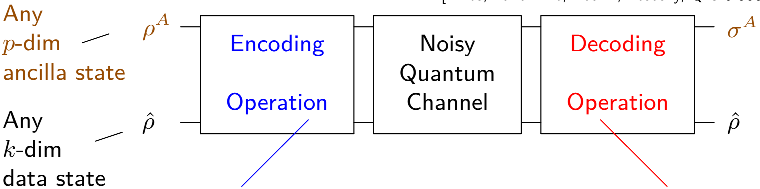
A subsystem B of $A \otimes B$ is a NS for $\mathcal{E}$ if and only if

$$\begin{aligned} F_j P_{AB} &= P_{AB} F_j P_{AB} \quad \forall 1 \le j \le r \\ P_{kk} F_j P_{\ell\ell} &= \lambda_{jk\ell} P_{k\ell} \quad \forall 1 \le j \le r, 1 \le k, \ell \le p, \end{aligned}$$

where $P_{k\ell} = |x_k\rangle\langle x_\ell| \otimes I_B$ for $1 \le k, \ell \le p$ and $P_{AB} = \sum_{k=1}^{p} P_{kk}$.

Operator Quantum Error Correction (OQEC)

[Kribs, Laflamme, Poulin, Lesosky, QIC 6:383-399 (2006)]



$$\rho^A \otimes \hat{\rho} \to U(\rho^A \otimes \hat{\rho})U^\dagger \to \mathcal{E}\left(U\left(\rho^A \otimes \hat{\rho}\right)U^\dagger\right) \to R^\dagger \mathcal{E}\left(U\left(\rho^A \otimes \hat{\rho}\right)U^\dagger\right) R = \sigma^A \otimes \hat{\rho}$$

Unitarily Recoverable Subsystem (URS) [Kribs, Spekkens, PRA 74:042329 (2006)]

A subsystem B of $A \otimes B$ is a CS for $\mathcal{E}$ if and only if

$$P_{kk} F_i^* F_j P_{\ell\ell} = \lambda_{ijk\ell} P_{k\ell} \quad \forall 1 \le i,j \le r, 1 \le k,\ell \le p$$

where $P_{k\ell} = |x_k\rangle\langle x_\ell| \otimes I_B$ for $1 \le k,\ell \le p$ and $P_{AB} = \sum_{k=1}^{p} P_{kk}$.

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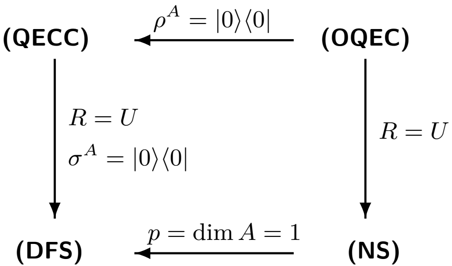
Quantum Error Correction



The Textbook Example: Three Qubit Bit-flip Quantum Channel

$$\mathcal{E} : \rho \mapsto F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger + F_3 \rho F_3^\dagger + F_4 \rho F_4^\dagger$$

$$\begin{aligned} F_1 &= \sqrt{q_1}\, I \otimes I \otimes I \\ F_2 &= \sqrt{q_2}\, X \otimes I \otimes I \\ F_3 &= \sqrt{q_3}\, I \otimes X \otimes I \\ F_4 &= \sqrt{q_4}\, I \otimes I \otimes X \end{aligned}$$

Assume that at most one of qubits can be flipped!

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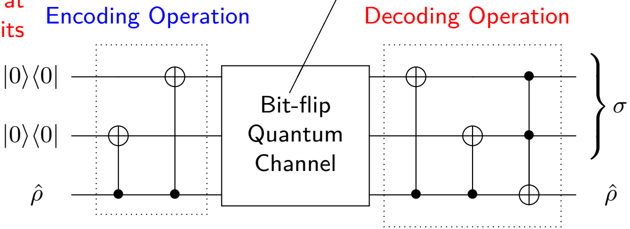

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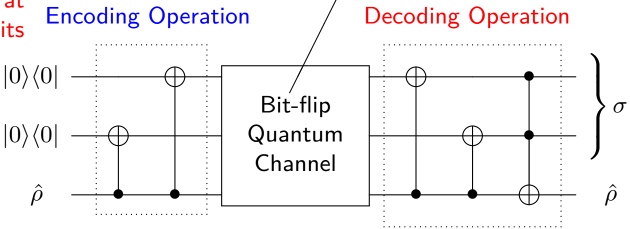

$\mathbf{V} = \text{span}\{|000\rangle, |111\rangle\}$

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$$F_1 = \sqrt{q_1}\, I \otimes I \otimes I$$
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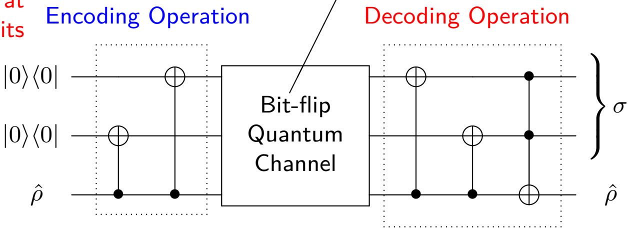

$\mathbf{V} = \mathrm{span}\{|000\rangle, |111\rangle\}$

[Nakahara, Tomita arXiv:1101.0413 (2011)]

Let $P_\mathbf{V} = |000\rangle\langle 000| + |111\rangle\langle 111|$. Then

$$P_\mathbf{V} F_i^\dagger F_j P_\mathbf{V} = \lambda_{ij} P_\mathbf{V} \quad \text{with} \quad [\lambda_{ij}] = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}$$

Bit-flip Quantum Channel

The Textbook Example: Three Qubit Bit-flip Quantum Channel

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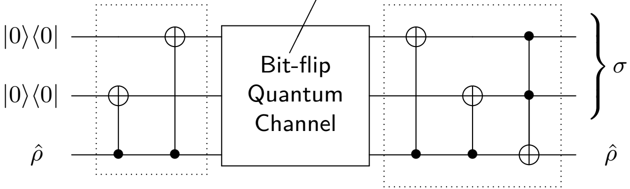
Encoding Operation

Decoding Operation



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$$\begin{aligned} |000\rangle &\longrightarrow |000\rangle \\ |001\rangle &\longrightarrow |111\rangle \end{aligned}$$

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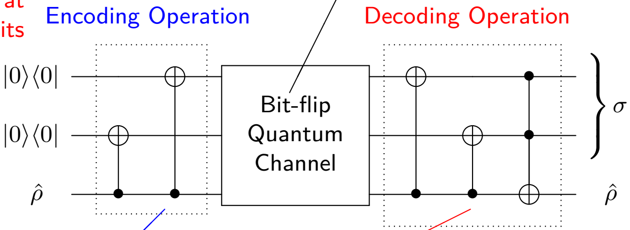

$$\mathbf{V} = \operatorname{span}\{|000\rangle, |111\rangle\}$$

[Nakahara, Tomita arXiv:1101.0413 (2011)]

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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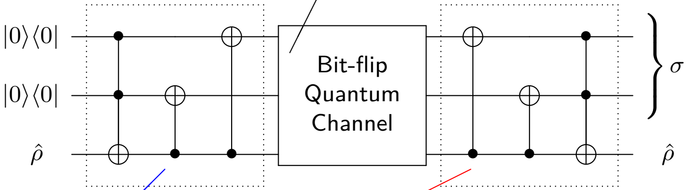


$\mathbf{V} = \mathrm{span}\{|000\rangle, |111\rangle\}$

[Nakahara, Tomita arXiv:1101.0413 (2011)]

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Textbook Example: Three Qubit Bit-flip Quantum Channel

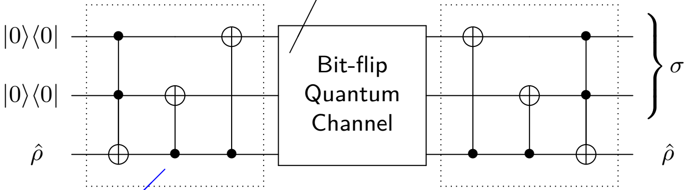
$$\mathcal{E} : \rho \mapsto F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger + F_3 \rho F_3^\dagger + F_4 \rho F_4^\dagger$$

$$\begin{aligned} F_1 &= \sqrt{q_1}\, I \otimes I \otimes I \\ F_2 &= \sqrt{q_2}\, X \otimes I \otimes I \\ F_3 &= \sqrt{q_3}\, I \otimes X \otimes I \\ F_4 &= \sqrt{q_4}\, I \otimes I \otimes X \end{aligned}$$

Assume that at most one of qubits can be flipped!

Encoding Operation

Decoding Operation



$\mathbf{V} = \mathrm{span}\{|000\rangle, |111\rangle\}$

[Nakahara, Tomita arXiv:1101.0413 (2011)]

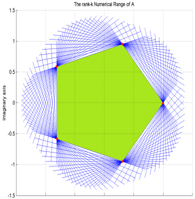
$$R = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ |000\rangle & |111\rangle & F_2|000\rangle & F_2|111\rangle & F_1|000\rangle & F_1|111\rangle & F_3|000\rangle & F_3|111\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

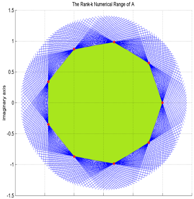
In connection to Quantum Error Correction, Choi, et al suggested

Rank-k numerical range [Choi, Kribs, and Zyczkowski LAA 418:828-839 (2006)]

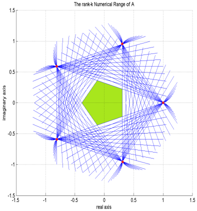
The rank-k numerical range of A on $\mathcal{B}(\mathcal{H})$ is defined by

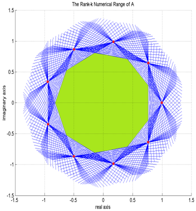
$$\Lambda_k(A) = \{\mu \in \mathbb{C} : PAP = \mu P \text{ for some rank-}k \text{ orthogonal projection P}\}.$$

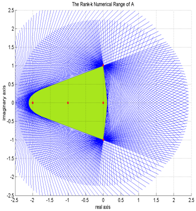


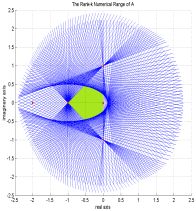


[Choi, Giesinger, Holbrook, Kribs, LAMA 56:53-64 (2008)]
[Choi, Holbrook, Kribs, Zyczkowski, OAM 1:409-426 (2007)]
[Choi, Kribs, Zyczkowski, RMP 58:77-91 (2006)]
[Li, Poon, S., JMAA 348:843-855 (2008)]
[Li, Poon, S., LAMA 57:365-368 (2009)]
[Li, S., PAMS 136:3013-3023 (2008)]
[Woerdeman, LAMA 56:65-67 (2008)]





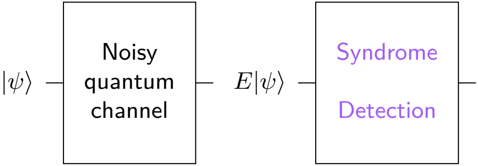




The $[n, k, d]$ code

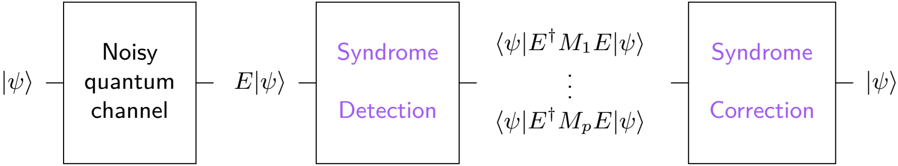
E: Error operators

M_j: Measurement operators



E: Error operators

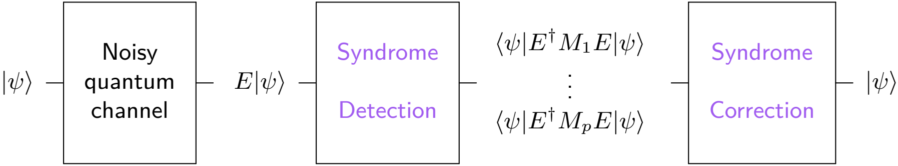
M_j: Measurement operators



The $[n, k, d]$ code

E: Error operators

M_j: Measurement operators



$$E_a \neq E_b \quad \Longleftrightarrow \quad \begin{pmatrix} \langle\psi|E_a^\dagger M_1 E_a|\psi\rangle \\ \vdots \\ \langle\psi|E_a^\dagger M_p E_a|\psi\rangle \end{pmatrix} \neq \begin{pmatrix} \langle\psi|E_b^\dagger M_1 E_b|\psi\rangle \\ \vdots \\ \langle\psi|E_b^\dagger M_p E_b|\psi\rangle \end{pmatrix}$$

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The $[n, k, d]$ code

- Consider the following local operations on an n-qubit system

$$E = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_n \quad \text{with} \quad \sigma_j \in \{I, X, Y, Z\}.$$

The $[n, k, d]$ code

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$$E = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_n \quad \text{with} \quad \sigma_j \in \{I, X, Y, Z\}.$$

- The weight of the operator E is defined to be the number of states σ_j which it differs from I, i.e.,

$$w(E) = \#\{j : \sigma_j \neq I\}.$$

For example, in a 5-qubit system,

$$w(X \otimes Y \otimes I \otimes I \otimes I) = 2 \quad \text{and} \quad w(X \otimes I \otimes I \otimes Y \otimes Y) = 3.$$

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$$w(X \otimes Y \otimes I \otimes I \otimes I) = 2 \quad \text{and} \quad w(X \otimes I \otimes I \otimes Y \otimes Y) = 3.$$

- The distance between two operators E_a and E_b is defined to be

$$d(E_a, E_b) = w(E_a^\dagger E_b).$$

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The $[n, k, d]$ code

- Let S be a set of commuting Pauli matrices in the n-qubit system and $\{M_1, M_2, \ldots, M_p\}$ are the generators of the set. Let

$$\mathbf{V} = \{|\psi\rangle : M|\psi\rangle = |\psi\rangle, \forall M \in S\}.$$

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- For any error E and $|\psi\rangle \in \mathbf{V}$, if

$$ME|\psi\rangle = -E|\psi\rangle \quad \Longleftrightarrow \quad ME = -EM,$$

then M can detect E. Otherwise,

$$ME|\psi\rangle = E|\psi\rangle \quad \Longleftrightarrow \quad ME = EM.$$

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then M can detect E. Otherwise,

$$ME|\psi\rangle = E|\psi\rangle \quad \Longleftrightarrow \quad ME = EM.$$

- Then

$$\langle\psi|E^\dagger ME|\psi\rangle = \begin{cases} 1 & \text{if } ME = EM \\ -1 & \text{if } ME = -EM \end{cases}$$

Set

$$f_M(E) = \begin{cases} 1 & \text{if } ME = EM \\ -1 & \text{if } ME = -EM \end{cases}$$

The $[n, k, d]$ code

- The generators $\{M_1, \ldots, M_p\}$ can distinguish E_a and E_b if
$$\exists M_j \in S \quad \text{s.t.} \quad f_{M_j}(E_a) \neq f_{M_j}(E_b).$$

- The subspace $\mathbf{V}$ of $\mathbb{C}^{2^n}$ with stabilizer S is an $[n, k, d]$ code if
  - $\dim(\mathbf{V}) = 2^k$,
  - $\{M_1, \ldots, M_p\}$ can distinguish E_a and E_b for any $d(E_a, E_b) < d$.

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  - $\{M_1, \ldots, M_p\}$ can distinguish E_a and E_b for any $d(E_a, E_b) < d$.

- An $[n, k, d]$ code $\mathbf{V}$ is a QECC for the error pattern E with

$$w(E) < \frac{d}{2}.$$

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Calderbank-Shor-Steane $[7,1,3]$ code

[Calderbank and Shor, PRA 54:1098 (1996) and Steane PRL 77: 793(1996)]

$$
\begin{aligned}
M_1 &= I \otimes I \otimes I \otimes X \otimes X \otimes X \otimes X \\
M_2 &= I \otimes X \otimes X \otimes I \otimes I \otimes X \otimes X \\
M_3 &= X \otimes I \otimes X \otimes I \otimes X \otimes I \otimes X \\
M_4 &= I \otimes I \otimes I \otimes Z \otimes Z \otimes Z \otimes Z \\
M_5 &= I \otimes Z \otimes Z \otimes I \otimes I \otimes Z \otimes Z \\
M_6 &= Z \otimes I \otimes Z \otimes I \otimes Z \otimes I \otimes Z
\end{aligned}
$$

$\mathbf{V} = \text{span } \{|v_1\rangle, |v_2\rangle\}$ with

$$
\begin{aligned}
|v_1\rangle &= \frac{1}{\sqrt{8}}(|0000000\rangle + |1111000\rangle + |1100110\rangle + |1010101\rangle \\
&\quad + |0011110\rangle + |0101101\rangle + |0110011\rangle + |1001011\rangle) \\
|v_2\rangle &= \frac{1}{\sqrt{8}}(|0000111\rangle + |1111111\rangle + |1100001\rangle + |1010010\rangle \\
&\quad + |0011001\rangle + |0101010\rangle + |0110100\rangle + |1001100\rangle)
\end{aligned}
$$

The $[n, k, d]$ code

			M_1	M_2	M_3	M_4	M_5	M_6
X_1	$=$	$X \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I$	1	1	1	1	1	-1
X_2	$=$	$I \otimes X \otimes I \otimes I \otimes I \otimes I \otimes I$	1	1	1	1	-1	1
X_3	$=$	$I \otimes I \otimes X \otimes I \otimes I \otimes I \otimes I$	1	1	1	1	-1	-1
X_4	$=$	$I \otimes I \otimes I \otimes X \otimes I \otimes I \otimes I$	1	1	1	-1	1	1
X_5	$=$	$I \otimes I \otimes I \otimes I \otimes X \otimes I \otimes I$	1	1	1	-1	1	-1
X_6	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes X \otimes I$	1	1	1	-1	-1	1
X_7	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes X$	1	1	1	-1	-1	-1
Z_1	$=$	$Z \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I$	1	1	-1	1	1	1
Z_2	$=$	$I \otimes Z \otimes I \otimes I \otimes I \otimes I \otimes I$	1	-1	1	1	1	1
Z_3	$=$	$I \otimes I \otimes Z \otimes I \otimes I \otimes I \otimes I$	1	-1	-1	1	1	1
Z_4	$=$	$I \otimes I \otimes I \otimes Z \otimes I \otimes I \otimes I$	-1	1	1	1	1	1
Z_5	$=$	$I \otimes I \otimes I \otimes I \otimes Z \otimes I \otimes I$	-1	1	-1	1	1	1
Z_6	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes Z \otimes I$	-1	-1	1	1	1	1
Z_7	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes Z$	-1	-1	-1	1	1	1
Y_1	$=$	$Y \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I$	1	1	-1	1	1	-1
Y_2	$=$	$I \otimes Y \otimes I \otimes I \otimes I \otimes I \otimes I$	1	-1	1	1	-1	1
Y_3	$=$	$I \otimes I \otimes Y \otimes I \otimes I \otimes I \otimes I$	1	-1	-1	1	-1	-1
Y_4	$=$	$I \otimes I \otimes I \otimes Y \otimes I \otimes I \otimes I$	-1	1	1	-1	1	1
Y_5	$=$	$I \otimes I \otimes I \otimes I \otimes Y \otimes I \otimes I$	-1	1	-1	-1	1	-1
Y_6	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes Y \otimes I$	-1	-1	1	-1	-1	1
Y_7	$=$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes Y$	-1	-1	-1	-1	-1	-1

The $[n, k, d]$ code

- Shor $[9, 1, 3]$ code [Shor, PRA 52:2493 (1995)]

  $\mathbf{V} = \mathrm{span}\ \{|v_1\rangle, |v_2\rangle\}$ with

$$
\begin{aligned}
|v_1\rangle &= \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \\
|v_2\rangle &= \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)
\end{aligned}
$$

The $[n, k, d]$ code

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  $\mathbf{V} = \text{span } \{|v_1\rangle, |v_2\rangle\}$ with

$$|v_1\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

$$|v_2\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$

- Calderbank-Shor-Steane $[7, 1, 3]$ code

  [Calderbank and Shor, PRA 54:1098 (1996)and Steane PRL 77: 793(1996)]

  $\mathbf{V} = \text{span } \{|v_1\rangle, |v_2\rangle\}$ with

$$|v_1\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |1111000\rangle + |1100110\rangle + |1010101\rangle + |0011110\rangle + |0101101\rangle + |0110011\rangle + |1001011\rangle)$$

$$|v_2\rangle = \frac{1}{\sqrt{8}}(|0000111\rangle + |1111111\rangle + |1100001\rangle + |1010010\rangle + |0011001\rangle + |0101010\rangle + |0110100\rangle + |1001100\rangle)$$

The $[n, k, d]$ code

- The $[5,1,3]$ code [DiVincenzo & Shor, PRL 77:3260 (1996)]

  $\mathbf{V} = \text{span}\,\{|v_1\rangle, |v_2\rangle\}$ with

$$\begin{aligned} |v_1\rangle &= \frac{1}{4}(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ &\quad +|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ &\quad -|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\ &\quad -|10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle) \end{aligned}$$

$$\begin{aligned} |v_2\rangle &= \frac{1}{4}(|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\ &\quad +|10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\ &\quad -|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle \\ &\quad -|01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle) \end{aligned}$$

$$\begin{aligned} M_1 &= Z \otimes X \otimes X \otimes Z \otimes I & M_2 &= I \otimes Z \otimes X \otimes X \otimes Z \\ M_3 &= Z \otimes I \otimes Z \otimes X \otimes X & M_4 &= X \otimes Z \otimes I \otimes Z \otimes X \end{aligned}$$

The $[n, k, d]$ code

- $[8, 3, 3]$ code [Calderbank et al., PRL 78:405 (1997)]
- $[2^r, 2^r - j - 2, 3]$ code [Gottesman, PRA 54:1862 (1996)]
- $\left[2^r,\ 2^r - {}_rC_p - 2\sum_{j=0}^{p} {}_rC_j,\ 2^p + 2^p + 2^{p-1}\right]$ code [Steane, PRL 77:793 (1996)]
- $((9, 12, 3))$ code [Yu, Chen, Lai, Oh, PRL 101:090501 (2008)]
- $((10, 20, 3))$ code [Cross, Smith, Smolin, Zeng IEEE TIT 55:433-438 (2009)]
- $[16, 7, 4]$ code [Looi, Yu, Gheorghiu, Griffiths, PRA 78:042303 (2008)]
- $[85, 77, 3]$ code [Grassl, Shor, Smith, Smolin, Zeng, PRA 79.050306 (2009)]
- $\cdots\cdots$

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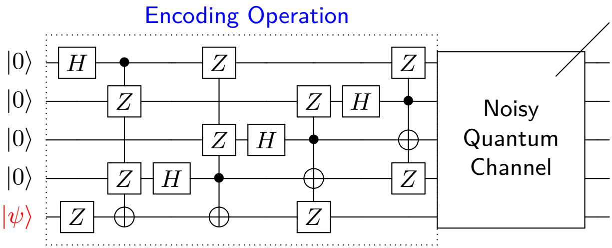
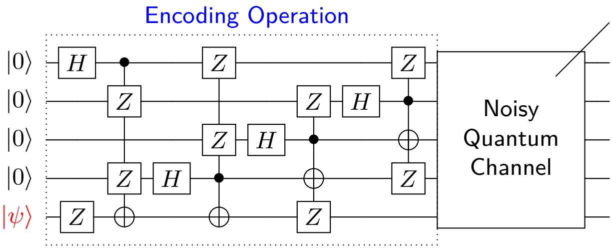


All error has the form $E = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_4 \otimes \sigma_5$ with $\sigma_j \in \{I, X, Y, Z\}$ and at most one of $\sigma_j \neq I$.



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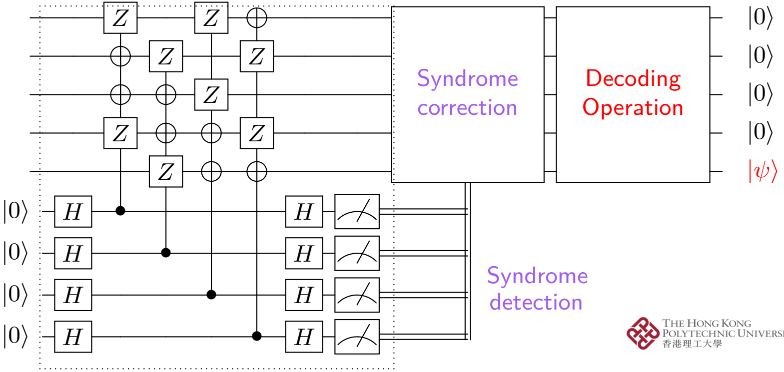
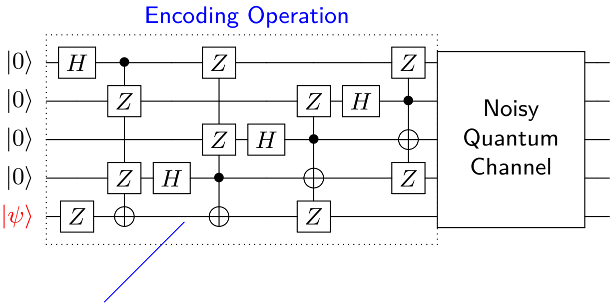
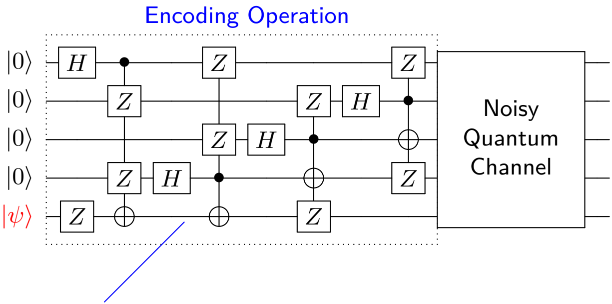




$U :$ $|v_0\rangle = U|00000\rangle$ and $|v_1\rangle = U|00001\rangle$

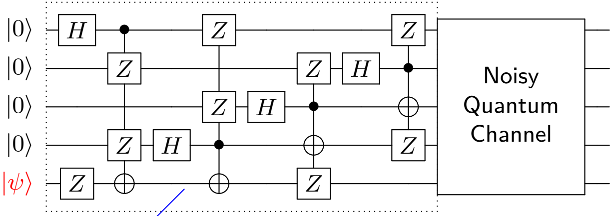


U : $|v_0\rangle = U|00000\rangle$ and $|v_1\rangle = U|00001\rangle$

Clearly, $\langle v_1|v_2\rangle = 0$

The $[5, 1, 3]$ code

Encoding Operation



U : $\quad |v_0\rangle = U|00000\rangle$ and $|v_1\rangle = U|00001\rangle$

Clearly, $\langle v_1|v_2\rangle = 0$

Also $\langle v_i|E_a^\dagger E_b|v_j\rangle = \langle 0000i|\, U^\dagger E_a^\dagger E_b U \,|0000j\rangle$

for all $i, j = 0, 1$ and $E_a, E_b \in \{X_i, Y_j, Z_j\}$

The $[5,1,3]$ code

$$
\begin{aligned}
X_1|v_0\rangle &= -U|10111\rangle & X_1|v_1\rangle &= -U|10110\rangle \\
X_2|v_0\rangle &= -U|11101\rangle & X_2|v_1\rangle &= -U|11100\rangle \\
X_3|v_0\rangle &= U|00111\rangle & X_3|v_1\rangle &= -U|00110\rangle \\
X_4|v_0\rangle &= U|10011\rangle & X_4|v_1\rangle &= -U|10010\rangle \\
X_5|v_0\rangle &= U|11111\rangle & X_5|v_1\rangle &= -U|11110\rangle
\end{aligned}
$$

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The $[5,1,3]$ code

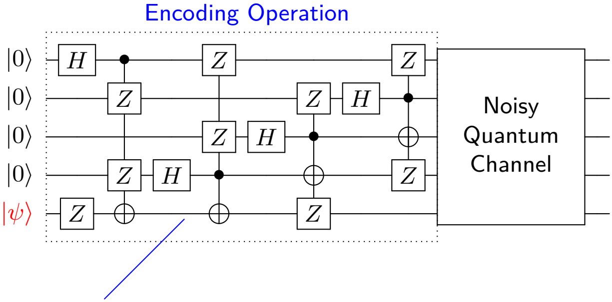
$$
\begin{aligned}
X_1|v_0\rangle &= -U|10111\rangle & X_1|v_1\rangle &= -U|10110\rangle \\
X_2|v_0\rangle &= -U|11101\rangle & X_2|v_1\rangle &= -U|11100\rangle \\
X_3|v_0\rangle &= U|00111\rangle & X_3|v_1\rangle &= -U|00110\rangle \\
X_4|v_0\rangle &= U|10011\rangle & X_4|v_1\rangle &= -U|10010\rangle \\
X_5|v_0\rangle &= U|11111\rangle & X_5|v_1\rangle &= -U|11110\rangle \\
Y_1|v_0\rangle &= -U|10101\rangle & Y_1|v_1\rangle &= U|10100\rangle \\
Y_2|v_0\rangle &= U|01101\rangle & Y_2|v_1\rangle &= U|01100\rangle \\
Y_3|v_0\rangle &= -U|01001\rangle & Y_3|v_1\rangle &= U|01000\rangle \\
Y_4|v_0\rangle &= -U|01011\rangle & Y_4|v_1\rangle &= U|01010\rangle \\
Y_5|v_0\rangle &= U|11011\rangle & Y_5|v_1\rangle &= U|11010\rangle
\end{aligned}
$$

The $[5,1,3]$ code

$$
\begin{aligned}
X_1|v_0\rangle &= -U|10111\rangle & X_1|v_1\rangle &= -U|10110\rangle \\
X_2|v_0\rangle &= -U|11101\rangle & X_2|v_1\rangle &= -U|11100\rangle \\
X_3|v_0\rangle &= U|00111\rangle & X_3|v_1\rangle &= -U|00110\rangle \\
X_4|v_0\rangle &= U|10011\rangle & X_4|v_1\rangle &= -U|10010\rangle \\
X_5|v_0\rangle &= U|11111\rangle & X_5|v_1\rangle &= -U|11110\rangle \\
Y_1|v_0\rangle &= -U|10101\rangle & Y_1|v_1\rangle &= U|10100\rangle \\
Y_2|v_0\rangle &= U|01101\rangle & Y_2|v_1\rangle &= U|01100\rangle \\
Y_3|v_0\rangle &= -U|01001\rangle & Y_3|v_1\rangle &= U|01000\rangle \\
Y_4|v_0\rangle &= -U|01011\rangle & Y_4|v_1\rangle &= U|01010\rangle \\
Y_5|v_0\rangle &= U|11011\rangle & Y_5|v_1\rangle &= U|11010\rangle \\
Z_1|v_0\rangle &= U|00010\rangle & Z_1|v_1\rangle &= -U|00011\rangle \\
Z_2|v_0\rangle &= U|10000\rangle & Z_2|v_1\rangle &= U|10001\rangle \\
Z_3|v_0\rangle &= U|01110\rangle & Z_3|v_1\rangle &= U|01111\rangle \\
Z_4|v_0\rangle &= U|11000\rangle & Z_4|v_1\rangle &= U|11001\rangle \\
Z_5|v_0\rangle &= U|00100\rangle & Z_5|v_1\rangle &= U|00101\rangle
\end{aligned}
$$

The $[5, 1, 3]$ code

Error	Vector	Action
X_1	$U\|1011j\rangle$	$-X$
X_2	$U\|1110j\rangle$	$-X$
X_3	$U\|0011j\rangle$	Y
X_4	$U\|1001j\rangle$	Y
X_5	$U\|1111j\rangle$	Y
Y_1	$U\|1010j\rangle$	Y
Y_2	$U\|0110j\rangle$	X
Y_3	$U\|0100j\rangle$	Y
Y_4	$U\|0101j\rangle$	Y
Y_5	$U\|1101j\rangle$	X
Z_1	$U\|0001j\rangle$	Z
Z_2	$U\|1000j\rangle$	I
Z_3	$U\|0111j\rangle$	I
Z_4	$U\|1100j\rangle$	I
Z_5	$U\|0010j\rangle$	I

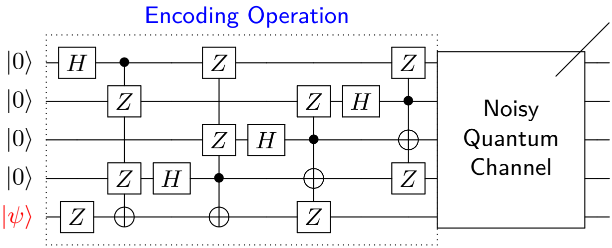


U : $|v_0\rangle = U|00000\rangle$ and $|v_1\rangle = U|00001\rangle$

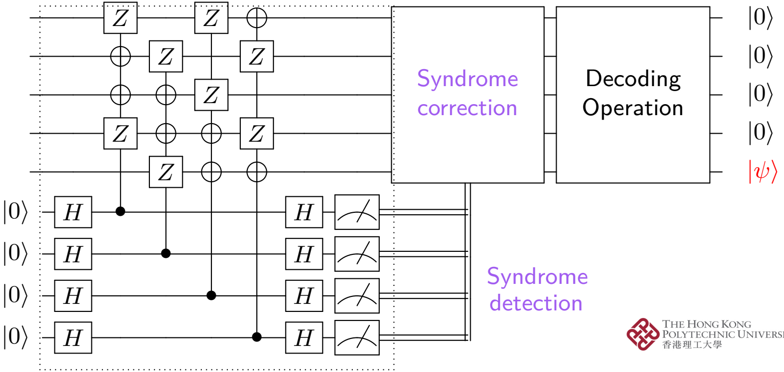
Construct a recovery operation R such that

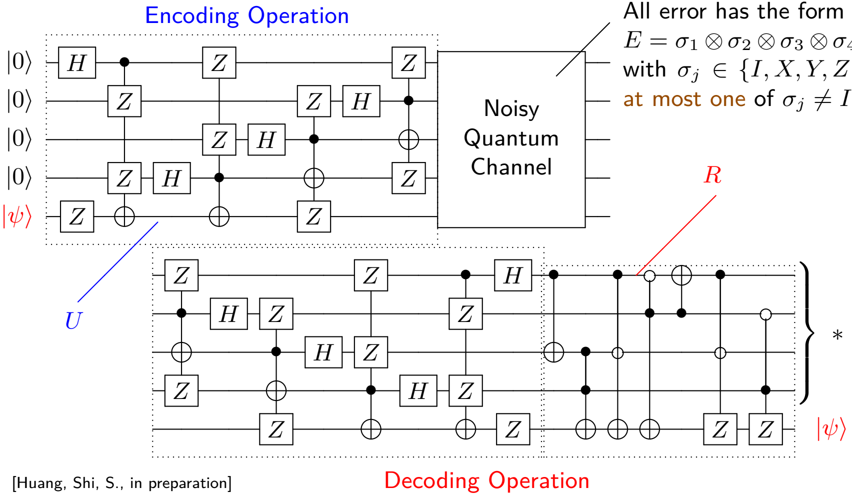
$$REU|0000\rangle|j\rangle = U|x_1x_2x_3x_4\rangle|j\rangle \quad \text{for all } i = 0, 1, \text{ and } E \in \{X_i, Y_j, Z_k\}.$$

The $[5, 1, 3]$ code



All error has the form $E = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_4 \otimes \sigma_5$ with $\sigma_j \in \{I, X, Y, Z\}$ and at most one of $\sigma_j \neq I$.





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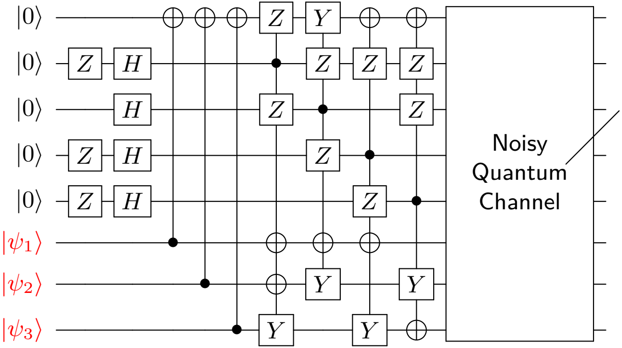
[Huang, Shi, S., in preparation]

No syndrome detection and correction and no additional ancilla qubit is needed!

The $[8, 3, 3]$ code

Encoding Operation



All error has the form $E = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_8$ with $\sigma_j \in \{I, X, Y, Z\}$ and at most one of $\sigma_j \neq I$.

[Calderbank et al., PRL 78:405 (1997)]

The $[8, 3, 3]$ code can be constructed by the following stabilizer code with operators

$$
\begin{aligned}
M_1 &= X \otimes Z \otimes X \otimes Z \otimes I \otimes I \otimes X \otimes X \otimes Y \\
M_2 &= X \otimes Y \otimes Z \otimes X \otimes Z \otimes I \otimes X \otimes Y \otimes I \\
M_3 &= X \otimes X \otimes Z \otimes I \otimes X \otimes Z \otimes X \otimes I \otimes Y \\
M_4 &= X \otimes X \otimes Z \otimes Z \otimes I \otimes X \otimes I \otimes Y \otimes X \\
M_5 &= Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z
\end{aligned}
$$

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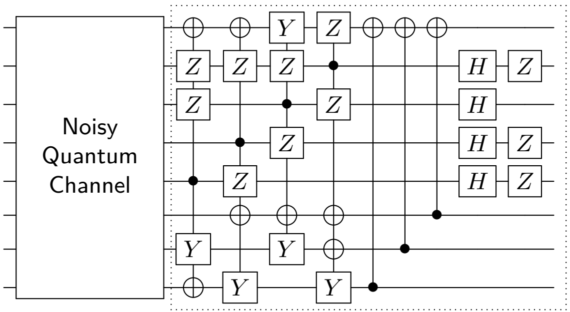
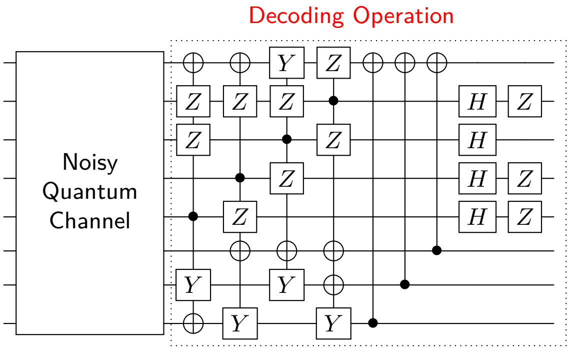
The $[8,3,3]$ code

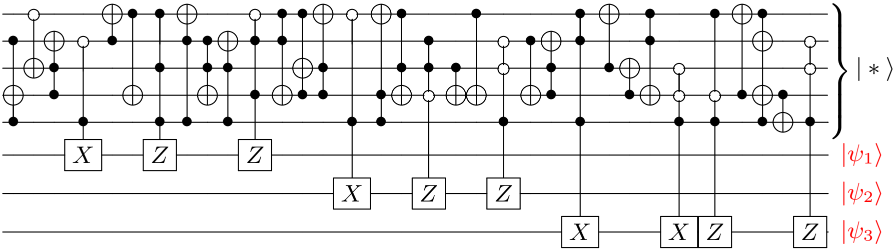
Decoding Operation



The $[8, 3, 3]$ code



Decoding Operation
Noisy Quantum Channel



$|*\rangle$
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$

Three Qubit Fully Correlated Quantum Channel (QECC)

A noisy quantum channel is called fully correlated when all the qubits constituting the codeword are subject to the same error operators.

- Size of the system $= \sim$ a few micrometers
- The wavelength of external disturbance $= \sim$ a few millimeters

$$\rho \mapsto \sum_{j=1}^{4} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{ I^{\otimes n}, X^{\otimes n}, Y^{\otimes n}, Z^{\otimes n} \right\}$$

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Fully Correlated Quantum Channel

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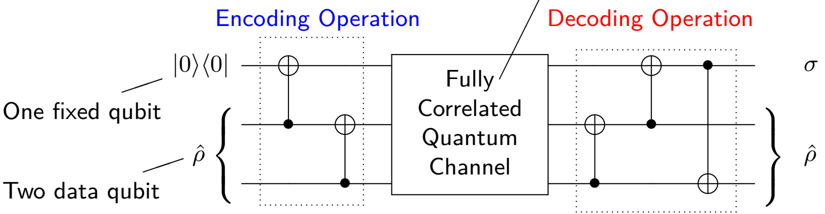

[Li, Nakahara, Poon, S., Tomita, PLA 375:3255-3258 (2011)]

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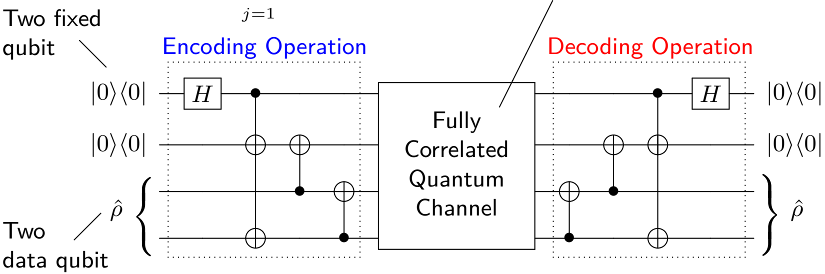
Fully Correlated Quantum Channel

Four Qubit Fully Correlated Quantum Channel (DFS)

$$\rho \mapsto \sum_{j=1}^{4} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{ I^{\otimes n}, X^{\otimes n}, Y^{\otimes n}, Z^{\otimes n} \right\}$$

Four Qubit Fully Correlated Quantum Channel (DFS)

$$\rho \mapsto \sum_{j=1}^{4} p_j F_j \rho F_j^{\dagger} \qquad F_j \in \left\{ I^{\otimes n}, X^{\otimes n}, Y^{\otimes n}, Z^{\otimes n} \right\}$$



Fully Correlated Quantum Channel

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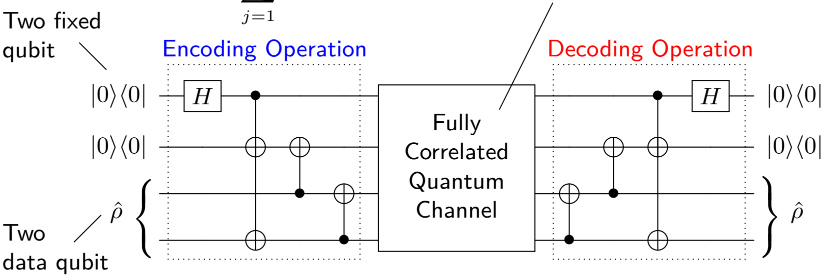
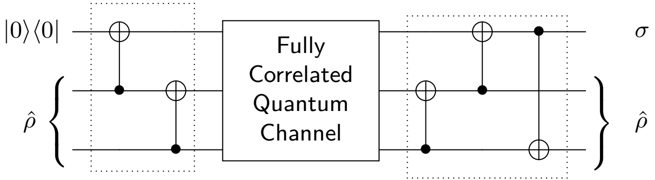


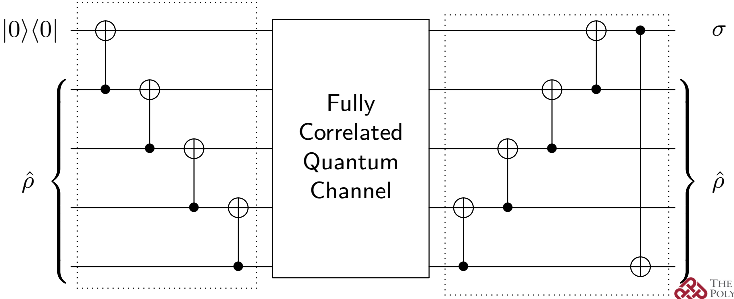
Fully Correlated Channel [Li, Nakahara, Poon, S., Tomita, PLA 375:3255-3258 (2011)]

Odd n: one can encode $(n-1)$-data qubit states to n-qubit codewords

Even n: one can encode $(n-2)$-data qubit states to n-qubit codewords

Fully Correlated Quantum Channel





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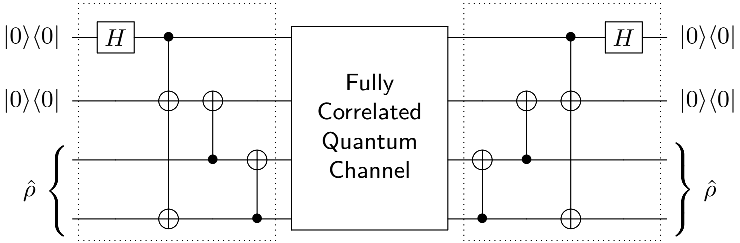
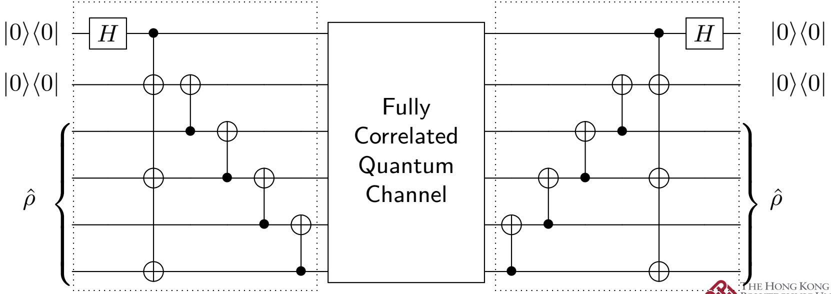
Fully Correlated Quantum Channel





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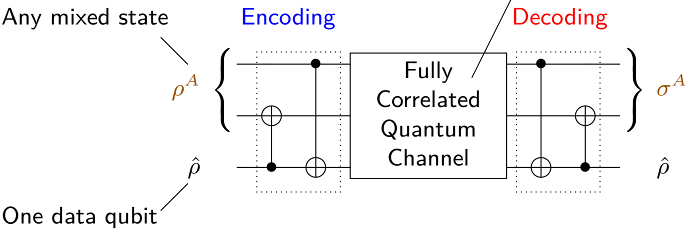
Three Qubit Fully Correlated Quantum Channel (NS)

$$\rho \mapsto \sum_{j=1}^{4} p_j F_j \rho F_j^\dagger \qquad F_j \in \{I^{\otimes n}, X^{\otimes n}, Y^{\otimes n}, Z^{\otimes n}\}$$



- The scheme is implemented **experimentally** by making use of a three-qubit NMR quantum computer with mixed states as ancilla states.

[Kondo, Bagnasco, Nakahara, PRA 88:022314 (2013)]

Three Qubit General Fully Correlated Quantum Channel (NS)

$$\rho \mapsto \sum_{j=1}^{r} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{U^{\otimes n} : U \text{ is } 2 \times 2 \text{ unitary}\right\} = SU(2)^{\otimes n}$$

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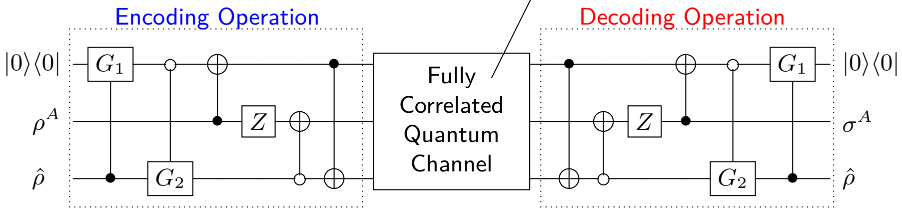

[Li, Nakahara, Poon, S., Tomita, PRA 84:044301 (2011)]

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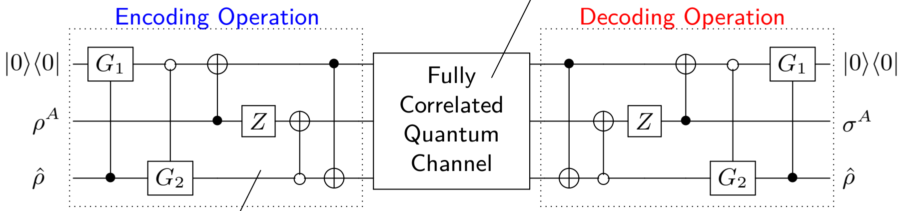

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-2}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

$$G_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

[Li, Nakahara, Poon, S., Tomita, PRA 84:044301 (2011)]

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Fully Correlated Quantum Channel

$$\mathcal{E} : \rho \mapsto \sum_{j=1}^{r} p_j F_j \rho F_j^\dagger \qquad F_j \in \left\{ U^{\otimes n} : U \text{ is } 2 \times 2 \text{ unitary} \right\} = SU(2)^{\otimes n}.$$

Fully Correlated Quantum Channel

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Notice that $SU(2)^{\otimes n}$ admits the unique decomposition into irreducible representations up to unitary similarity as

$$\bigotimes_{j=0}^{[n/2]} M_{n_j} \otimes I_{r_j} \quad \text{with} \quad \sum_{j=0}^{[n/2]} r_j n_j = n, \quad \text{where}$$

$$(r_0, n_0) = (1, n+1) \quad \text{and} \quad (r_j, n_j) = \left({}_nC_j - {}_nC_{j-1},\ n+1-2j \right), \quad j > 0.$$

Write

$$SU(2)^{\otimes n} = U_n^\dagger \left(M_{n_k} \otimes I_{n_k} \oplus K \right) U_n \quad \text{with} \quad K = \bigotimes_{j \neq k} M_{n_j} \otimes I_{r_j}.$$

Then for any $\hat{\rho} \in M_{r_k}$, there is a density matrix $\sigma \in M_{n_k}$ such that

$$\mathcal{E}\left(U_n \left(|0\rangle\langle 0| \otimes \tilde{\rho} \oplus 0_K \right) U_n^\dagger \right) = U_n \left(\sigma \otimes \tilde{\rho} \oplus 0_K \right) U_n^\dagger.$$

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Fully Correlated Quantum Channel

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General Fully Correlated Channel Li, Nakahara, Poon, S., Tomita, PRA 84:044301, 2011]

An n-qubit general fully correlated quantum channel has a r_k-dimensional NS. Hence, Furthermore, it can encode at most $[\log_2(r_k)]$ qubits.

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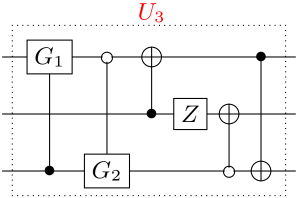
Fully Correlated Quantum Channel

Total Qubit	Dimension of NS	Data Qubit	Total Qubit	Dimension of NS	Data Qubit
2	1	0	19	16796	14
3	2	1	20	16796	14
4	2	1	21	58786	15
5	5	2	22	58786	15
6	5	2	23	208012	17
7	14	3	24	208012	17
8	14	3	25	742900	19
9	42	5	26	742900	19
10	42	5	27	2674440	21
11	132	7	28	2674440	21
12	132	7	29	9694845	23
13	429	8	30	9694845	23
14	429	8	31	35357670	25
15	1430	10	32	35357670	25
16	1430	10	33	129644790	26
17	4862	12	34	129644790	26
18	4862	12	35	477638700	28

Fully Correlated Quantum Channel



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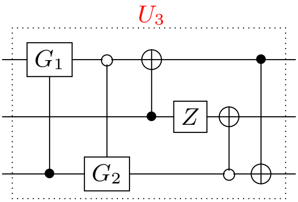
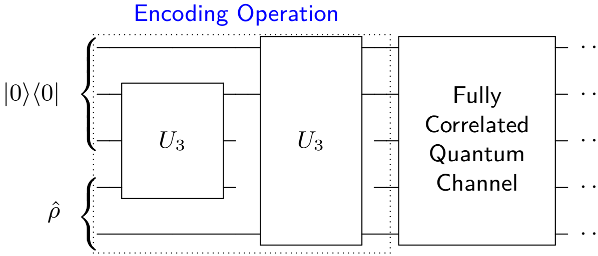
Fully Correlated Quantum Channel





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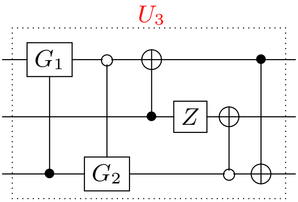
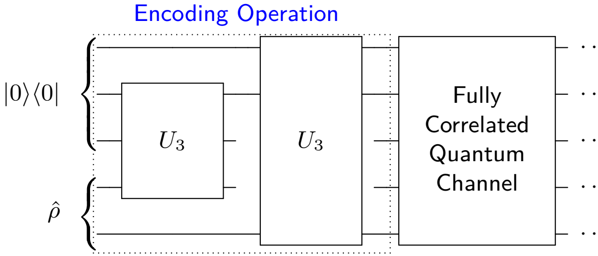
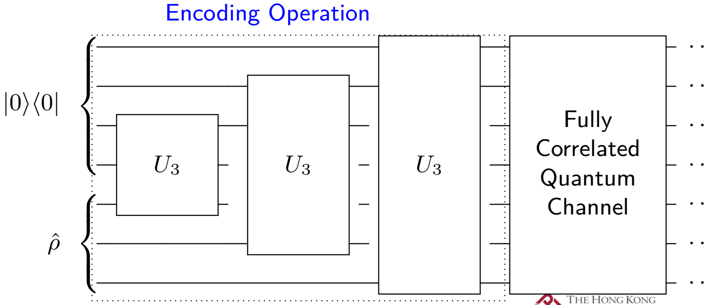
Fully Correlated Quantum Channel







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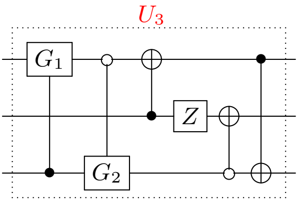
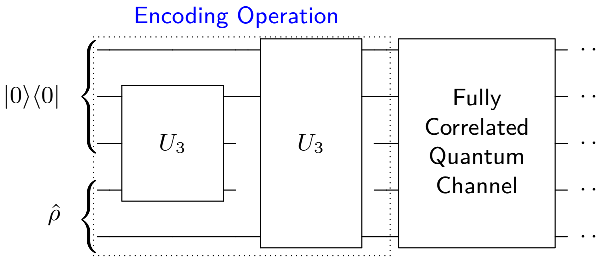
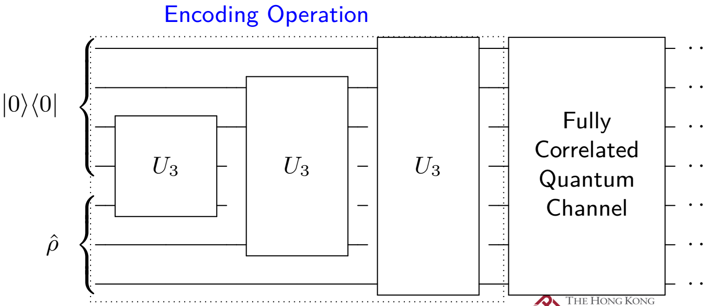
Fully Correlated Quantum Channel







This gives a recursive way to construct circuits that can correct $\frac{1}{2}(n-1)$ qubits in a n qubit system for odd integer n. However, $\frac{1}{2}(n-1) < \lceil \log_2(r_{[n/2]}) \rceil$ in general.

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Summary

- Quantum error correction is one of the strategies to flight against decoherence in quantum system. There are different error correction models including decoherence-free subspace, noiseless subsystem, quantum error correction code, and operator quantum error correction.
- These schemes are applied to fully correlated noise. Encoding and decoding circuits of these schemes are constructed too.
- Implementation for the $[5,1,3]$ code and $[8,3,3]$ code is presented. It will be of interested to investigate the encoding and decoding circuits for $[n,k,3]$ code and even in general $[n,k,d]$ code.
- Currently, we are working on $[10,4,3]$ code and $[11,1,5]$ code.

  Difficulty:
  - $[10,4,3]$ code: 16 dimension QECC.
  - $[11,1,5]$ code: 529 different error operators E_j.

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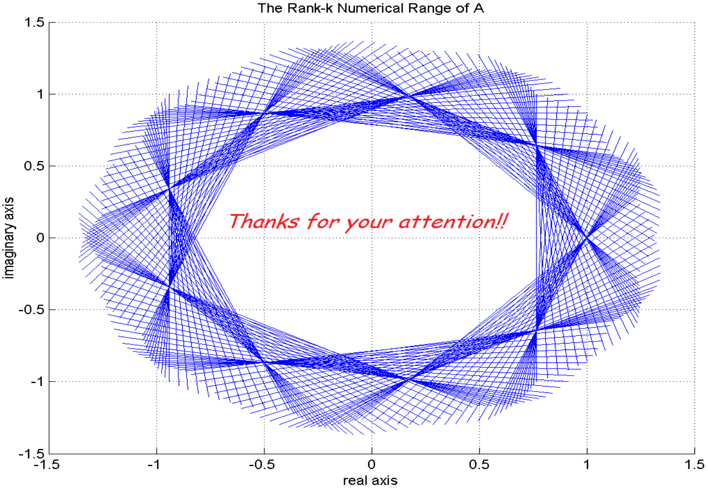

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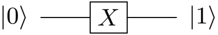
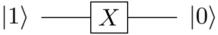
- A NOT gate acting on one qubit:

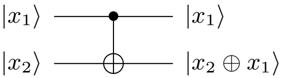
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

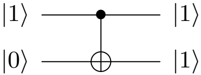


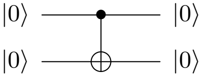


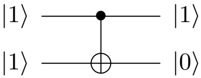
- A controlled-NOT (CNOT) gate acting on 2 qubits:

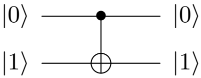
$$\begin{aligned} U_{\text{CNOT}} &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$







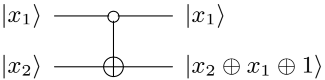


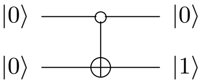


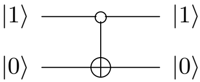
Quantum Gate

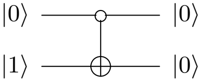
- An negative controlled-NOT (CNOT) gate acting on 2 qubits:

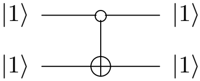
$$U = |0\rangle\langle 0| \otimes X + |1\rangle\langle 1| \otimes I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

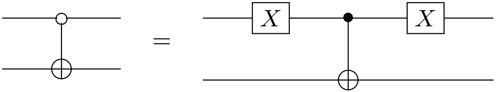












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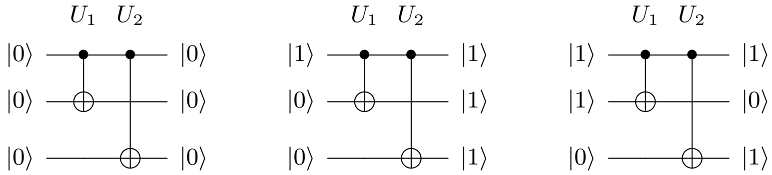
Quantum Gate

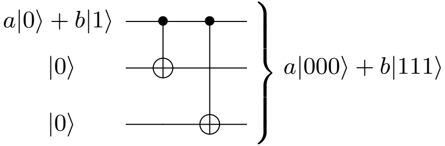


The corresponding unitary operators are

$$U_1 = |0\rangle\langle 0| \otimes I_2 \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_x \otimes I_2$$

$$U_2 = |0\rangle\langle 0| \otimes I_2 \otimes I_2 + |1\rangle\langle 1| \otimes I_2 \otimes \sigma_x .$$

In general,



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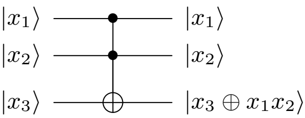
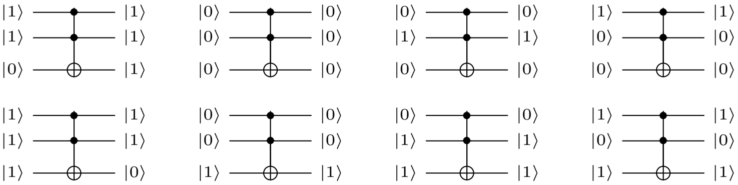
- An controlled controlled-NOT (CCNOT) (Toffolie) gate acting on 3 qubits:

$$U_{\mathrm{CCNOT}} = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I + |11\rangle\langle 11| \otimes X$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$





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Quantum Gate

- A controlled-Unitary gate acting on 2 qubits:

$$U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w_{11} & w_{12} \\ 0 & 0 & w_{21} & w_{22} \end{bmatrix}$$

with unitary $W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$.