Stability theorem of depolarizing channel for output quantum Rényi entropy

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Outline

- Introduction
  - Definitions
  - Maximal output purity
  - Stability theorem of depolarizing channel for maximal output purity & its application
- Our result
  - E. Bae, G. Gour, **SL**, J. Park & B.C. Sanders, J. Phys. A: Math. Theor. **49** (2016) 115304.
  - Applications
    - A polygraph test for pure product states
    - A generalized product test: p-copy product test

INTRODUCTION

Definitions

- D_λ: d-dimensional depolarizing channel with noise rate $1-\lambda$
  - $D_\lambda(\rho) = \lambda \rho + (1-\lambda) I/d, \quad 0 \le \lambda \le 1$
- S_p: quantum Rényi entropy of order $p \ge 0$
  - $S_p(\rho) = \log \mathrm{Tr}\, \rho^p/(1-p)$
- S_p^{min}: minimal output Rényi entropy
  - $S_p^{min}(D_\lambda^{\otimes n}) = \min_\rho S_p(D_\lambda^{\otimes n}(\rho))$, ρ: an n-qudit state
- P^{max}: maximal output purity
  - $P^{max}(D_\lambda^{\otimes n}) = \max_\rho \mathrm{Tr}(D_\lambda^{\otimes n}(\rho)^2)$, ρ: an n-qudit state

Maximal output purity

- P^{max}: maximal output purity
  - $P^{max}(D_\lambda^{\otimes n}) = \max_\rho Tr(D_\lambda^{\otimes n}(\rho)^2)$, ρ: an n-qudit state
- The maximal output purity is obtained only for pure product states.
  - $P^{max}(D_\lambda^{\otimes n}) = Tr(D_\lambda^{\otimes n}(|\varphi\rangle\langle\varphi|)^2)$ for any n-qudit product state $|\varphi\rangle = |\varphi_1\rangle|\varphi_2\rangle\ldots|\varphi_n\rangle$
- Its stronger version is called the stability theorem of depolarizing channel for output purity.
  - If $P^{max}(D_\lambda^{\otimes n}) \approx Tr(D_\lambda^{\otimes n}(|\psi\rangle\langle\psi|)^2)$ then $|\psi\rangle \approx |\varphi\rangle$ for some product $|\varphi\rangle$.

Stability theorem

Theorem (A. W. Harrow and A. Montanaro (2013))

Given $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, *let* $1-\epsilon = \max\left\{ |\langle\psi|\phi_1,\cdots,\phi_n\rangle|^2 : |\phi_i\rangle \in \mathbb{C}^d \right\}$.

Then $\operatorname{Tr}\left(\mathcal{D}_\lambda^{\otimes n}|\psi\rangle\langle\psi|\right)^2 \leq$

$$\mathrm{OPP}(\lambda)\left(1-4\epsilon(1-\epsilon)\left(\frac{d\lambda^2(1-\lambda^2)}{(1+(d-1)\lambda^2)^2}\right)+4\epsilon^{3/2}\left(\frac{(1-\lambda^2)^2+d^2\lambda^4}{(1+(d-1)\lambda^2)^2}\right)^2\right),$$

where $\mathrm{OPP}(\lambda) \equiv \operatorname{Tr}\left(\mathcal{D}_\lambda^{\otimes n}|\phi\rangle\langle\phi|\right)^2$ *is called the Output Purity of Product states for an arbitrary product state* $|\phi\rangle = |\phi_1,\cdots,\phi_n\rangle$.

- If $\mathrm{OPP}(\lambda) = P^{max}(D_\lambda^{\otimes n}) \approx \mathrm{Tr}(D_\lambda^{\otimes n}(|\psi\rangle\langle\psi|)^2)$ then $|\psi\rangle \approx |\varphi\rangle$ for some product $|\varphi\rangle$.

Application

Figure: Swap test

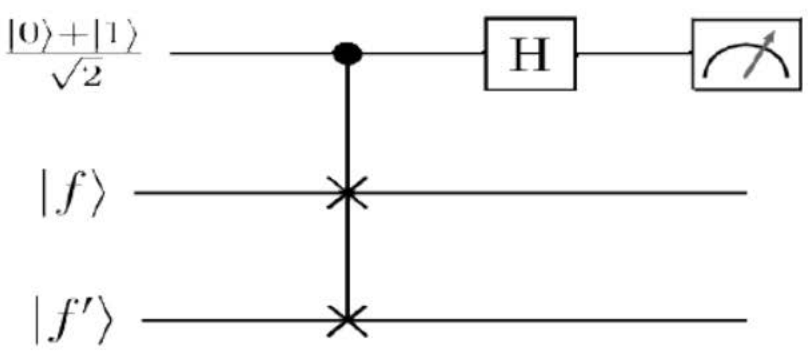
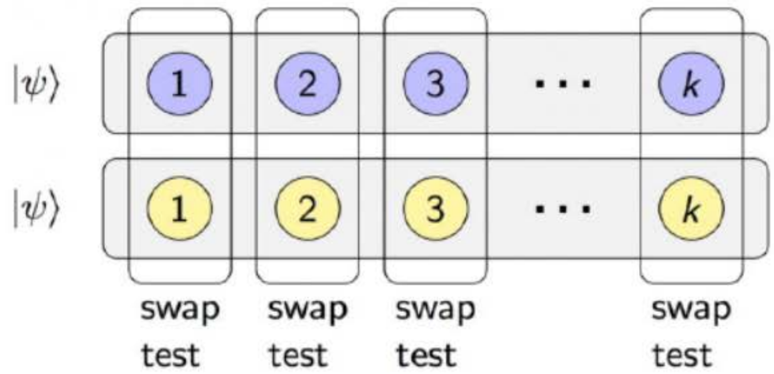


Figure: Product test



- A. W. Harrow & A. Montanaro, Journal of the ACM, vol. 60 no. 1 (2013)
- It was shown that *the complexity class QMA(k) has the same complexity as QMA(2) for k>2*, by using the 2-copy product test.

Motivation

- A. W. Harrow & A. Montanaro, Journal of the ACM, vol. 60 no. 1 (2013)
- A. W. Harrow & A. Montanaro raised an open question
  - Does a (similar) stability theorem hold for the depolarizing channel with respect to output quantum Rényi entropies?
- Our work
  - We prove a stability theorem for the depolarizing channel with respect to minimal output p-Rényi entropies for $p \geq 2$ (and $p = 1$).
  - Furthermore, we discuss two applications.

OUR RESULTS:

E. BAE, G. GOUR, SL, J. PARK & B.C. SANDERS

Minimal output Rényi entropy

- S_p^{min}: minimal output Rényi entropy
  - $S_p^{min}(D_\lambda^{\otimes n}) = \min_\rho S_p(D_\lambda^{\otimes n}(\rho))$, ρ: an n-qudit state
- The minimal output Rényi entropy is obtained only for pure product states.
  - $S_p^{min}(D_\lambda^{\otimes n}) = S_p(D_\lambda^{\otimes n}(|\varphi\rangle\langle\varphi|))$
    for any n-qudit product state $|\varphi\rangle = |\varphi_1\rangle|\varphi_2\rangle\ldots|\varphi_n\rangle$
- Its stronger version is called the stability theorem of depolarizing channel for output Rényi entropy.
  - If $S_p^{min}(D_\lambda^{\otimes n}) \approx S_p(D_\lambda^{\otimes n}(|\psi\rangle\langle\psi|))$ then $|\psi\rangle \approx |\varphi\rangle$ for some product $|\varphi\rangle$.

Our stability theorem

Theorem (Our stability theorem)

Let $p \geq 2$, $\epsilon > 0$, $r = \frac{1+(d-1)\lambda}{1-\lambda}$ *and* $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$. *Then*

1. $S_p\left(\mathcal{D}_\lambda^{\otimes n}|\psi\rangle\langle\psi|\right) \leq S_p^{min}\left(\mathcal{D}_\lambda^{\otimes n}\right)$

$$+\epsilon\frac{2p(r-1)(r^{p-1}-1)(2r^p+dr+d-2)}{(p-1)(r+1)(r^p+d-1)^2} + O(\epsilon^{3/2})$$

holds only if a pure product state $|\phi\rangle$ *exists such that* $|\psi\rangle$ *satisfies*

$$|\langle\psi|\phi\rangle|^2 \geq 1-\epsilon.$$

2. $S_p\left(\mathcal{D}_\lambda^{\otimes n}|\psi\rangle\langle\psi|\right) \geq S_p^{min}\left(\mathcal{D}_\lambda^{\otimes n}\right) + \epsilon\frac{p}{p-1} + O(\epsilon^{3/2})$ *implies that*

$$|\langle\psi|\phi\rangle|^2 < 1-\epsilon$$

for any product state $|\phi\rangle$.

Stability theorem

Theorem (Our stability theorem)

Let $p \geq 2$, $\epsilon > 0$, $r = \frac{1+(d-1)\lambda}{1-\lambda}$ *and* $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$. *Then*

1. $S_p\left(\mathcal{D}_\lambda^{\otimes n}|\psi\rangle\langle\psi|\right) \leq S_p^{min}\left(\mathcal{D}_\lambda^{\otimes n}\right)$

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2. $S_p\left(\mathcal{D}_\lambda^{\otimes n}|\psi\rangle\langle\psi|\right) \geq S_p^{min}\left(\mathcal{D}_\lambda^{\otimes n}\right) + \epsilon\frac{p}{p-1} + O(\epsilon^{3/2})$ *implies that*

$$|\langle\psi|\phi\rangle|^2 < 1-\epsilon$$

for any product state $|\phi\rangle$.

- If $S_p^{min}(D_\lambda^{\otimes n}) \approx S_p(D_\lambda^{\otimes n}(|\psi\rangle\langle\psi|))$ then $\exists$ product $|\varphi\rangle$ such that $|\psi\rangle \approx |\varphi\rangle$.
- If $\exists$ product $|\varphi\rangle$ s. t. $|\psi\rangle \approx |\varphi\rangle$ then $S_p^{min}(D_\lambda^{\otimes n}) \approx S_p(D_\lambda^{\otimes n}(|\psi\rangle\langle\psi|))$.
- The original stability theorem is essentially equivalent to our stability theorem for the case when $p = 2$.

Application 1: A polygraph test

- Polygraph test for pure product states
  - Sender Alice transmits an n-qudit state through n-copies of the depolarizing channel to receiver Bob.
  - Bob measures the output states in order to estimate its p-Rényi entropies.
  - Bob wants to know whether or not Alice prepared a state close to a pure product state.

Application 1: A polygraph test

- Alice's task
  - Prepare an n-qudit state that is close to a product state (within some ε specified by Bob).
- Bob's task
  - Test whether Alice's preparation satisfies the requirements. If the first inequality in our stability theorem holds, Bob answers "True". If the second inequality holds in our theorem, Bob answers "False".

Application 1: A polygraph test

- There is some gap between the right hand sides of the first and second inequalities, that is, Bob cannot detect Alice's lie for some cases.
- The gap, gap(p), is close to zero when Bob chooses small enough ε.
- When ε is fixed, Bob will get $gap(p) < gap(2)$ for some $p > 2$.
  - $\lim_{p\to\infty} gap(p) < gap(2)$

Application 2: a product test

- A. W. Harrow's comment
  - A generalized product test: p-copy product test
  - If $p>2$ is a prime then p-copy product test is better than 2-copy product test.
- Not yet proved.

Conclusion

- Stability theorem of depolarizing channel for minimal output p-Rényi entropies ($p \geq 2$ or $p = 1$).
- Applications
  - A polygraph test for pure product states
  - A generalized product test: p-copy product test?
- Future works
  - For $1 < p < 2$?
  - Stability theorem of channels satisfying additivity?
  - Other applications?

Thank you for your attention